Comparing Two Approaches to Testing Linearity against Markov-switching Type Non-linearity

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Content

Markov-switching models

- random variable s_t in case of N possible states, can attain values from set $\{1, 2, 3, \ldots, N\}$
- stochastic process $\{s_t\}$ - a first-order ergodic Markov process (Hamilton 1989)
- $Pr(s_t = j | s_{t-1} = i, s_{t-2} = k, ...) = Pr(s_t = j | s_{t-1} = i) = p_{ij}$
- $p_{ij} > 0,\ i, j = 1, ..., N$
- $p_{i1} + p_{i2} + ... + p_{iN} = 1, \qquad i = 1, ..., N$
- Complete probability distribution of Markov chain is defined by the initial distribution $\pi_i = Pr(s_1 = i)$ and the state transition probability matrix $P = (p_{ij})_{i,j=1,...,N}$

Markov-switching models

- observable time series $\{y_1, \dots, y_T\}$
- $y_t = \phi_{0,s_t} + \phi_{1,s_t} y_{t-1} + \dots + \phi_{q,s_t} y_{t-q} + \epsilon_t, \qquad s_t = 1, \dots, N$
  - $\epsilon_t \sim N(0, \sigma^2)$
  - ϕ_{j,s_t} are autoregressive coefficients of an appropriate regime $s_t = 1, \dots, N$, $j = 0, 1, \dots, q$
  - q is model order

The general non-linear modeling procedure (Granger 1993)

- Model identification
- Testing linearity against non-linearity
- Parameters estimation
- Diagnostic control
- Model modification, if it is needed
- Description and prediction of an examined time series

Classical approach - Likelihood ratio test

Testing of a linear model against a 2-regime model

$H_0 : \boldsymbol{\varphi_1} = \boldsymbol{\varphi_2}$
against
$H_1 : \phi_{i,1} \neq \phi_{i,2}$ for at least one $i \in \{0, 1, 2, ..., q\}$

$\boldsymbol{\varphi_1}$, $\boldsymbol{\varphi_2}$ represents AR coefficients of a Markov-switching model in both regimes

Classical approach - Likelihood ratio test

Likelihood ratio test

$L = L_{MSW} - L_{AR}$

- L_{MSW} and L_{AR} are loglikelihood functions for the corresponding Markov-switching model and AR model
- this test statistic has non-standard distribution (Hansen 1992)
- simulation has to be carried out

Our proposed testing

Using score function

- score function of tth observation

$$\mathbf{h}_t(\boldsymbol{\theta}) \equiv \frac{\partial \ln f(y_t|\Omega_{t-1};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}$$

  $\boldsymbol{\theta}$ is the parameter vector, Ω_{t-1} represents observation history

- parameter vector for a 2-regime model

$$\boldsymbol{\theta} = (\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2', \sigma^2, p_{11}, p_{22})$$

Our proposed testing

Score function for the Markov-switching model was derived by Hamilton(1996):

$$\frac{\partial \ln f(y_t|\Omega_{t-1};\boldsymbol{\theta})}{\partial \boldsymbol{\alpha}} = \sum_{j=1}^{N} \frac{\partial \ln f(y_t|\mathbf{X_t}, s_t = j;\boldsymbol{\theta})}{\partial \boldsymbol{\alpha}} Pr(s_t = j|\Omega_t) + \\ + \sum_{\tau=1}^{t-1} \sum_{s_\tau=1}^{N} \frac{\partial \ln f(y_\tau|\mathbf{X}_\tau, s_\tau = j;\boldsymbol{\theta})}{\partial \boldsymbol{\alpha}} \{Pr(s_\tau|\Omega_t) - Pr(s_\tau|\Omega_{t-1})\}$$

for $t = 1, 2, \ldots, T$, $\alpha = (\boldsymbol{\varphi}_1', \boldsymbol{\varphi}_2', \sigma^2)$, $\mathbf{X_t} = (1, y_{t-1}, y_{t-2}, ..., y_{t-q})$

Our proposed testing

$$\frac{\partial \ln f(y_t|\Omega_{t-1};\boldsymbol{\theta})}{\partial p_{ij}} = p_{ij}^{-1} Pr(s_t = j, s_{t-1} = i|\Omega_t) - p_{iN}^{-1} Pr(s_t = N, s_{t-1} = i|\Omega_t) +$$
$$+ p_{ij}^{-1} \left\{ \sum_{\tau=2}^{t-1} [Pr(s_\tau = j, s_{\tau-1} = i|\Omega_t) - Pr(s_\tau = j, s_{\tau-1} = i|\Omega_{t-1})] \right\} -$$
$$- p_{iN}^{-1} \left\{ \sum_{\tau=2}^{t-1} [Pr(s_\tau = N, s_{\tau-1} = i|\Omega_t) - Pr(s_\tau = N, s_{\tau-1} = i|\Omega_{t-1})] \right\} +$$
$$+ \sum_{s_1=1}^{N} \frac{\partial \ln Pr(s_1;\mathbf{p})}{\partial p_{ij}} [Pr(s_1|\Omega_t) - Pr(s_1|\Omega_{t-1})]$$

for $i = 1, 2, \ldots, N$, $j = 1, 2, \ldots, N-1$ and $t = 2, \ldots, T$ where
$\mathbf{p} = (p_{11}, p_{12}, \ldots, p_{1,N-1}, p_{21}, p_{22}, \ldots, p_{N,N-1})$

For $t = 1$

$$\frac{\partial \ln f(y_1|\Omega_0;\boldsymbol{\theta})}{\partial p_{ij}} = \sum_{s_1=1}^{N} \frac{\partial \ln Pr(s_1;\mathbf{p})}{\partial p_{ij}} Pr(s_1|\Omega_1).$$

Using Newey-Tauchen-White test

- test statistic

$$\left[T^{-\frac{1}{2}}\sum_{t=1}^{T}\mathbf{c_t}(\hat{\boldsymbol{\theta}})\right].\left[T^{-1}\sum_{t=1}^{T}\mathbf{c_t}(\hat{\boldsymbol{\theta}}).\mathbf{c_t}(\hat{\boldsymbol{\theta}})'\right]^{-1}.\left[T^{-\frac{1}{2}}\sum_{t=1}^{T}\mathbf{c_t}(\hat{\boldsymbol{\theta}})\right]\rightarrow\chi^2(k).$$

- to carry out this test, we need to construct $(k \times 1)$ vector $\mathbf{c_t}(\boldsymbol{\theta})$ consisting of elements of $(m \times m)$ matrix $[\mathbf{h_t}(\boldsymbol{\theta})].[\mathbf{h_{t-1}}(\boldsymbol{\theta})]'$, which correspond to testing examined properties, where m is a number of estimated parameters

Testing Markov assumptions

- $Pr(s_t = j | s_{t-1} = i) = Pr(s_t = j | s_{t-1} = i, y_{t-1}),$

$$\frac{\partial \ln f(y_t|\Omega_{t-1};\boldsymbol{\theta})}{\partial p_{ij}} \cdot \frac{\partial \ln f(y_{t-1}|\Omega_{t-2};\boldsymbol{\theta})}{\partial \phi_{0,i}}, \qquad i,j = 1,...,N$$

- $Pr(s_t = j | s_{t-1} = i) = Pr(s_t = j | s_{t-1} = i, s_{t-2} = k),$

$$\frac{\partial \ln f(y_t|\Omega_{t-1};\boldsymbol{\theta})}{\partial p_{ij}} \cdot \frac{\partial \ln f(y_{t-1}|\Omega_{t-2};\boldsymbol{\theta})}{\partial p_{ij}}, \qquad i,j = 1,...,N$$

HAMILTON, J.D. (1996): Specification testing in Markov-switching time series models. *Journal of Econometrics* 70, 127-157.

Application

- Comparing simulations with our proposed testing
  - linearity against Markov-switching type non-linearity
  - remaining non-linearity (comparing the 2-regime with the 3-regime Markov-switching model)

Data

- 100 various economic and financial time series - exchange rates, macroeconomic indicators, stock market indexes,...

Results - simulations vs. the proposed test

- 100 time series
  - Testing linearity against Markov-switching type of nonlinearity - the same conclusion in 72%
  - Testing remaining nonlinearity - the same conclusion in 79%

Simulation procedure

- Generating at least 5000 artificial time series according to model representing the null hypothesis
- Parameters estimation of the best AR and MSW model for each artificial time series
- Calculation of the corresponding likelihood ratio statistics to get critical values

Calculating time

Example: Rouble/EUR exchange rate

for q=5 and T=130

- testing linearity by
  - simulations: 14 802.7 sec (cca 4.11 h)
  - new test: 68.5 sec
- testing remaining non-linearity
  - simulations: 53 372 sec (cca 14.83 h)
  - new test: 1031.7 sec

What next?

- Calculating of power properties for the proposed test
- Investigation of the efficiency of the proposed technique
- Comparing the proposed test with other types of tests for an investigation of non-linear properties
- Testing independence of residuals and modeling dependence of residuals with auto-copulas

Thank you for your attention!