Comparing Two Approaches to Testing Linearity against Markov-switching Type Non-linearity

Jana Lenčuchová, Anna Petričková and Magdaléna Komorníková

Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology Bratislava

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Markov-switching models

- random variable $s_t$ in case of $N$ possible states, can attain values from set \{1, 2, 3, \ldots, N\}
- stochastic process \{s_t\} - a first-order ergodic Markov process (Hamilton 1989)
  \[ Pr(s_t = j|s_{t-1} = i, s_{t-2} = k, \ldots) = Pr(s_t = j|s_{t-1} = i) = p_{ij} \]
  \[ p_{ij} > 0, \; i, j = 1, \ldots, N \]
  \[ p_{i1} + p_{i2} + \ldots + p_{iN} = 1, \; i = 1, \ldots, N \]
- Complete probability distribution of Markov chain is defined by the initial distribution $\pi_i = Pr(s_1 = i)$ and the state transition probability matrix $P = (p_{ij})_{i,j=1,\ldots,N}$
Markov-switching models

- observable time series \{y_1, ..., y_T\}
- \( y_t = \phi_{0,s_t} + \phi_{1,s_t}y_{t-1} + \ldots + \phi_{q,s_t}y_{t-q} + \epsilon_t \), \( s_t = 1, ..., N \)
  - \( \epsilon_t \sim N(0, \sigma^2) \)
  - \( \phi_{j,s_t} \) are autoregressive coefficients of an appropriate regime
  - \( s_t = 1, ..., N, j = 0, 1, ..., q \)
- \( q \) is model order
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The general non-linear modeling procedure (Granger 1993)

- Model identification
- Testing linearity against non-linearity
- Parameters estimation
- Diagnostic control
- Model modification, if it is needed
- Description and prediction of an examined time series
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Classical approach - Likelihood ratio test

Testing of a linear model against a 2-regime model

$H_0 : \varphi_1 = \varphi_2$

against

$H_1 : \phi_{i,1} \neq \phi_{i,2}$ for at least one $i \in \{0, 1, 2, \ldots, q\}$

$\varphi_1, \varphi_2$ represents AR coefficients of a Markov-switching model in both regimes
Classical approach - Likelihood ratio test

\[ L = L_{MSW} - L_{AR} \]

- \( L_{MSW} \) and \( L_{AR} \) are loglikelihood functions for the corresponding Markov-switching model and AR model.
- This test statistic has non-standard distribution (Hansen 1992).
- Simulation has to be carried out.
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- Testing for MSW type of nonlinearity
- Our proposed testing

Our proposed testing

Using score function

- score function of \( t \)th observation
  \[
  h_t(\theta) \equiv \frac{\partial \ln f(y_t|\Omega_{t-1};\theta)}{\partial \theta}
  \]

\( \theta \) is the parameter vector, \( \Omega_{t-1} \) represents observation history

- parameter vector for a 2-regime model
  \[
  \theta = (\varphi_1', \varphi_2', \sigma^2, p_{11}, p_{22})
  \]
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Testing for MSW type of nonlinearity

Our proposed testing

Score function for the Markov-switching model was derived by Hamilton (1996):

\[
\frac{\partial \ln f(y_t | \Omega_{t-1}; \theta)}{\partial \alpha} = \sum_{j=1}^{N} \frac{\partial \ln f(y_t | X_t, s_t = j; \theta)}{\partial \alpha} Pr(s_t = j | \Omega_t) + \\
+ \sum_{\tau=1}^{t-1} \sum_{s_\tau=1}^{N} \frac{\partial \ln f(y_\tau | X_\tau, s_\tau = j; \theta)}{\partial \alpha} \{Pr(s_\tau | \Omega_t) - Pr(s_\tau | \Omega_{t-1})\}
\]

for \( t = 1, 2, \ldots, T \), \( \alpha = (\varphi'_1, \varphi'_2, \sigma^2) \), \( X_t = (1, y_{t-1}, y_{t-2}, \ldots, y_{t-q}) \)
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Our proposed testing

\[
\frac{\partial \ln f(y_t | \Omega_{t-1}; \theta)}{\partial p_{ij}} = p_{ij}^{-1} \Pr(s_t = j, s_{t-1} = i | \Omega_t) - p_{iN}^{-1} \Pr(s_t = N, s_{t-1} = i | \Omega_t) + \\
+ p_{ij}^{-1} \left\{ \sum_{\tau=2}^{t-1} \left[ \Pr(s_\tau = j, s_{\tau-1} = i | \Omega_t) - \Pr(s_\tau = j, s_{\tau-1} = i | \Omega_{t-1}) \right] \right\} - \\
- p_{iN}^{-1} \left\{ \sum_{\tau=2}^{t-1} \left[ \Pr(s_\tau = N, s_{\tau-1} = i | \Omega_t) - \Pr(s_\tau = N, s_{\tau-1} = i | \Omega_{t-1}) \right] \right\} + \\
+ \sum_{s_1=1}^{N} \frac{\partial \ln \Pr(s_1; p)}{\partial p_{ij}} \left[ \Pr(s_1 | \Omega_t) - \Pr(s_1 | \Omega_{t-1}) \right]
\]

for \( i = 1, 2, \ldots, N \), \( j = 1, 2, \ldots, N - 1 \) and \( t = 2, \ldots, T \) where
\[
p = (p_{11}, p_{12}, \ldots, p_{1,N-1}, p_{21}, p_{22}, \ldots, p_{N,N-1})
\]

For \( t = 1 \)

\[
\frac{\partial \ln f(y_1 | \Omega_0; \theta)}{\partial p_{ij}} = \sum_{s_1=1}^{N} \frac{\partial \ln \Pr(s_1; p)}{\partial p_{ij}} \Pr(s_1 | \Omega_1).
\]
Using Newey-Tauchen-White test

- test statistic

\[
\left[ T^{-\frac{1}{2}} \sum_{t=1}^{T} c_t(\hat{\theta}) \right] \cdot \left[ T^{-1} \sum_{t=1}^{T} c_t(\hat{\theta}).c_t(\hat{\theta})' \right]^{-1} \cdot \left[ T^{-\frac{1}{2}} \sum_{t=1}^{T} c_t(\hat{\theta}) \right] \to \chi^2(k).
\]

- to carry out this test, we need to construct \((k \times 1)\) vector \(c_t(\theta)\) consisting of elements of \((m \times m)\) matrix \([h_t(\theta)].[h_{t-1}(\theta)]'\), which correspond to testing examined properties, where \(m\) is a number of estimated parameters
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Testing for MSW type of nonlinearity

Our proposed testing

Testing Markov assumptions

- \( Pr(s_t = j | s_{t-1} = i) = Pr(s_t = j | s_{t-1} = i, y_{t-1}) , \)
  \[ \frac{\partial \ln f(y_t | \Omega_{t-1}; \theta)}{\partial p_{ij}} \cdot \frac{\partial \ln f(y_{t-1} | \Omega_{t-2}; \theta)}{\partial \phi_{0,i}} , \quad i, j = 1, ..., N \]

- \( Pr(s_t = j | s_{t-1} = i) = Pr(s_t = j | s_{t-1} = i, s_{t-2} = k) , \)
  \[ \frac{\partial \ln f(y_t | \Omega_{t-1}; \theta)}{\partial p_{ij}} \cdot \frac{\partial \ln f(y_{t-1} | \Omega_{t-2}; \theta)}{\partial p_{ij}} , \quad i, j = 1, ..., N \]

Comparing simulations with our proposed testing
  - linearity against Markov-switching type non-linearity
  - remaining non-linearity (comparing the 2-regime with the 3-regime Markov-switching model)
Data

- 100 various economic and financial time series - exchange rates, macroeconomic indicators, stock market indexes,...
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Applications

Results - simulations vs. the proposed test

- 100 time series
  - Testing linearity against Markov-switching type of nonlinearity - the same conclusion in 72%
  - Testing remaining nonlinearity - the same conclusion in 79%
Simulation procedure

- Generating at least 5000 artificial time series according to model representing the null hypothesis
- Parameters estimation of the best AR and MSW model for each artificial time series
- Calculation of the corresponding likelihood ratio statistics to get critical values
Calculating time

Example: Rouble/EUR exchange rate

for $q=5$ and $T=130$

- testing linearity by
  - simulations: 14 802.7 sec (cca 4.11 h)
  - new test: 68.5 sec

- testing remaining non-linearity
  - simulations: 53 372 sec (cca 14.83 h)
  - new test: 1031.7 sec
What next?

- Calculating of power properties for the proposed test
- Investigation of the efficiency of the proposed technique
- Comparing the proposed test with other types of tests for an investigation of non-linear properties
- Testing independence of residuals and modeling dependence of residuals with auto-copulas
Thank you for your attention!