Multivariate Value at Risk Based on Extremality

Henry Laniado
Rosa E. Lillo
Juan Romo

Department of Statistics
Universidad Carlos III de Madrid

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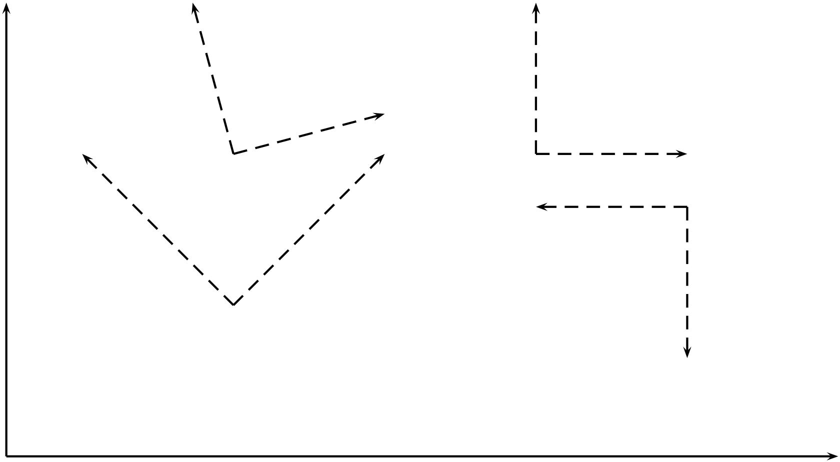
Outline

1. **Extremality Measure**
2. **Multivariate Value at Risk**

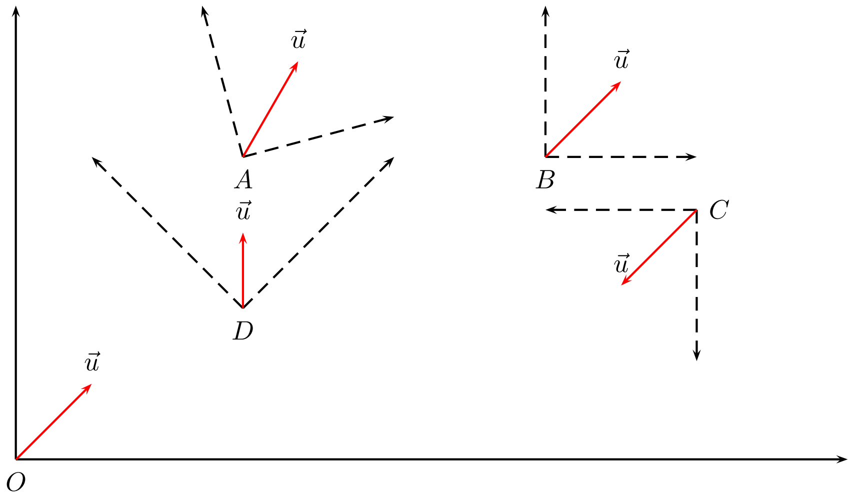
Extremality Measure

Multivariate Value at Risk

Oriented Sub-Orthants $\mathcal{C}_x^{\vec{u}}$



Oriented Sub-Orthants $\mathcal{C}_x^{\vec{u}}$



Oriented Sub-Orthants $\mathcal{C}_v^{\vec{u}}$

Given a unit director vector $\vec{u} \in \mathbb{R}^n$ and a vertex $v \in \mathbb{R}^n$; a Oriented Sub-Orthant $\mathcal{C}_v^{\vec{u}}$ is the convex cone given by

$$\mathcal{C}_v^{\vec{u}} = \{x \in \mathbb{R}^n \mid \mathcal{R}_{\vec{u}}(x - v) \geq 0\}\,. \tag{1}$$

The inequality in (1) is componentwise.

Oriented Sub-Orthants $\mathcal{C}_v^{\vec{u}}$

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$$\mathcal{C}_v^{\vec{u}} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \ : \ \frac{\sqrt{2}}{2} \begin{pmatrix} u_1 + u_2 & u_2 - u_1 \\ u_1 - u_2 & u_1 + u_2 \end{pmatrix} \begin{pmatrix} x_1 - v_1 \\ x_2 - v_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

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In $\mathbb{R}^2$, the director vector $\vec{u}$ can be determined by an angle $0 \leq \theta \leq 2\pi$ as $\vec{u} = [\cos\theta, \sin\theta]'$ that indicates the direction of the cone.

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$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} \cos(\theta - \frac{\pi}{4}) & \sin(\theta - \frac{\pi}{4}) \\ -\sin(\theta - \frac{\pi}{4}) & \cos(\theta - \frac{\pi}{4}) \end{pmatrix} \begin{pmatrix} x_1 - v_1 \\ x_2 - v_2 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Extremality Measure

Definition (Extremality Measure)

The Extremality of $x \in \mathbb{R}^n$ respect to a distribution function F in direction $\vec{u}$ is a mapping $\mathcal{E}_{\vec{u}}(x, F) : \mathbb{R}^n \times \mathfrak{F} \longrightarrow R^+ \cup \{0\}$, defined by

$$\mathcal{E}_{\vec{u}}(x, F) = 1 - P_{x,\vec{u}},$$

where $P_{x,\vec{u}}$ is given by $P_F\left(\mathcal{C}_x^{\vec{u}}\right)$.

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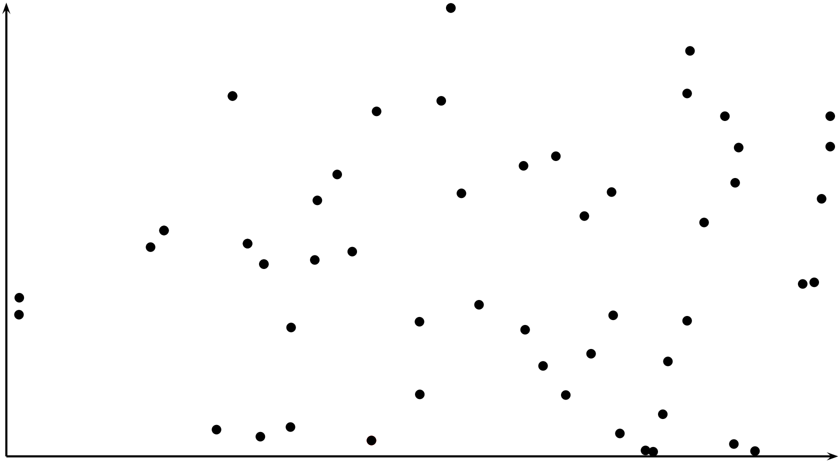
A natural estimator for $\mathcal{E}_{\vec{u}}(x,F)$ is given by

$$\mathcal{E}_{\vec{u}}(x,\hat{F}) = 1 - \frac{1}{m}\sum_{j=1}^{m} 1_{\{x_j, \in \mathcal{C}_x^{\vec{u}}\}} = 1 - \frac{1}{m}\sum_{j=1}^{m} 1_{\{\mathcal{R}_{\vec{u}}(x_j - x) \geq 0\}},$$

where $x_1, \ldots, x_m$ is a sample of the random vector X, that is, $1-$ the proportion of the point cloud that belong to $\mathcal{C}_x^{\vec{u}}$.

Extremality Measure



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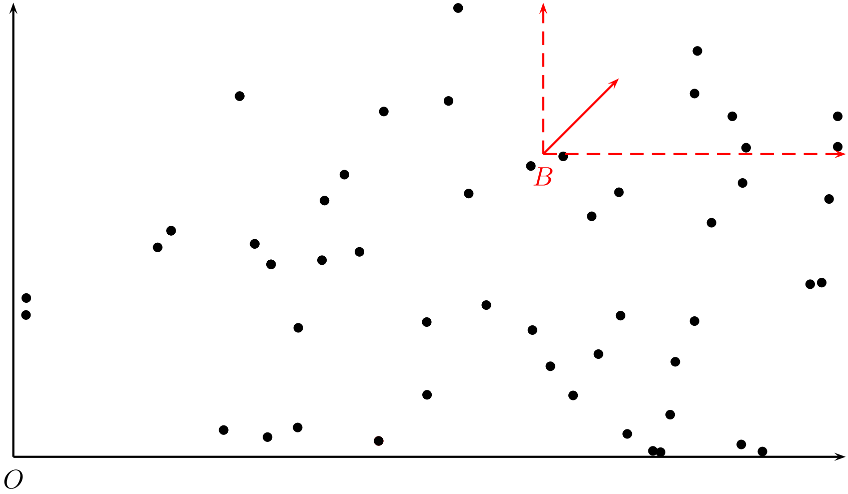

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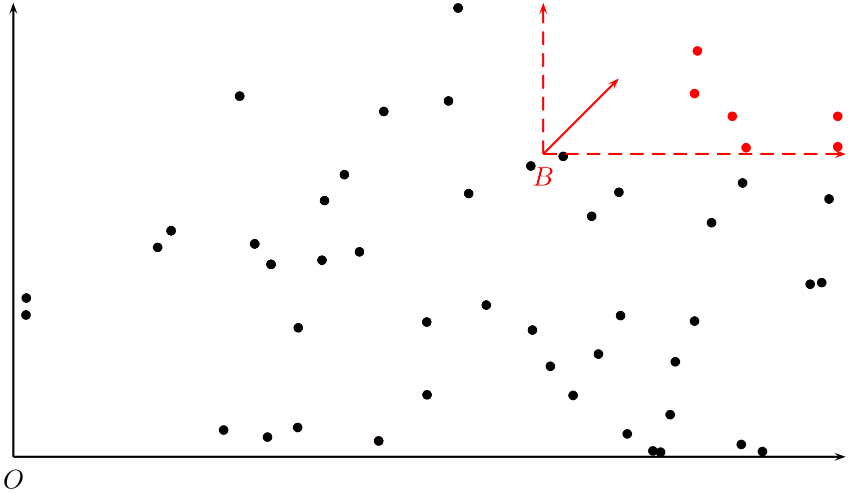

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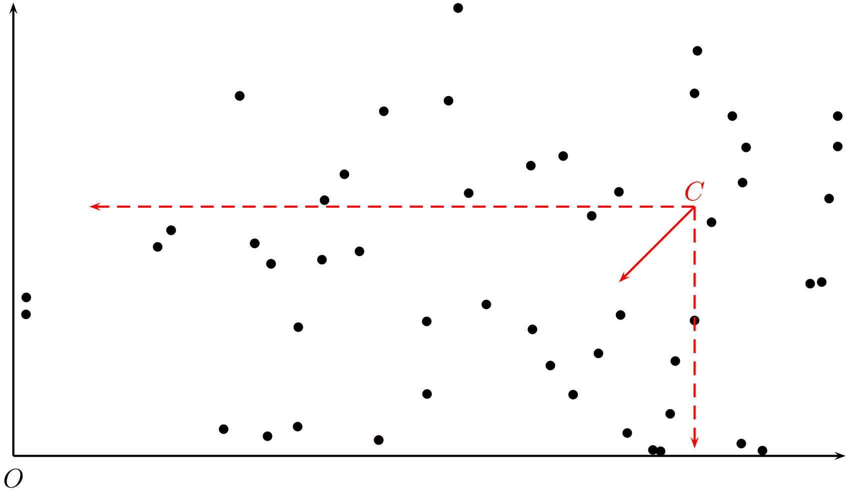

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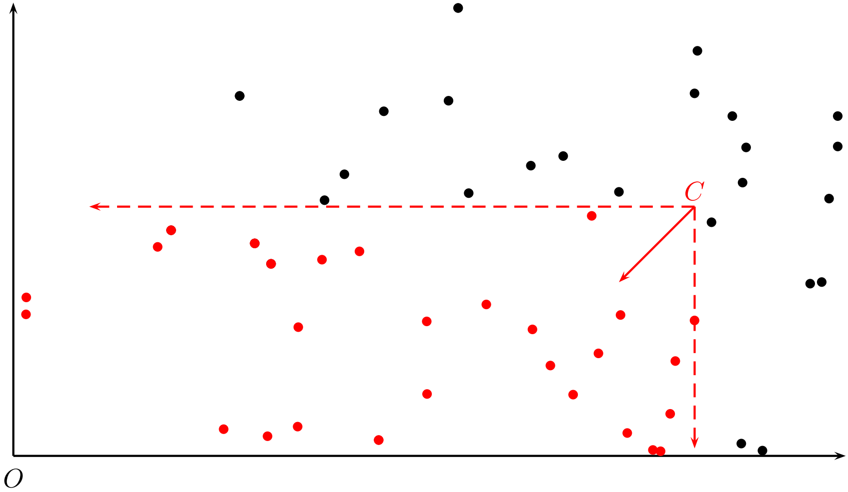

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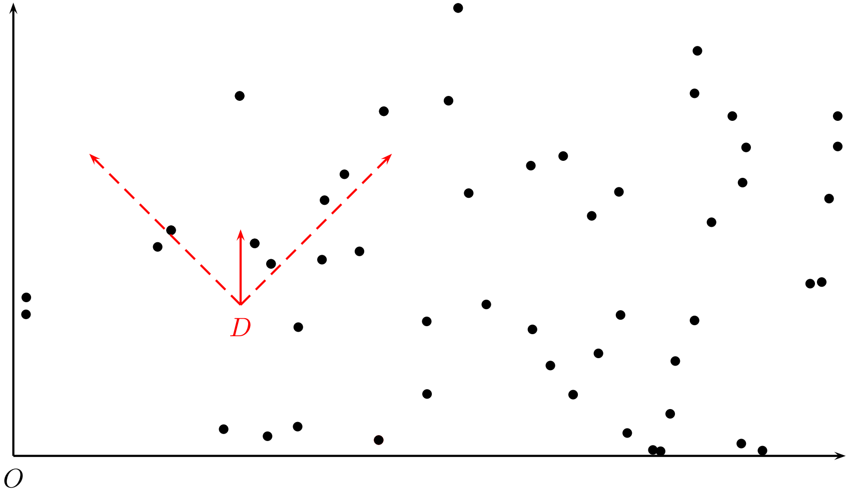

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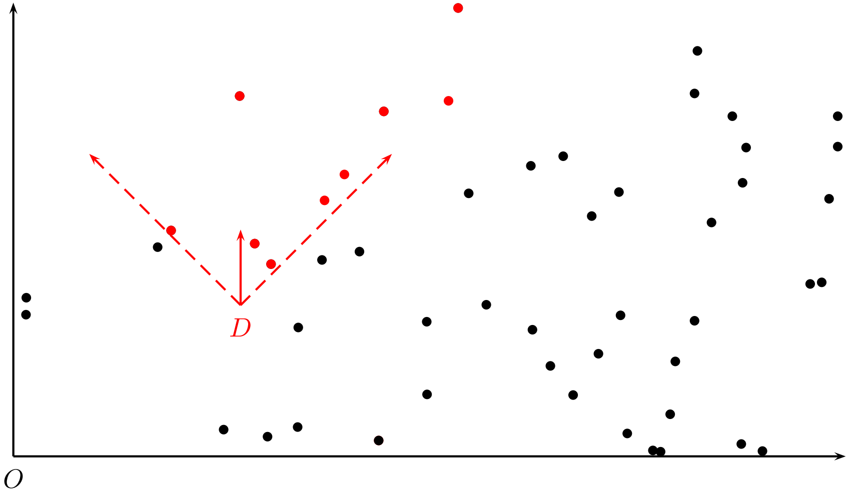

Extremality Measure

- **High extremality of a point x means that the convex cone $\mathcal{C}_x^{\vec{u}}$ contains a small part of the total mass and possibly x belongs to some tail of the distribution.**

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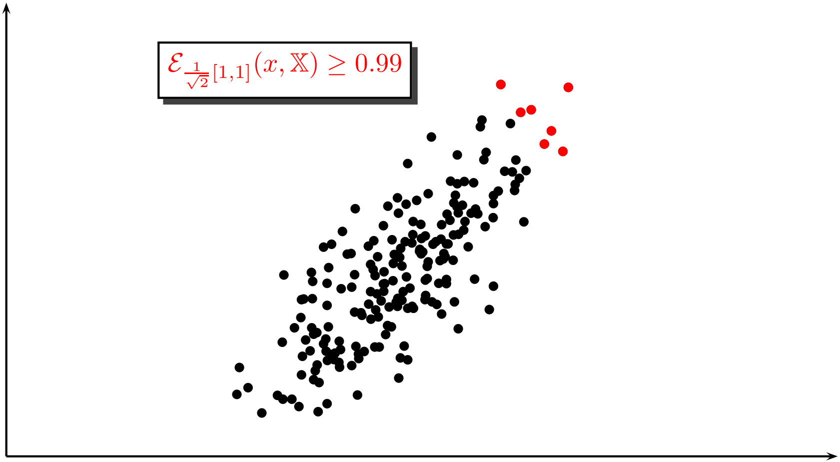
- **Hence, high extremality can be interpreted as ”farness” regarding distribution**

Extremality Measure



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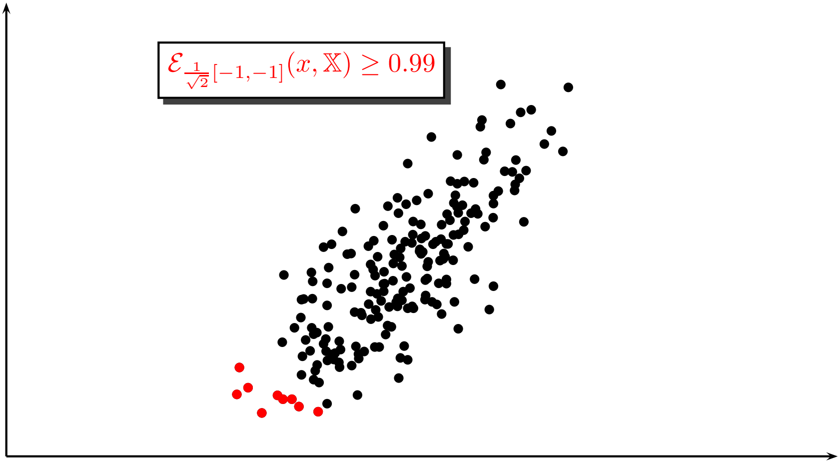

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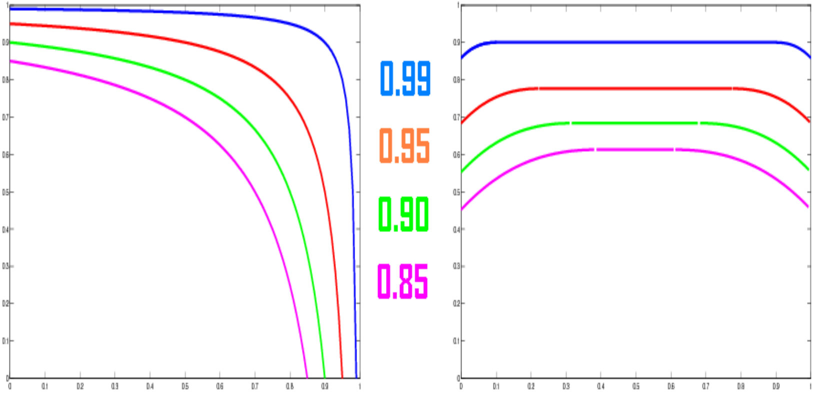

Figure: $\mathcal{E}_{\frac{1}{\sqrt{2}}[1,\,1]}(x, F) = \alpha$ and $\mathcal{E}_{[0,\,1]}(x, F) = \alpha$

Extremality Measure

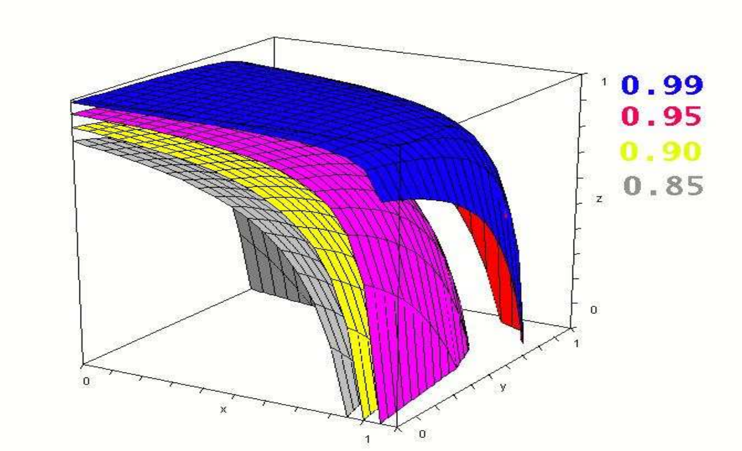

Figure: $\mathcal{E}_{\frac{1}{\sqrt{3}}[1,\,1,\,1]}(x, F) = \alpha$

Properties

Property

For any $x_0 \in \mathbb{R}^n$ and any absolutely continuous $F \in \mathfrak{F}$

$$\mathcal{E}_{\vec{u}}(x_0, F) \quad \textbf{is continuous in} \quad \vec{u}.$$

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Property

$$\mathcal{E}_{\vec{u}}(x, F) \leq \mathcal{E}_{\vec{u}}(x^*, F) \quad \textit{for all} \;\; x^* \in \mathcal{C}_x^{\vec{u}}$$

Properties

Property

Let X be a $n-$ multivariate random variable with distribution function F. Let A a $n-$orthogonal matrix and let b be a vector in $\mathbb{R}^n$. Then

$$\mathcal{E}_{A\vec{u}}(Ax+b, F_{AX+B}) = \mathcal{E}_{\vec{u}}(x, F_X)$$

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Property

For $x \in \mathbb{R}^n - \{0\}$ and $\vec{u} = \frac{x}{\|x\|}$, where $\|\cdot\|$ is the Euclidean norm.

$$\|x\| \longrightarrow \infty \Longrightarrow \mathcal{E}_{\vec{u}}(x, F) \longrightarrow 1.$$

Extremality Measure

Multivariate Value at Risk

Value at Risk

Suppose that F is the loss distribution and $\alpha \in [0,1]$ then

$$VaR_{\alpha}(X) := \inf\{x \in \mathbb{R} \mid F(x) \geq \alpha\}.$$

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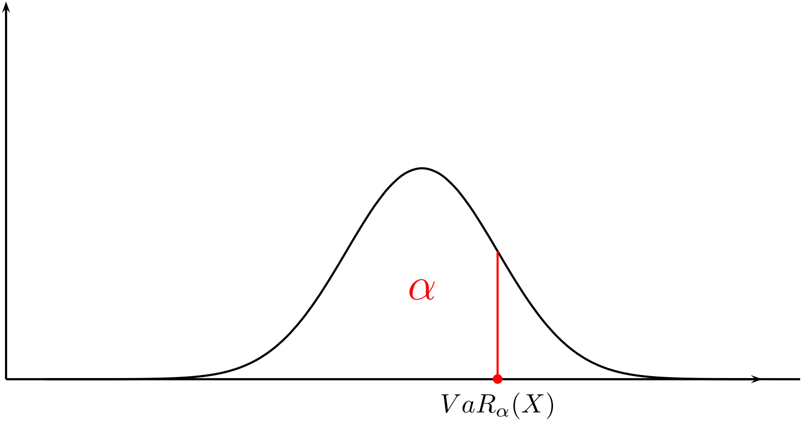

Value at Risk

A natural idea to study risk for portfolio vectors

$$X = (X_1, \ldots, X_n)$$

is to consider a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ **and one-dimensional risk measure on** $f(X)$**.**

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- **The VaR of the joint portfolio is that associated to $f(X)$.**
- **In Burgert and Rüschendorf (2006),**

$$f(X) = \sum_{i=1}^{n} X_i \quad \text{or} \quad f(X) = \max_{i \le n} X_i.$$

Multivariate Value at Risk

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- **Multivariate upper-orthant Value at Risk**

$$\overline{VaR}_\alpha(\bar{F}) := \partial\{x \in \mathbb{R}^n \ : \ \bar{F}(x) \leq 1 - \alpha\}$$

Multivariate Value at Risk Based on Extremality

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We define the *Oriented Multivariate Value at Risk*

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Particulary for $\vec{u} = \frac{1}{\sqrt{n}}[1, \ldots, 1]$ and $\vec{u} = -\frac{1}{\sqrt{n}}[1, \ldots, 1]$

$$VaR_\alpha^{\vec{u}}(X)$$

is the upper-orthant value at risk and the lower-orthant value at risk respectively.

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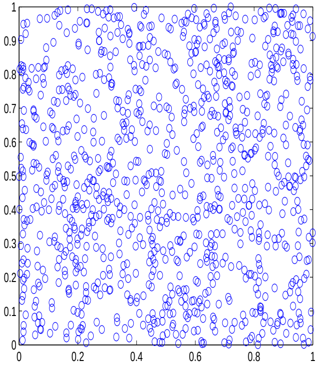

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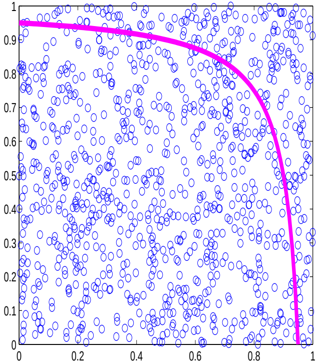
$$\boxed{VaR_{0.05}^{\frac{1}{\sqrt{2}}[1,1]}(X)}$$



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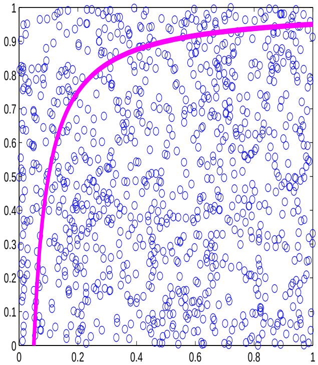

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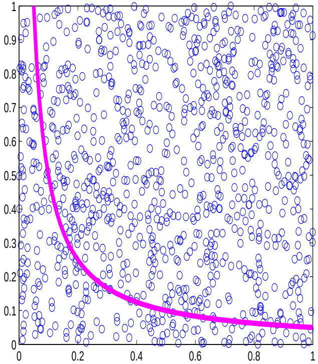

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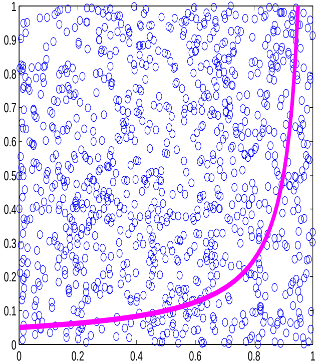

Estimation Method Proposed

Given $\{x_1 \ldots, x_m\}$

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$$S_\alpha^{\vec{u}}(F_m) = \left\{ x_i : \mathcal{E}_{\vec{u}}\left(x_i, \hat{F_m} \right) = 1 - \alpha \right\}.$$

However, it may be possible that $S_\alpha^{\vec{u}}(F_m) = \emptyset$**.**

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$$S_{\alpha,h}^{\vec{u}}(F_m) = \left\{x_i : \left|\mathcal{E}_{\vec{u}}\left(x_i, \hat{F_m}\right) - 1 + \alpha\right| \leq h\right\},$$

where h **is a slack.**

$$S_{\alpha}^{\vec{u}}(F_m) \subset S_{\alpha,h}^{\vec{u}}(F_m).$$

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$$S_{\alpha}^{\vec{u}}(F_m) \subset S_{\alpha,h}^{\vec{u}}(F_m).$$

The direction given by $\vec{u}$ **can have influence in the estimation of** $S_{\alpha,h}^{\vec{u}}(F_m)$**.**

Estimation Method Proposed

To estimate $VaR_{\alpha}^{\vec{u}}(X)$, we propose to change the original coordinates. Suppose

$$S_{\alpha,h}^{\vec{u}}(F) = \{x_1, x_2 \ldots, x_k\}.$$

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$$\mathcal{R}_{\vec{u}} S^{\vec{u}}_{\alpha,h}(F) = \{\mathcal{R}_{\vec{u}} x_1, \mathcal{R}_{\vec{u}} x_2 \ldots, \mathcal{R}_{\vec{u}} x_k\}\,. \tag{2}$$

Smoothing is done using the points in (2). The resulting surface is returned to original system.

Estimation Method Proposed

Input:

$\vec{u},\ \alpha,\ \ h,$ and the multivariate sample $\mathbb{X} = (x_1, \ldots, x_m)$

for $i = 1$ to m

$\mathcal{E}_i = \mathcal{E}_{\vec{u}}(x_i, \hat{F_m})$

if $|\mathcal{E}_i - 1 + \alpha| \leq h$

$x_i \in S^{\vec{u}}_{\alpha,h}(\hat{F_m})$

end

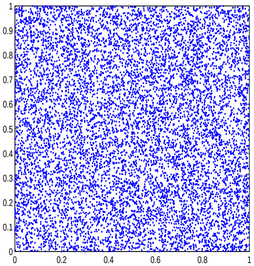
end

Fitting a function f on $\mathcal{R}_{\vec{u}} S^{\vec{u}}_{\alpha,h}(\hat{F_m})$

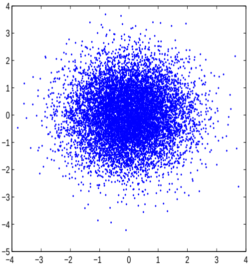
$VaR^{\vec{u}}_{\alpha}(X) = \mathcal{R}^{-1}_{\vec{u}} f$

Estimation Method

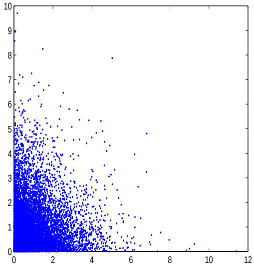
Uniforms



Normals



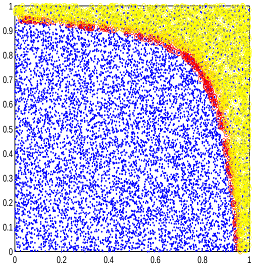
Exponentials



$\mathcal{E}_{\vec{e}}\left(x_i, \hat{F}\right) > 0.95$ (yellow) $S^{\vec{u}}_{\alpha,h}(F)$ (red) $\widehat{VaR}^{\vec{e}}_{0.05}$ (green) $VaR^{\vec{e}}_{0.05}$ (magenta)

Estimation Method

Uniforms



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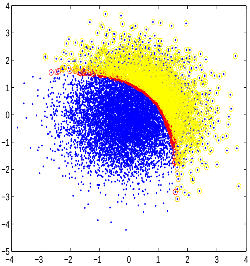

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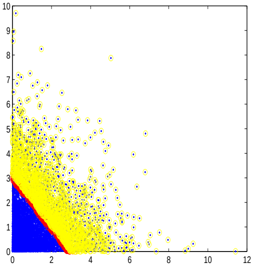

$\mathcal{E}_{\vec{e}}\left(x_i, \hat{F}\right) > 0.95$ $\quad S^{\vec{u}}_{\alpha,h}(F)$ $\quad \widehat{VaR}^{\vec{e}}_{0.05}$ $\quad VaR^{\vec{e}}_{0.05}$

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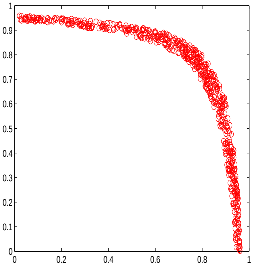

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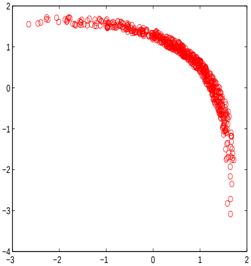

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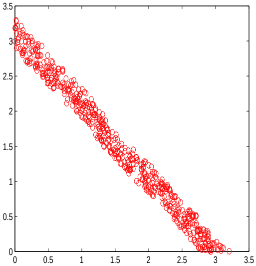

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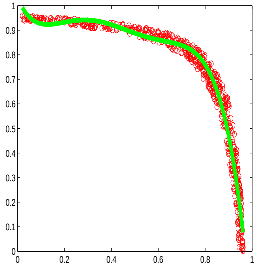
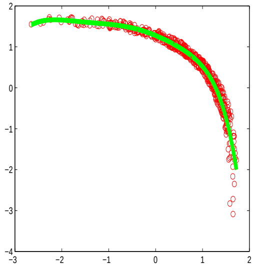
Estimation Method

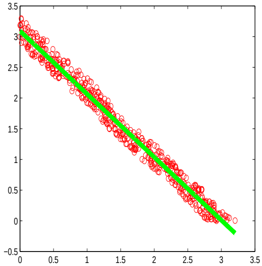
Uniforms



Normals



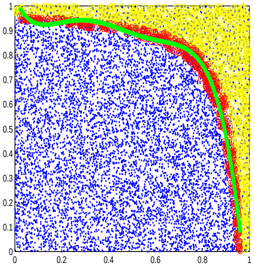
Exponentials



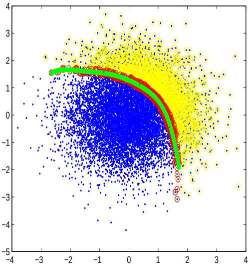
$\mathcal{E}_{\vec{e}}\left(x_i, \hat{F}\right) > 0.95$ $\quad S^{\vec{u}}_{\alpha,h}(F)$ $\quad \widehat{VaR}^{\vec{e}}_{0.05}$ $\quad VaR^{\vec{e}}_{0.05}$

Estimation Method

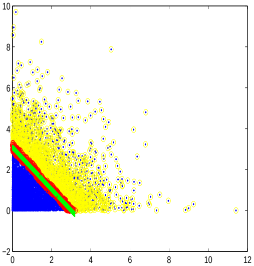
Uniforms



Normals



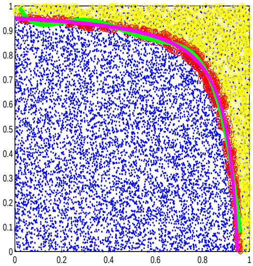
Exponentials



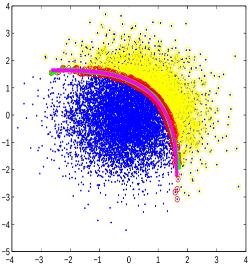
$\mathcal{E}_{\vec{e}}\left(x_i, \hat{F}\right) > 0.95$ $\quad S^{\vec{u}}_{\alpha,h}(F)$ $\quad \widehat{VaR}^{\vec{e}}_{0.05}$ $\quad VaR^{\vec{e}}_{0.05}$

Estimation Method

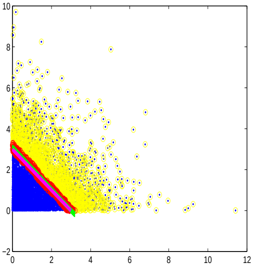
Uniforms



Normals



Exponentials



$\mathcal{E}_{\vec{e}}\left(x_i, \hat{F}\right) > 0.95$ $\quad S^{\vec{u}}_{\alpha,h}(F)$ $\quad \widehat{VaR}^{\vec{e}}_{0.05}$ $\quad VaR^{\vec{e}}_{0.05}$

Oriented Multivariate Value at Risk

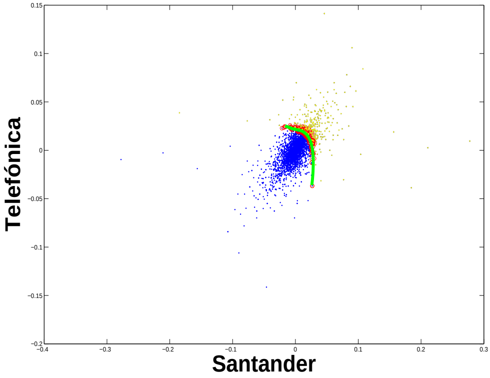


Figure: $\mathcal{E}_{\vec{e}}\left(x_i, \hat{F}\right) > 0.95$ Estimated $VaR^{\vec{e}}_{0.05}$

Oriented Multivariate Value at Risk

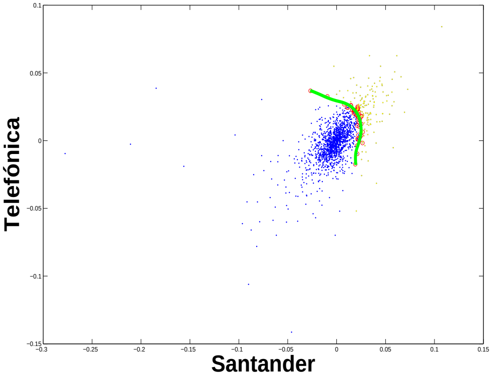


Figure: $\mathcal{E}_{\vec{e}}\left(x_i, \hat{F}\right) > 0.95$ Estimated $VaR^{\vec{pc}}_{0.05}$

thanks for your attention