Using Auxiliary Information Under a Generic Sampling Design

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1 Theoretical results
   - Auxiliary information
   - A class of estimators
   - The best estimator

2 Simulation results
   - Simulated $\pi_i$ and $\pi_{ij}$
   - Accuracy of $\pi_i$ and $\pi_{ij}$

3 Conclusions
Auxiliary information

- Auxiliary information plays a relevant role in sampling to obtain improved design and/or more efficient estimators

- When auxiliary information is used at the estimation stage, the ratio, product and regression methods are widely employed in many situations

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Motivated by Bacanli and Kadilar (BK, 2008), we intend to show how the problem of finding the best estimator for the mean of a study variable can be treated under a generic sampling design by means of a very simple class of estimators.

- The class is not exhaustive and a more general discussion can be found, among others, in Diana and Perri (2007).
- The best estimator in the class is compared with BK estimators according to UPS, where inclusion probability are computed on the basis of a limited numbers of samples.
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Notation

- $U = \{1, 2, \ldots, N\}$ a finite population
- $Y$ a study variable with unknown mean $\bar{Y} = N^{-1} \sum_{i=1}^{N} y_i$
- $X$ an auxiliary variable with $\bar{X} = N^{-1} \sum_{i=1}^{N} x_i$ known
- $p(s)$ a generic sampling design
- $s$ a sample of size $n$ from $p(s)$
- $\pi_i = \sum_{s \ni i} p(s)$ and $\pi_{ij} = \sum_{s \ni (i,j)} p(s)$ the first and second order inclusion probabilities
- $\hat{Y}, \hat{X}$ two unbiased estimators of $\bar{Y}, \bar{X}$ under $p(s)$
- $\tau$ a constant that may be related to population parameters
A class of estimators for $\bar{Y}$

We introduce a very simple class of estimators for $\bar{Y}$ as

$$\hat{Y}_{pr} = \frac{\bar{Y}}{\hat{X} + \tau} \bar{X} + \tau$$

Expanding it in a Taylor’s series ($\delta$-method) and retaining only terms up to the second degree, we get - for $n$ sufficiently large - the first order approximation of the bias (B) and mean square error (MSE)

$$B(\hat{Y}_{pr}) = \frac{1}{\bar{X} + \tau} \left[ \frac{\bar{Y} \text{Var}(\hat{X})}{\bar{X} + \tau} - \text{Cov}(\hat{X}, \hat{Y}) \right]$$

$$\text{MSE}(\hat{Y}_{pr}) = \text{Var}(\hat{Y}) + \frac{\bar{Y}^2 \text{Var}(\hat{X})}{(\bar{X} + \tau)^2} - \frac{2\bar{Y} \text{Cov}(\hat{X}, \hat{Y})}{\bar{X} + \tau}$$
Optimality of the class

Minimization of $MSE(\hat{Y}_{pr})$ is achieved for

$$\tau = \bar{X} \left[ \frac{C(\hat{X})^2 - C(\hat{X}, \hat{Y})}{C(\hat{X}, \hat{Y})} \right]$$

with $C(\hat{X}) = \sqrt{Var(\hat{X})/\bar{X}}$, $C(\hat{X}, \hat{Y}) = Cov(\hat{X}, \hat{Y})/\bar{X}\bar{Y}$. For this optimum choice, we get

$$\min MSE(\hat{Y}_{pr}) = Var(\hat{Y})(1 - \rho_{\bar{X},\bar{Y}}^2)$$

which is the variance of the regression estimator

$$\hat{Y}_{lr} = \hat{Y} + \beta_{\hat{Y},\hat{X}}(\bar{X} - \hat{X})$$
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$$\hat{Y}_{lr} = \hat{Y} + \beta_{\hat{Y},\hat{X}}(\bar{X} - \hat{X})$$

Optimality of $\hat{Y}_{lr}$ is well-known in sampling theory but this aspect is very often overlooked. Why?
Efficiency considerations

- All the estimators belonging to the class can be only, at best, as efficient as $\hat{Y}_{lr}$. They are equivalent to it only when

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- For instance, the following estimators (in SRSWOR) are not optimum in the class

<table>
<thead>
<tr>
<th>Authors</th>
<th>Estimators</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sisodia and Dwivedi (1981)</td>
<td>$\hat{Y}_{SD} = \hat{Y} \frac{\hat{X} + C_x}{\hat{X} + C_x}$</td>
<td>$C_x$</td>
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<tr>
<td>Singh and Kakran (1993)</td>
<td>$\hat{Y}_{SK} = \hat{Y} \frac{\hat{X} + \beta_2(x)}{\hat{X} + \beta_2(x)}$</td>
<td>$\beta_2(x)$</td>
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<tr>
<td>Upadhyaya and Singh (1999)</td>
<td>$\hat{Y}_{US1} = \hat{Y} \frac{\bar{X}/\beta_2(x) + C_x}{\bar{X}/\beta_2(x) + C_x}$</td>
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Bacanli-Kadilar estimators

Previous estimators have been considered by Bacanli and Kadilar (2008) under UPSWOR by replacing \( \hat{Y} \) and \( \hat{X} \) with Horvitz-Thompson estimator

\[
\hat{T}_{HT} = \frac{1}{N} \sum_{i \in s} \frac{t_i}{\pi_i}, \quad t = x, y
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with

\[
Var(\hat{T}_{HT}) = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \frac{\pi_{ij} - \pi_i \pi_j}{\pi_i \pi_j} \right) t_i t_j, \quad t = x, y
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- The modified estimators have been analytically compared with the ratio estimator $\hat{Y}_r = (\hat{Y}_{HT}/\hat{X}_{HT})\hat{X}$
- Numerical comparisons have been performed by using exact expressions for $\pi_i$ and $\pi_{ij}$ inherited from the adaptive cluster sampling
Some questions

1. BK estimators belong to the proposed class but they are not optimum

2. Why to compare these estimators with $\hat{Y}_r = (\hat{Y}_{HT}/\hat{X}_{HT})\bar{X}$ and not with $\hat{Y}_{lr} = \hat{Y}_{HT} + \beta_{\hat{Y}_{HT},\hat{X}_{HT}} (\bar{X} - \hat{X}_{HT})$?

   It is well-known that $MSE(\hat{Y}_r) \geq MSE(\hat{Y}_{lr})$

   BK estimators can not outperform $\hat{Y}_{lr}$

3. The use of the exact expressions for $\pi_i$ and $\pi_{ij}$ from adaptive cluster sampling seems to be rather questionable
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   Possible solution: $\hat{\pi}_i$ and $\hat{\pi}_{ij}$
The explicit derivation of $\pi_i$ and $\pi_{ij}$ becomes prohibitive when $N$ and/or $n$ increase: $\binom{N}{n}$ samples are to be investigated.

To overcome the problem, a solution can be adopted by simulating $\pi_i$ and $\pi_{ij}$.

implement in R the procedure drawing the PPS samples by
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sample(U, n, replace=FALSE, prob=p)
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Computing $\pi_i$ and $\pi_{ij}$

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### Algorithm

1. Each unit has a selection probability $p_i = z_i / \sum_{j=1}^{N} z_j$
2. $M < \binom{N}{n}$ samples WOR are independently drawn from $U$
3. $M_i$ and $M_{ij}$ are the number of samples that contain unit $i$ and units $(i,j)$
4. Estimate $\pi_i$ and $\pi_{ij}$ with $\hat{\pi}_i = M_i / M$ and $\hat{\pi}_{ij} = M_{ij} / M$
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Numerical examples

- Numerical study on real data from Cochran (1977, p. 34): weekly expenditure on food ($y$), weekly family income ($x$) and number of persons per family ($z$)
- $N = 32$ and $n = 10, 15, 20$.
- $M = 100,000$ samples are considered instead of $\binom{32}{n}$.
- To evaluate the performance of $\hat{\pi}_i$ and $\hat{\pi}_{ij}$, we compare the efficiency of $\hat{Y}_{lr}$ with that of BK estimators.
- Two situations are considered
  1. **EPS**: units are selected according to SRSWOR for which $\pi_i$ and $\pi_{ij}$ are known in advance: $\pi_i = n/N$ and $\pi_{ij} = \frac{n(n-1)}{N(N-1)}$.
  2. **UPS**: units are selected according to Midzuno scheme: $\pi_i = p_i + (1 - p_i) \frac{n-1}{N-1}$ and $\pi_{ij} = \frac{n-1}{N-1} \left[ \frac{N-n}{N-2} (p_i + p_j) + \frac{n-2}{N-2} \right]$.
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References


SINGH, H.P., KAKRAN, M.S.(1993): A modified ratio estimator using known coefficient of kurtosis of an auxiliary character. *Unpublished manuscript*

