Symbolic Data Analysis: Principal Component Analysis (PCA)

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PRINCIPAL COMPONENT ANALYSIS for interval-valued data:

Trivial and nontrivial intervals

ω_u	$Y_1 = [a_1, b_1]$	$Y_2 = [a_2, b_2]$	$Y_3 = [a_3, b_3]$	
ω_1	[1,3]	[1,4]	[2,3]	cube
ω_2	[2,4]	[3,4]	[4,6]	cube
ω_3	[1,3]	[2,4]	[3,3]	plane
ω_4	[3,5]	[3,3]	[2,2]	line
ω_5	[3,3]	[2,2]	[5,5]	point

For observed $Y_{uj} = [a_{uj},\ b_{uj}]$, $u = 1, \ldots, n$, $j = 1, \ldots, p$, when

$$a_{uj} < b_{uj},\ \text{nontrivial interval},$$
$$a_{uj} = b_{uj},\ \text{trivial interval}.$$

Then,

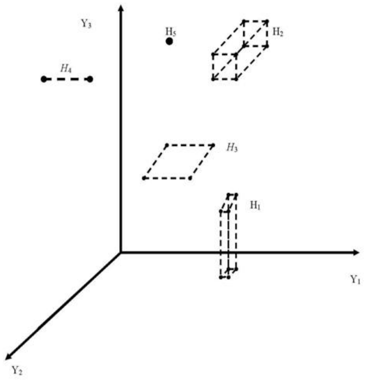
$$\#\ \text{vertices} = v_u = 2^{q_u} \quad (1)$$

where $q_u = \#$ nontrivial intervals. If there are no trivial intervals,

$$q_u = p, \quad v_u = 2^p.$$

Clouds of Vertices

H_u	Y_1	Y_2	Y_3	
H_1	[1,3]	[1,4]	[2,3]	cube
H_2	[2,4]	[3,4]	[4,6]	cube
H_3	[1,3]	[2,4]	[3,3]	plane
H_4	[3,5]	[3,3]	[2,2]	line
H_5	[3,3]	[2,2]	[5,5]	point



Set up an $n \times p$ data matrix of the vertices

$$\mathbf{M} = \begin{pmatrix} \mathbf{M}_1 \\ \vdots \\ \mathbf{M}_m \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} a_{11} & \dots & a_{1q_1} \\ \vdots & & \vdots \\ b_{11} & \dots & b_{1q_1} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} a_{m1} & \dots & a_{mq_m} \\ \vdots & & \vdots \\ b_{m1} & \dots & b_{mq_m} \end{bmatrix} \end{pmatrix}$$

where

$$n = \sum_u 2^{q_u}, \quad q_u = \# \text{ non-trivial intervals in } \omega_u \tag{2}$$

Construction of **M**:

E.g.,

ω_u	$[a_1, b_1]$	$[a_2, b_2]$	$[a_3, b_3]$
ω_1	[1,3]	[1,4]	[2,3]
ω_2	[2,4]	[3,4]	[4,6]
ω_3	[1,3]	[2,4]	[3,3]
ω_4	[3,5]	[3,3]	[2,2]
ω_5	[3,3]	[2,2]	[5,5]

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_1 \\ \mathbf{M}_2 \\ \mathbf{M}_3 \\ \mathbf{M}_4 \\ \mathbf{M}_5 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \\ 3 & 1 & 2 \\ 3 & 1 & 3 \\ 3 & 4 & 2 \\ 3 & 4 & 3 \end{pmatrix} \\ \begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & 6 \\ 2 & 4 & 4 \\ 2 & 4 & 6 \\ 4 & 3 & 4 \\ 4 & 3 & 6 \\ 4 & 4 & 4 \\ 4 & 4 & 6 \end{pmatrix} \\ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 3 \\ 3 & 2 & 3 \\ 3 & 4 & 3 \end{pmatrix} \\ \begin{pmatrix} 3 & 3 & 2 \\ 5 & 3 & 2 \end{pmatrix} \\ \begin{pmatrix} 3 & 2 & 5 \end{pmatrix} \end{bmatrix}$$

Symbolic Principal Component Analysis - Vertices Method

Treat data matrix **M** as input for n classical data points with p variables
[$n =$ total number of vertices]

Chouakria **(1998)** - shows classical theory holds
———–

Classical Principal Components (PC)
Observations: $X_i = (X_{i1}, ..., X_{ip}), \quad i = 1, ..., n$
Then, for $\nu = 1, \ldots, p$,

$$PC\nu = e_{\nu_1} X_1 + \cdots + e_{\nu_p} X_p$$

$$PC\nu = e_{\nu_1} X_1 + \cdots + e_{\nu_p} X_p$$

where

$$\lambda_\nu = (\lambda_{\nu_1}, \ldots, \lambda_{\nu_p}), \mathbf{e}_\nu = (\mathbf{e}_{\nu_1}, \ldots, \mathbf{e}_{\nu_p})$$

is ν^{th} eigenvalue and ν^{th} eigenvector of variance-covariance matrix $\mathbf{\Sigma}$, and with

$$\sum_j \lambda_{\nu_j} = 1.$$

Suppose $\mathbf{X}' = (X_1, \ldots, X_p)$ has variance-covariance matrix Σ with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$.

Take $Y_i = \mathbf{a}\mathbf{X} = a_{i1}X_1 + \cdots + a_{ip}X_p$ Then,

$$Var(Y_j) = \mathbf{a}_j' \boldsymbol{\Sigma} \mathbf{a}_j, \quad j = 1, \ldots, p,$$

$$Cov(Y_j, Y_k) = \mathbf{a}_j' \boldsymbol{\Sigma} \mathbf{a}_k, \quad j, \ k = 1, \ldots, p.$$

First $PC \equiv PC1$ is linear combination $\mathbf{a}_1'\mathbf{X}$ that
Maximizes $var(\mathbf{a}_1'\mathbf{X})$ subject to $\mathbf{a}_1'\mathbf{a}_1 = 1$

Second $PC \equiv PC2$ is linear combination $\mathbf{a}_2'\mathbf{X}$ that
Maximizes $var(\mathbf{a}_2'\mathbf{X})$ subject to $\mathbf{a}_2'\mathbf{a}_2 = 1$ and $Cov(\mathbf{a}_1'\mathbf{X}, \ \mathbf{a}_2'\mathbf{X}) = 0$

...$PC \equiv PC\nu$ is linear combination $\mathbf{a}_\nu'\mathbf{X}$ that
Maximizes $var(\mathbf{a}_\nu'\mathbf{X})$ subject to $\mathbf{a}_\nu'\mathbf{a}_\nu = 1$ and
$Cov(\mathbf{a}_\nu'\mathbf{X}, \ \mathbf{a}_k'\mathbf{X}) = 0, \ k < \nu$

Hence, for $\nu = 1, \ldots, p$, we have

$$\begin{aligned} PC\nu = Y_\nu &= \mathbf{e}'_\nu \mathbf{X} \qquad (3) \\ Var(Y_\nu) &= \mathbf{e}'_\nu bf\Sigma \mathbf{e}_\nu = \lambda_\nu, \\ Cov(Y_\nu,\ Y_k) &= \mathbf{e}'_\nu bf\Sigma \mathbf{e}_k = 0, \quad \nu \neq k, \end{aligned}$$

where $\mathbf{e}'_\nu = (e_{\nu 1}, \ldots, e_{\nu p})$ is the ν^{th} eigenvector of $\mathbf{\Sigma}$; λ_ν is ν^{th} eigenvalue. I.e., the $PC\nu \equiv Y_\nu$'s are uncorrelated.

Total (population) variance $= \lambda_1 + \cdots + \lambda_p$;
Proportion of variation due to $PC\nu = \lambda_\nu / \lambda, \quad \nu = 1, \ldots, p$

Each observation $Y_i = (Y_{i1}, \ldots, Y_{ip})$ has a principal component representation $(PC1, \ldots, PCp)$ with, for each $i = 1, \ldots, n$,

$$PC\nu = e_{\nu 1} X_1 + \cdots + e_{\nu p} X_p. \qquad (3)$$

I.e., for each observation i, (3) gives

$$PC(\nu, i) = e_{\nu 1} X_{i1} + \cdots + e_{\nu p} X_{ip}$$

To standardize/normalize, replace

$$X_{ij} \text{ by } (X_{ij} - \bar{X}_j)/SD(X_j)$$

Construction of PCs for interval observations:

Observed intervals $\omega_u = \{[a_{uj},\ b_{uj}],\ j = 1, \ldots, p\},\ u = 1, \ldots, m,$

- Each ω_u has $v_u = 2^{q_u}$ vertices [eqn(1)].
- Each ω_u is represented by the set L_u of rows in $\mathbf{M}_u$.
- Let $y_{\nu uk}$ be the $PC\nu$ for vertex $k (\equiv k_u)$ in $L_u, k = 1, \ldots, v_u,\ \nu = 1, \ldots, s \leq p$.

Then, the $PC\nu$ interval for observation ω_u is

$$Y_{\nu u} \equiv y\nu u = [y^a_{\nu u},\ y^b_{\nu u}],\quad \nu = 1, \ldots, s,$$

where

$$y^a_{\nu u} = \min_{k_u \in L_u} \{y_{\nu u k_u}\} \text{ and } y^b_{\nu u} = \max_{k_u \in L_u} \{y_{\nu u k_u}\}. \tag{4}$$

[See (5.13) and (5.14), p. 142, where $k \equiv k_u$]

u	Species	Head	Tail	Height	Forearm
1	PIPC	[33, 52]	[26, 33]	[4, 7]	[27, 32]
2	PRH	[35, 43]	[24, 30]	[8, 11]	[34, 41]
3	MOUS	[38, 50]	[30, 40]	[7, 8]	[32, 37]
4	PIPS	[43, 48]	[34, 39]	[6, 7]	[31, 38]
5	PIPN	[44, 48]	[34, 44]	[7, 8]	[31, 36]
6	MDAUB	[41, 51]	[30, 39]	[8, 11]	[33, 41]
7	MNAT	[42, 50]	[32, 43]	[8, 9]	[36, 42]
8	MDEC	[40, 45]	[39, 44]	[9, 9]	[36, 42]
9	MGP	[45, 53]	[35, 38]	[10, 12]	[39, 44]
10	OCOM	[41, 51]	[34, 50]	[9, 10]	[34, 50]
11	MBEC	[46, 53]	[34, 44]	[9, 11]	[39, 44]
12	SBOR	[48, 54]	[38, 47]	[9, 11]	[37, 42]
13	BARB	[44, 58]	[41, 54]	[6, 8]	[35, 41]
14	OGRIS	[47, 53]	[43, 53]	[7, 9]	[37, 41]
15	SBIC	[50, 63]	[40, 45]	[8, 10]	[40, 47]
16	FCHEV	[50, 69]	[30, 43]	[11, 13]	[51, 61]
17	MSCH	[52, 60]	[50, 60]	[10, 11]	[42, 48]
18	SCOM	[62, 80]	[46, 57]	[9, 12]	[48, 56]
19	NOCT	[69, 82]	[41, 59]	[10, 12]	[45, 55]
20	GMUR	[65, 80]	[48, 60]	[12, 16]	[55, 68]
21	MGES	[82, 87]	[46, 57]	[11, 12]	[58, 63]

Bats:

$$\mathbf{M} = \begin{bmatrix} 33 & 26 & 4 & 27 \\ 33 & 26 & 4 & 32 \\ & \dots & & \\ 87 & 57 & 12 & 58 \\ 87 & 57 & 12 & 63 \end{bmatrix}$$

$q_u = \#$ nontrivial intervals $= p = 4$, $\quad v_u = (\#$ vertices $=) 2^{q_u} = 16$, for all u;
$n =$ total $\#$ vertices $= \sum_u 2^{q_u} = \sum_u 2^p = 21x16 = 336$

Construction of PC for Intervals

Then, construct interval PC from PCs of vertices of ω_u.

Recall
Observed intervals $\omega_u = \{[a_{uj},\ b_{uj}],\ j = 1, \ldots, p\},\ u = 1, \ldots, m,$

- Each ω_u has $v_u = 2^{q_u}$ vertices [eqn(1)].
- Each ω_u is represented by the set L_u of rows in $\mathbf{M}_u$.
- Let $y_{\nu uk}$ be the $PC\nu$ for vertex $k(\equiv k_u)$ in $L_u, k = 1, \ldots, v_u,\ \nu = 1, \ldots, s \leq p$.

Then, the $PC\nu$ interval for observation ω_u is

$$Y_{\nu u} \equiv y_{\nu u} = [y^a_{\nu u},\ y^b_{\nu u}],\quad \nu = 1, \ldots, s,$$

where

$$y^a_{\nu u} = \min_{k_u \in L_u} \{y_{\nu u k_u}\} \text{ and } y^b_{\nu u} = \max_{k_u \in L_u} \{y_{\nu u k_u}\}; \tag{4}$$

$$y_{\nu i} = PC(\nu, i) = e_{\nu 1} X_{i1} + \cdots + e_{\nu p} X_{ip} \tag{3}$$

Take **BARB**: $\omega_{13} = ([44, 58], [41, 54], [6, 8], [35, 41])$

Sorted by **First** Principal Component $(\nu = 1)$

Y_1	Y_2	Y_3	Y_4	$PC1$	$PC2$	$PC3$	$PC4$
44	41	6	35	**-1.599**	0.910	-0.240	-0.299
44	41	6	41	-1.246	0.762	-0.360	-0.799
44	41	8	35	-1.164	0.395	0.251	0.043
58	41	6	35	-1.043	1.046	-0.952	0.223
44	41	8	41	-0.811	0.246	0.132	-0.458
58	41	6	41	-0.690	0.897	-1.071	-0.277
44	54	6	41	-0.641	1.849	0.282	-0.850
58	41	8	35	-0.608	0.530	-0.460	0.565
44	54	8	35	-0.560	1.482	0.894	-0.008
58	54	6	35	-0.438	2.133	-0.310	0.173
58	41	8	41	-0.256	0.381	-0.580	0.064
44	54	8	41	-0.207	1.334	0.774	-0.509
58	54	6	41	-0.086	1.985	-0.429	-0.328
58	54	8	35	-0.004	1.618	0.182	0.514
58	54	8	41	**0.349**	1.469	0.062	0.014

$$Y_{1,13} = [\min_{k \in L_{13}} \{y_{1,13,k}\}, \max_{k \in L_{13}} \{y_{1,13,k}\}] = [-\mathbf{1.599}, \mathbf{0.349}]$$

Detail – Y_1 = Head, Y_2 = Tail, Y_3 = Height, Y_4 = Forearm

	Eigenvectors			
	$PC1$	$PC2$	$PC3$	$PC4$
Y_1	0.5286	0.1287	-0.6764	0.4964
Y_2	0.4312	0.7763	0.4583	-0.0361
Y_3	0.4816	-0.5719	0.5455	0.3787
Y_4	0.5501	-0.2316	-0.1868	-0.7803

	Basic Statistics			
	Y_1	Y_2	Y_3	Y_4
Mean	53.5000	41.7619	9.4048	42.5952
STD	13.3119	9.2792	2.2186	9.3517

Take: $u = 13,\ k_u = k = 1,\ (Y_1, Y_2, Y_3, Y_4) = (44, 41, 6, 35)$
Then, $PC1$ for this vertex, after normalizing, $y_{\nu uk} = y_{1,13,1}$ is

$$\begin{aligned} y_{\nu uk} &= 0.5286(44 - 53.5)/13.3119 + 0.4312(41 - 41.7619)/9.2792 \\ &\quad + 0.4816(6 - 9.4048)/2.2186 + 0.5501(35 - 42.5952)/9.3517 \\ &= \mathbf{-1.599} \end{aligned}$$

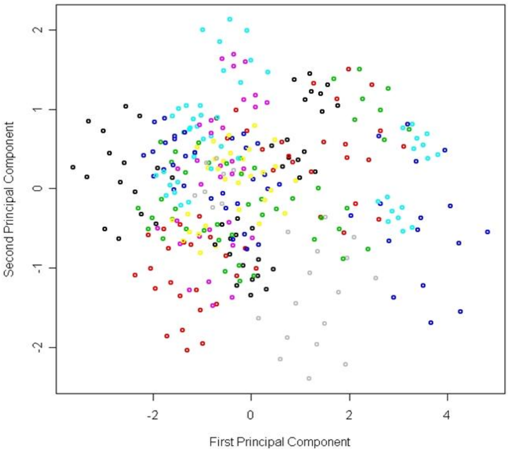
where, from (3), $y_{\nu i} = PC(\nu, i) = e_{\nu 1}X_{i1} + \cdots + e_{\nu p}X_{ip}$

All Bats:

u	Species	$PC1$	$PC2$
1	PIPC	[-3.637, -1.612]	[-0.635, 1.032]
2	PRH	[-2.371, -0.711]	[-2.037, -0.511]
3	MOUS	[-2.307, -0.855]	[-0.633, 0.701]
4	PIPS	[-2.199, -1.139]	[-0.017, 0.881]
5	PIPN	[-1.942, -0.807]	[-0.216, 1.041]
6	MDAUB	[-1.912, 0.025]	[-1.477, 0.344]
7	MNAT	[-1.603, -0.204]	[-0.809, 0.595]
8	MDEC	[-1.140, -0.356]	[-0.243, 0.373]
9	MGP	[-0.734, 0.451]	[-1.352, -0.384]
10	OCOM	[-1.451, 0.848]	[-1.107, 0.982]
11	MBEC	[-0.958, 0.513]	[-1.168, 0.376]
12	SBOR	[-0.810, 0.575]	[-0.764, 0.686]
13	**BARB**	**[-1.599, 0.349]**	**[0.246, 2.133]**
14	OGRIS	[-1.052, 0.321]	[0.185, 1.694]
15	SBIC	[-0.678, 0.916]	[-0.444, 0.789]
16	FCHEV	[0.155, 2.536]	[-2.400, -0.366]
17	MSCH	[0.418, 1.770]	[0.130, 1.450]
18	SCOM	[0.765, 3.112]	[-0.564, 1.502]
19	NOCT	[0.851, 3.226]	[-0.890, 1.505]
20	GMUR	[2.040, 4.826]	[-1.696, 0.806]
21	MGES	[2.581, 3.802]	[-0.544, 0.806]

Construction of PC Envelope

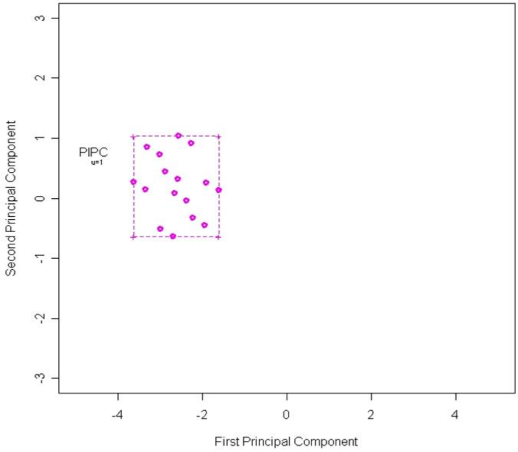
To construct the *PC* envelope, consider



This is a classical PCA of the vertices: Bats

Construction of PC Envelope

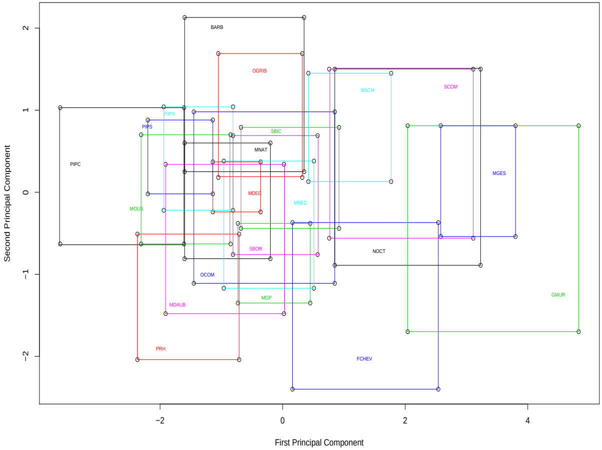
For $u = 1$, bat **PIPC**, the PCs of the vertices are:



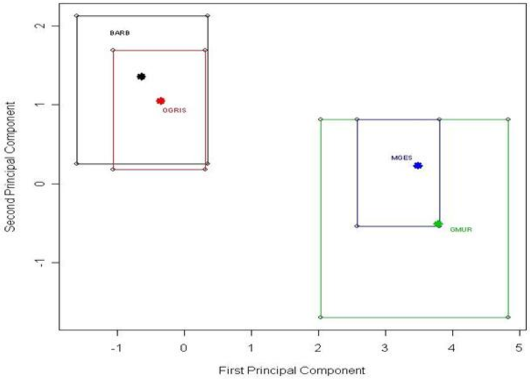
$PC1 = [-3.637, -1.612]$ $PC2 = [-0.635, 1.032]$

Construction of PC Envelope

For all observations u, the plot of $PC1 \times PC2$ is:



PC hypercubes reflect relative data hypercubes H_u
Bats: GMUR, MGES, BARB, OGRIS – $PC1 \times PC2$



Classical *PC* based on midpoints:
GMUR: $PC = (3.778,-0.508)$
MGES: $PC = (3.483,0.233)$

For $u = 20$ GMUR:
$Y_{20} = ([65, 80],[48,60],[12,16],[55,68])$; $PC_{1,2} = ([2.040, 4.826],[-1.696, 0.806])$
For $u = 21$ MGES:
$Y_{21} = ([82, 87],[46,57],[11,12],[58,63])$; $PC_{1,2} = ([2.581, 3.802],[-0.544, 0.806])$.

Visualization - Interpretation

The observation ω_u has vertices $k \equiv k_u \; = 1, \ldots, v_u$

Contribution of vertex k_u is (eqn(5.25), (5.18))

$$\begin{aligned} Con(k_u, PC\nu) &= \frac{y^2_{\nu u k_u}}{[d(k_u, \mathbf{G})]^2} \\ d^2(k_u, \mathbf{G}) &= \sum_{j=1}^{p} \left(\frac{X_{k_u j} - \bar{X}_{Gj}}{S_{Gj}} \right)^2 \\ \bar{X}_{Gj} &= \frac{1}{n} \sum_{u=1}^{m} \sum_{k_u=1}^{v_u} X_{k_u j} \\ S^2_{Gj} &= \frac{1}{n-1} \sum_{u=1}^{m} \sum_{k_u=1}^{v_u} (X_{k_u j} - \bar{X}_{Gj})^2 \end{aligned} \tag{5}$$

$y_{\nu u k_u}$ is the $PC\nu$ for vertex k_u of observation u, and
$X_{k_u j}$ is the value of variable Y_j at vertex k_u of hypercube H_u

For prespecified α, $PC\nu(\alpha)$ is (see eqn(5.24))

$$\begin{aligned} Y^*_{\nu u}(\alpha) &= [y^a_{\nu u}(\alpha), y^b_{\nu u}(\alpha)], \\ y^a_{\nu u}(\alpha) &= \min_{k_u \in L_u} \{y_{\nu u k_u} | Con(k_u, \nu) \geq \alpha\}, \\ y^b_{\nu u}(\alpha) &= \max_{k_u \in L_u} \{y_{\nu u k_u} | Con(k_u, \nu) \geq \alpha\}, \end{aligned} \tag{6}$$

If for a particular $\nu = \nu_1$ (say)

$$Con(k_u, PC\nu_1) < \alpha, \quad \text{for all} \quad k_u,$$

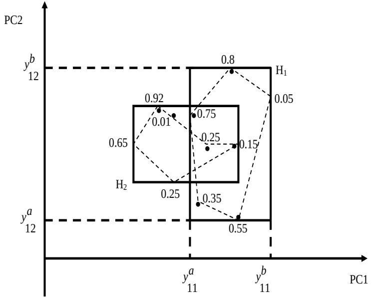
then project H_u onto the center of the ν_1 axis at

$$y^a_{\nu_1 u} \equiv y^b_{\nu_1 u} = \frac{1}{v_u} \sum_{k=1}^{v_u} y_{\nu_1 uk}$$

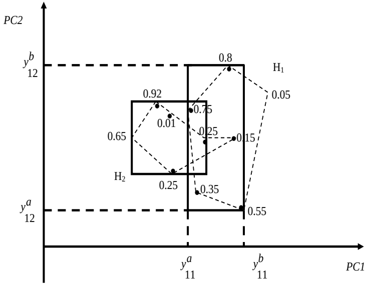
where $v_u = 2^{q_u} = \#$ vertices in H_u (L_u)

PC Envelope, based on contributions of vertices to *PC*

$\alpha = 0$



$\alpha = 0.2$



When $\alpha = 0.2$, notice, vertex with
Con $= 0.05$ (in H_1), and Con $= 0.15$ (in H_2) are not considered

However, vertex with Con=0.01 in H_2 kept, as it is inside H_2 envelope

For $u = 13$, BARB, $\alpha = 0.4$,
(Y_1 = Head, Y_2 = Tail, Y_3 = Height, Y_4 = Forearm)

k_u	Y_1	Y_2	Y_3	Y_4	$PC1$	$PC2$	$CON1$	$CON2$
1	44	41	6	35	-1.59858	0.91015	0.72377	0.23462
2	44	41	6	41	-1.24563	0.76159	0.53499	0.19999
3	44	41	8	35	-1.16440	0.39457	0.85998	0.09875
4	58	41	6	35	-1.04265	1.04555	0.34669	0.34862
5	44	54	6	35	-0.99441	1.99778	0.18787	0.75828
6	44	41	8	41	-0.81145	0.24601	0.69601	0.06397
7	58	41	6	41	-0.68970	0.89699	0.18988	0.32116
8	44	54	6	41	-0.64146	1.84922	0.08882	0.73812
9	58	41	8	35	-0.60847	0.52997	0.31334	0.23771
10	44	54	8	35	-0.56023	1.48221	0.09484	0.66388
11	58	54	6	35	-0.43848	2.13318	0.03949	0.93469
12	58	41	8	41	-0.25552	0.38141	0.11849	0.26400
13	44	54	8	41	-0.20728	1.33364	0.01604	0.66398
14	58	54	6	41	-0.08553	1.98462	0.00173	0.92940
15	58	54	8	35	-0.00430	1.61760	0.00001	0.89788
16	58	54	8	41	0.34865	1.46904	0.05323	0.94499

Calculation of Contributions $Con(k_u, PC\nu)$ - **First,**

$$\bar{X}_{Gj} = \frac{1}{n}\sum_{u=1}^{m}\sum_{k_u=1}^{v_u} X_{k_u j}, \qquad S_{Gj}^2 = \frac{1}{n-1}\sum_{u=1}^{m}\sum_{k_u=1}^{v_u}(X_{k_u j} - \bar{X}_{Gj})^2$$

$\bar{X}_{Gj}$, S_{Gj}^2 are mean and variance over all vertices for each Y_j

	Basic Statistics			
	Y_1	Y_2	Y_3	Y_4
Mean	53.5000	41.7619	9.4048	42.5952
STD	13.3119	9.2792	2.2186	9.3517

Then, calculate the Euclidean distances

$$d^2(k_u, \mathbf{G}) = \sum_{j=1}^{p}\left(\frac{X_{k_u j} - \bar{X}_{Gj}}{S_{Gj}}\right)^2 = \sum_{j=1}^{p} d_j^2(k_u, \mathbf{G}), \qquad \text{(say)}$$

For $u = 13$, BARB, $k_u = 1$ with $Y_u = (44, 41, 6, 35) \equiv X_1$,

$$\sum_{j=1}^{p} d_j^2(k_u, \mathbf{G}) = [(44 - 53.5)/13.31]^2 + [(41 - 41.72)/9.28]^2$$

$$+ [(6 - 9.41)/2.22]^2 + [(35 - 42.6)/9.32]^2 = \mathbf{3.531}$$

Finally, we calculate the contributions
Hence, for vertex **Y** $= (44, 41, 6, 35)$ (i.e., $k_u = 1$), we have, for $\nu = 1$,

$$Con(k_u, PC\nu) = \frac{y^2_{\nu u k_u}}{d^2(k_u, \mathbf{G})} = (-1.599)^2/3.531 = \mathbf{0.724}$$

For all vertices of $u = 13$, **BARB**, **contributions** from $PC1$ are:

k_u	Y_1	Y_2	Y_3	Y_4	d_1^2	d_2^2	d_3^2	d_4^2	$\sum d_j^2$	$PC1$	$Con1$
1	44	41	6	35	0.509	0.007	2.355	0.660	**3.531**	**-1.599**	**0.724**
2	44	41	6	41	0.509	0.007	2.355	0.029	2.900	-1.246	0.535
3	44	41	8	35	0.509	0.007	0.401	0.660	1.577	-1.164	0.860
4	58	41	6	35	0.114	0.007	2.355	0.660	3.136	-1.043	0.347
5	44	54	6	35	0.509	1.739	2.355	0.660	5.263	-0.994	0.188
6	44	41	8	41	0.509	0.007	0.401	0.029	0.946	-0.811	0.696
7	58	41	6	41	0.114	0.007	2.355	0.029	2.505	-0.690	0.190
8	44	54	6	41	0.509	1.739	2.355	0.029	4.633	-0.641	0.089
9	58	41	8	35	0.114	0.007	0.401	0.660	1.182	-0.608	0.313
10	44	54	8	35	0.509	1.739	0.401	0.660	3.309	-0.560	0.095
11	58	54	6	35	0.114	1.739	2.355	0.660	4.868	-0.438	0.039
12	58	41	8	41	0.114	0.007	0.401	0.029	0.551	-0.256	0.118
13	44	54	8	41	0.509	1.739	0.401	0.029	2.679	-0.207	0.016
14	58	54	6	41	0.114	1.739	2.355	0.029	4.238	-0.086	0.002
15	58	54	8	35	0.114	1.739	0.401	0.660	2.914	-0.004	0.000
16	58	54	8	41	0.114	1.739	0.401	0.029	2.284	0.349	0.053

For both $PC1$ and $PC2$

For $u = 13$, BARB, $\alpha = 0.4$,
(Y_1 = Head, Y_2 = Tail, Y_3 = Height, Y_4 = Forearm)

k_u	Y_1	Y_2	Y_3	Y_4	$PC1$	$PC2$	$CON1$	$CON2$
1	44	41	6	35	-1.59858	0.91015	0.72377	0.23462
2	44	41	6	41	-1.24563	0.76159	0.53499	0.19999
3	44	41	8	35	-1.16440	0.39457	0.85998	0.09875
4	58	41	6	35	-1.04265	1.04555	0.34669	0.34862
5	44	54	6	35	-0.99441	1.99778	0.18787	0.75828
6	44	41	8	41	-0.81145	0.24601	0.69601	0.06397
7	58	41	6	41	-0.68970	0.89699	0.18988	0.32116
8	44	54	6	41	-0.64146	1.84922	0.08882	0.73812
9	58	41	8	35	-0.60847	0.52997	0.31334	0.23771
10	44	54	8	35	-0.56023	1.48221	0.09484	0.66388
11	58	54	6	35	-0.43848	2.13318	0.03949	0.93469
12	58	41	8	41	-0.25552	0.38141	0.11849	0.26400
13	44	54	8	41	-0.20728	1.33364	0.01604	0.66398
14	58	54	6	41	-0.08553	1.98462	0.00173	0.92940
15	58	54	8	35	-0.00430	1.61760	0.00001	0.89788
16	58	54	8	41	0.34865	1.46904	0.05323	0.94499

I.e., for $u = 13$, we now have
$PC1(\alpha = 0.4) = [-1.599, -0.811]$, $PC2(\alpha = 0.4) = [1.334, 2.133]$

The complete set of Principal Components $(\nu = 1, \nu = 2)$:

u	Species	$\alpha = 0$		$\alpha = 0.4$			
		$PC1$	$PC2$	$PC1$	$PC2$	$ve*$	
1	PIPC	[-3.637, -1.612]	[-0.635, 1.032]	[-3.637, -1.612]	0.199	16	0
2	PRH	[-2.371, -0.711]	[-2.037, -0.511]	[-2.371, -1.362]	[-2.037, -1.284]	10	7
3	MOUS	[-2.307, -0.855]	[-0.633, 0.701]	[-2.307, -0.855]	0.034	16	0
4	PIPS	[-2.199, -1.139]	[-0.017, 0.881]	[-2.199, -1.139]	0.432	16	0
5	PIPN	[-1.942, -0.807]	[-0.216, 1.041]	[-1.942, -0.807]	[0.917, 0.917]	16	1
6	MDAUB	[-1.912, 0.025]	[-1.477, 0.344]	[-1.912, -0.626]	[-1.477, -0.627]	8	5
7	MNAT	[-1.603, -0.204]	[-0.809, 0.595]	[-1.603, -0.557]	[-0.731, -.447]	13	2
8	MDEC	[-1.140, -0.356]	[-0.243, 0.373]	[-1.140, -0.589]	0.065	12	0
9	MGP	[-0.734, 0.451]	[-1.352, -0.384]	[-0.734, -0.595]	[-1.352, -0.508]	2	14
10	OCOM	[-1.451, 0.848]	[-1.107, 0.982]	[-1.451, 0.848]	[-1.107, 0.982]	5	4
11	MBEC	[-0.958, 0.513]	[-1.168, 0.376]	[-0.958, 0.513]	[-1.168, 0.376]	5	7
12	SBOR	[-0.810, 0.575]	[-0.764, 0.686]	[-0.810, -0.516]	[-0.764, 0.686]	3	6
13	**BARB**	**[-1.599, 0.349]**	**[0.246, 2.133]**	**[-1.599, -0.811]**	**[1.334, 2.133]**	**4**	**8**
14	OGRIS	[-1.052, 0.321]	[0.185, 1.694]	[-1.052, -0.382]	[0.243, 1.694]	5	11
15	SBIC	[-0.678, 0.916]	[-0.444, 0.789]	[-0.678, 0.916]	[-0.444, 0.789]	3	3
16	FCHEV	[0.155, 2.536]	[-2.400, -0.366]	[0.759, 2.536]	[-2.400, -1.453]	8	7
17	MSCH	[0.418, 1.770]	[0.130, 1.450]	[0.735, 1.770]	[1.373, 1.450]	10	2
18	SCOM	[0.765, 3.112]	[-0.564, 1.502]	[0.764, 3.112]	[1.323, 1.323]	16	1
19	NOCT	[0.851, 3.226]	[-0.890, 1.505]	[0.851, 3.226]	0.307	15	0
20	GMUR	[2.040, 4.826]	[-1.696, 0.806]	[2.040, 4.826]	-0.445	16	0
21	MGES	[2.581, 3.802]	[-0.544, 0.806]	[2.581, 3.802]	0.131	16	0

$ve* = (ve_1, ve_2) = \#$ vertices with $Con(k_u, PC\nu) > \alpha$, $\nu = 1, 2$

Let us look at $PC2$, $\alpha = 0.4$, for species PIPS and PIPN

		$\alpha = 0$		$\alpha = 0.4$			
u	Species	$PC1$	$PC2$	$PC1$	$PC2$	$ve*$	
1	PIPC	[-3.637,-1.612]	[-0.635, 1.032]	[-3.637,-1.612]	0.199	16	0
2	PRH	[-2.371,-0.711]	[-2.037,-0.511]	[-2.371,-1.362]	[-2.037,-1.284]	10	7
3	MOUS	[-2.307,-0.855]	[-0.633, 0.701]	[-2.307,-0.855]	0.034	16	0
4	PIPS	[-2.199,-1.139]	[-0.017, 0.881]	[-2.199,-1.139]	**0.432**	**16**	**0**
5	PIPN	[-1.942,-0.807]	[-0.216, 1.041]	[-1.942,-0.807]	**[0.917, 0.917]**	**16**	**1**
.	...						

$ve* = (ve_1, ve_2) = \#$ vertices with $Con(k_u, PC\nu) > \alpha$, $\nu = 1, 2$

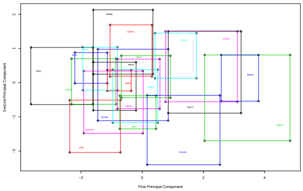
Now, for $u = 5$, PIPN, there is one vertex with $Con2 > \alpha = 0.4$
(at vertex Y = (48,44,7,36) with distance $d^2 = 1.901$, $PC2 = 0.917$, $Con1 = 0.443$)

However, for $u = 4$, PIPS, all vertices have $Con2 \leq 0.4$
In this case, we project H_u onto center of $PC\nu$ axis, for $\nu = 2$, at

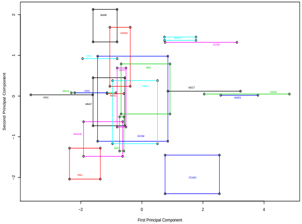
$$y^a_{\nu u} \equiv y^b_{\nu u} = \frac{1}{v_u} \sum_{k=1}^{v_u} y_{\nu_1 uk}$$

Here, this gives the point $y^a_{\nu u} \equiv y^b_{\nu u} = 0.432$

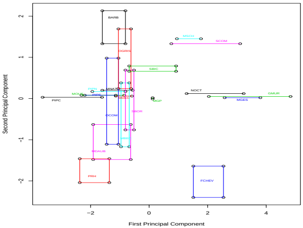
Bats: Symbolic Principal Components



$\alpha = 0$



$\alpha = 0.4$



$\alpha = 0.5$

Visualization - Interpretation

We have shown how, for prespecified $\alpha, PC\nu(\alpha)$ is (see eqn(5.24))

$$\begin{aligned} Y^*_{\nu u}(\alpha) &= [y^a_{\nu u}(\alpha), y^b_{\nu u}(\alpha)], \\ y^a_{\nu u}(\alpha) &= \min_{k_u \in L_u} \{y_{\nu u k_u} | Con(k_u, \nu) \geq \alpha\}, \\ y^b_{\nu u}(\alpha) &= \max_{k_u \in L_u} \{y_{\nu u k_u} | Con(k_u, \nu) \geq \alpha\}, \end{aligned} \quad (6)$$

Alternatively, we could replace $Con(k_u, \nu) \geq \alpha$ by

$$Con(k_u, \nu_1, \nu_2) = Con(k_u, \nu_1) + Con(k_u, \nu_2)$$

Relative contribution between $PC\nu$ and observation u (ω_u) i.e., H_u, is (eqn(5.17))

$$C_{\nu u} = Con_1(PC\nu, H_u) = \frac{\sum_{k_u=1}^{v_u} p^u_{k_u} y^2_{\nu u k_u}}{\sum_{k_u=1}^{v_u} p^u_{k_u} [d(k_u, \mathbf{G})]^2}$$

with weight $p^u_{k_u}$ at vertex k_u.

Compare this with **contribution** between $PC\nu$ and vertex k_u of ω_u

Relative contribution

For $u = 13$, BARB,

k_u	Y_1	Y_2	Y_3	Y_4	$\sum d_j^2$	$PC1$	$PC2$
1	44	41	6	35	**3.531**	-1.599	**0.910**
2	44	41	6	41	2.900	-1.246	0.762
3	44	41	8	35	1.577	-1.164	0.395
4	58	41	6	35	3.136	-1.043	1.046
5	44	54	6	35	5.263	-0.994	1.998
6	44	41	8	41	0.946	-0.811	0.246
7	58	41	6	41	2.505	-0.690	0.897
8	44	54	6	41	4.633	-0.641	1.849
9	58	41	8	35	1.182	-0.608	0.530
10	44	54	8	35	3.309	-0.560	1.482
11	58	54	6	35	4.868	-0.438	2.133
12	58	41	8	41	0.551	-0.256	0.381
13	44	54	8	41	2.679	-0.207	1.334
14	58	54	6	41	4.238	-0.086	1.985
15	58	54	8	35	2.914	-0.004	1.618
16	58	54	8	41	**2.284**	0.349	**1.469**

$$C_{\nu u} = \frac{\sum_{k_u=1}^{v_u} p_{k_u}^u y_{\nu u k_u}^2}{\sum_{k_u=1}^{v_u} p_{k_u}^u [d(k_u, \mathbf{G})]^2}$$

Vertices equally weighted:
for $PC\nu = PC1$,

$$C_{\nu u} = \frac{(-1.599)^2 + \cdots + (0.349)^2}{3.531 + \cdots + 2.284} = 0.219;$$

for $PC\nu = PC2$,

$$C_{\nu u} = \frac{(0.910)^2 + \cdots + (1.469)^2}{3.531 + \cdots + 2.284} = \mathbf{0.615};$$

u	Species	C_{1u}	C_{2u}	C_{3u}
1	PIPC	0.829	0.033	0.110
2	PRH	0.534	0.383	0.046
3	MOUS	0.841	0.064	0.063
4	PIPS	0.831	0.074	0.063
5	PIPN	0.786	0.145	0.036
6	MDAUB	0.492	0.271	0.122
7	MNAT	0.661	0.172	0.093
8	MDEC	0.571	0.050	0.282
9	MGP	0.088	0.619	0.189
10	OCOM	0.252	0.256	0.223
11	MBEC	0.213	0.449	0.235
12	SBOR	0.179	0.274	0.349
13	**BARB**	**0.219**	**0.615**	**0.104**
14	OGRIS	0.158	0.679	0.115
15	SBIC	0.244	0.201	0.324
16	FCHEV	0.401	0.423	0.082
17	MSCH	0.452	0.282	0.233
18	SCOM	0.743	0.108	0.096
19	NOCT	0.715	0.118	0.094
20	GMUR	0.866	0.053	0.045
21	MGES	0.888	0.022	0.085

Relative contribution

All BATS:

$$C_{\nu u} = Con_1(PC\nu, H_u) = \frac{\sum_{k_u=1}^{v_u} p_{k_u}^u y_{\nu u k_u}^2}{\sum_{k_u=1}^{v_u} p_{k_u}^u [d(k_u, \mathbf{G})]^2}$$

Other interpretation aids:

Eigenvalues of the Correlation Matrix:

ν	Eigenvalue	Difference	Proportion	Cumulative
1	**2.7081**	2.0215	**0.6770**	**0.6770**
2	0.6866	0.2947	0.1716	**0.8487**
3	0.3919	0.1784	0.0980	0.9466
4	0.2135		0.0534	1.0000

E.g.,

$Var(PC1) = \lambda_1 = \mathbf{2.7081}$

Also,
Total variation $= \lambda = \lambda_1 + \cdots + \lambda_p = p = 4$
Proportion variation explained by $PC\nu = \lambda_\nu / \lambda$

Hence,
Proportion variation explained by $PC1 = \lambda_1/\lambda = 2.7081/4 = \mathbf{0.6770}$
Proportion variation explained by $PC1$ and $PC2 = \mathbf{0.8487}$

Other interpretation aids:

Correlations between X_j and $PC\nu$ (eqn(5.19))

$$C_{\nu j} = Cor(PC\nu, X_j) = e_{\nu j}\sqrt{\lambda_\nu/\sigma_j^2}$$

	Eigenvectors			
	$PC1$	$PC2$	$PC3$	$PC4$
Y_1	0.5286	0.1287	-0.6764	0.4964
Y_2	0.4312	0.7763	0.4583	-0.0361
Y_3	0.4816	-0.5719	0.5455	0.3787
Y_4	0.5501	-0.2316	-0.1868	-0.7803

ν	Eigenvalue
1	**2.7081**
2	0.6866
3	0.3919
4	0.2135

E.g., correlation between $PC\nu = PC1$ and Y_1 = Head is

$$Cor(PC1, Y_1) = 0.5286\sqrt{(2.7081/1)} = 0.8699 \qquad (\sigma_j^2 = 1)$$

Similarly, for all ν, j to give $C\nu j$:

Y_j	$PC1$	$PC2$	$PC3$
Head	**0.8699**	0.1067	-0.4234
Tail	0.7097	0.6433	0.2869
Height	0.7926	-0.4739	0.3415
Forearm	0.9053	-0.1919	-0.1170

Inertia:

Contribution of observation/hypercube H_u to variance λ_ν is

$$I_{\nu u} = \text{Inertia}(PC\nu, H_u) = [\sum_{k_u=1}^{v_u} p^u_{k_u} y^2_{\nu u k_u}]/\lambda_\nu$$

where $p^u_{k_u}$ is weight associated with vertex k_u

Contribution of observation/hypercube H_u to total variance λ_ν is

$$I_u = \text{Inertia}(H_u) = \{\sum_{k_u=1}^{v_u} p^u_{k_u} [d(k_u, \mathbf{G})]^2\}/I_T,$$

where $I_T = \Sigma\lambda_\nu$

If all observations and all vertices have equal weight, then weights $p^u_{k_u} = p_u/q_u, \quad k_u = 1, \ldots, v_u$, with $p_u = 1/m, \quad u = 1, \ldots, m$

For $u = 13$, BARB,

k_u	Y_1	Y_2	Y_3	Y_4	$\sum d_j^2$	$PC1$
1	44	41	6	35	**3.531**	-1.599
2	44	41	6	41	2.900	-1.246
3	44	41	8	35	1.577	-1.164
4	58	41	6	35	3.136	-1.043
5	44	54	6	35	5.263	-0.994
6	44	41	8	41	0.946	-0.811
7	58	41	6	41	2.505	-0.690
8	44	54	6	41	4.633	-0.641
9	58	41	8	35	1.182	-0.608
10	44	54	8	35	3.309	-0.560
11	58	54	6	35	4.868	-0.438
12	58	41	8	41	0.551	-0.256
13	44	54	8	41	2.679	-0.207
14	58	54	6	41	4.238	-0.086
15	58	54	8	35	2.914	-0.004
16	58	54	8	41	**2.284**	0.349

Inertia(H_{13}):
$\nu = 1, \lambda_1 = 2.708, u = 13$

$$I_u = \{ \sum_{k_u=1}^{v_u} p^u_{k_u} [d(k_u, \mathbf{G})]^2 \} / I_T$$

I.e., Contribution of H_{13} to total variance λ_1 is

$$I_{1u} = \frac{(3.351)^2 + \cdots + (2.284)^2}{21 \times 16 \times 4}$$
$$= \mathbf{0.035}$$

$$(p^u_{k_u} = \frac{1}{21}\frac{1}{16})$$

For all BATS, Inertias

u	Species	I_{1u}	I_{2u}	I_{3u}	I_{4u}	I_u
1	PIPC	**0.126**	0.020	0.116	0.053	**0.103**
2	PRH	0.045	**0.128**	0.027	0.039	0.057
3	MOUS	0.046	0.014	0.024	0.023	0.037
4	PIPS	0.050	0.018	0.026	0.025	0.041
5	PIPN	0.035	0.025	0.011	0.019	0.030
6	MDAUB	0.020	0.043	0.034	0.059	0.027
7	MNAT	0.017	0.017	0.016	0.024	0.017
8	MDEC	0.011	0.004	0.037	0.023	0.013
9	MGP	0.002	0.058	0.031	0.032	0.016
10	OCOM	0.009	0.035	0.054	0.119	0.024
11	MBEC	0.003	0.028	0.026	0.021	0.011
12	SBOR	0.002	0.015	0.033	0.035	0.009
13	BARB	0.011	**0.124**	0.037	0.040	0.035
14	OGRIS	0.005	0.078	0.023	0.018	0.020
15	SBIC	0.003	0.011	0.030	0.039	0.009
16	FCHEV	0.038	**0.159**	0.054	0.115	0.065
17	MSCH	0.023	0.057	0.083	0.021	0.035
18	SCOM	0.072	0.042	0.065	0.065	0.066
19	NOCT	0.080	0.052	0.072	0.102	0.075
20	GMUR	**0.216**	0.052	0.078	0.113	**0.169**
21	MGES	**0.181**	0.017	0.120	0.013	**0.138**

E.g.,
PIPC accounts for 10.3% of total variation I_u,
GMUR for 16.9%,
MGES for 13.8%,
and so on...;

PIPC accounts for 12.6% of λ_1 variation I_{1u},
GMUR for 21.6%,
MGES for 18.1%,
and so on...;

And,
PRH accounts for **12.8%** of λ_2 variation I_{2u},
BARB for **12.4%,**
FCHEV for **15.9%,**
and so on...

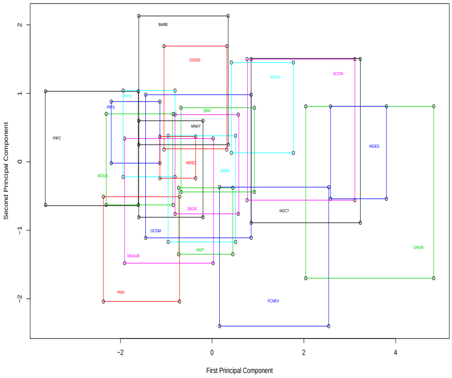
Visualization - Interpretation

u	Species	I_{1u}	I_{2u}	I_u
1	**PIPC**	**0.126**	0.020	**0.103**
2	**PRH**	0.045	**0.128**	0.057
3	MOUS	0.046	0.014	0.037
4	PIPS	0.050	0.018	0.041
5	PIPN	0.035	0.025	0.030
6	MDAUB	0.020	0.043	0.027
7	MNAT	0.017	0.017	0.017
8	MDEC	0.011	0.004	0.013
9	MGP	0.002	0.058	0.016
10	OCOM	0.009	0.035	0.024
11	MBEC	0.003	0.028	0.011
12	SBOR	0.002	0.015	0.009
13	**BARB**	0.011	**0.124**	0.035
14	OGRIS	0.005	0.078	0.020
15	SBIC	0.003	0.011	0.009
16	FCHEV	0.038	**0.159**	0.065
17	MSCH	0.023	0.057	0.035
18	SCOM	0.072	0.042	0.066
19	NOCT	0.080	0.052	0.075
20	GMUR	**0.216**	0.052	**0.169**
21	MGES	**0.181**	0.017	**0.138**

$I_{\nu u}$ = Contribution of H_u to variance λ_u

I_u = Contribution of H_u to total variance

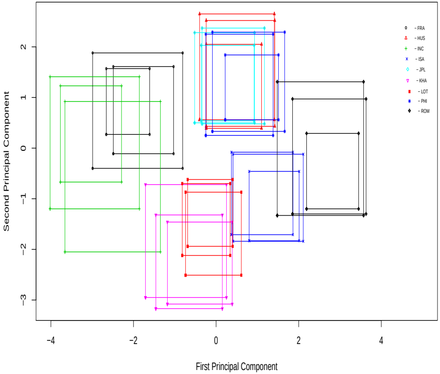
For all BATS, Inertias



Faces dataset

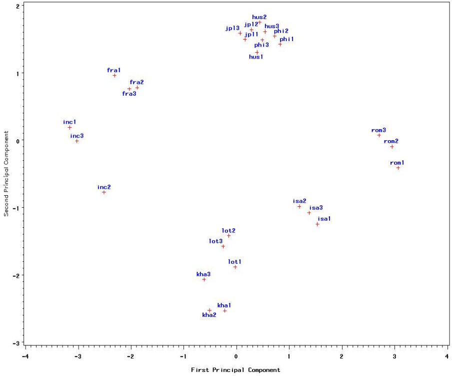
Subject	X_1 = AD	X_2 = BC	X_3 = AH	X_4 = DH	X_5 = EH	X_6 = GH
FRA1	[155.00, 157.00]	[58.00, 61.01]	[100.45, 103.28]	[105.00, 107.30]	[61.40, 65.73]	[64.20, 67.80]
FRA2	[154.00, 160.01]	[57.00, 64.00]	[101.98, 105.55]	[104.35, 107.30]	[60.88, 63.03]	[62.94, 66.47]
FRA3	[154.01, 161.00]	[57.00, 63.00]	[99.36, 105.65]	[101.04, 109.04]	[60.95, 65.60]	[60.42, 66.40]
HUS1	[168.86, 172.84]	[58.55, 63.39]	[102.83, 106.53]	[122.38, 124.52]	[56.73, 61.07]	[60.44, 64.54]
HUS2	[169.85, 175.03]	[60.21, 64.38]	[102.94, 108.71]	[120.24, 124.52]	[56.73, 62.37]	[60.44, 66.84]
HUS3	[168.76, 175.15]	[61.40, 63.51]	[104.35, 107.45]	[120.93, 125.18]	[57.20, 61.72]	[58.14, 67.08]
INC1	[155.26, 160.45]	[53.15, 60.21]	[95.88, 98.49]	[91.68, 94.37]	[62.48, 66.22]	[58.90, 63.13]
INC2	[156.26, 161.31]	[51.09, 60.07]	[95.77, 99.36]	[91.21, 96.83]	[54.92, 64.20]	[54.41, 61.55]
INC3	[154.47, 160.31]	[55.08, 59.03]	[93.54, 98.98]	[90.43, 96.43]	[59.03, 65.86]	[55.97, 65.80]
ISA1	[164.00, 168.00]	[55.01, 60.03]	[120.28, 123.04]	[117.52, 121.02]	[54.38, 57.45]	[50.80, 53.25]
ISA2	[163.00, 170.00]	[54.04, 59.00]	[118.80, 123.04]	[116.67, 120.24]	[55.47, 58.67]	[52.43, 55.23]
ISA3	[164.01, 169.01]	[55.00, 59.01]	[117.38, 123.11]	[116.67, 122.43]	[52.80, 58.31]	[52.20, 55.47]
JPL1	[167.11, 171.19]	[61.03, 65.01]	[118.23, 121.82]	[108.30, 111.20]	[63.89, 67.88]	[57.28, 60.83]
JPL2	[169.14, 173.18]	[60.07, 65.07]	[118.85, 120.88]	[108.98, 113.17]	[62.63, 69.07]	[57.38, 61.62]
JPL3	[169.03, 170.11]	[59.01, 65.01]	[115.88, 121.38]	[110.34, 112.49]	[61.72, 68.25]	[59.46, 62.94]
KHA1	[149.34, 155.54]	[54.15, 59.14]	[111.95, 115.75]	[105.36, 111.07]	[54.20, 58.14]	[48.27, 50.61]
KHA2	[149.34, 155.32]	[52.04, 58.22]	[111.20, 113.22]	[105.36, 111.07]	[53.71, 58.14]	[49.41, 52.80]
KHA3	[150.33, 157.26]	[52.09, 60.21]	[109.04, 112.70]	[104.74, 111.07]	[55.47, 60.03]	[49.20, 53.41]
LOT1	[152.64, 157.62]	[51.35, 56.22]	[116.73, 119.67]	[114.62, 117.41]	[55.44, 59.55]	[53.01, 56.60]
LOT2	[154.64, 157.62]	[52.24, 56.32]	[117.52, 119.67]	[114.28, 117.41]	[57.63, 60.61]	[54.41, 57.98]
LOT3	[154.83, 157.81]	[50.36, 55.23]	[117.59, 119.75]	[114.04, 116.83]	[56.64, 61.07]	[55.23, 57.80]
PHI1	[163.08, 167.07]	[66.03, 68.07]	[115.26, 119.60]	[116.10, 121.02]	[60.96, 65.30]	[57.01, 59.82]
PHI2	[164.00, 168.03]	[65.03, 68.12]	[114.55, 119.60]	[115.26, 120.97]	[60.96, 67.27]	[55.32, 61.52]
PHI3	[161.01, 167.00]	[64.07, 69.01]	[116.67, 118.79]	[114.59, 118.83]	[61.52, 68.68]	[56.57, 60.11]
ROM1	[167.15, 171.24]	[64.07, 68.07]	[123.75, 126.59]	[122.92, 126.37]	[51.22, 54.64]	[49.65, 53.71]
ROM2	[168.15, 172.14]	[63.13, 68.07]	[122.33, 127.29]	[124.08, 127.14]	[50.22, 57.14]	[49.93, 56.94]
ROM3	[167.11, 171.19]	[63.13, 68.03]	[121.62, 126.57]	[122.58, 127.78]	[49.41, 57.28]	[50.99, 60.46]

Faces dataset: Symbolic Principal Components

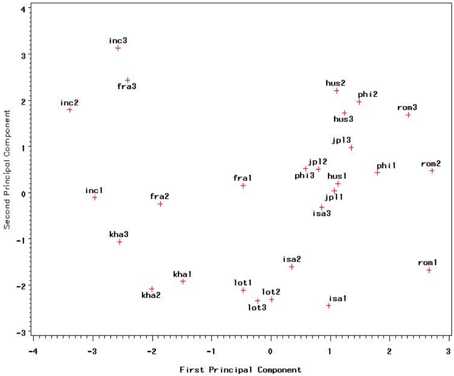


Notice the coherency for the sets of (3) faces

Faces dataset: Classical surrogates PCA



Interval Centers/Midpoints



Interval Midpoint and Ranges

Not coherent

(Fuzzy Analysis)