Monotone Graphical Multivariate Markov Chains

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Outline

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Graphical models for Markov chains

The idea of graphical models is to represent the dependence structure of a multivariate random vector by a graph, where the nodes correspond to the variables and the edges between nodes describe the association structure among the variables

We apply a graphical approach to analyze the dynamic relationships among the marginal processes of a multivariate Markov chain

Graphical models for Markov chains

We apply a graphical approach to analyze the dynamic relationships among the marginal processes of a multivariate Markov chain

Our graphical approach offers:

- a graphical representation that allows a direct and intuitive understanding of the dynamic associations which can exist among the processes of an MMC
- the possibility to investigate potential causal, monotone dependence and contemporaneous relationships by testing simple hypotheses on parameters

Multivariate Markov chain: basic notation

$$\mathbf{A}_{\mathcal{V}} = \{A_j(t) : t = 0, 1, 2..., \quad j \in \mathcal{V}\} \qquad \mathcal{V} = \{1, ..., q\}$$

- an **univariate process** $A_j(t)$ takes values on $\mathcal{A}_j = \{1, 2, ..., s_j\}$ $j \in \mathcal{V}$
- for $\mathcal{S} \subset \mathcal{V}$, a **marginal process** is $\mathbf{A}_{\mathcal{S}} = \{A_j(t) : t = 0, 1, 2..., j \in \mathcal{S}\}$
- $\overline{A}_{\mathcal{V}}(t-1) = \{A_{\mathcal{V}}(r) : r \leq t-1\}$ is the **past** history up to $t-1$ of $\mathbf{A}_{\mathcal{V}}$
- $\times_{j \in \mathcal{V}} \mathcal{A}_j$ is the **joint state space**

$\mathbf{A}_{\mathcal{V}}$ is a **first order multivariate Markov chain** (with q components)

$$A_{\mathcal{V}}(t) \perp\!\!\!\perp \overline{A}_{\mathcal{V}}(t-2) | A_{\mathcal{V}}(t-1) \qquad \forall t = 1, 2, ...$$

In general, different types of dependence relations are relevant when the time dimension of the variables is taken into account:

- the effect of the past of a process on the present of another
  - $\hookrightarrow$ **Granger non-causality**
  - $\hookrightarrow$ **monotone dependence coherent with a stochastic ordering**
- the relation among processes at the same time
  - $\hookrightarrow$ **contemporaneous dependence**

Dynamic relations

Given 2 disjoint marginal processes $\mathbf{A}_{\mathcal{T}}$ and $\mathbf{A}_{\mathcal{S}}$ of an MMC $\mathbf{A}_{\mathcal{V}}$

- i) *Granger non-causality condition*

$\mathbf{A}_{\mathcal{T}}$ is not Granger caused by $\mathbf{A}_{\mathcal{S}}$ with respect to $\mathbf{A}_{\mathcal{V}}$

$$\Leftrightarrow \quad A_{\mathcal{T}}(t) \perp\!\!\!\perp A_{\mathcal{S}}(t-1) | A_{\mathcal{V}\setminus\mathcal{S}}(t-1) \quad \forall t = 1, 2, \ldots$$

the past of $\mathbf{A}_{\mathcal{S}}$ does not contain additional information on the present of $\mathbf{A}_{\mathcal{T}}$, given the past of $\mathbf{A}_{\mathcal{V}\setminus\mathcal{S}}$

Dynamic relations

Given 2 disjoint marginal processes $\mathbf{A}_{\mathcal{T}}$ and $\mathbf{A}_{\mathcal{S}}$
of an MMC $\mathbf{A}_{\mathcal{V}}$

- ii) *Contemporaneous independence condition*

  $\mathbf{A}_{\mathcal{T}}$ and $\mathbf{A}_{\mathcal{S}}$ are contemporaneously conditionally independent with respect to $\mathbf{A}_{\mathcal{V}}$

$$\Leftrightarrow \quad A_{\mathcal{T}}(t) \perp\!\!\!\perp A_{\mathcal{S}}(t) | A_{\mathcal{V}}(t-1) \quad \forall t = 1, 2, \ldots$$

two marginal processes are independent at each time point, given past information on all processes of the chain

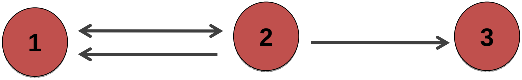
ME graphs

a **Multi Edge graph**
encodes the G-noncausal and contemporaneous independence relations among the marginal processes of an MMC

in an ME graph $G = (\mathcal{V}, \mathcal{E})$, the nodes in the set $\mathcal{V}$ represent the univariate components of an MMC and directed and bi-directed edges in the set $\mathcal{E}$ describe the dependence structure among them

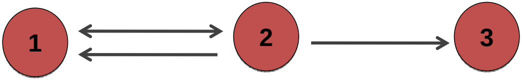
In a multi edge graph

- there exists a one-to-one correspondence between the nodes $j \in \mathcal{V}$ and the univariate processes $\mathbf{A}_j$, $j \in \mathcal{V}$, of the MMC $\mathbf{A}_\mathcal{V}$
- any pair of nodes, $i, k \in \mathcal{V}$, may be joined by directed edges $i \rightarrow k$, $i \leftarrow k$, and by bi-directed edge $i \leftrightarrow k$
- each pair of distinct nodes can be connected by up to all the 3 types of edges
- sets of G-noncausality and contemporaneous independence restrictions are associated with missing directed and bi-directed edges, respectively
- Example $\mathcal{V} = \{1, 2, 3\}$ $\mathcal{E} = \{2 \rightarrow 1, 2 \rightarrow 3, 1 \leftrightarrow 2\}$



Graph terminology

- ♢ if there is $i \to j$, then i is a *parent* of j,
  $Pa(\mathcal{S}) = \{i \in \mathcal{V} : i \to j \in \mathcal{E}, j \in \mathcal{S}\}$ is the *set of parents* of $\mathcal{S} \subset \mathcal{V}$
- ♢ if there is $i \leftrightarrow j$, the nodes i, j are *neighbors*,
  $Nb(\mathcal{S}) = \{i \in \mathcal{V} : i \leftrightarrow j \in \mathcal{E}, j \in \mathcal{S}\}$ is the *set of neighbors* of $\mathcal{S} \subset \mathcal{V}$



✓ in the example: $Pa(1) = \{1, 2\}$, $Pa(2) = \{2\}$, $Pa(3) = \{2, 3\}$;
$Nb(1) = \{1, 2\}$, $Nb(2) = \{1, 2\}$, $Nb(3) = \{3\}$

Graphical models

Graphical models associate missing edges of a graph with some conditional independence restrictions imposed on a multivariate probability distribution

In the multi edge graphical models for MMC missing edges have a direct significance in terms of G-noncausal and contemporaneous independence restrictions imposed on the transition probabilities

Markov properties of ME graphs

Graphical MMC

An MMC is graphical with respect to an ME graph $G = (\mathcal{V}, \mathcal{E})$ iff its transition probabilities satisfy the following conditional independencies for all $t = 1, 2, \ldots$

C1) $A_{\mathcal{S}}(t) \perp\!\!\!\perp A_{\mathcal{V} \setminus Pa(\mathcal{S})}(t-1) | A_{Pa(\mathcal{S})}(t-1) \qquad \forall \mathcal{S} \in \mathcal{P}(\mathcal{V})$

C2) $A_{\mathcal{S}}(t) \perp\!\!\!\perp A_{\mathcal{V} \setminus Nb(\mathcal{S})}(t) | A_{\mathcal{V}}(t-1) \qquad \forall \mathcal{S} \in \mathcal{P}(\mathcal{V})$

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Condition $\mathbf{C1}$)

- the past of $\mathbf{A}_{\mathcal{V}\setminus Pa(\mathcal{S})}$ is not informative for the present of $\mathbf{A}_{\mathcal{S}}$ as long as we know the past of $Pa(\mathcal{S})$
- is a **G-noncausality** condition
- $\mathbf{A}_{\mathcal{V}\setminus Pa(\mathcal{S})} \nrightarrow \mathbf{A}_{\mathcal{S}}$, i.e. $\mathbf{A}_{\mathcal{S}}$ is not G-caused by $\mathbf{A}_{\mathcal{V}\setminus Pa(\mathcal{S})}$ wrt $\mathbf{A}_{\mathcal{V}}$
- corresponds to missing directed edges
- refers to processes at two consecutive time-points

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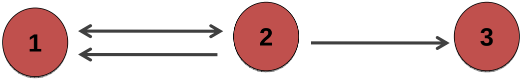
C2) $A_{\mathcal{S}}(t) \perp\!\!\!\perp A_{\mathcal{V}\setminus Nb(\mathcal{S})}(t) | A_{\mathcal{V}}(t-1) \qquad \forall \mathcal{S} \in \mathcal{P}(\mathcal{V})$

Condition C2)

- $\mathbf{A}_{\mathcal{S}}$ and $\mathbf{A}_{\mathcal{V}\setminus Nb(\mathcal{S})}$ are independent of each other at any point in time as long as we know the past of $\mathbf{A}_{\mathcal{V}}$
- is a **contemporaneous independence** condition
- $\mathbf{A}_{\mathcal{S}} \not\leftrightarrow \mathbf{A}_{\mathcal{V}\setminus Nb(\mathcal{S})}$, i.e. $\mathbf{A}_{\mathcal{S}}$ and $\mathbf{A}_{\mathcal{V}\setminus Nb(\mathcal{S})}$ are contemporaneously independent wrt $\mathbf{A}_{\mathcal{V}}$
- corresponds to missing bi-directed edges
- refers to processes at the same time-points

Example: Reading G-noncausal and CI restrictions **C1**) and **C2**) off an ME graph



- the G-noncausality conditions associated to the missing directed edges in the graph are:

$$\mathbf{A}_{\{1,3\}} \nrightarrow \mathbf{A}_2; \quad \mathbf{A}_1 \nrightarrow \mathbf{A}_{\{2,3\}}; \quad \mathbf{A}_3 \nrightarrow \mathbf{A}_{\{1,2\}}$$

- the contemporaneous independence condition associated to the missing bi-directed edges in the graph is:

$$\mathbf{A}_3 \nleftrightarrow \mathbf{A}_{\{1,2\}}$$

Monotone dependence

Given 2 variables A_j and A_k
with ordered categories in the sets $\mathcal{A}_j$ and $\mathcal{A}_k$

if a **monotone dependence of A_j on A_k** exists:

the conditional distributions of A_j given A_k
become stochastically greater in a coherent way with the order of the categories of A_k in $\mathcal{A}_k$

Monotone dependence

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if a monotone dependence of A_j on A_k exists:

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Stochastic orderings

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Stochastic orderings

Simple

$$(A_j|A_k = k) \preceq_s (A_j|A_k = k+1) \leftrightsquigarrow P\left[A_j \leq j|A_k = k\right] \geq P\left[A_j \leq j|A_k = k+1\right]$$

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Stochastic orderings

Uniform

$$(A_j|A_k = k) \preceq_u (A_j|A_k = k+1) \leftrightsquigarrow P\left[A_j > j|A_j \geq j, A_k = k\right] \leq P\left[A_j > j|A_j \geq j, A_k = k+1\right]$$

Monotone dependence

Given 2 variables A_j and A_k
with ordered categories in the sets $\mathcal{A}_j$ and $\mathcal{A}_k$

if a monotone dependence of A_j on A_k exists:

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Stochastic orderings

Likelihood ratio

$$(A_j|A_k = k) \preceq_{lr} (A_j|A_k = k+1) \leftrightsquigarrow \frac{P[A_j{=}j|A_k{=}k]}{P[A_j{=}j{+}1|A_k{=}k]} \leq \frac{P[A_j{=}j|A_k{=}k{+}1]}{P[A_j{=}j{+}1|A_k{=}k{+}1]}$$

Monotone Graphical MMC

A graphical MMC is monotone with respect to an ME graph $G = (\mathcal{V}, \mathcal{E})$ iff there exists at least one $\mathbf{A}_j$, $j \subseteq \mathcal{V}$, whose dependence on its parents is monotone

- the dependence of $A_j(t)$ on $A_k(t-1)$, $\forall k \in Pa(j)$, is monotone $\forall t$
- the distributions of $A_j(t)$ conditioned by $A_{Pa(j)}(t-1)$ can be partially ordered coherently with the orderings on the sets $\mathcal{A}_k$, $k \in Pa(j)$ according to a stochastic dominance criterion (*simple, uniform, LR*)
- NB. the dominance criterion concerns only the marginal processes in an MMC and does not refer to their joint behavior

A multivariate logistic model for transition probabilities

G-noncausality, CI and monotone dependence conditions are equivalent to equality and inequality constraints on interactions of a multivariate logistic model which parameterize the transition probabilities

- $\mathcal{I} = \times_{j \in \mathcal{V}} \mathcal{A}_j$ is the joint state space
- $\boldsymbol{i} = (i_1, i_2, ..., i_q)' \in \mathcal{I}$ is a state
- for a pair of states $\boldsymbol{i} \in \mathcal{I}$, $\boldsymbol{i}' \in \mathcal{I}$, $p(\boldsymbol{i} \,|\, \boldsymbol{i}')$ are the joint transition probabilities

A multivariate logistic model for transition probabilities

Given $\boldsymbol{i'} \in \mathcal{I}$,

for $p(\boldsymbol{i}\,|\boldsymbol{i'})$, $\boldsymbol{i} \in \mathcal{I}$, we adopt a Gloneck-McCullagh multivariate logistic model whose marginal interaction parameters

$$\eta^P(\boldsymbol{i}_P|\boldsymbol{i'}) \qquad P \subseteq \mathcal{V}, P \neq \emptyset,\ \boldsymbol{i}_P \in \times_{j\in P}\mathcal{A}_j$$

are contrasts of logarithms of marginal transition probabilities $p(\boldsymbol{i}_P\,|\boldsymbol{i'})$

- G-noncausality and CI relations correspond to equality constraints on the $\eta^P(\boldsymbol{i}_P|\boldsymbol{i'})$ while hypotheses of monotone dependence impose inequality constraints on $\eta^P(\boldsymbol{i}_P|\boldsymbol{i'})$

Testing dynamic relations

For an MMC with $p(\boldsymbol{i}\,|\boldsymbol{i'}) > 0$ it holds that, for $\boldsymbol{i}_P \in \times_{j\in P}\mathcal{A}_j$

- G-noncausality condition $\mathbf{C1})$ $\Leftrightarrow$ $\eta^{P}(\boldsymbol{i}_P|\boldsymbol{i'}) = \eta^{P}(\boldsymbol{i}_P|\boldsymbol{i'}_{Pa(P)}), P \subseteq \mathcal{V}, P \neq \emptyset$
- CI condition $\mathbf{C2})$ $\Leftrightarrow$ $\eta^{P}(\boldsymbol{i}_P|\boldsymbol{i'}) = 0$ P is not a bi-connected set

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- the requirements $\mathbf{C1}), \mathbf{C2})$ for a graphical MMC correspond to simple linear constraints on the $\eta^P(\boldsymbol{i}_P|\boldsymbol{i}')$ parameters
- testing the hypotheses $\mathbf{C1}), \mathbf{C2})$ is a standard parametric problem
- the restrictions under $\mathbf{C1}), \mathbf{C2})$ can be rewritten as $\boldsymbol{C}\ln(\boldsymbol{M\pi}) = \mathbf{0}$

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Testing monotone dependence

For a graphical MMC with $p(\boldsymbol{i} \mid \boldsymbol{i}') > 0$ it holds that

- positive (negative) monotone dependence

$$\Leftrightarrow$$

$$\eta^j(\boldsymbol{i}_j|\boldsymbol{i}'_{Pa(j)\setminus k}, i'_k) \leq (\geq)\, \eta^j(\boldsymbol{i}_j|\boldsymbol{i}'_{Pa(j)\setminus k}, i'_k + 1),$$

$$k \in Pa(j), j \in \mathcal{M}, \mathcal{M} \subseteq \mathcal{V}, \mathcal{M} \neq \emptyset$$

Testing monotone dependence

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$$k \in Pa(j), j \in \mathcal{M}, \mathcal{M} \subseteq \mathcal{V}, \mathcal{M} \neq \emptyset$$

- The simple, uniform and likelihood ratio orderings are obtained when the logits $\eta^j(\boldsymbol{i}_j|\boldsymbol{i}'_{Pa(j)})$ subjected to inequality constraints are of global, continuation and local types
- the inequality constraints for monotone dependence have a compact expression given by $\boldsymbol{K} \ln(\boldsymbol{M}\boldsymbol{\pi}) \geq \boldsymbol{0}$

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Likelihood ratio tests

- H_G: $\boldsymbol{C}\ln(\boldsymbol{M}\boldsymbol{\pi}) = \boldsymbol{0}$ (graphical MMC)
  H_M: $\boldsymbol{C}\ln(\boldsymbol{M}\boldsymbol{\pi}) = \boldsymbol{0},\ \boldsymbol{K}\ln(\boldsymbol{M}\boldsymbol{\pi}) \geq \boldsymbol{0}$ (monotone graphical MMC)
  H_U: unrestricted model
- L_G, L_M, L_U denote the max log-likelihood functions under H_G, H_M, H_U

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Testing equality constraints: H_G vs H_U

under suitable assumptions (*Fahrmeir and Kaufmann, 1987*)

$\Rightarrow$ ***the null asymptotic distribution of the statistic***

$$LRT = 2(L_U - L_G) \text{ is } \chi^2$$

Likelihood ratio tests

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Testing inequality constraints: H_M vs H_G and H_M vs H_U

- under suitable assumptions (*Fahrmeir and Kaufmann, 1987*)
- the parametric space under H_M is defined by linear inequality constraints

$\Rightarrow$ the statistics $LRT = 2(L_G - L_M)$ and $LRT = 2(L_U - L_M)$ are asymptotically chi-bar-squared $\bar{\chi}^2$ distributed

($\bar{\chi}^2$ is a mixture of χ^2's, *Silvapulle and Sen, 2005*)

Computational procedures:

- ML estimation methods for multinomial data under equality and inequality constraints are adapted to the MMC context
- Monte Carlo methods to simulate the asymptotic p-values of the LRT statistics $2(L_G - L_M)$ and $2(L_U - L_M)$
- procedures for computing ML estimates and p-values are implemented in the R-package *hmmm* by Colombi

In summary, our approach provides

missing directed and bi-directed edges in ME graph

$\Leftrightarrow$

G-noncausality and CI conditions

$\Leftrightarrow$

linear constraints on interactions which parameterize the transition probabilities

monotone dependence

$\Leftrightarrow$

inequality constraints on interactions which parameterize the transition probabilities

- Example: ME graph for pasta spaghetti data

  DATA:

  - binary data collected on 365 days (Dec. 2006 - Jan. 2009) by an Italian wholesale dealer
  - a 3-dimensional binary time series of sales levels of 3 Italian brands (*Amato* A_1, *Barilla* A_2, *Divella* A_3) of pasta (*spaghetti*)
  - 3-variate Markov chain of spaghetti data $\{A_1, A_2, A_3\}$ with categories: low and high level

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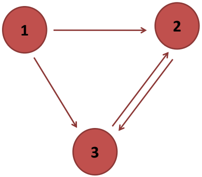
the MC of spaghetti data is monotone graphical wrt the ME graph



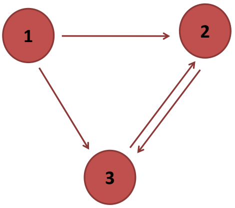
- test for CI: $LRT = 44.81, p = 0.44$
- test for G-noncausality and monotone dep. $LRT = 0, p = 1$

- Example: ME graph for pasta spaghetti data

the graph encodes:

the G-noncausality and CI relations:

$$\mathbf{A}_{\{2,3\}} \not\rightarrow \mathbf{A}_1, \; \mathbf{A}_1 \not\leftrightarrow \mathbf{A}_2 \not\leftrightarrow \mathbf{A}_3$$



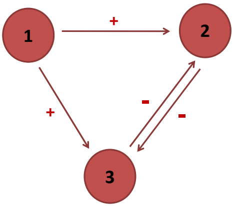
- the current sales level of *Amato* does not depend on previous sales of *Divella* or *Barilla*
- there is no influence between the contemporaneous sales of all 3 brands

- Example: ME graph for pasta spaghetti data

the graph encodes:

\+ monotone dependence
for $1 \to 1$, $2 \to 2$, $3 \to 3$,
$1 \to 2$, $1 \to 3$

\- monotone dependence
for $2 \to 3$ and $3 \to 2$



- sales of all brands depend positively on their own previous sales levels
- a high level of sales of *Barilla* and *Divella* on one day is more probable when the quantity which *Amato* previously sold was high
- given previous high sales level of *Divella*, a low level of *Barilla* sales is more probable, and vice versa

THANKS FOR

YOUR ATTENTION!!