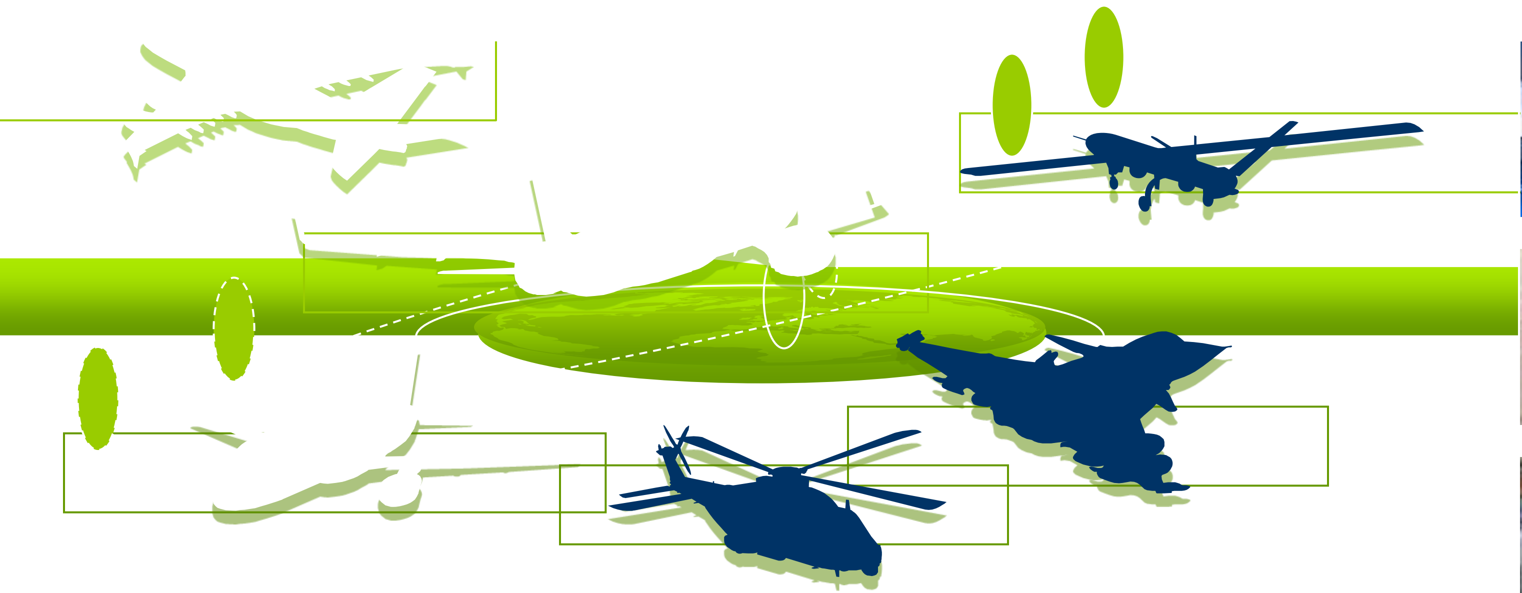

Use of Monte Carlo when estimating reliability of complex systems

COMPSTAT 2010 : August 27, 2010

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- Clusters and scan statistics : simple example
- Simulation methods
  - ❖ Monte-Carlo
  - ❖ Petri nets
- Markov approach
  - ❖ Simplified Markov chain - one scan window
  - ❖ Simplified Markov chain - double scan window
  - ❖ Complete Markov chain
- Simulation results and comparison
- Conclusions

August 2nd
Flight 358 of Air France went out of runways during landing in Toronto

August 6
Flight 1153 of Tuninter landed on see close to Palermo

August 14
Flight 522 of Hélios crashed into a mountain close to Athens

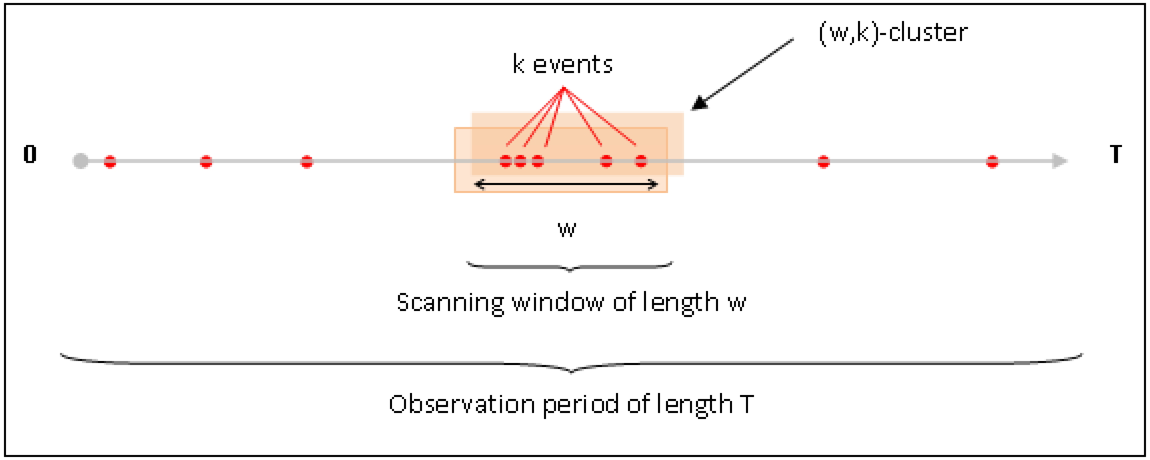
August 16
Flight 1153 lof West Caribbean Crached in Venezuela

August 23
Flight 204 of Trans Cracked approaching Amazonie

Unreliable series, isn't it, BUT…

Scan statistics allow to evaluate or approximer probability of occurence of such a "cluster" of events.

Goal : calculate **probability that we will observe** a **cluster of** k or more events in a scanning windows of length **w** moving during **a fixed period of length T**.



Problems

- **Any window** of length w can constain a cluster
- Windows overlap

Example:

$$\begin{cases} \text{T} = \text{one year, i.e. 365 days} \\ \lambda \text{ or p correspond to 8 events per year (on mean)} \\ (\text{w},\text{k}) = (10,3) \ : \ \text{3 events in 10 day} \end{cases}$$

Solutions

- Monte Carlo simulations
    - direct (implemented using a specific algorithm)
    - supported by Petri nets
- Markov chains

Two probability models :

- Bernoulli $\text{Be}(p)$
- Poisson $\text{P}(\lambda)$

Direct Monte-Carlo simulation

- Dates of accidents are generated along the considered distribuion to cover given period of observation [0,T[

$$0 < \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_S \leq T$$

- List of dates is scanned until the cluster is observed
- Counter of clusters - Nb_Cluster – is incremented by 1

We estimate unknown parameter using the quantity

$$\frac{\text{Nb\_Cluster}}{N}$$

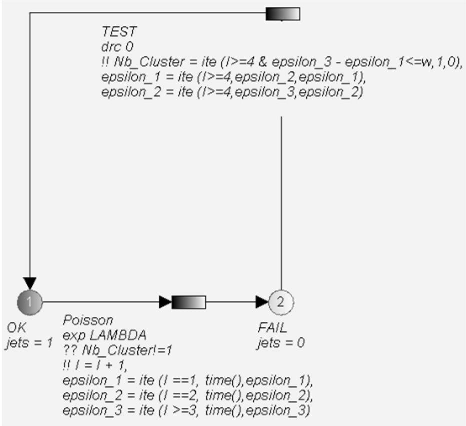
N est is number of repetitions of the simulation.

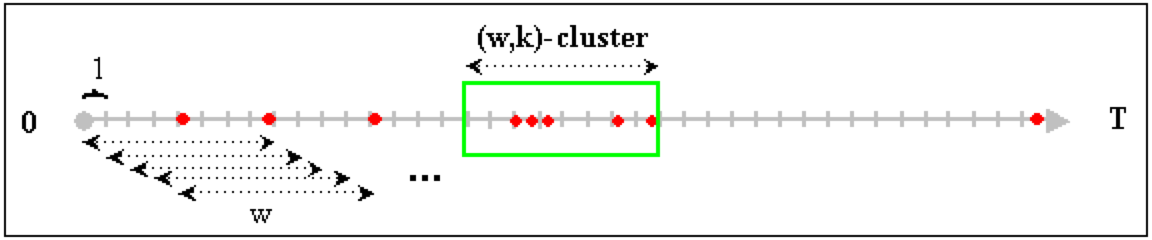
w	10	Number of years with a cluster	116
p	8/365	Number of clusters	122
k	3	Mean number of accidents per year	8,09
T	365		

n°simu	CPT	CPT_ACC	CPT_YES													
simu_749	0	8	0	102	109	161	177	205	228	233	300					
simu_750	0	9	0	11	66	71	81	98	199	226	256	319				
simu_751	0	7	0	28	31	129	132	160	191	237						
simu_752	2	10	1	50	54	55	57	62	91	197	265	282	319			
simu_753	0	8	0	48	60	150	175	208	229	278	348					
simu_754	0	5	0	227	248	295	312	313								
simu_755	0	5	0	7	59	75	307	311								
simu_756	0	8	0	76	95	104	224	272	288	293	327					
simu_757	0	8	0	92	126	139	170	214	226	230	346					
simu_758	0	4	0	71	94	173	303									
simu_759	1	11	1	30	55	128	181	210	288	310	314	316	333	348		
simu_760	0	9	0	68	77	201	228	255	305	317	325	339				
simu_761	0	6	0	14	171	214	242	249	257							
simu_762	0	3	0	37	208	271										
simu_763	0	8	0	52	57	87	167	177	197	223	322					
simu_764	0	5	0	166	174	334	340	347								
simu_765	1	9	1	38	157	166	175	176	180	247	285	310				

Use of Petri nets stimulating Monte Carlo simulation

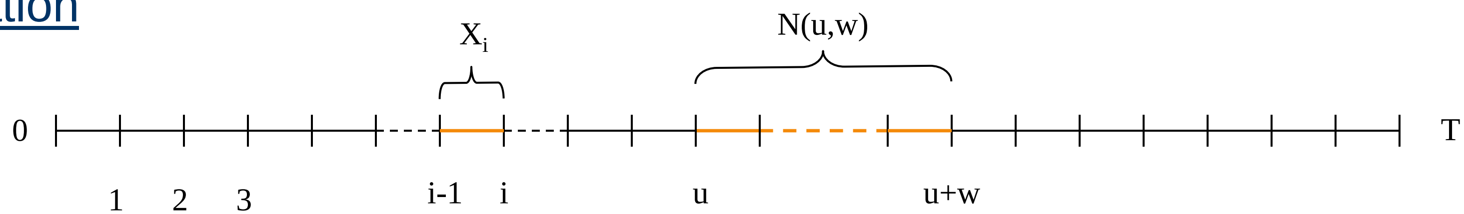
- Counting processes *(simple counting medium)*
- 2 places and 2 transitions
- Initialization
    - place 1 is set to one
    - Nb_Cluster = 0
- Variables ε_i (i =1,...,k) indicates dates of k consecutive accidents
- Index I allows to calculate continuously time elapsed between eventsl i and (i+k-1)
- Nb_Cluster passe à 1 when k accidents appear within a window of lentgh w





Scanning observation period

Notation



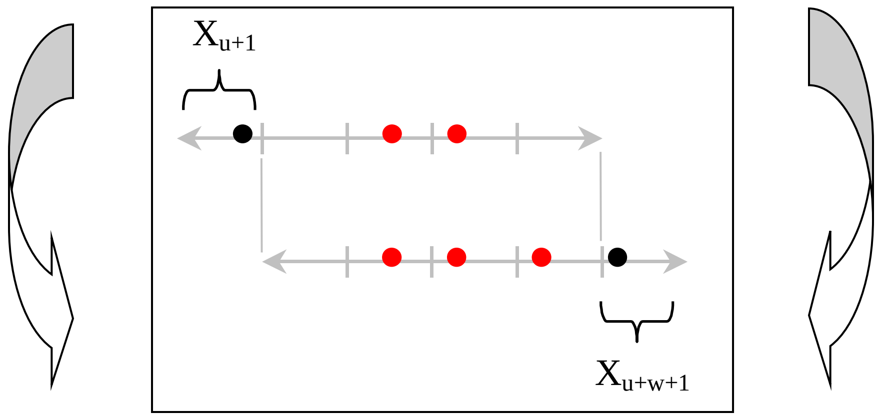
- X_i… random variable denoting number of events in interval [i-1,i[
- N(u,w)… random variable counting number of events in window [u,u+w[
- p probability that an event will appear in a subinterval of the length equal t o1

Bernoulli model

$$X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q = 1 - p \end{cases}$$

FIRST MARKOV MODEL

“Lost” of random variable X_{u+1}



“Arrival” of random variable X_{u+w+1}

From window N(u,w) à to window N(u+1,w)

independent

$$N(u+1,w) = N(u,w) - X_{u+1} + X_{u+w+1}$$

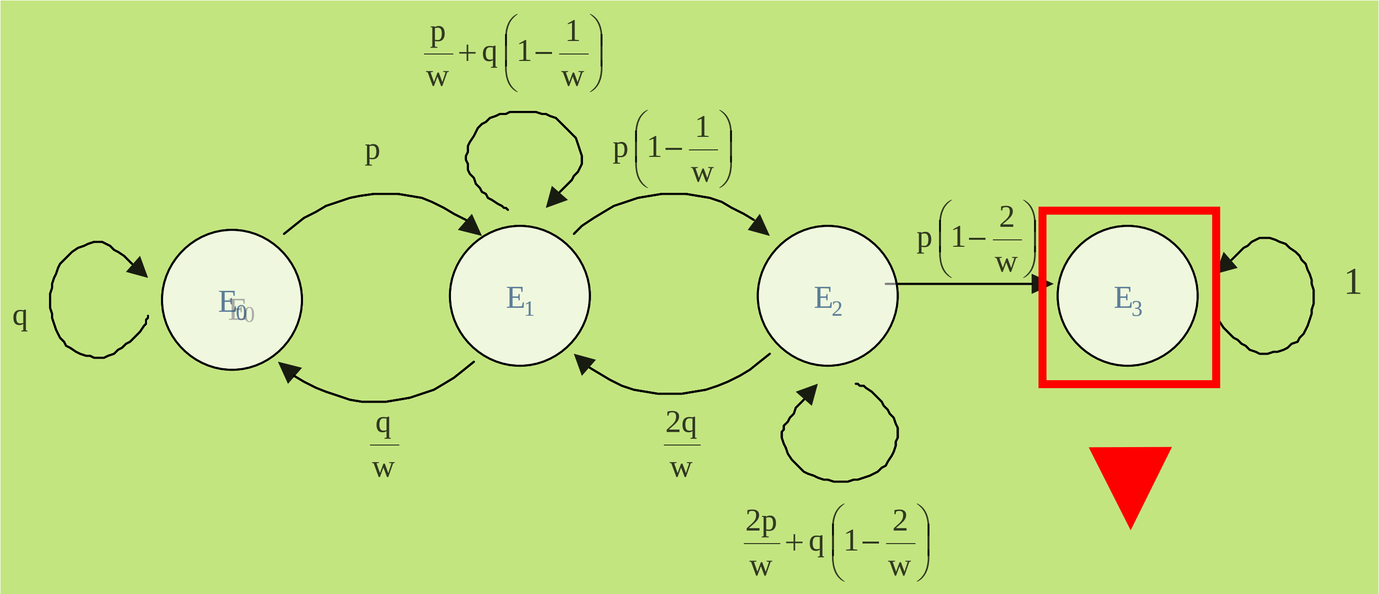
dependent

$$P(X_{u+1} = 1 \mid N(u,w) = n) = \frac{n}{w} \qquad P(X_{u+1} = 0 \mid N(u,w) = n) = 1 - \frac{n}{w}$$

States:

E_0, E_1, E_2 : 0, 1 or 2 events in current window
E_3 : 3 events or more in current window

Markov chain



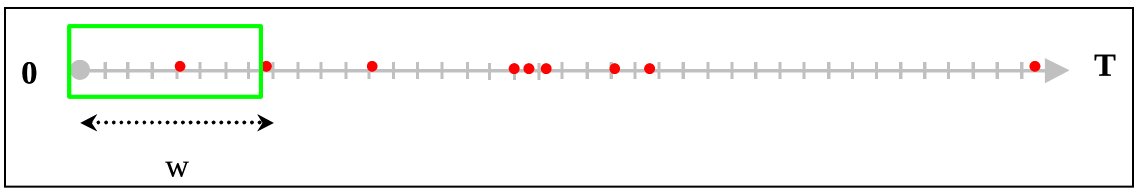
Probability of one **cluster** of 3 events or more **in a window of size** w=10

FIRST MARKOV MODEL

$$M = \begin{bmatrix} q & p & 0 & 0 \\ \frac{q}{w} & \frac{p}{w} + q\frac{w-1}{w} & p\frac{w-1}{w} & 0 \\ 0 & \frac{2q}{w} & \frac{2p}{w} + q\frac{w-2}{w} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

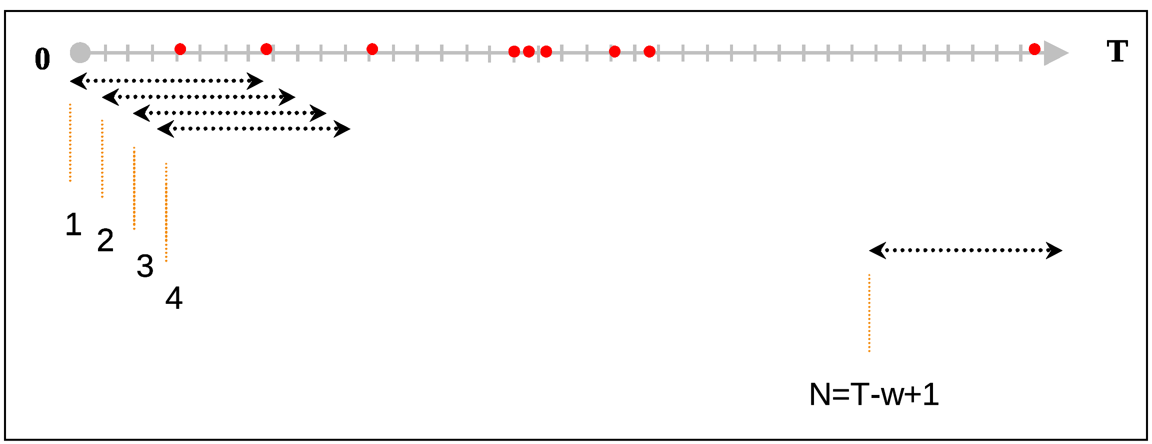
Transition matrix

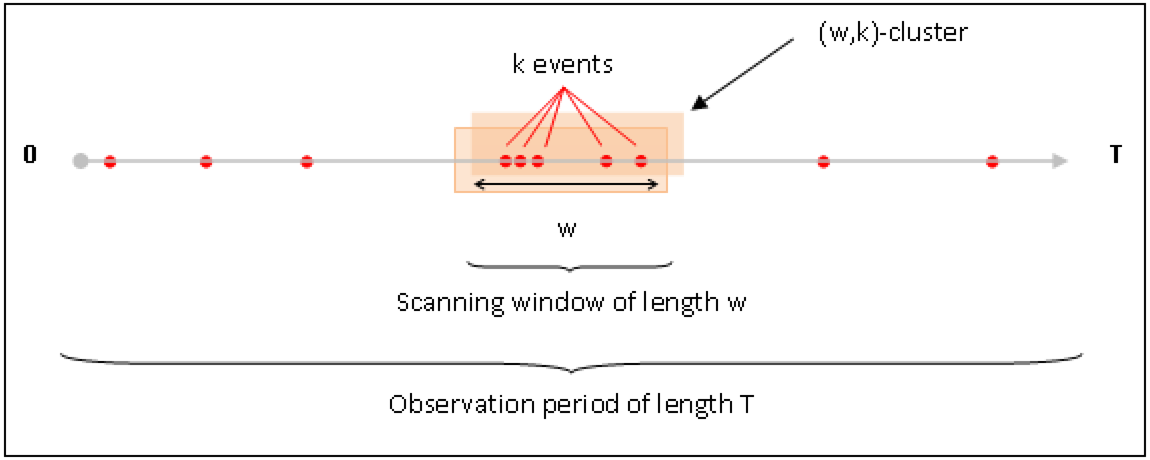
Vector of initial probabilities



$$X = \begin{bmatrix} q^{w} \\ p q^{w-1} \\ p^{2} q^{w-2} \\ 1 - q^{w} - p q^{w-1} - p^{2} q^{w-2} \end{bmatrix}$$

Number of iterations

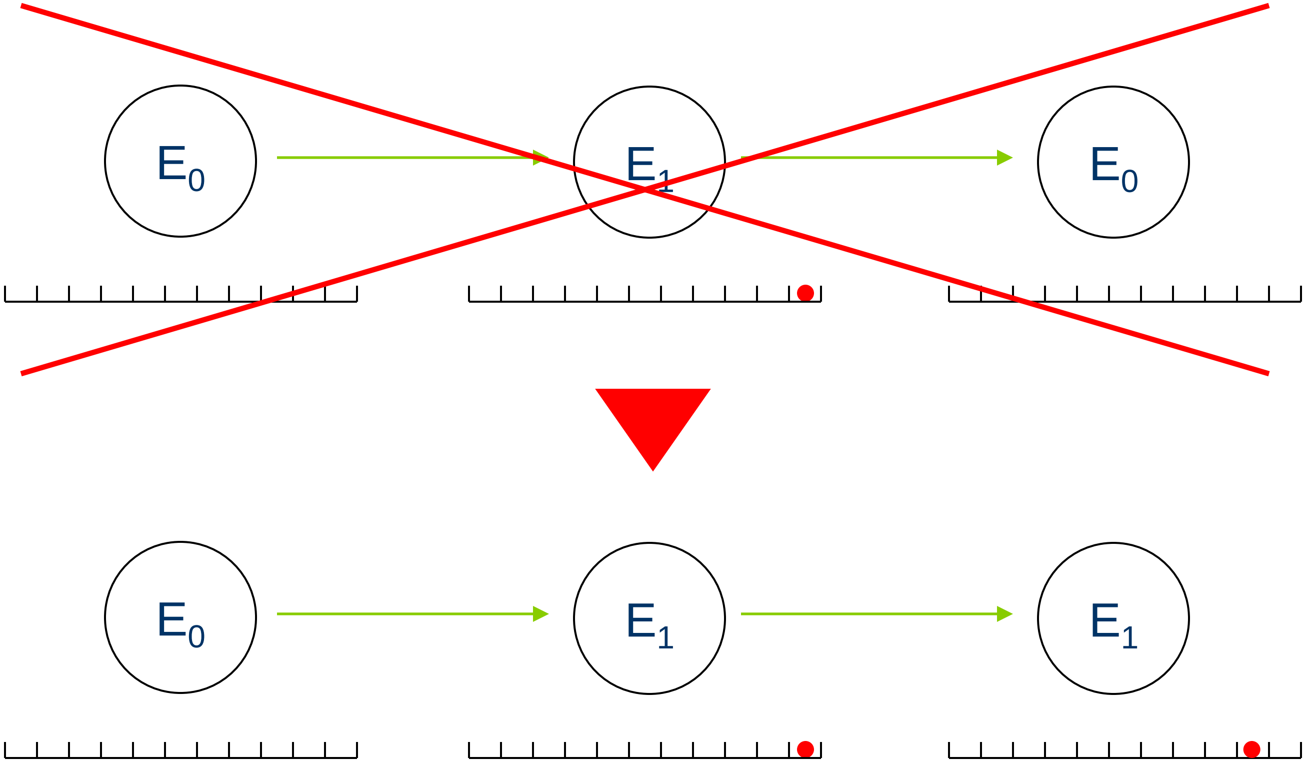


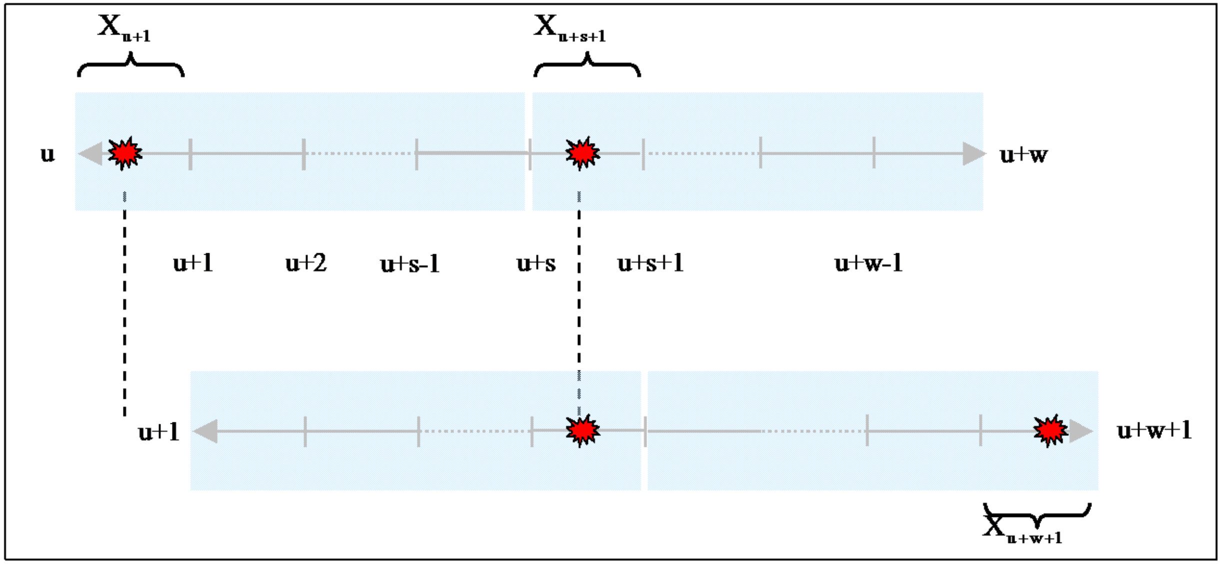


Probability to find cluster consisting of k=3 events or more in window of size w=10 scanning the period of length T=365 is given by product

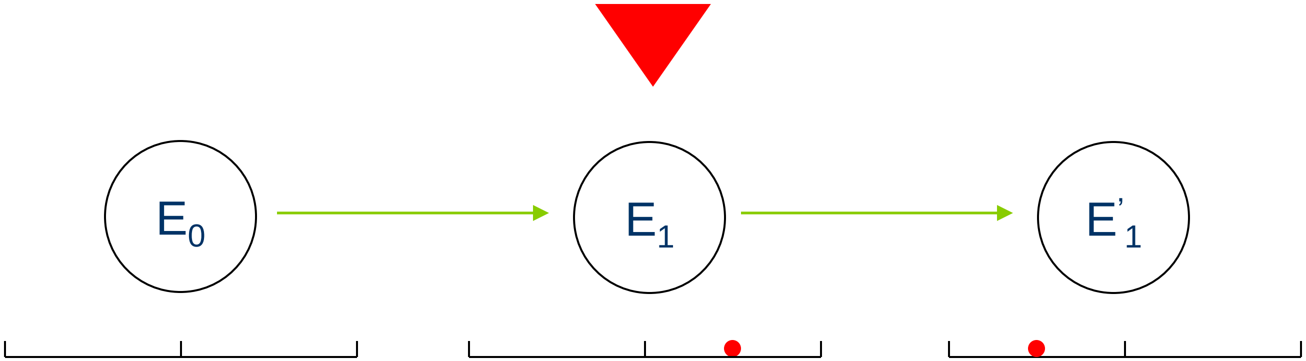
$M^N X$ N=356

Problem : Model allows the "ways" which cannot be realized in practice





Division of the scanning window into two sub-windows



State is:

- ***either pair (i,j) if i+j<k***
- ***absorbing state if i+j=k***

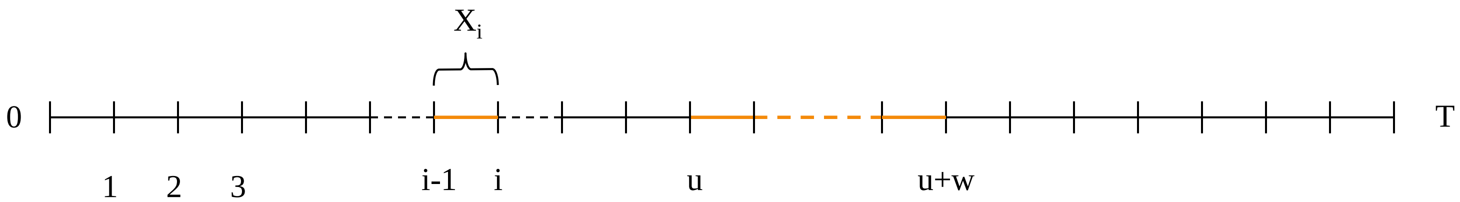
<u>Transition matrix</u> **is matrix of size D×D avec D=k(k-1)+1**

Transition probabilities and vector of initial probabilities are calculated analogously as before

$$M = \begin{bmatrix} q & 0 & q\cdot\left(\frac{2}{w}\right) & 0 & 0 & 0 & 0 \\ p & q\cdot\left(1-\frac{2}{w}\right) & p\cdot\left(\frac{2}{w}\right) & 0 & q\left(\frac{2}{w}\right)\cdot\left(1-\frac{2}{w}\right) & 0 & 0 \\ 0 & q\cdot\left(\frac{2}{w}\right) & q\cdot\left(1-\frac{2}{w}\right) & 0 & q\cdot\left(\frac{2}{w}\right)\cdot\left(\frac{2}{w}\right) & q\cdot\left(\frac{4}{w}\right) & 0 \\ 0 & p\cdot\left(1-\frac{2}{w}\right) & 0 & q\cdot\left(1-\frac{4}{w}\right) & p\cdot\left(\frac{2}{w}\right)\cdot\left(1-\frac{2}{w}\right) & 0 & 0 \\ 0 & q\cdot\left(\frac{2}{w}\right) & p\cdot\left(1-\frac{2}{w}\right) & q\cdot\left(\frac{4}{w}\right) & p\cdot\left(\frac{2}{w}\right)\cdot\left(\frac{2}{w}\right)+q\cdot\left(1-\frac{2}{w}\right)\cdot\left(1-\frac{2}{w}\right) & p\cdot\left(\frac{4}{w}\right) & 0 \\ 0 & 0 & 0 & 0 & q\cdot\left(\frac{2}{w}\right)\cdot\left(1-\frac{2}{w}\right) & q\cdot\left(1-\frac{4}{w}\right) & 0 \\ 0 & 0 & 0 & p & p\cdot\left(1-\frac{2}{w}\right) & p\cdot\left(1-\frac{4}{w}\right) & 1 \end{bmatrix}$$

$$X = \begin{bmatrix} b\left(0,\frac{w}{2},p\right)^2 \\ b\left(0,\frac{w}{2},p\right)b\left(1,\frac{w}{2},p\right) \\ b\left(1,\frac{w}{2},p\right)b\left(0,\frac{w}{2},p\right) \\ b\left(0,\frac{w}{2},p\right)b\left(2,\frac{w}{2},p\right) \\ b\left(1,\frac{w}{2},p\right)b\left(1,\frac{w}{2},p\right) \\ b\left(2,\frac{w}{2},p\right)b\left(0,\frac{w}{2},p\right) \\ 1-B(2,w,p) \end{bmatrix}$$

"Complete" model



- **State is:**
  - ***either w-uplet $(X_1, X_2,\dots, X_w)$ if $X_1 + X_2 +\dots+ X_w < k$***
  - ***Or absorbing A if $X_1 + X_2 +\dots+ X_w = k$***

- **Space of states is** $$E = \left\{ (X_1, X_2, \dots, X_w) \middle| X_i \in \{0,1\} \text{ and } \sum_{i=1}^{w} X_i < k \right\} \bigcup A$$

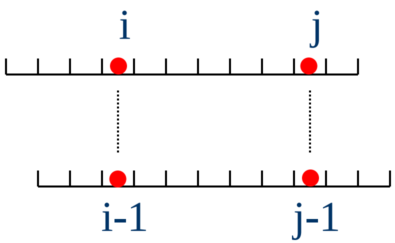
With dimension $1 + \binom{w}{1} + \binom{w}{2} + \dots + \binom{w}{k-1} + 1$

<u>Notation</u>: state $(i_1, i_2, \dots, i_m)$ if $i_1 = i_2 = \dots = i_m = 1$ and $i_l = 0$ otherwise

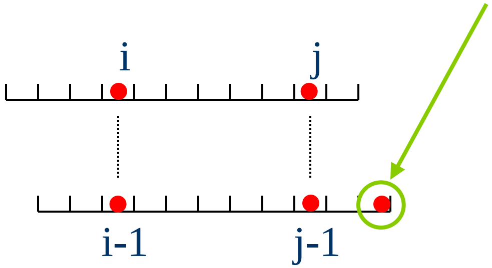
Transition matrix

Etats	0	1	2	3	4	5	6	7	8	9	10	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)	(1,7)	(1,8)	(1,9)	(1,10)	(2,3)	(2,4)	(2,5)	(2,6)	(2,7)	(2,8)	(2,9)	(2,10)	...	(i,j)	...	(8,9)	(8,10)	(9,10)	A
0	q	q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
1	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
2	0	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
3	0	0	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
4	0	0	0	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
5	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
6	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
7	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
8	0	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
10	p	p	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(1,2)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(1,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	0	0	0	0	0	...	0	...	0	0	0	0
(1,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	0	0	0	0	...	0	...	0	0	0	0
(1,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	0	0	0	...	0	...	0	0	0	0
(1,6)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	0	0	...	0	...	0	0	0	0
(1,7)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	0	...	0	...	0	0	0	0
(1,8)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	0	...	0	...	0	0	0	0
(1,9)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	q	...	0	...	0	0	0	0
(1,10)	0	0	p	0	0	0	0	0	0	0	0	p	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(2,3)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(2,4)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(2,5)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(2,6)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(2,7)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(2,8)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(2,9)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(2,10)	0	0	0	p	0	0	0	0	0	0	0	0	p	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	0	...	...	...	...	...
(i-1,j-1)	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	q	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	...	0	...	...	...	...	...
(8,9)	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	...	0	...	0	0	q	0
(8,10)	0	0	0	0	0	0	0	0	0	p	0	0	0	0	0	0	0	0	p	0	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
(9,10)	0	0	0	0	0	0	0	0	0	0	p	0	0	0	0	0	0	0	0	p	0	0	0	0	0	0	0	0	...	0	...	0	0	0	0
A	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	p	p	p	p	p	p	p	p	...	p	...	p	p	p	1

Transition of state (i,j) to state (i-1,j-1) with probability q:

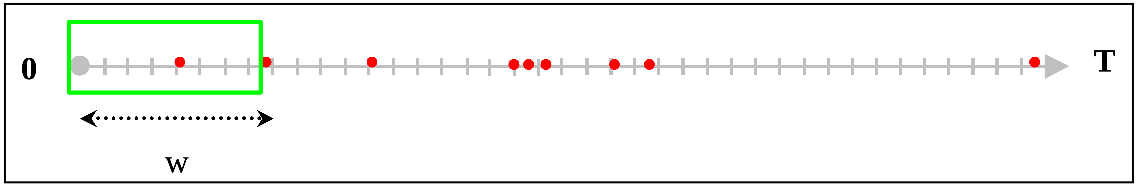


Transition of state (i,j) to absorbing state with probability p:



THALES

Vector of initial probabilities



$$X = \left[b(0,10,p) \quad \frac{1}{10} b(1,10,p) \ldots \frac{1}{10} b(1,10,p) \quad \frac{1}{45} b(2,10,p) \ldots \frac{1}{45} b(2,10,p) \quad 1 - B(2,10,p) \right]^t$$

with $b(i,10,p) = C_{10}^{i} p^{i} q^{10-i}$ and $B(i,10,p) = \sum^{i} C_{10}^{i} p^{i} q^{10-i}$

Probability to observe a cluster of k=3 events or more in a window of size w=10 scanning the period of length T=365 is given by a product

$M^N X$ with N=356

Results

Discretization	Day		Hour	
Method	Bernoulli	Poisson	Bernoulli	Poisson
Monte Carlo direct	0.1250	0.1329	0.1310	0.1329
RdP, Monte Carlo	0.1225	0.1317	0.1251	0.1317
Simple Markov model	0.0991	0.1176	0.1274	0.1280
Double scanning window	0.1014	NaN	0.1296	NaN
Complete Markov model	0.1028	0.1217	NaN	NaN

Conclusions

❖Results obtained using Bernoulli model converge to those using Poisson models if discretization step converges to zero

❖ As far as we know there does not exist exact method enabling to solve in « short » time the problem to estimate the probability of existence of a cluster of events.

❖ … Our method allow to find an approximation of this probability in acceptable time.Obtained results are almost identical provided the discretization is fine enough.

❖Proposed method are different and range from simulations and combinatorics to the use of Markov chains.

Assume n linearly (serially) arranged components

each component is associated with a failure indicator I_i

MODEL I

k-within-r-out-of-n system

system failed if exist window of size r (covering r objects) with at least k failed components

MODEL II

k-out-of-n $r=n$

MODEL III

consecutive k-out-of-n $r=k$

Denote

K_n^k Unreability of k-out-of-n system

$T_n^{l,h}$ Unreability of l-to-h-out-of-n system

C_n^k Unreability of consecutive k-out-of-n system

$P_i \,, Q_i$ Reliability and unreability of k-th component

It holds

$$K_i^j = 0,\ j > i \qquad K_i^j = 1,\ j \le 0 \qquad K_i^j = Q_i K_{i-1}^{j-1} + P_i K_{i-1}^j, otherwise$$

$$T_i^{l,h} = 0, (l > i) \vee (h < 0) \qquad T_i^{l,h} = 1, (l \le 0) \wedge (h \ge i)$$
$$T_i^{l,h} = Q_i T_{i-1}^{l-1,h-1} + P_i T_{i-1}^{l,h}, otherwise$$

$$C_i^j = 0,\ j > i \qquad C_i^j = 1,\ j \le 0 \qquad C_i^j = Q_i C_{i-1}^{j-1} + P_i C_{i-1}^k, otherwise$$

Implementation

Using binary decision diagrams of Bryant, i.e. Shannon like decomposition of Boolean formulas

What can we get

MODEL I

k-within-r-out-of-n system

system failed if exist window of size r (covering r objects) with at least k failed components

Provided all components have the same reliability, for k-within-r-out-of-n system the complexity is O(2^h.k.n), 0<=h<=r, so that for small r (tenths) we are able to calculate exact results thousands of components on "ordinary" PC computer