Computational treatment of the error distribution in nonparametric regression with right-censored and selection-biased data

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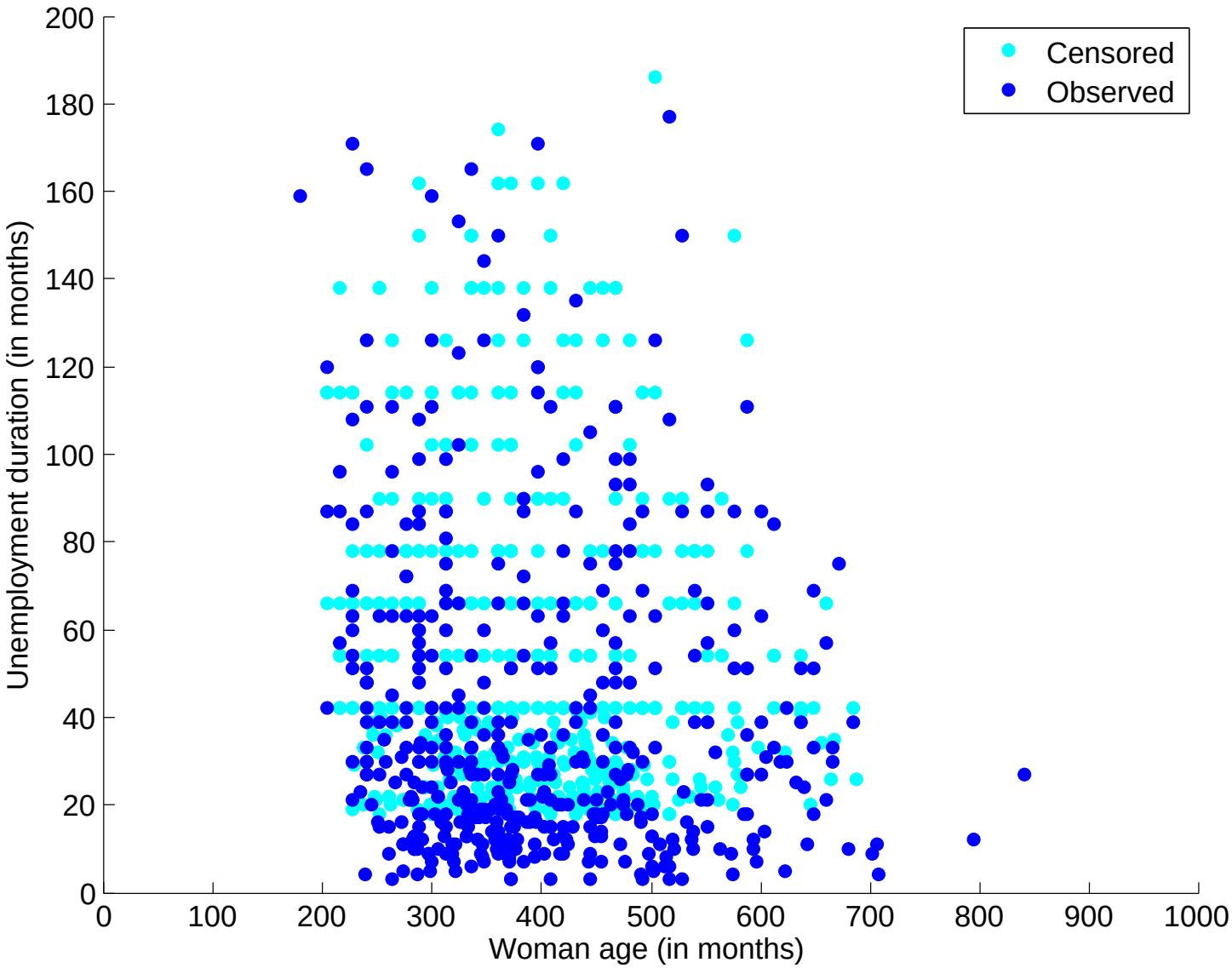
The Spanish Institute for Statistics studied between 1987 and 1997 the unemployment of active people, and more especially the married women.

For these data, we note that

- the time of unemployment will not be completely observed,
- the age of the woman acts on the future job.


Censored
Observed
Unemployment duration (in months)
Woman age (in months)
0
20
40
60
80
100
120
140
160
180
200
0
100
200
300
400
500
600
700
800
900
1000

Estimation

We consider the nonparametric regression model

$$Y = m(X) + \sigma(X)\varepsilon$$

where

- Y is the response variable
- X is the covariate
- $m(\cdot) = E[Y|\cdot]$ and $\sigma^2(\cdot) = Var[Y|\cdot]$ are unknown smooth functions
- ε is independent of X, with $E[\varepsilon] = 0$ and $Var[\varepsilon] = 1$

Particularity of (X, Y)

- (X, Y) is obtained from cross-sectional sampling
- Y is subject to right censoring.

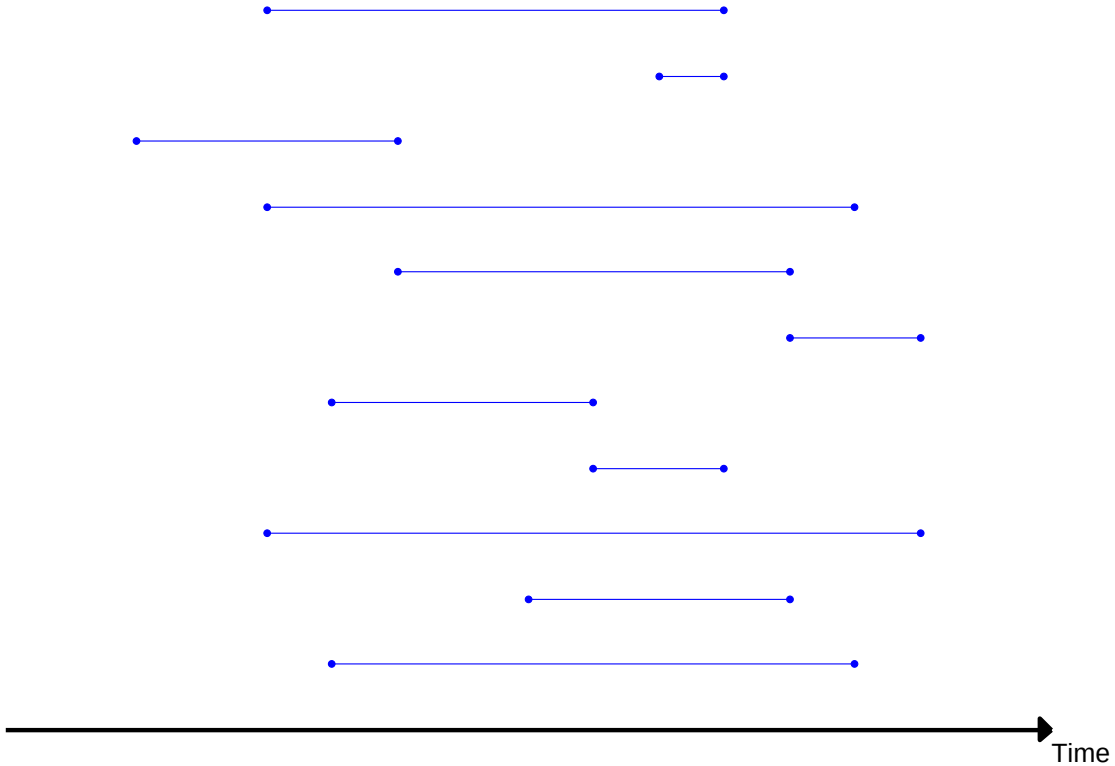
We study the variable Y delimited by

$$T \leq Y \leq C$$

where

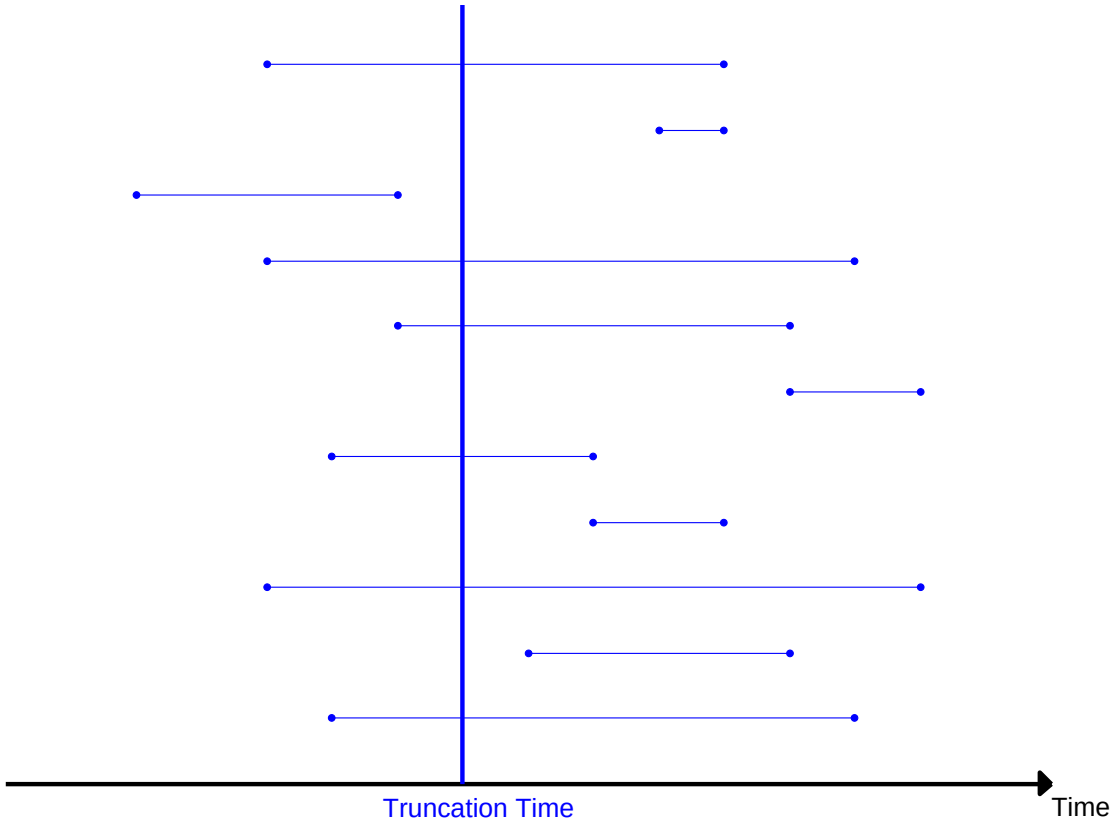
- T is the truncation variable
- C is the censoring variable.

Real World



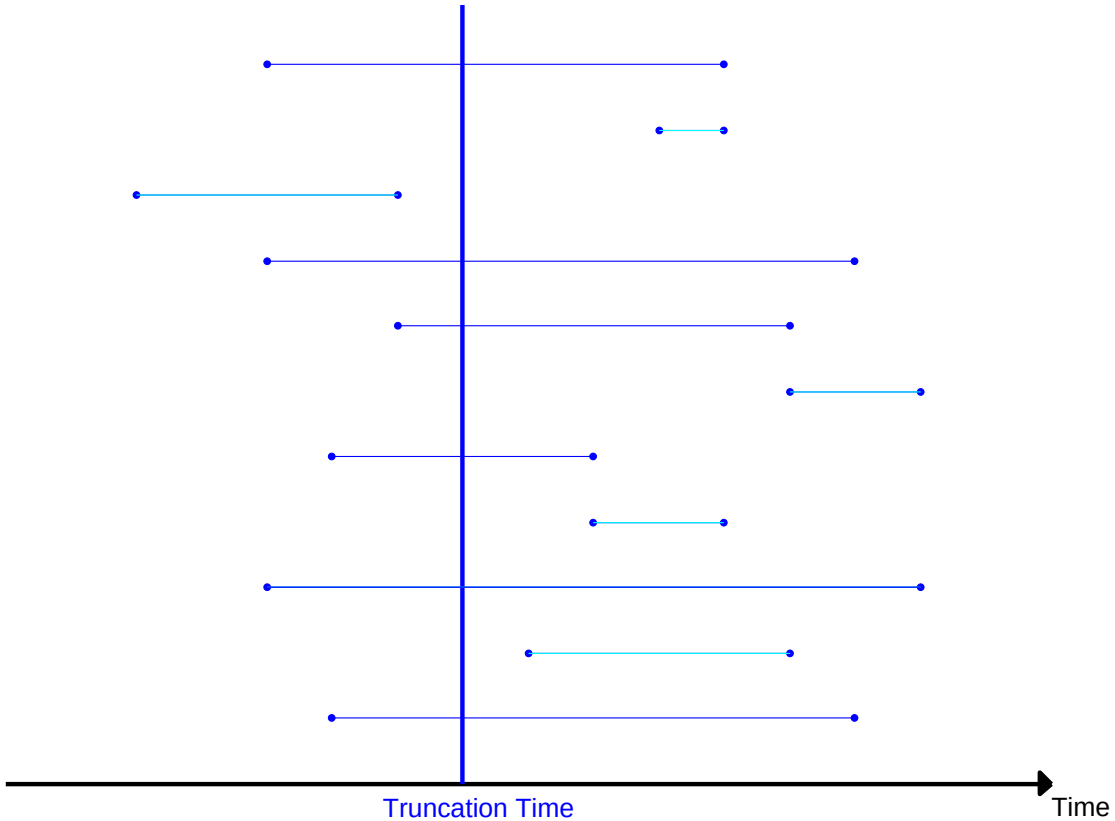
We use as notation F for cdf

Real World



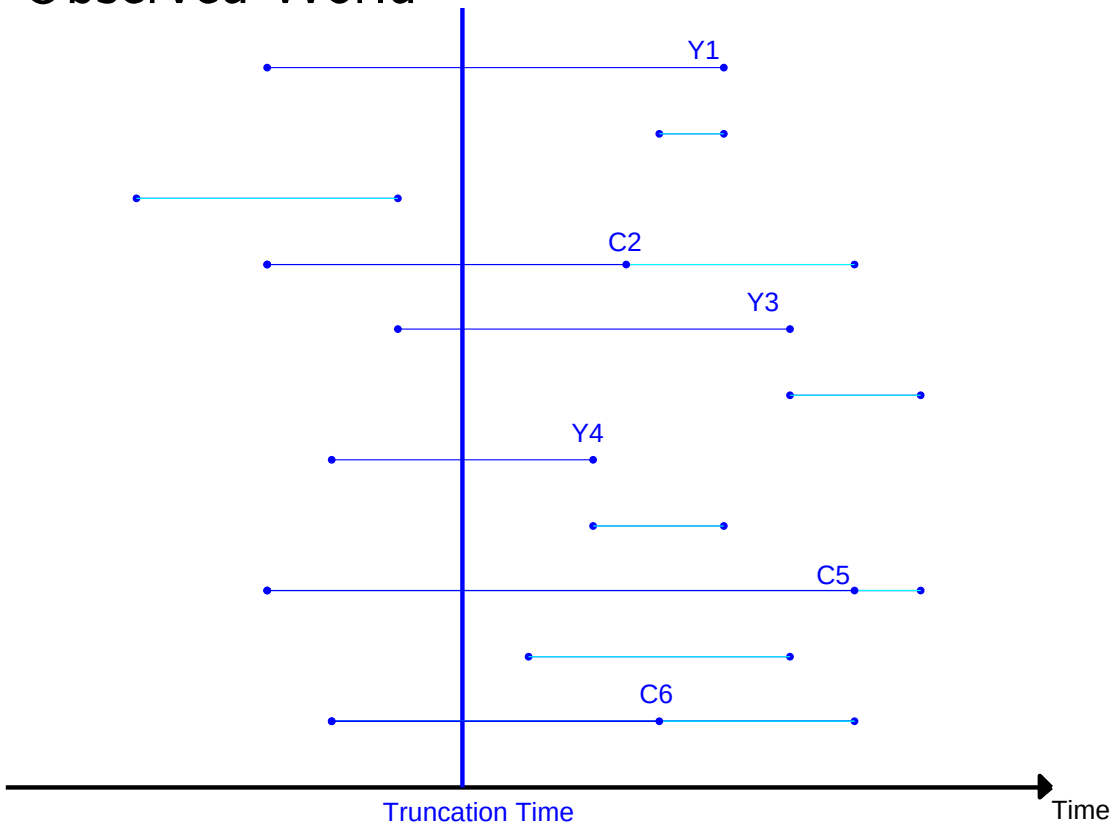
We use as notation F for cdf

Real World



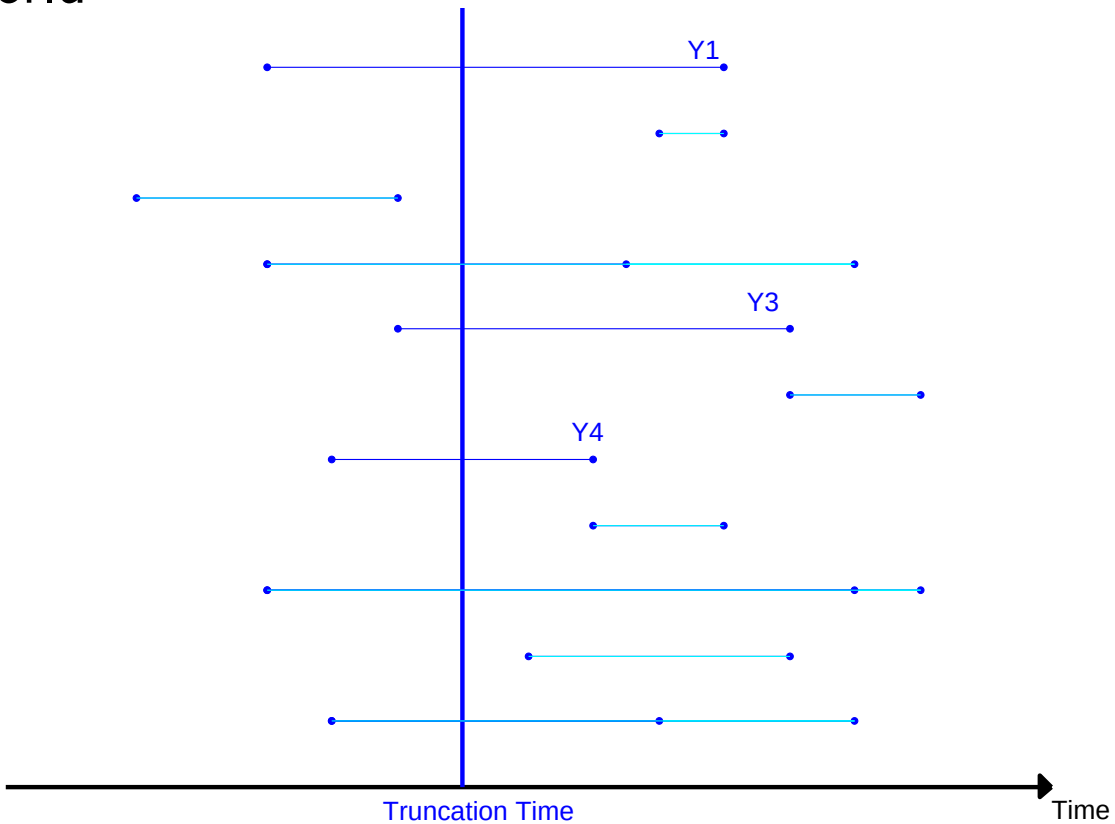
We use as notation F for cdf

Intermediate Observed World



We use as notation $\mathcal{H}$ for cdf, n the sample size

Observed World



We use as notation H for cdf

Aim : Estimation of the error distribution

$$F_\varepsilon(e) = \mathbf{P}(\varepsilon \leq e)$$

with

$$(X, Y) \text{ where } T \leq Y \leq C$$

where

- the distribution $F_{T|X}$ is a parametric distribution
- the distribution $F_{C-T|X}$ is completely unknown

Assumptions:

- the variables Y and T are independent, conditionally on X
- for each value x, the support of $F_{Y|X}(\cdot|x)$ is included into the support of $F_{T|X}(\cdot|x)$
- the lower bound of the T support is zero
- the variables (T, Y) and $C - T$ are independent, conditionally on $T \leq Y$, X

We have

$$\begin{aligned} H_{X,Y}(x,y) &= \mathbf{P}(X \leq x, Y \leq y | T \leq Y \leq C) \\ &= (E[w(X,Y)])^{-1} \int_{r \leq x} \int_{s \leq y} w(r,s) dF_{X,Y}(r,s), \end{aligned}$$

the weight function $w(x,y)$ is defined by

$$w(x,y) = \int_{t \leq y} \{1 - \mathcal{G}(y-t|x)\} \, dF_{T|X}(t|x)$$

where $\mathcal{G}(z|x) = \mathbf{P}(C - T \leq z | X = x, T \leq Y)$.

In particular, if $C = T + \tau$ where τ is a positive constant, the weight function is

$$w(x, y) = \int_{0 \vee y - \tau}^{y} dF_{T|X}(t|x)$$

by applying the same procedure.

We obtain

$$F_{X,Y}(x,y) = \int_{r\leq x}\int_{s\leq y} \frac{E[w(X,Y)]}{w(r,s)} dH_{X,Y}(r,s)$$

Therefore,

$$\begin{aligned} F_{\varepsilon}(e) &= \mathbf{P}\left(\frac{Y-m(X)}{\sigma(X)} \leq e\right) \\ &= \iint_{\left\{(x,y):\frac{y-m(x)}{\sigma(x)}\leq e\right\}} dF_{X,Y}(x,y) \\ &= \iint_{\left\{(x,y):\frac{y-m(x)}{\sigma(x)}\leq e\right\}} \frac{E[w(X,Y)]}{w(x,y)} dH_{X,Y}(x,y) \end{aligned}$$

Thus, the estimator is

$$\hat{F}_{\varepsilon}(e) = \frac{1}{M} \sum_{i=1}^{n} \frac{\hat{E}[w(X,Y)]}{\hat{w}(X_i, Y_i)} I\{\hat{\varepsilon}_i \leq e, \Delta_i = 1\}$$

with

$$\hat{\varepsilon}_i = \frac{Y_i - \hat{m}(X_i)}{\hat{\sigma}(X_i)}, \qquad M = \sum_{i=1}^{n} \Delta_i,$$

$$\hat{E}[w(X,Y)] = \left(\frac{1}{M} \sum_{i=1}^{n} \frac{\Delta_i}{\hat{w}(X_i, Y_i)} \right)^{-1}$$

where the functions $\hat{m}(\cdot)$, $\hat{\sigma}(\cdot)$ and $\hat{w}(\cdot,\cdot)$ are nonparametric estimators.

For $\mathcal{G}(t|x)$, we use the Beran (1981) estimator defined by

$$\hat{\mathcal{G}}(t|x) = 1 - \prod_{Z_i \leq t, \Delta_i = 0} \left(1 - \frac{W_i(x, h_n)}{\sum_{j=1}^{n} W_j(x, h_n) I\{Z_j \geq Z_i\}} \right)$$

where

- $Z_i = \min(C_i - T_i, Y_i - T_i)$ and $\Delta_i = I\{Y_i \leq C_i\}$
- $W_i(x, h_n) = \frac{K\left(\frac{x - X_i}{h_n}\right)}{\sum_{j=1}^{n} K\left(\frac{x - X_j}{h_n}\right)}$ are the Nadaraya-Watson weights
- K is a kernel function
- h_n is a bandwidth sequence tending to 0 when $n \to \infty$

$$=> \hat{w}(x, y) = \int_{t \leq y} \left\{ 1 - \hat{\mathcal{G}}(y - t|x) \right\} dF_{T|X}(t|x)$$

The estimators of $m(\cdot)$ and $\sigma(\cdot)$ are given by

$$\hat{m}(x) = \frac{\sum_{i=1}^{n} \frac{W_i(x,h_n) Y_i \Delta_i}{\hat{w}(x,Y_i)}}{\sum_{i=1}^{n} \frac{W_i(x,h_n) \Delta_i}{\hat{w}(x,Y_i)}},$$

$$\hat{\sigma}^2(x) = \frac{\sum_{i=1}^{n} \frac{W_i(x,h_n) \Delta_i (Y_i - \hat{m}(x))^2}{\hat{w}(x,Y_i)}}{\sum_{i=1}^{n} \frac{W_i(x,h_n) \Delta_i}{\hat{w}(x,Y_i)}},$$

extension of the estimators in de Uña-Alvarez and Iglesias-Pérez (2008).

Asymptotic results

Under some assumptions,

$$\hat{F}_\varepsilon(e) - F_\varepsilon(e) = \sum_{i=1}^{n} V(X_i, Y_i, Z_i, \Delta_i, e) + o_p(n^{\frac{-1}{2}})$$

uniformly in e.
$=>$ Weak convergence of the process

$$\sqrt{n}(\hat{F}_\varepsilon(e) - F_\varepsilon(e)) \rightarrow \Omega(e)$$

where Ω is a Gaussian process with zero mean and complex covariance.

Bandwidth selection

We want to determine the smoothing parameter h_n which minimizes

$$MISE = E\left[\int \left\{\hat{F}_{\varepsilon,h_n}(e) - F_\varepsilon(e)\right\}^2 de\right]$$

We consider bootstrap procedure which is an extension of Li and Datta (2001).

For $b = 1, \ldots, B$,
For $i = 1, \ldots, n$

Step 1 Generate $X^*_{i,b}$ from

$$\hat{F}_X(\cdot) = \sum_{j=1}^{n} \frac{\hat{E}[w(X,Y)]}{\hat{E}[w(X,Y)|X=\cdot]} I\{X_j \leq \cdot, \Delta_j = 1\},$$

where $\hat{E}[w(X,Y)|X=\cdot] = \sum_{j=1}^{n} W_j(\cdot, g_n)\Delta_j / \sum_{j=1}^{n} \frac{W_j(\cdot, g_n)\Delta_j}{\hat{w}(\cdot, Y_j)}$

and g_n is a pilot bandwidth

Step 2 Generate $Y^*_{i,b}$ from

$$\hat{F}_{Y|X}(\cdot|X^*_{i,b}) = \sum_{j=1}^{n} \frac{\hat{E}[w(X,Y)|X = X^*_{i,b}]W_j(X^*_{i,b}, g_n)}{\hat{w}(X^*_{i,b}, Y_j)(\sum_{k=1}^{n} W_k(X^*_{i,b}, g_n)\Delta_k)} I\{Y_j \leq \cdot, \Delta_j = 1\}$$

Step 3 Draw $T^*_{i,b}$ from the distribution $F_{T|X}(\cdot|X^*_{i,b})$.

- If $T^*_{i,b} > Y^*_{i,b}$, then reject $(X^*_{i,b}, Y^*_{i,b}, T^*_{i,b})$ and go to Step 1.
- Otherwise, go to Step 4.

Step 4 Select at random $V^*_{i,b}$ from $\hat{\mathcal{G}}(\cdot|X^*_{i,b})$ calculated with g_n

Step 5 Define

- $Z^*_{i,b} = \min(Y^*_{i,b} - T^*_{i,b}, V^*_{i,b})$
- $\Delta^*_{i,b} = I\{Y^*_{i,b} - T^*_{i,b} \leq V^*_{i,b}\}$.

Compute $\hat{F}^*_{\varepsilon,h_n,b}$, the error distribution based on

- bandwidth h_n
- resample $\{(X^*_{i,b}, T^*_{i,b}, Z^*_{i,b}, \Delta^*_{i,b}) : i = 1, \ldots, n\}$.

The expression of the MISE can be approximated by

$$\text{argmin}_{h_n} B^{-1} \sum_{b=1}^{B} \int \{\hat{F}^*_{\varepsilon,h_n,b}(e) - \hat{F}_{\varepsilon,g_n}(e)\}^2 de.$$

Simulations

We consider

- model $Y = X + \varepsilon$ where
  - $X \sim U([1.7321; 2])$
  - $\varepsilon \sim U\left(\left[-\sqrt{3}; \sqrt{3}\right]\right)$
- model $\log Y = X + \varepsilon$ where
  - $X \sim U([0; 1])$
  - $\varepsilon \sim N(0; 1)$
- model $Y = X^2 + X * \varepsilon$ where
  - $X \sim U\left(\left[2; 2 * \sqrt{3}\right]\right)$
  - $\varepsilon \sim U\left(\left[-\sqrt{3}; \sqrt{3}\right]\right)$
- model $\log Y = X^2 + X * \varepsilon$ where
  - $X \sim U([0; 1])$
  - $\varepsilon \sim N(0; 1)$

where X and ε are independent in each model

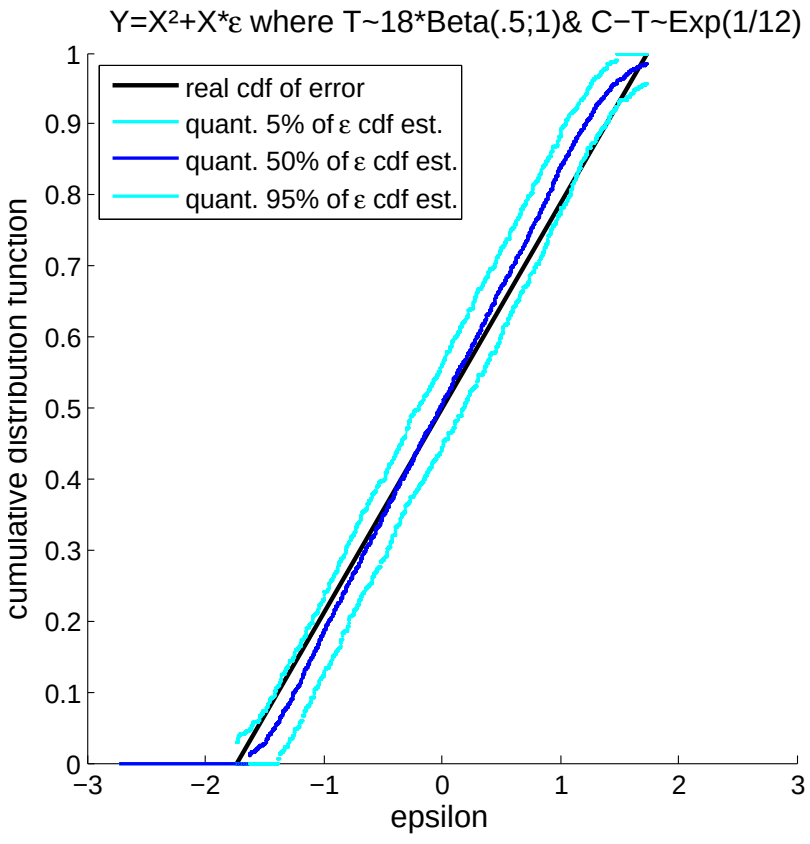
Homoscedastic model : $Y = X + \varepsilon$

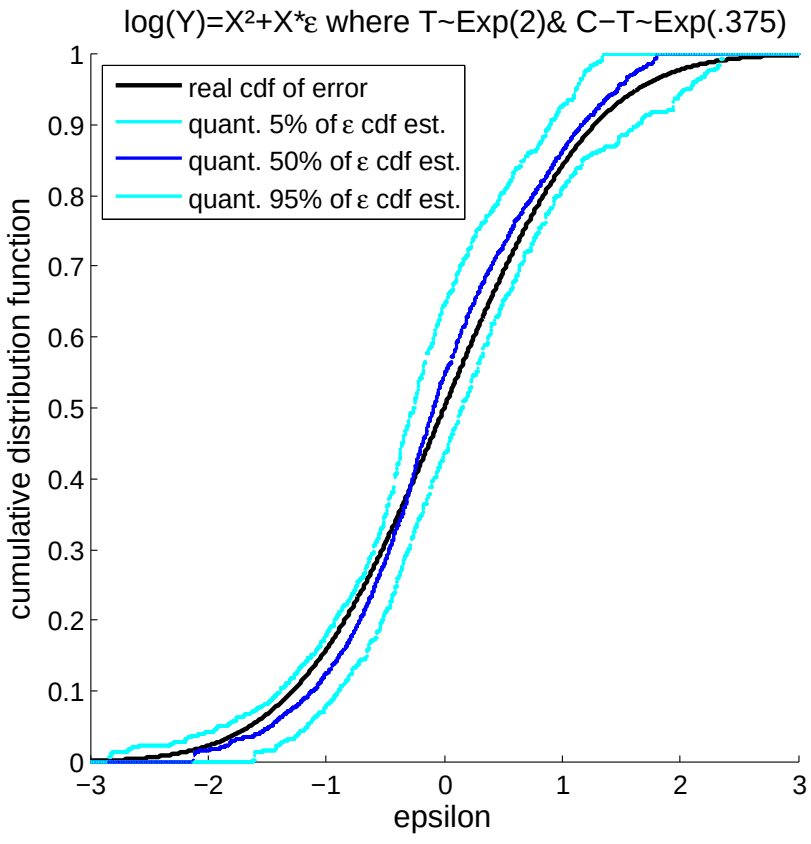
Dist. of T	Dist. of $C - T$	% Censor.	$\widehat{MISE}$ $(*10^{-3})$
Unif$([0; 4])$	Exp$(2/5)$	0.37	5.5
Unif$([0; 4])$	Exp$(2/7)$	0.28	4.9
Unif$([0; X + 2])$	Exp$(2/5)$	0.36	5.2
Unif$([0; X + 2])$	Exp$(2/7)$	0.29	5.0
$4 *$ Beta$(0.5; 1)$	Exp$(2/7)$	0.34	4.2
$4 *$ Beta$(0.5; 1)$	Exp$(2/9)$	0.29	4.0
Unif$([0; 4])$	Exp$(1/(X + 1.5))$	0.28	4.6
$4 *$ Beta$(0.5; 1)$	Exp$(1/(X^2 - 1))$	0.34	4.5

Heteroscedastic model : $Y = X^2 + X * \varepsilon$

Dist. of T	Dist. of $C - T$	% Censor.	$\widehat{MISE}$ $(*10^{-3})$
Unif$([0; 18])$	Exp(0.1)	0.34	6.9
Unif$([0; 18])$	Exp(0.05)	0.19	6.2
$18 *$ Beta$(0.5; 1)$	Exp$(1/12)$	0.35	6.3
$18 *$ Beta$(0.5; 1)$	Exp$(1/15)$	0.29	6.2
Unif$([0; X + 16])$	Exp$(1/12)$	0.29	6.2
$18 *$ Beta$(0.5; 1)$	Exp$(1/(2X^2 - 1))$	0.30	6.6

Interval containing 90% of value of $\hat{F}_\varepsilon$





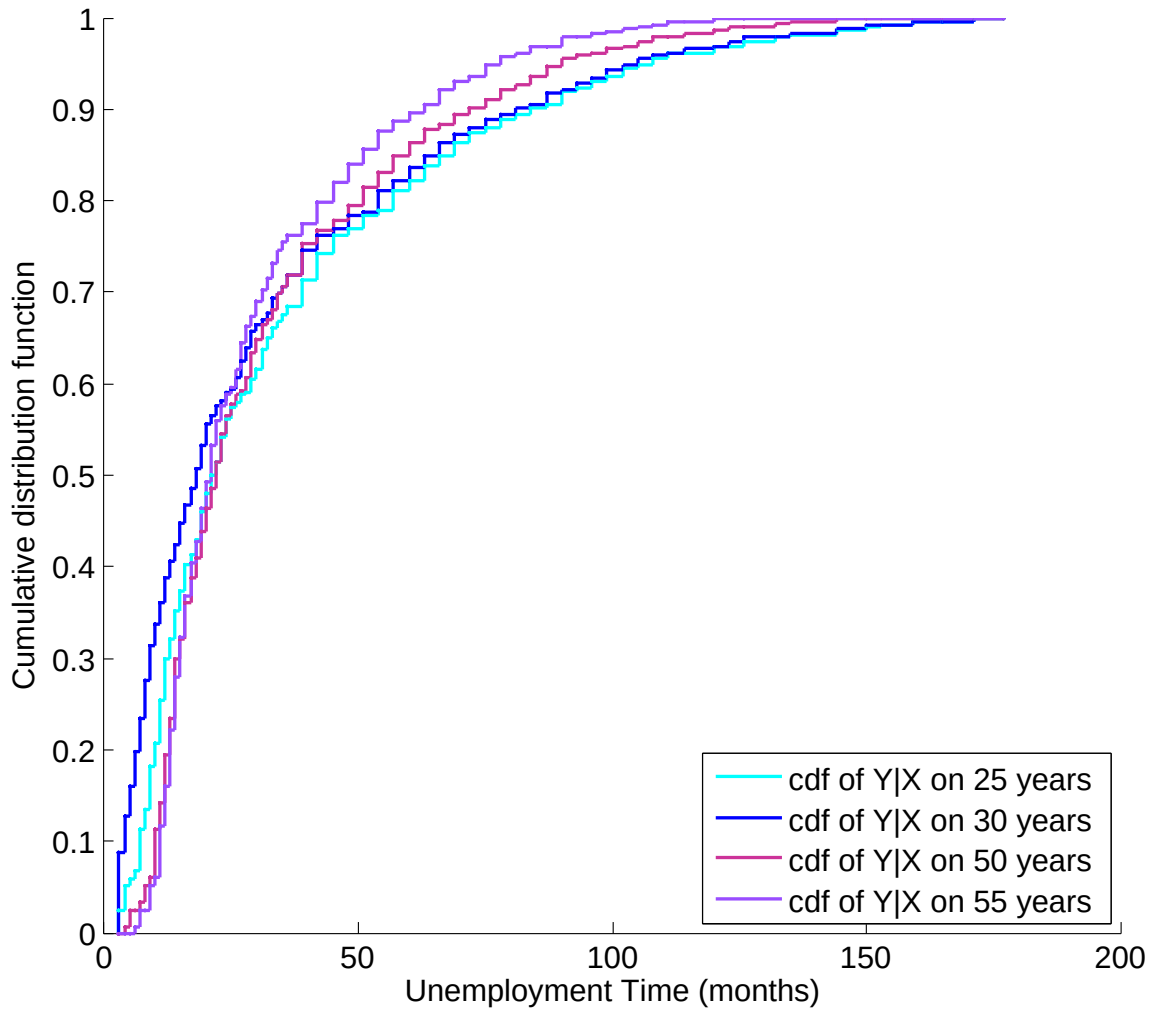
Data analysis

For the real data, we suppose that

- the number of time periods is equal to 1009 but only 446 aren't censored.
- the distribution of T is a uniform one (Wang, 1991);
- the variable C is defined by $C = T + \tau$ where τ is a constant equal to 18 months;

The Bootstrap approximation gives the value of 70 months as the optimal bandwidth.

Representation of $\hat{F}_{Y|X}$ for various ages.



Thank you for your attention

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