Submodular Function Optimization - An Overview

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April 23, 2013

Submodularity

Definition (submodular function)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A, B \subseteq V$, we have that:

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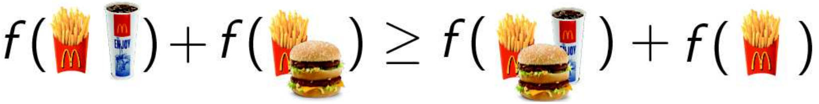

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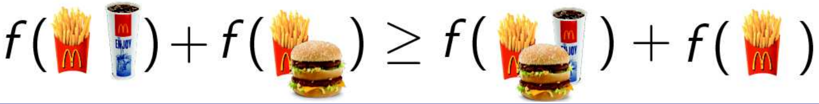

Definition (diminishing returns)

A function $f : 2^V \to \mathbb{R}$ is submodular if for any $A \subseteq B \subset V$, and $v \in V \setminus B$, we have that:

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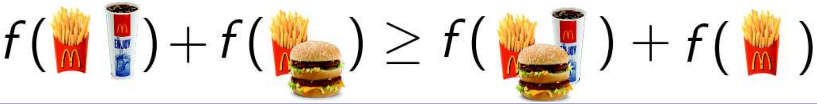

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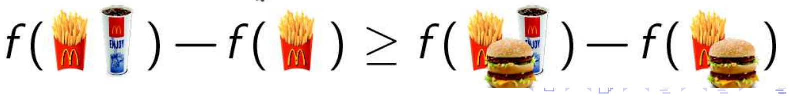

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- ρ-Approximate algorithms:
  - $OPT \leq f(x) \leq \rho OPT$, if $\rho > 1$.
  - $\rho OPT \leq f(x) \leq OPT$, if $\rho < 1$.

Graph Cuts

- MINIMUM CUT : Given a graph $G = (V, E)$, find a set of vertices $S \subseteq V$ that minimizes the cut (set of edges) between S and $V \setminus S$.
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- Weighted versions.
- Eg :- Segmentation in Computer Vision.

Image Segmentation

- An image needing to be segmented.

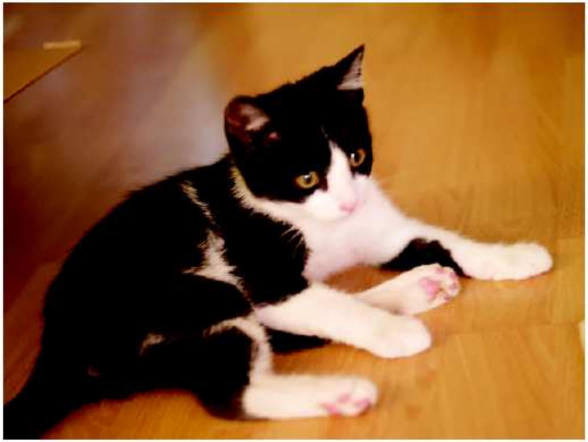


Image Segmentation

- User marks foreground(red) and background(blue).

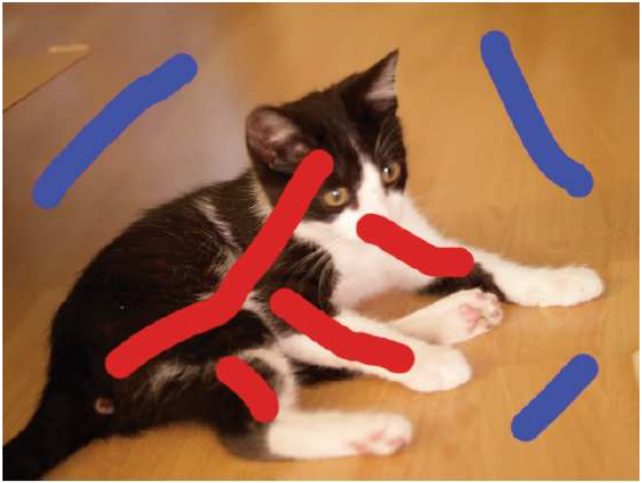
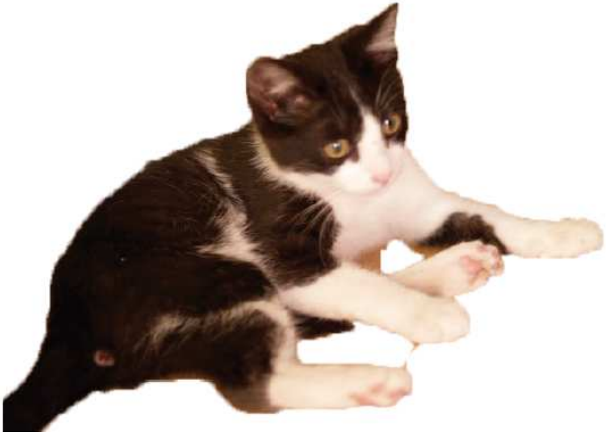


Image Segmentation

- Goal.

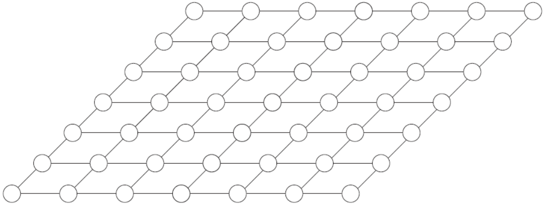


Markov random fields and image segmentation

Markov random field

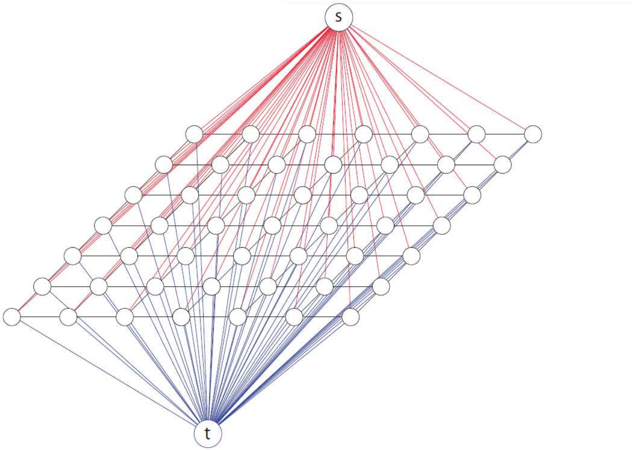
$$\log p(x) \propto \sum_{v \in V(G)} e_v(x_v) + \sum_{(i,j) \in E(G)} e_{ij}(x_i, x_j)$$

where G is a 2D grid graph, we have



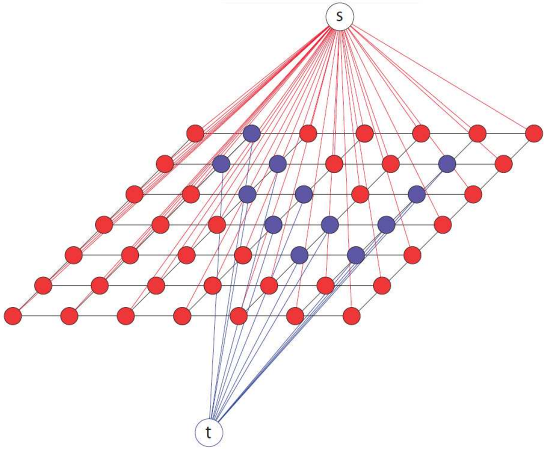
Markov random fields and image segmentation

Augmented graph-cut graph. The edge weights of graph are derived from $\{e_v\}_{v \in V}$ and $\{e_{ij}\}_{(i,j) \in E(G)}$.



Markov random fields and image segmentation

Augmented graph-cut graph with indicated cut corresponding to particular vector $\bar{x} \in \{0,1\}^n$. Each cut $\bar{x}$ has a score corresponding to $p(\bar{x})$.



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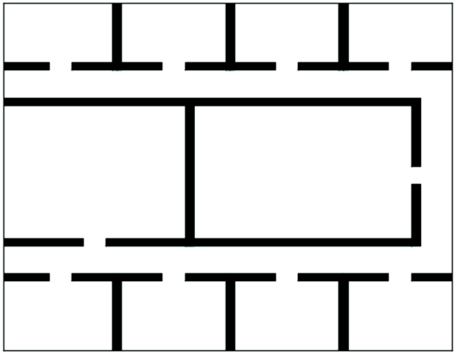
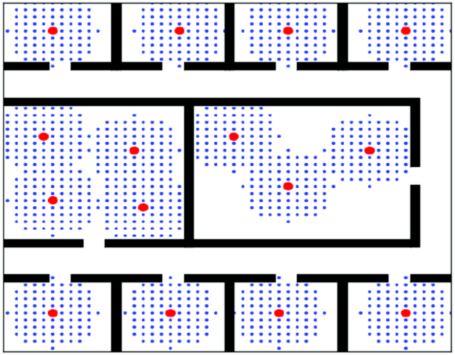
Sensor Placement in Buildings

- An example of a room layout.



Sensor Placement in Buildings

- Small range sensors.



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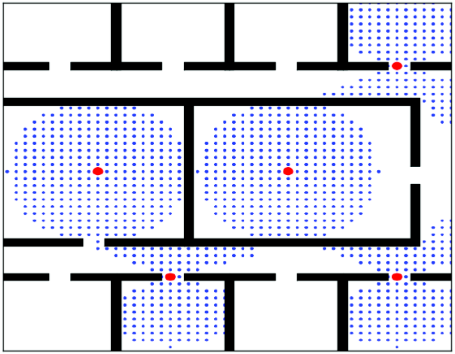
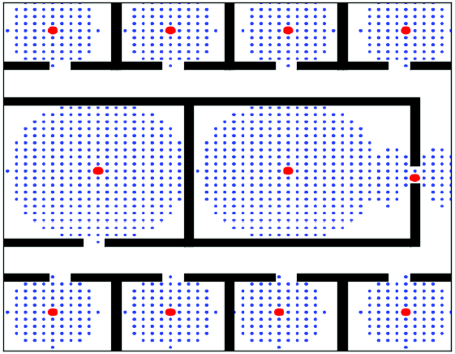


Sensor Placement in Buildings

- Sensors with mixed ranges.



A model of Influence in Social Networks

- Given a graph $G = (V, E)$, each $v \in V$ corresponds to a person, to each v we have an activation function $f_v : 2^V \rightarrow [0, 1]$ dependent only on its neighbours, i.e, $f_v(A) = f_v(A \cup \Gamma(v))$.

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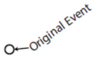
- Goal - Viral Marketing: find a small subset $S \subseteq V$ of individuals to direct influence, and thus indirectly influence the greatest number of possible other individuals (via the social network G).
- We define a function $f : 2^V \to \mathbb{Z}^+$ that models the ultimate influence of an initial set S of nodes based on the following iterative process: At each step, a given set of nodes S are activated, and we activate a new node $v \in V \setminus S$ if $f_v(S) \geq U[0, 1]$(where $U[0, 1]$ is a uniform random number between 0 and 1).

Modeling Information Cascade



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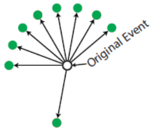

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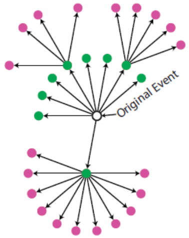

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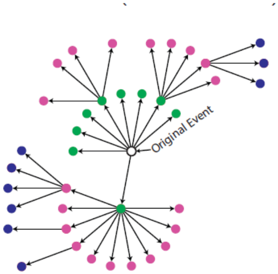

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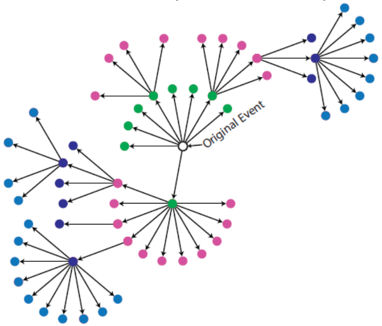

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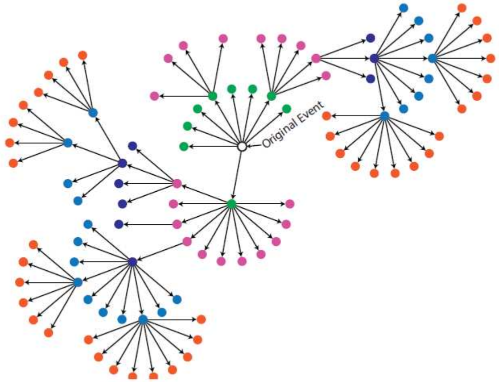

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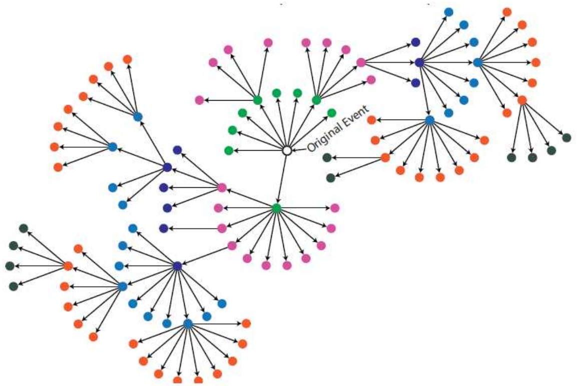

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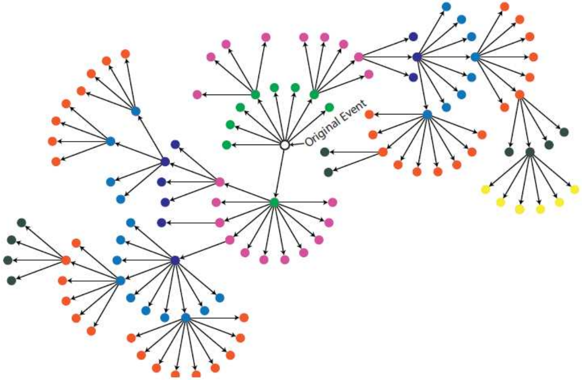

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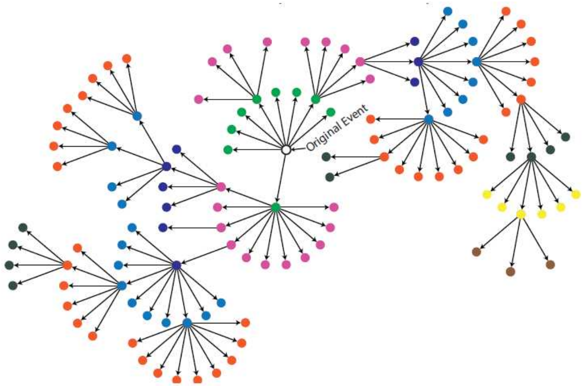

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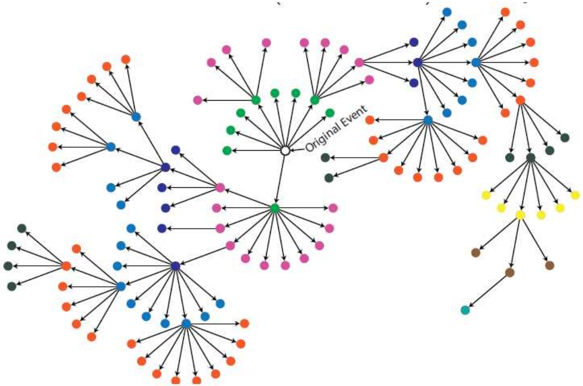

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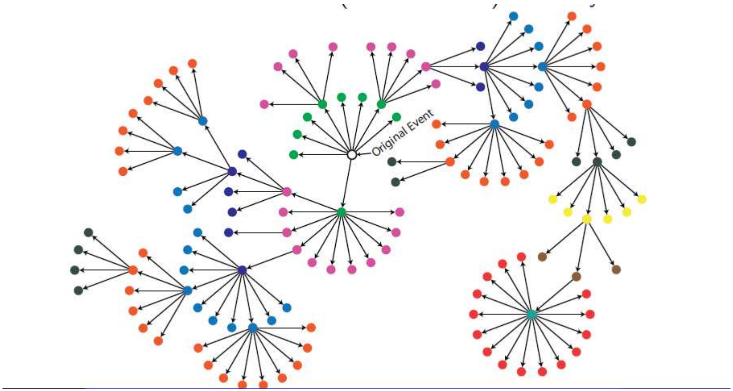
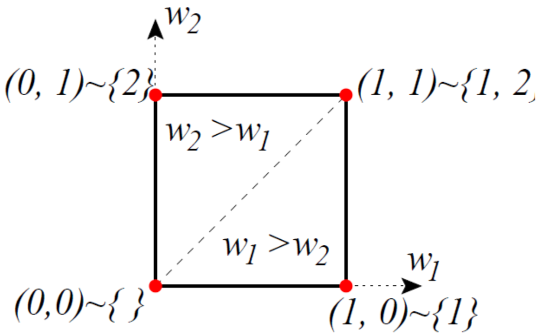


Submodular Minimization - Lovász extension



- Given any set function f and w such that $w_{j_1} \geq \ldots \geq w_{j_p}$, define:

$$\begin{aligned}\hat{f}(w) &= \sum_{k=1}^{p} w_{j_k}[f(\{j_1,\ldots,j_k\})] - f(\{j_1,\ldots,j_{k-1}\})] \\ &= \sum^{p-1}(w_{j_k} - w_{j_{k+1}})[f(\{j_1,\ldots,j_k\})] + w_{j_p} f(\{j_1,\ldots,j_p\})]\end{aligned}$$

Submodular Minimization

- if $w = 1_A, \hat{f}(w) = f(A) \implies$ extension from $\{0,1\}^p$ to $\mathbb{R}^p$
- $\hat{f}$ is piecewise affine and positively homogeneous
- f is submodular if and only if $\hat{f}$ is convex.
  - Minimizing $\hat{f}(w)$ on $w \in [0,1]^p$ is equivalent to minimizing f on 2^V.
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- Minimizing symmetric submodular functions.
    - A submodular function f is said to be symmetric if for all $B \subset V$, $f(V \setminus B) = f(B)$.
    - Example: undirected cuts, mutual information
    - Minimization in $O(p^3)$ over all non-trivial subsets of V, where $|V| = p$.

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- Maximizing non-decreasing submodular functions with cardinality constraint
  - A submodular function f is said to be non-decreasing if for all $A \subseteq B$, $f(A) \leq f(B)$.
  - $\max_{\substack{A \subset V \\ |A| \leq k}} f(A)$.
  - Greedy algorithm achieves $(1 - 1/e)$ of the optimal value.(Nemhauser et al., 1978).

Maximization with cardinality constraint

- Let $A^* = \{b_1, \dots, b_k\}$ be a maximizer of F with k elements, and a_j the j-th selected element. Let $\rho_j = F(\{a_1, \dots, a_j\}) - F(\{a_1, \dots, a_{j-1}\})$

$$\begin{aligned} f(A^*) &\leq f(A^* \cup A_{j-1}) \text{because } f \text{ is non-decreasing,} \\ &= f(A_{j-1}) + \sum_{i=1}^{k} [f(A_{j-1} \cup \{b_1, \dots, b_i\}) \\ &\qquad - f(A_{j-1} \cup \{b_1, \dots, b_{i-1}\})] \\ &\leq f(A_{j-1}) + \sum_{i=1}^{k} [f(A_{j-1} \cup \{b_i\} - f(A_{j-1})] \text{by submodularity,} \\ &\leq f(A_{j-1}) + k\rho_j \text{ by definition of the greedy algorithm,} \\ &= \sum_{i=1}^{j-1} \rho_i + k\rho_j \end{aligned}$$

- Minimize $\sum_{i=1}^{k} \rho_i : \rho_j = (k-1)^{j-1} k^{-j} f(A^*)$

Courtesy and References

- Course on "Submodular Functions" by Prof. Jeff Bilmes, University of Washington
    - *http://j.ee.washington.edu/ bilmes/classes/ee596a_fall_2012/*
- "Learning with Submodular Functions: A Convex Optimization Perspective" by Prof. Francis Bach, INRIA.
    - *http://hal.archives-ouvertes.fr/docs/00/64/52/71/PDF/submodular_fot.pdf*
    - *http://www.di.ens.fr/ fbach/submodular_fbach_mlss2012.pdf*
- Tutorials by Prof. Andreas Krause, ETH, Zurich.
    - *http: submodularity.org*

Thank You. Questions?