A simulation study of the Bayes estimator of parameters in an extension of the exponential distribution

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Samira Sadeghi

An Extension of Exponential Distribution

Density function

$$f(t) = \alpha\lambda(1+\lambda t)^{\alpha-1}\exp\{1-(1+\lambda t)^{\alpha}\}$$

The two-parameter extension of Exponential distribution

↓

The three-parameter Power Generalized Weibull distribution, introduced by Nikulin and Haghighi (2006).

An Extension of Exponential Distribution

Density function

$$f(t) = \alpha\lambda(1+\lambda t)^{\alpha-1}\exp\{1-(1+\lambda t)^{\alpha}\}$$

Hazard function

$$h(t) = \alpha\lambda(1+\lambda t)^{\alpha-1}$$

Estimation and Fitting

Method of maximum likelihood

$$l(\alpha,\lambda)=\prod_{i=1}^{n}\alpha\lambda(1+\lambda t_i)^{\alpha-1}\,e^{1-(1+\lambda t_i)^{\alpha}}$$

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$$\begin{cases}\dfrac{n}{\alpha}+\displaystyle\sum_{i=1}^{n}\log(1+\lambda t_i)-\sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}\log(1+\lambda t_i)=0\\[2ex]\dfrac{n}{\lambda}+(\alpha-1)\displaystyle\sum_{i=1}^{n}t_i(1+\lambda t_i)^{-1}-\alpha\sum_{i=1}^{n}t_i(1+\lambda t_i)^{\alpha-1}=0\end{cases}$$

Estimation and Fitting

Bayes Estimator under SEL loss function

$$\pi_1(\lambda) \propto \lambda^{b-1} e^{-a\lambda} \qquad \pi_2(\alpha) \propto \alpha^{d-1} e^{-c\alpha}$$

Estimation and Fitting

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$$\pi_1(\lambda) \propto \lambda^{b-1} e^{-a\lambda} \qquad \pi_2(\alpha) \propto \alpha^{d-1} e^{-c\alpha}$$

$$l(\alpha, \lambda) = \prod_{i=1}^{n} \alpha\lambda (1+\lambda t_i)^{\alpha-1} e^{1-(1+\lambda t_i)^{\alpha}}$$

$$\pi(\alpha, \lambda | data) = \frac{l(\alpha, \lambda)\, \pi_1(\lambda) \pi_2(\alpha)}{\int_0^{\infty}\int_0^{\infty} l(\alpha, \lambda)\, \pi_1(\lambda) \pi_2(\alpha)\, d\alpha\, d\lambda}$$

Estimation and Fitting

Bayes Estimator under SEL loss function

$$\hat{\theta}_B = E(\theta \mid T)$$

$$E(g(\alpha,\lambda) \mid T = t) = \frac{\int_0^\infty \int_0^\infty g(\alpha,\lambda)\, l(\alpha,\lambda)\, \pi_1(\lambda) \pi_2(\alpha)\, d\alpha\, d\lambda}{\int_0^\infty \int_0^\infty l(\alpha,\lambda)\, \pi_1(\lambda) \pi_2(\alpha)\, d\alpha\, d\lambda}$$

Lindley's procedure

$$\frac{\int w(\theta)\, e^{L(\theta)}\, d(\theta)}{\int v(\theta)\, e^{L(\theta)}\, d(\theta)}\theta \quad \Rightarrow \quad I = E(g(\theta|t)) = \frac{\int g(\theta)\, e^{L(\theta)+\rho(\theta)}\, d(\theta)}{\int e^{L(\theta)+\rho(\theta)}\, d(\theta)}$$

$$w(\theta) = v(\theta)\, g(\theta)$$

$$\rho(\theta) = \ln(v(\theta))$$

Lindley's procedure

$$I = g(\hat{\theta}) + \frac{1}{2}\sum_{ij}[g_{ij}(\hat{\theta}) + 2g_i(\hat{\theta})\rho_j(\hat{\theta})]\sigma_{ij} + \frac{1}{2}\sum_{ijkL} L_{ijk}(\hat{\theta})\, g_L(\hat{\theta})\, \sigma_{ij}\, \sigma_{kL}$$

On MLE point

$$\hat{g}_i = \frac{\partial g(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_i}, \quad \hat{g}_{ij} = \frac{\partial^2 g(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_i \partial \hat{\theta}_j}, \quad \hat{L}_{ijk} = \frac{\partial^3 L(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_i \partial \hat{\theta}_j \partial \hat{\theta}_k}$$

$$\hat{\rho}_j = \frac{\partial \rho(\hat{\theta}_1, \hat{\theta}_2)}{\partial \hat{\theta}_j}, \quad \hat{\sigma}_{ij} = \frac{-1}{\hat{L}_{ij}}$$

The approximate Bayes estimators of **λ**, under Lindley's procedure (both parameters are unknown)

$$\hat{\lambda}_B = E(\lambda \mid T = t) \longrightarrow \quad g(\theta) = \lambda$$

The approximate Bayes estimators of **λ**, under Lindley's procedure (both parameters are unknown)

$$I = \hat{\lambda} + \hat{\rho}_1 \hat{\sigma}_{11} + \frac{1}{2} \hat{\sigma}_{11}^2 \hat{L}_{111} + \frac{1}{2} \hat{\sigma}_{11} \hat{\sigma}_{22} \hat{L}_{221}$$

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$$\lambda_B = \hat{\lambda} + \frac{\left(\frac{b-1}{\hat{\lambda}}\right) - a}{\frac{n}{\hat{\lambda}^2} + \hat{\alpha}(\hat{\alpha}-1)\sum_{i=1}^n t_i^2 (1+\hat{\lambda}t_i)^{\hat{\alpha}-2} + (\hat{\alpha}-1)\sum_{i=1}^n \frac{t_i^2}{(1+\hat{\lambda}t_i)^2}}$$
$$+ \frac{\frac{2n}{\hat{\lambda}^3} - \hat{\alpha}(\hat{\alpha}-1)(\hat{\alpha}-2)\sum_{i=1}^n t_i^3 (1+\hat{\lambda}t_i)^{\hat{\alpha}-3} + (\hat{\alpha}-1)\sum_{i=1}^n \frac{2t_i^3}{(1+\hat{\lambda}t_i)^3}}{2\left[\frac{n}{\hat{\lambda}^2} + \hat{\alpha}(\hat{\alpha}-1)\sum_{i=1}^n t_i^2 (1+\hat{\lambda}t_i)^{\hat{\alpha}-2} + (\hat{\alpha}-1)\sum_{i=1}^n \frac{t_i^2}{(1+\hat{\lambda}t_i)^2}\right]^2}$$
$$+ \frac{-\hat{\alpha}\sum_{i=1}^n t_i (1+\hat{\lambda}t_i)^{\hat{\alpha}-1} \ln^2(1+\hat{\lambda}t_i) - 2\sum_{i=1}^n t_i (1+\hat{\lambda}t_i)^{\hat{\alpha}-1} \ln(1+\hat{\lambda}t_i)}{2\left[\frac{n}{\hat{\lambda}^2} + \hat{\alpha}(\hat{\alpha}-1)\sum_{i=1}^n t_i^2 (1+\hat{\lambda}t_i)^{\hat{\alpha}-2} + (\hat{\alpha}-1)\sum_{i=1}^n \frac{t_i^2}{(1+\hat{\lambda}t_i)^2}\right]\left[\frac{n}{\hat{\alpha}^2} + \sum_{i=1}^n (1+\hat{\lambda}t_i)^{\hat{\alpha}} \ln^2(1+\hat{\lambda}t_i)\right]}$$

The approximate Bayes estimators of α , under Lindley's procedure (both parameters are unknown)

$$\hat{\alpha}_B = E(\alpha \mid T = t) \longrightarrow g(\theta) = \alpha$$

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$$I = \hat{\alpha} + \hat{\rho}_2 \hat{\sigma}_{22} + \frac{1}{2} \hat{\sigma}_{22} (\hat{L}_{211} \hat{\sigma}_{11} + \hat{L}_{222} \hat{\sigma}_{22})$$

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$$\alpha_B = \hat{\alpha} + \frac{\frac{d-1}{\hat{\alpha}} - c}{\frac{n}{\hat{\alpha}^2} + \sum_{i=1}^{n} (1+\hat{\lambda}t_i)^{\hat{\alpha}} \ln^2(1+\hat{\lambda}t_i)} + \frac{\frac{2n}{\hat{\alpha}^3} - \sum_{i=1}^{n} (1+\hat{\lambda}t_i)^{\hat{\alpha}} \ln^3(1+\hat{\lambda}t_i)}{2\left[\frac{n}{\hat{\alpha}^2} + \sum_{i=1}^{n} (1+\hat{\lambda}t_i)^{\hat{\alpha}} \ln^2(1+\hat{\lambda}t_i)\right]^2}$$

$$+ \frac{-\sum_{i=1}^{n} t_i^2 (1+\hat{\lambda}t_i)^{\hat{\alpha}-2} [2\hat{\alpha} - 1 + \hat{\alpha}(\hat{\alpha}-1) \ln(1+\hat{\lambda}t_i)] - \sum_{i=1}^{n} \frac{t_i^2}{(1+\hat{\lambda}_i)^2}}{2\left[\frac{n}{\hat{\lambda}^2} + \hat{\alpha}(\hat{\alpha}-1) \sum_{i=1}^{n} t_i^2 (1+\hat{\lambda}t_i)^{\hat{\alpha}-2} + (\hat{\alpha}-1) \sum_{i=1}^{n} \frac{t_i^2}{(1+\hat{\lambda}t_i)^2}\right] \left[\frac{n}{\hat{\alpha}^2} + \sum_{i=1}^{n} (1+\hat{\lambda}t_i)^{\hat{\alpha}} \ln^2(1+\hat{\lambda}t_i)\right]}$$

The approximate Bayes estimators of λ, under Lindley's procedure (α is known)

$$I = g(\hat{\theta}) + \frac{1}{2}[(\hat{g}_{11} + 2\hat{g}_1\hat{\rho}_1)\hat{\sigma}_{11}] + \frac{1}{2}\hat{g}_1\hat{\sigma}_{11}\hat{L}_{111}$$

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$$+ \frac{\frac{2n}{\lambda^3} - \alpha(\alpha-1)(\alpha-2)\sum_{i=1}^{n} t_i^3 (1+\lambda t_i)^{\alpha-3} + (\alpha-1)\sum_{i=1}^{n} \frac{2t_i^3}{(1+\lambda t_i)^3}}{2[\frac{n}{\lambda^2} + \alpha(\alpha-1)\sum_{i=1}^{n} t_i^2 (1+\lambda t_i)^{\alpha-2} + (\alpha-1)\sum_{i=1}^{n} \frac{t_i^2}{(1+\lambda t_i)^2}]^2}$$

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$$I = \hat{\alpha} + \frac{1}{2}[\hat{g}_{22} + 2\hat{\rho}_2\hat{g}_2]\hat{\sigma}_{22} + \frac{1}{2}\hat{L}_{222}\hat{g}_2\hat{\sigma}_{22}^2$$

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$$I = \hat{\alpha} + \frac{\frac{d-1}{\hat{\alpha}} - a}{\frac{n}{\alpha^2} + \sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}\ln^2(1+\lambda t_i)}$$

$$+2\frac{\frac{2n}{\alpha^3} - \sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}\ln^3(1+\lambda t_i)}{[\frac{n}{\alpha^2} + \sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}\ln^2(1+\lambda t_i)]^2}$$

The approximate Bayes estimators of parameters, with MCMC method (Gibbs sampler)

With joint posterior density function of **λ** and α :

$$\pi(\alpha,\lambda \mid data) \propto \alpha^{n+d-1}\lambda^{n+b-1}e^{n-\sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}} \prod_{i=1}^{n} (1+\lambda t_i)^{\alpha-1} e^{-a\lambda-c\alpha}$$

The approximate Bayes estimators of parameters, with MCMC method (Gibbs sampler)

posterior density function of α given **λ** :

$$\pi(\alpha|\lambda, data) \propto \alpha^{n+d-1} e^{-\sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}} e^{-\alpha(c-\sum_{i=1}^{n}\ln(1+\lambda t_i))}$$

posterior density function of **λ** given α :

$$\pi(\lambda|\alpha, data) \propto \lambda^{n+b-1} e^{-\sum_{i=1}^{n}(1+\lambda t_i)^{\alpha}} e^{(\alpha-1)\sum_{i=1}^{n}(1+\lambda t_i)^{\alpha-1} - a\lambda}$$

The approximate Bayes estimators of parameters, with MCMC method (Gibbs sampler)

- ✓ start with α_0 as initial value for α
- ✓ generate $\boldsymbol{\lambda}_1$ using $\pi(\boldsymbol{\lambda} \mid \alpha = \alpha_0)$
- ✓ generate α_1 using $\pi(\alpha \mid \boldsymbol{\lambda} = \boldsymbol{\lambda}_1)$

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$$\alpha_i \sim \pi(\alpha \mid \lambda = \lambda_i, data)$$

$$\lambda_i \sim \pi(\lambda \mid \alpha = \alpha_{i-1}, data)$$

Numerical Comparisons

Bayes estimators → compute approximated Bayes estimators using Lindley's approximation

↓

Under non-informative priors on both α and **λ**

Comparing

Numerical Comparisons

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- Bayes estimators → compute approximated Bayes estimators using Lindley's approximation
  - → Under non-informative priors on both α and $\boldsymbol{\lambda}$
- MLE estimators

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- MLE estimators

average estimates (AE)

square root of the mean squared error (RMS)

The (AE) and (RMS) for the MLE's and the approximate Bayes estimate of α when λ is known

n	α	0.2	0.5	1	1.5	2	2.5	3
20	MLE	1.0387	1.0453	1.0406	1.0410	1.0335	1.0374	1.0361
		(0.1563)	(0.1575)	(0.1588)	(0.1588)	(0.1561)	(0.1574)	(0.1550)
	ABAYES	1.0193	1.0285	1.0216	1.0216	1.0143	1.0180	1.0167
		(0.1498)	(0.1503)	(0.1544)	(0.1520)	(0.1503)	(0.1511)	(0.1488)
50	MLE	1.0150	1.0176	1.0115	1.0165	1.0090	1.0151	1.0114
		(0.0933)	(0.0911)	(0.0905)	(0.0917)	(0.0887)	(0.0922)	(0.0888)
	ABAYES	1.0074	1.0099	1.0038	1.0089	1.0014	1.0075	1.0038
		(0.0917)	(0.0893)	(0.0892)	(0.0900)	(0.0876)	(0.0906)	(0.0875)
100	MLE	1.0059	1.0095	1.0055	1.0096	1.0092	1.0085	1.0052
		(0.0616)	(0.0659)	(0.0608)	(0.0633)	(0.0640)	(0.0643)	(0.0638)
	ABAYES	1.0021	1.0057	1.0017	1.0057	1.0054	1.0047	1.0014
		(0.0611)	(0.0652)	(0.0604)	(0.0626)	(0.0633)	(0.0637)	(0.0633)

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20	MLE	1.1578	1.0656	1.0559	1.0442	1.0489	1.0415	1.0473
		(0.7326)	(0.3756)	(0.2621)	(0.2269)	(0.2065)	(0.1917)	(0.1942)
	ABAYES	1.3450	1.0972	1.0559	1.0363	1.0375	1.0283	1.0328
		(0.8996)	(0.3931)	(0.2621)	(0.2238)	(0.2020)	(0.1870)	(0.1886)
50	MLE	1.0438	1.0122	1.0149	1.0137	1.0165	1.0157	1.0189
		(0.3900)	(0.2087)	(0.1527)	(0.1280)	(0.1179)	(0.1125)	(0.1100)
	ABAYES	1.1108	1.0241	1.0149	1.0106	1.0121	1.0105	1.0133
		(0.4271)	(0.2122)	(0.1527)	(0.1273)	(0.1168)	(0.1114)	(0.1086)
100	MLE	1.0279	1.0144	1.0122	1.0083	1.0122	1.0109	1.0119
		(0.2636)	(0.1454)	(0.1007)	(0.0862)	(0.0801)	(0.0794)	(0.0784)
	ABAYES	1.0608	1.0204	1.0122	1.0067	1.0100	1.0083	1.0093
		(0.2774)	(0.1470)	(0.1007)	(0.0860)	(0.0797)	(0.0789)	(0.0779)

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The (AE) and (RMS) for the MLE's and the approximate Bayes estimate of α , λ when both are unknown

n		$\alpha=0.2, \lambda=0.2$		$\alpha=0.2, \lambda=0.5$		$\alpha=0.2, \lambda=1$	
		MLE	ABAYES	MLE	ABAYES	MLE	ABAYES
20	α	0.2228	0.2238	0.2247	0.2259	0.2246	0.2257
		(0.0629)	(0.0613)	(0.0687)	(0.0667)	(0.0762)	(0.0740)
	λ	0.2448	0.2760	0.6317	0.7275	1.2345	1.4285
		(0.2828)	(0.3425)	(0.7409)	(0.9342)	(1.4233)	(1.7872)
50	α	0.2104	0.2112	0.2084	0.2091	0.2085	0.2093
		(0.0319)	(0.0318)	(0.0297)	(0.0296)	(0.0318)	(0.0317)
	λ	0.2078	0.2170	0.5385	0.5633	1.0826	1.1327
		(0.1167)	(0.1249)	(0.3164)	(0.3408)	(0.6455)	(0.6951)
100	α	0.2038	0.2042	0.2036	0.2040	0.2036	0.2041
		(0.0193)	(0.0192)	(0.0188)	(0.0188)	(0.0189)	(0.0189)
	λ	0.2062	0.2107	0.5167	0.5281	1.0332	1.0561
		(0.0767)	(0.0795)	(0.1906)	(0.1977)	(0.3817)	(0.3959)

The (AE) and (RMS) for the MLE's and the approximate Bayes estimate of α, λ when both are unknown

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		(0.0629)	(0.0613)	(0.0687)	(0.0667)	(0.0762)	(0.0740)
	λ	0.2448	0.2760	0.6317	0.7275	1.2345	1.4285
		(0.2828)	(0.3425)	(0.7409)	(0.9342)	(1.4233)	(1.7872)
50	α	0.2104	0.2112	0.2084	0.2091	0.2085	0.2093
		(0.0319)	(0.0318)	(0.0297)	(0.0296)	(0.0318)	(0.0317)
	λ	0.2078	0.2170	0.5385	0.5633	1.0826	1.1327
		(0.1167)	(0.1249)	(0.3164)	(0.3408)	(0.6455)	(0.6951)
100	α	0.2038	0.2042	0.2036	0.2040	0.2036	0.2041
		(0.0193)	(0.0192)	(0.0188)	(0.0188)	(0.0189)	(0.0189)
	λ	0.2062	0.2107	0.5167	0.5281	1.0332	1.0561
		(0.0767)	(0.0795)	(0.1906)	(0.1977)	(0.3817)	(0.3959)

The (AE) and (RMS) for the MLE's and the approximate Bayes estimate of α, λ when both are unknown

n		$\alpha=0.2, \lambda=0.2$		$\alpha=0.2, \lambda=0.5$		$\alpha=0.2, \lambda=1$	
		MLE	ABAYES	MLE	ABAYES	MLE	ABAYES
20	α	0.2228	0.2238	0.2247	0.2259	0.2246	0.2257
		(0.0629)	(0.0613)	(0.0687)	(0.0667)	(0.0762)	(0.0740)
	λ	0.2448	0.2760	0.6317	0.7275	1.2345	1.4285
		(0.2828)	(0.3425)	(0.7409)	(0.9342)	(1.4233)	(1.7872)
50	α	0.2104	0.2112	0.2084	0.2091	0.2085	0.2093
		(0.0319)	(0.0318)	(0.0297)	(0.0296)	(0.0318)	(0.0317)
	λ	0.2078	0.2170	0.5385	0.5633	1.0826	1.1327
		(0.1167)	(0.1249)	(0.3164)	(0.3408)	(0.6455)	(0.6951)
100	α	0.2038	0.2042	0.2036	0.2040	0.2036	0.2041
		(0.0193)	(0.0192)	(0.0188)	(0.0188)	(0.0189)	(0.0189)
	λ	0.2062	0.2107	0.5167	0.5281	1.0332	1.0561
		(0.0767)	(0.0795)	(0.1906)	(0.1977)	(0.3817)	(0.3959)

The (AE) and (RMS) for the MLE's and the approximate Bayes estimate of α , λ when both are unknown

n		$\alpha=0.5,\lambda=0.2$		$\alpha=1,\lambda=0.2$		$\alpha=1,\lambda=1$	
		MLE	ABAYES	MLE	ABAYES	MLE	ABAYES
20	α	0.6855	0.6643	1.8029	1.7212	1.7970	1.7160
		(0.4743)	(0.4443)	(1.4501)	(1.3504)	(1.5022)	(1.4013)
	λ	0.2209	0.2211	0.2106	0.2061	1.0775	1.0548
		(0.2011)	(0.2079)	(0.2473)	(0.2469)	(1.2175)	(1.2118)
50	α	0.5483	0.5429	1.2951	1.2730	1.3050	1.2827
		(0.1506)	(0.1460)	(0.7394)	(0.7167)	(0.7409)	(0.7181)
	λ	0.2054	0.2050	0.2010	0.1988	0.9899	0.9792
		(0.0967)	(0.0973)	(0.1246)	(0.1238)	(0.6013)	(0.5975)
100	α	0.5248	0.5223	1.13434	1.1249	1.1354	1.1261
		(0.0893)	(0.0878)	(0.3895)	(0.3823)	(0.3919)	(0.3846)
	λ	0.2010	0.2008	0.1978	0.1967	0.9891	0.9835
		(0.0714)	(0.0716)	(0.0837)	(0.0834)	(0.4188)	(0.4174)

The (AE) and (RMS) for the MLE's and the approximate Bayes estimate of α , λ when both are unknown

n		$\alpha=0.5,\lambda=0.2$		$\alpha=1,\lambda=0.2$		$\alpha=1,\lambda=1$	
		MLE	ABAYES	MLE	ABAYES	MLE	ABAYES
20	α	0.6855	0.6643	1.8029	1.7212	1.7970	1.7160
		(0.4743)	(0.4443)	(1.4501)	(1.3504)	(1.5022)	(1.4013)
	λ	0.2209	0.2211	0.2106	0.2061	1.0775	1.0548
		(0.2011)	(0.2079)	(0.2473)	(0.2469)	(1.2175)	(1.2118)
50	α	0.5483	0.5429	1.2951	1.2730	1.3050	1.2827
		(0.1506)	(0.1460)	(0.7394)	(0.7167)	(0.7409)	(0.7181)
	λ	0.2054	0.2050	0.2010	0.1988	0.9899	0.9792
		(0.0967)	(0.0973)	(0.1246)	(0.1238)	(0.6013)	(0.5975)
100	α	0.5248	0.5223	1.13434	1.1249	1.1354	1.1261
		(0.0893)	(0.0878)	(0.3895)	(0.3823)	(0.3919)	(0.3846)
	λ	0.2010	0.2008	0.1978	0.1967	0.9891	0.9835
		(0.0714)	(0.0716)	(0.0837)	(0.0834)	(0.4188)	(0.4174)

The (AE) and (RMS) for the MLE's and the approximate Bayes estimate of α , λ when both are unknown

n		$\alpha=1,\lambda=1.5$		$\alpha=1.5,\lambda=1$		$\alpha=0.5,\lambda=1$	
		MLE	ABAYES	MLE	ABAYES	MLE	ABAYES
20	α	1.6993	1.6233	2.9071	2.7643	0.6697	0.6496
		(1.4678)	(1.3697)	(2.4205)	(2.2501)	(0.4267)	(0.3995)
	λ	1.7080	1.6732	1.0634	1.0338	1.1119	1.1147
		(1.9380)	(1.9362)	(1.2540)	(1.2320)	(1.0622)	(1.1017)
50	α	1.3159	1.2933	2.1864	2.1438	0.5502	0.5448
		(0.7422)	(0.7191)	(1.4808)	(1.4347)	(0.1477)	(0.1430)
	λ	1.4553	1.4396	0.9935	0.9809	1.0144	1.0123
		(0.8964)	(0.8919)	(0.6812)	(0.6745)	(0.4865)	(0.4898)
100	α	1.1246	1.1154	1.8368	1.8191	0.5211	0.5188
		(0.3725)	(0.3657)	(0.8554)	(0.8399)	(0.0873)	(0.0859)
	λ	1.4823	1.4739	0.9778	0.9714	1.0156	1.0143
		(0.5923)	(0.5904)	(0.4704)	(0.4681)	(0.3489)	(0.3500)

The (AE) and (RMS) for the MLE's and the approximate Bayes estimate of α , λ when both are unknown

n		$\alpha=1, \lambda=1.5$		$\alpha=1.5, \lambda=1$		$\alpha=0.5, \lambda=1$	
		MLE	ABAYES	MLE	ABAYES	MLE	ABAYES
20	α	1.6993	1.6233	2.9071	2.7643	0.6697	0.6496
		(1.4678)	(1.3697)	(2.4205)	(2.2501)	(0.4267)	(0.3995)
	λ	1.7080	1.6732	1.0634	1.0338	1.1119	1.1147
		(1.9380)	(1.9362)	(1.2540)	(1.2320)	(1.0622)	(1.1017)
50	α	1.3159	1.2933	2.1864	2.1438	0.5502	0.5448
		(0.7422)	(0.7191)	(1.4808)	(1.4347)	(0.1477)	(0.1430)
	λ	1.4553	1.4396	0.9935	0.9809	1.0144	1.0123
		(0.8964)	(0.8919)	(0.6812)	(0.6745)	(0.4865)	(0.4898)
100	α	1.1246	1.1154	1.8368	1.8191	0.5211	0.5188
		(0.3725)	(0.3657)	(0.8554)	(0.8399)	(0.0873)	(0.0859)
	λ	1.4823	1.4739	0.9778	0.9714	1.0156	1.0143
		(0.5923)	(0.5904)	(0.4704)	(0.4681)	(0.3489)	(0.3500)

Data Analysis

Linhart and Zucchini (1986)

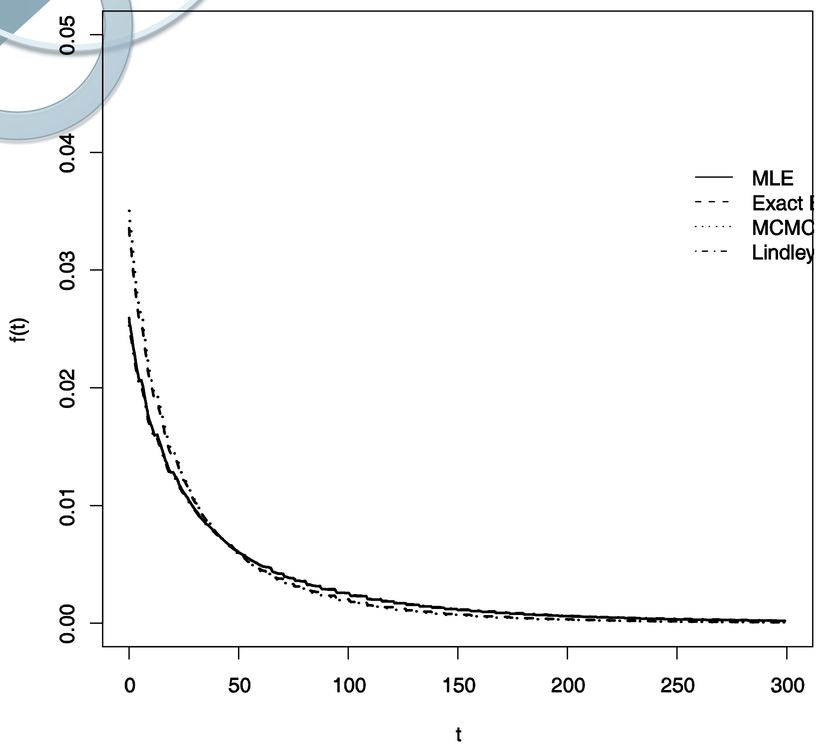
The failure times of the air conditioning system of an airplane

Data set: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95

Parameter estimations, Kolmogorov-Smirnov and Chi-squared statistics for the data set

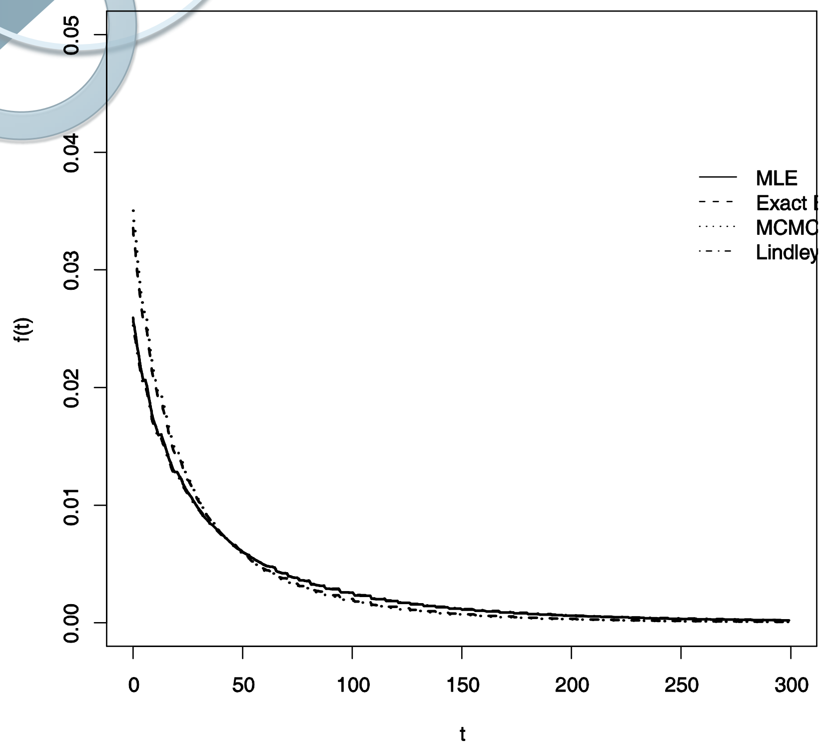
Methods	$(\hat{\alpha}, \hat{\lambda})$	K-S	χ^2
MLE	(0.5985,0.0434)	0.5301	10.44232
Lindley	(0.5895,0.0429)	0.5373	10.67144
MCMC	(0.6571,0.0534)	0.4718	11.82650
Exact Bayes	(0.6477,0.0519)	0.4710	11.10849

Comparing different methods

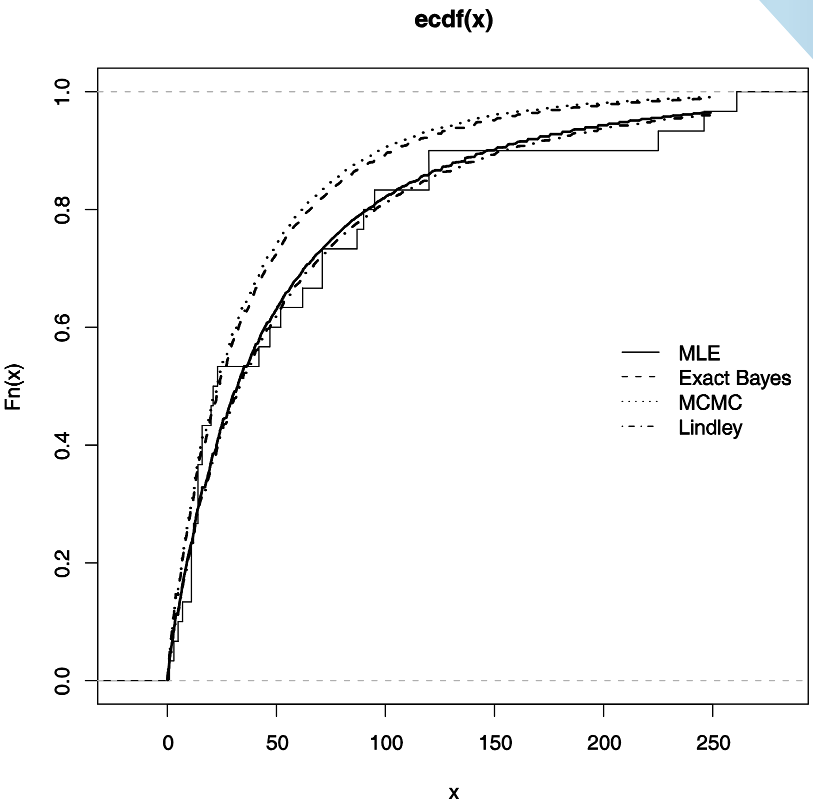


The estimated density function by different methods

Comparing different methods



The estimated density function by different methods



The estimated cumulative distribution functions and empirical distribution function


Thank you

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