As easy as a piece of cake

Analytically cutting infinite cakes (yum!)

Baptiste Plaquevent-Jourdain, with
Jean-Pierre Dussault, Université de Sherbrooke
Jean Charles Gilbert, INRIA Paris

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Outline

Plan

Who are you listening to? (1)

origin

French PhD student from Brittany (sea, crêpes, galettes, Mont Saint-Michel...), then from ENSTA Paris

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Who are you listening to? (2)

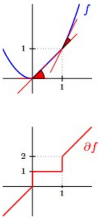
current status

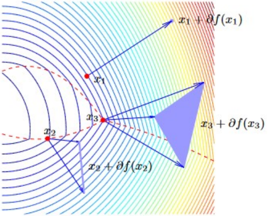
- starting 3rd year, finishing on December, 31st (unless...)
- "cotutelle" France-Québec, here during winter

Who are you listening to? (3)

My (first) subject

Initially doing nonsmooth optimization (theoretically)...





(Fragments d'Optimisation Différentiable - Théorie et Algorithmes)

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... but today: ~~computational/combinatorial geometry~~ cakes!

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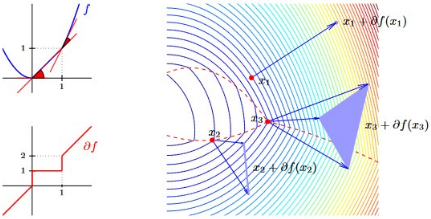

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Plan

1. My Personal Recipe
2. First part(s of the cake)
3. Formalism
4. An algorithm
5. Some improvements

Cutting cakes rules

main rule

cut:= line that completely cut the cake (no stopping in the middle)

second rule

We also assume the cakes are infinite (see later).

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A first taste - 1

One cut, 2 slices

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One cut, 2 slices

Two cuts, 4 slices

A first taste - 2

Three cuts, 6 slices

$$p \text{ cuts}, 2p \text{ slices}$$

'Proof': every cut makes 2 previous slices becoming 4 smaller slices
$2p \to (2p-2) + 2*2 = (2p-2) + 4 = 2(p+1)$.

A first taste - 2

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us around the pizzas

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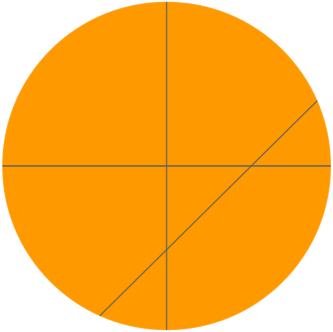
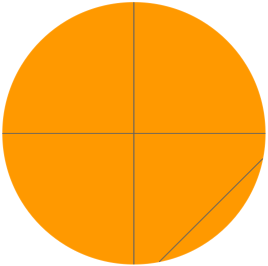
What about 7 parts ?



Asymmetric cuts - they don't all pass by the center/middle

Other possibilities - 2



Acually can't (really) have 5 slices: this is cheating. This does not respect the infinite cakes assumption.

But the 7-slices one still works: the $2p$ formula isn't valid...

Other possibilities - 3

Is it possible to get 8 slices in three cuts?

Other possibilities - 3

Summary

- symmetric cuts in 2D (all by the center): p cuts $\Rightarrow 2p$ slices
- cutting in a "new dimension" doubles ; 2^n slices!
- asymmetric cuts: it's harder

But what about a cake-shaped cake?

So here, p cuts mean $p+1$ slices... because they're all parallel!

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Parallel sets in each dimension

But parallel set of cuts in each dimension also work:

$$p_1, p_2 \to (p_1 + 1) \times (p_2 + 1)$$

(you can check the slices after the pizzas :3)

Conclusion

So maybe not completely a piece of cake...
Depends on: dimension n, number of cuts p, and which cuts.

Observations: new dimension means doubling the cuts, parallel cuts behave weirdly, 5 slices is hard to get...

Question

For a given set of cuts, how many slices do we get?

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Hyperplanes - 1

The cake n-dimensional, a 'cut' is an hyperplane.
$=$ linear (affine) subspace of dimension $n-1$ (codimension 1).
One hyperplane: $H = v^{\perp} = \{d \in \mathbb{R}^n : v^\top d = 0\}$.

p cuts: p hyperplanes: $H_i = v_i^{\perp}, \forall\ i \in [1:p]$, $(v_i)_i =$ problem data.

halfspaces of an hyperplane

$$\mathbb{R}^n = H_i^- \cup H_i \cup H_i^+, \qquad \begin{array}{l} H_i^- = \{d \in \mathbb{R}^n : v_i^\top d < 0\} \\ H_i^+ = \{d \in \mathbb{R}^n : v_i^\top d > 0\} \end{array}$$

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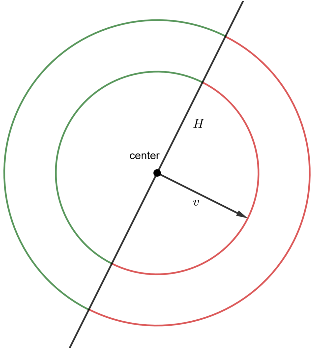
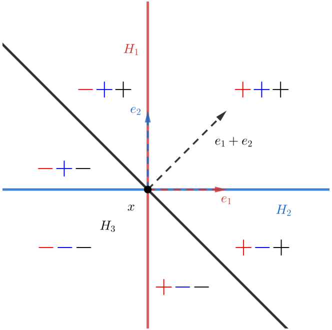
Hyperplanes - 2



Each cut: a − and a + side: each of the p cuts, intersection of each halfspaces...

Illustration



$$H_1 = e_1^{\perp},\ H_2 = e_2^{\perp},\ H_3 = (e_1 + e_2)^{\perp}.$$

Actually, # of slices and on which side of each cut it is.

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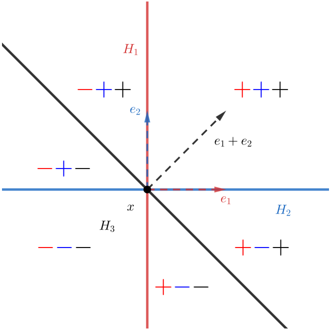

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Technical formalism

There are p cuts, 2^p potential slices $(\forall\ i \in [1:p], \{-1,+1\})$
Slice $s = (s_1, \dots, s_p) \in \{\pm 1\}^p$ exists $\Leftrightarrow H_1^{s_1} \cap H_2^{s_2} \cap \cdots \cap H_p^{s_p} \neq \varnothing$

$$\begin{cases} H_i^+ : v_i^\top d > 0 \Leftrightarrow +v_i^\top d > 0 \\ H_i^- : v_i^\top d < 0 \Leftrightarrow -v_i^\top d > 0 \end{cases} \Leftrightarrow s_i v_i^\top d > 0$$

slice s non-empty $\Leftrightarrow d_s \in$ slice $s \Leftrightarrow \forall\ i \in [1:p], s_i(v_i^\top d_s) > 0$
Verifying p linear equations = very simple...

But there are 2^p such systems.
Thus the interest of designing non-brute force algorithm.

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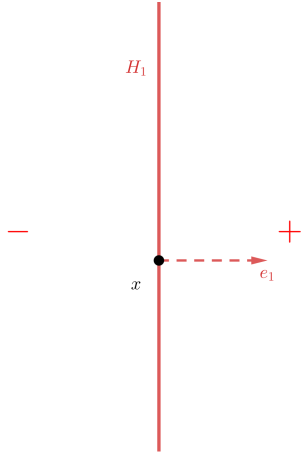
Main reasoning

Algorithm from [RČ18]:

- recursive binary tree that adds hyperplanes one at a time
- each node has descendant(s) $(s, +1)$ and/or $(s, -1)$
- checking one or two = main computational effort

Illustration of the regions and tree on the previous example



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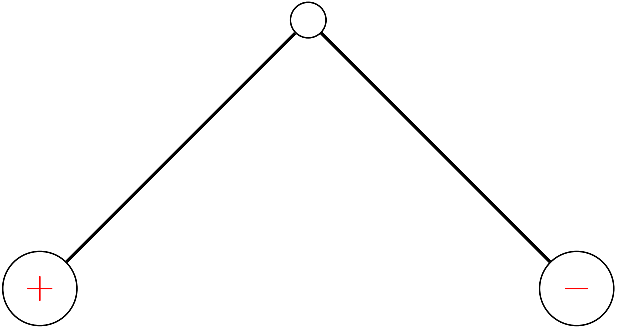

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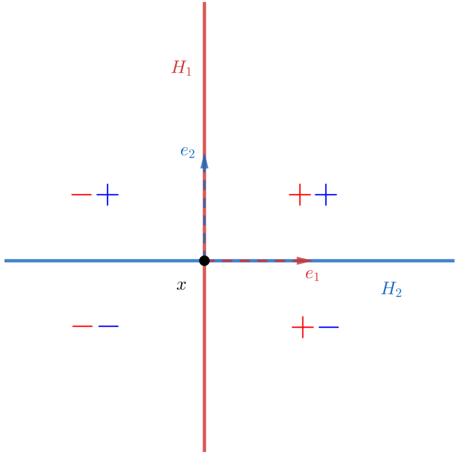

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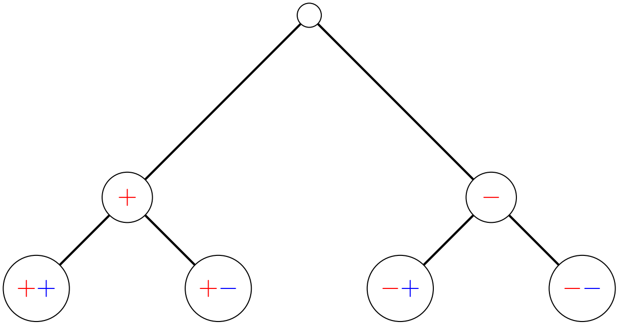

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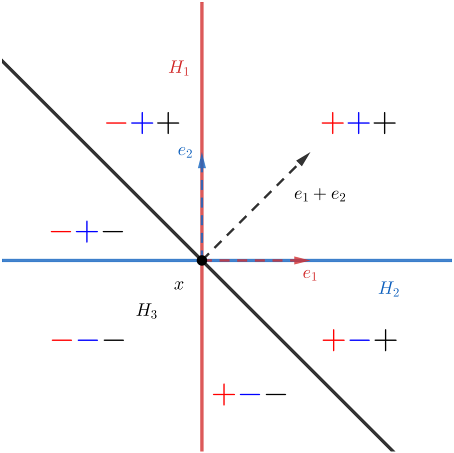

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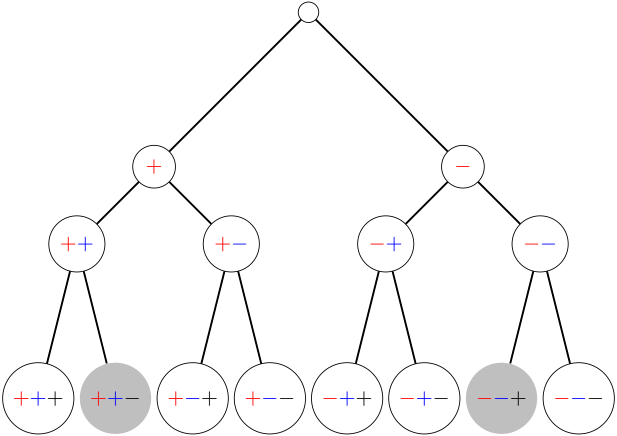

Important property

At level $k < p$, for a slice $s \in \{\pm 1\}^k$,

$$\forall\, i \in [1:k], \exists\, d_s, s_i v_i^\top d_s > 0 \Rightarrow \begin{cases} \forall\, i \in [1:k], s_i v_i^\top d > 0 \\ \qquad +v_{k+1}^\top d > 0 & ? \\ \forall\, i \in [1:k], s_i v_i^\top d > 0 \\ \qquad -v_{k+1}^\top d > 0 & ? \end{cases}$$

If $v_{k+1}^\top d_s > 0, (s, +1)$ verified with the same d_s (if $< 0, (s, -1)$ is).
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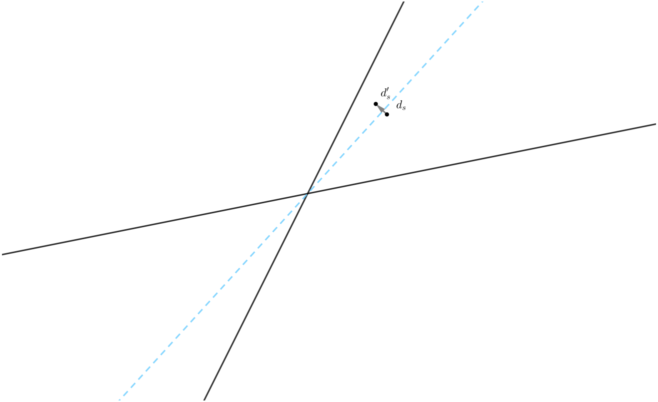

The point is "very close" to the new hyperplane, a small simple modification suffices.

Plan

Reducing the node count

So $|v_{k+1}^{\mathsf{T}} d_s|$ small $\Rightarrow$ probably 2 descendants.

idea: contrapositive

$|v_{k+1}^{\mathsf{T}} d_s|$ 'large' $\rightarrow$ less chance of both $(s, +1)$ and $(s, -1)$.

Only a heuristic, but reasonably efficient.
Also, this order change is local - for each s it can change.

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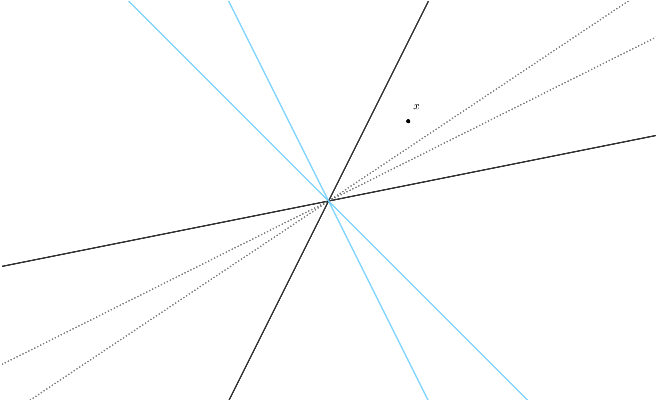
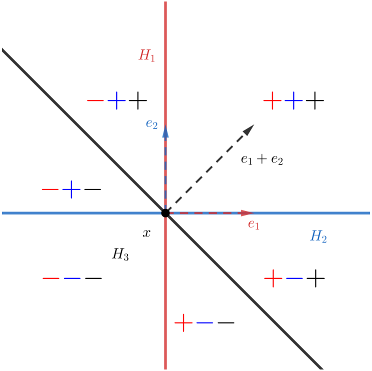


Black: hyperplanes already treated, x is the current point/region. Dotted and blue: remaining hyperplanes. Here, the blue hyperplanes are "far" from the point, so it's more likely there is only 1 descendant (thus less nodes and a faster algorithm).

Infeasibility, matroids and circuits - 1



$++-$ (and $--+$) corresponds to an empty region: $+$ means right to H_1, $+$ over H_2, $-$ down left H_3: such a point does not exist. The system is $+ : d_1 > 0, + : d_2 > 0, - : -d_1 - d_2 > 0$

Infeasibility, matroids and circuits - 2

With $p > 3$, $++-?\,?\ldots?\,?$ always infeasible, whatever the remaining signs are.

Idea

can be formalized through a (technical) ~~recipe~~ theorem

- before the tree, compute every "infeasible" combination
- linear optimization ($\simeq$ black-box) $\rightarrow$ linear algebra (nice!)
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- The RC algorithm
- some improvements on the tree structure
- some improvements with duality (the linear algebra)
- best : using a little bit (using it cleverly)

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Results; blue = times, black = time RC / time variant

Name	RC	ABC		ABCD2		ABCD3		AD4	
R-4-8-2	$1.70\ 10^{-2}$	$7.20\ 10^{-3}$	2.36	$6.53\ 10^{-3}$	2.60	$3.13\ 10^{-3}$	**5.43**	$8.03\ 10^{-3}$	2.12
R-7-8-4	$5.70\ 10^{-2}$	$3.38\ 10^{-2}$	1.69	$3.15\ 10^{-2}$	1.81	$2.24\ 10^{-2}$	**2.54**	$2.79\ 10^{-2}$	2.04
R-7-9-4	$9.97\ 10^{-2}$	$4.98\ 10^{-2}$	2.00	$4.96\ 10^{-2}$	2.01	$3.43\ 10^{-2}$	**2.91**	$5.16\ 10^{-2}$	1.93
R-7-10-5	$2.33\ 10^{-1}$	$1.16\ 10^{-1}$	2.01	$1.29\ 10^{-1}$	1.81	$1.05\ 10^{-1}$	**2.22**	$1.22\ 10^{-1}$	1.91
R-7-11-4	$2.36\ 10^{-1}$	$1.22\ 10^{-1}$	1.93	$1.20\ 10^{-1}$	1.97	$8.49\ 10^{-2}$	**2.78**	$1.32\ 10^{-1}$	1.79
R-7-12-6	$9.35\ 10^{-1}$	$5.05\ 10^{-1}$	**1.85**	$5.74\ 10^{-1}$	1.63	$5.13\ 10^{-1}$	1.82	$5.65\ 10^{-1}$	1.65
R-7-13-5	$9.11\ 10^{-1}$	$4.70\ 10^{-1}$	**1.94**	$5.41\ 10^{-1}$	1.68	$4.71\ 10^{-1}$	1.93	$5.33\ 10^{-1}$	1.71
R-7-14-7	3.69	2.15	**1.72**	2.39	1.54	2.42	1.52	2.42	1.52
R-8-15-7	6.43	3.56	**1.81**	3.92	1.64	4.30	1.50	4.57	1.41
R-9-16-8	$1.51\ 10^{+1}$	8.88	**1.70**	$1.03\ 10^{+1}$	1.47	$1.34\ 10^{+1}$	1.13	$1.41\ 10^{+1}$	1.07
R-10-17-9	$3.45\ 10^{+1}$	$2.08\ 10^{+1}$	**1.66**	$2.50\ 10^{+1}$	1.38	$4.04\ 10^{+1}$	0.85	$3.53\ 10^{+1}$	0.98
2d-20-4	$3.48\ 10^{-1}$	$1.76\ 10^{-1}$	1.98	$8.03\ 10^{-2}$	4.33	$6.96\ 10^{-2}$	**5.00**	$1.73\ 10^{-1}$	2.01
2d-20-5	$6.74\ 10^{-1}$	$3.54\ 10^{-1}$	1.90	$1.29\ 10^{-1}$	**5.22**	$1.32\ 10^{-1}$	5.11	$3.59\ 10^{-1}$	1.88
2d-20-6	1.19	$6.04\ 10^{-1}$	1.97	$2.23\ 10^{-1}$	**5.34**	$2.70\ 10^{-1}$	4.41	$6.52\ 10^{-1}$	1.83
2d-20-7	2.08	1.45	1.43	$5.40\ 10^{-1}$	**3.85**	$6.21\ 10^{-1}$	3.35	1.11	1.87
2d-20-8	3.69	1.85	1.99	$6.36\ 10^{-1}$	**5.80**	$7.95\ 10^{-1}$	4.64	1.92	1.92
sR-2	$1.71\ 10^{+1}$	4.26	4.01	3.11	**5.50**	4.14	4.13	$1.05\ 10^{+1}$	1.63
sR-4	$8.03\ 10^{+1}$	$3.68\ 10^{+1}$	2.18	$4.40\ 10^{+1}$	**1.83**	$1.41\ 10^{+2}$	0.57	$2.02\ 10^{+2}$	0.40
sR-6	$1.08\ 10^{+2}$	$1.54\ 10^{+2}$	0.70	$7.01\ 10^{+1}$	**1.54**	$2.58\ 10^{+2}$	0.42	$4.04\ 10^{+2}$	0.27
perm-5	$6.64\ 10^{-1}$	$1.89\ 10^{-1}$	3.51	$6.87\ 10^{-2}$	**9.67**	$8.53\ 10^{-2}$	7.78	$3.75\ 10^{-1}$	1.77
perm-6	5.80	1.32	4.39	$5.19\ 10^{-1}$	**11.18**	1.03	5.63	3.81	1.52
perm-7	$5.70\ 10^{+1}$	$1.10\ 10^{+1}$	5.18	4.16	**13.70**	$2.12\ 10^{+1}$	2.69	$6.37\ 10^{+1}$	0.89
perm-8	$5.98\ 10^{+2}$	$1.08\ 10^{+2}$	5.54	$4.41\ 10^{+1}$	**13.56**	$6.46\ 10^{+2}$	0.93	$1.59\ 10^{+3}$	0.38
r-3-7	$5.83\ 10^{-1}$	$3.16\ 10^{-1}$	1.84	$2.79\ 10^{-1}$	2.09	$2.27\ 10^{-1}$	**2.57**	$3.64\ 10^{-1}$	1.60
r-3-9	$3.31\ 10^{-1}$	$2.92\ 10^{-1}$	1.13	$1.96\ 10^{-1}$	1.69	$1.41\ 10^{-1}$	**2.35**	$1.77\ 10^{-1}$	1.87
r-4-7	3.13	1.62	1.93	1.37	**2.28**	2.21	1.42	3.01	1.04
r-4-9	2.76	1.36	2.03	1.13	**2.44**	1.85	1.49	2.87	0.96
r-5-7	8.92	4.72	1.89	3.94	**2.26**	8.64	1.03	$1.26\ 10^{+1}$	0.71
r-5-9	9.02	4.47	2.02	3.72	**2.42**	7.92	1.14	$1.06\ 10^{+1}$	0.85
r-6-7	$2.18\ 10^{+1}$	$1.20\ 10^{+1}$	1.82	$1.14\ 10^{+1}$	**1.91**	$2.89\ 10^{+1}$	0.75	$4.03\ 10^{+1}$	0.54
r-6-9	$2.63\ 10^{+1}$	$1.45\ 10^{+1}$	1.81	$1.17\ 10^{+1}$	**2.25**	$3.39\ 10^{+1}$	0.78	$4.89\ 10^{+1}$	0.54
r-7-7	$5.72\ 10^{+1}$	$3.30\ 10^{+1}$	**1.73**	$3.49\ 10^{+1}$	1.64	$1.17\ 10^{+2}$	0.49	$1.60\ 10^{+2}$	0.36
r-7-9	$4.68\ 10^{+1}$	$2.58\ 10^{+1}$	1.81	$2.45\ 10^{+1}$	**1.91**	$7.30\ 10^{+1}$	0.64	$8.74\ 10^{+1}$	0.54
median/mean			1.93/2.23		2.05/3.70		1.93/2.48		1.52/1.32

Conclusion

- Better improvement ratios on ”structured” instances
- ”real-world” instances are ”structured” (so good ratios!)
- next steps: articles, code details, convincing advisors of why/how it works (, writing the thesis.....................)

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Bibliographic elements I

[RČ18] Miroslav Rada and Michal Černý. "A New Algorithm for Enumeration of Cells of Hyperplane Arrangements and a Comparison with Avis and Fukuda's Reverse Search". In: SIAM Journal on Discrete Mathematics 32 (Jan. 2018), pp. 455–473. DOI: 10.1137/15M1027930.

Theoretical detour

Very well-known in algebra / combinatorics...
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Very impressive results / algorithms for the cardinal (number of feasible systems, number of $J \in \partial_B$)
Upper bound, formula (also combinatorial)...

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Circuits of matroids

We look at subsets $I \subset [1:p]$, $\dim(\mathcal{N}(V_{:,I})) = \mathbf{1}$ and $\forall\ I' \subsetneq I$, $\dim(\mathcal{N}(V_{:,I'})) = 0$

$$\dim(\mathcal{N}(V_{:,I})) = 1 \Rightarrow \mathcal{N}(V_{:,I}) = \mathrm{Vect}(\eta)$$
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$\mathcal{N}(V_{:,I})$ gives 'unsigned' η's which define the sign $s_J = 1$ because

if ≥ 2, smaller subsets are of $\dim(\mathcal{N}) = 1$

2^p LO feasibility $\leftrightarrow$ 2^p $\mathcal{N}$ searches; subsets of size $\leq 1 + \mathrm{rank}(V)$

Issue (unresolved): "optimal" way to compute efficiently: if I s.t. $\dim(\mathcal{N}(V_{:,I})) = 1$, $I' \supsetneq I$ useless to check

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