Inference for the difference of two percentile residual life functions

Alba M. Franco-Pereira

Department of Statistics
Universidad Carlos III de Madrid
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Joint work with Rosa E. Lillo and Juan Romo

Outline

1. Introduction
   - Motivation
   - Reliability measures
   - Stochastic orders
2. The problem
   - Description and previous results
   - Our methodology
3. A simulation study
   - The sampling models
   - The simulation mechanism
   - The simulation results
   - Conclusions of the simulation study
4. A real data example
5. References

Example

Kalbfleisch and Prentice (1982)

TYPE OF TREATMENT	PATIENTS	% CENSORED
T1 (Radiotherapy)	100	27%
T2 (Radiotherapy + Chemotherapeutic agent)	95	27.37%

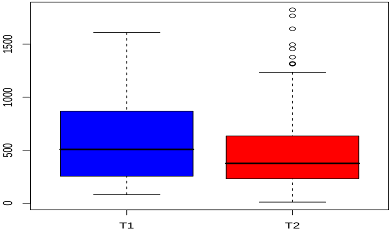

Figure: Box-and-whisker plots of the survival times of the two groups of patients

Example

Interests:

- To study the effectiveness of both treatments independently
- To compare both types of treatments

Tools:

- Reliability measures
- Stochastic orderings

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Definitions

Let X be the lifetime of an item or component and let $X_t = [X - t | X > t]$ represents its residual lifetime at time $t > 0$

Assume that X has an absolutely continuous distribution F_X and a density f_X

- The **survival function** of X is $\bar{F}_X(t) = P(X > t)$
- The **hazard rate function** of X is $r_X(t) = \frac{f_X(t)}{\bar{F}_X(t)}$
- The **mean residual life function** of X is $m_X(t) = E[X_t]$
- Fix $\gamma \in (0,1)$. The γ-**percentile residual life function** of X is the γ-percentile of X_t; i.e.,

$$q_{X,\gamma}(t) = \begin{cases} F_{X_t}^{-1}(\gamma), & t < u_X; \\ 0, & t \geq u_X, \end{cases}$$

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Motivation

RELIABILITY MEASURES	STOCHASTIC ORDERINGS
Hazard rate function	HR order
Survival function	ST order
Mean residual life function	MRL order
Percentile residual life function	γ-PRL, $\gamma \in (0, 1)$

Definitions

Let X and Y be two absolutely continuous random variables with survival functions $\bar{F}_X$ and $\bar{F}_Y$, hazard rate functions r_X and r_Y, and mean residual life functions m_X and m_Y, respectively

- X is said to be smaller than Y in the **usual stochastic order**, denoted by $X \leq_{st} Y$, if
$$\overline{F}_X(t) \leq \overline{F}_Y(t), \quad \text{for all} \quad t \in \mathbb{R}$$
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EXAMPLE

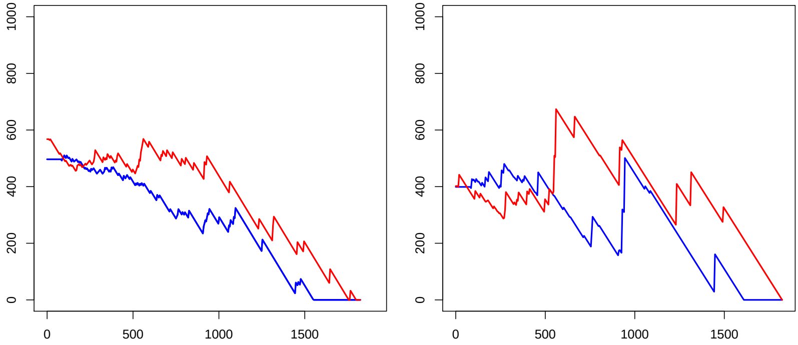

Figure: Comparison of the mrl's and merl's of the patients undergoing T1 and T2

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Description

Given $\gamma, \alpha \in (0,1)$, $X_1, X_2, \ldots X_n$ and $Y_1, Y_2, \ldots Y_m$

how to construct a $(1-\alpha)\cdot 100\%$-confidence band for $q_{Y,\gamma}(t) - q_{X,\gamma}(t)$?

The **empirical γ-percentile residual life function** of X is

$$q_{X,n,\gamma}(t) = Q_n(\gamma + (1-\gamma)\bar{F}_{X,n}(t)) - t, \quad t < u_X, \quad 0 < \gamma < 1$$

where $\bar{F}_{X,n}$ is the empirical survival of X and Q_n is its sample quantile function:

$$Q_n(x) = \begin{cases} X_k & \frac{(k-1)}{n} < x \leq \frac{k}{n} \quad (k = 1, \ldots, n) \\ X_1 & x = 0 \end{cases}$$

Previous results

Csörgo and Csörgo (1987)

- $q_{X,n,\gamma}(t) \quad \xrightarrow{a.s.} \quad q_{X,\gamma}(t)$
- $n^{\frac{1}{2}} f_X(q_{X,\gamma}(t)+t)\{q_{X,\gamma}(t)-q_{X,n,\gamma}(t)\} \quad \xrightarrow{d} \quad N(0,1)$

Our methodology is based on

- Bootstrap techniques
- Statistical depth

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Statistical depth

STATISTICAL DEPTH FOR MULTIVARIATE DATA

Measures the centrality of a d-dimensional observation with respect to a multivariate distribution F or with respect to a set of d-dimensional points

Mahalanobis (1936)
Oja (1983)
Singh (1991)
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Measures the centrality of a function with respect to a set of functions

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The modified band depth

López-Pintado and Romo (2009) ($J = 2$)

$$MBD_{B,2}(x) = \binom{B}{2}^{-1} \sum_{1 \leq i_1 < i_2 \leq B} \frac{\lambda(A(x; x_{i_1}, x_{i_2}))}{\lambda(I)}$$

where λ is the Lebesgue measure in I and

$$A(x; x_{i_1}, x_{i_2}) \equiv \{t \in I : \min_{r=i_1,i_2} x_r(t) \leq x(t) \leq \max_{r=i_1,i_2} x_r(t)\}$$

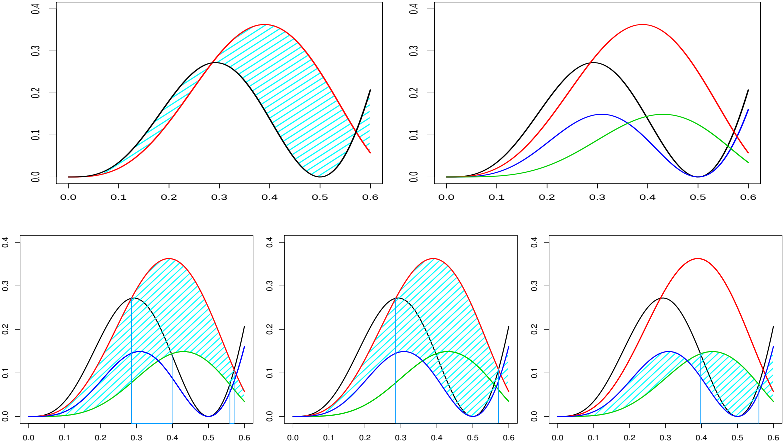

Figure: Illustration of how to compute the Modified Band Depth $J = 2$

The algorithm

Let B be the bootstrap size, $\alpha \in (0,1)$ the confidence level, $\gamma \in (0,1)$ the percentile.

$$X_1, X_2, \ldots X_n \quad \text{and} \quad Y_1, Y_2, \ldots Y_m$$

- For $b = 1, \ldots, B$;

  resample from $X_1, X_2, \ldots X_n$ and $Y_1, Y_2, \ldots Y_m$ to obtain $X_1^{*b}, X_2^{*b}, \ldots X_n^{*b}$ and $Y_1^{*b}, Y_2^{*b}, \ldots Y_m^{*b}$

- For $b = 1, \ldots, B$;

  compute $q_{X,n,\gamma}^{*b}$ and $q_{Y,m,\gamma}^{*b}$

- For every $t \in \mathbb{R}$;

  consider $q_b^*(t) = q_{Y,m,\gamma}^{*b}(t) - q_{X,n,\gamma}^{*b}(t)$, for $b = 1, \ldots, B$

- For $b = 1, \ldots, B$;

  order the sample curves q_b^*, from inner to outer using any notion of depth for curves and take the band given by the $(1-\alpha) \cdot 100\%$ deepest curves

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- For $b = 1, \dots, B$;
  compute $q_{X,n,\gamma}^{*b}$ and $q_{Y,m,\gamma}^{*b}$
- For every $t \in \mathbb{R}$;
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- For $b = 1, \dots, B$;
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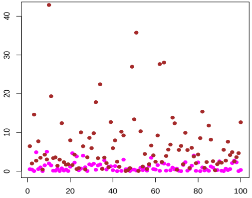
The algorithm

$X \sim$ Pareto(10,10) $\quad Y \sim$ Pareto(60,10)



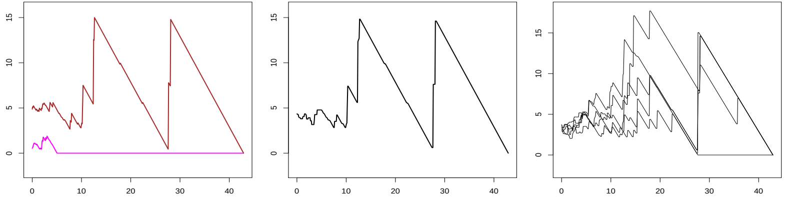


Figure: Illustration of the algorithm

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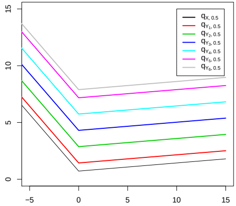
The sampling models

$$X \sim \text{Pareto}(10{,}10)$$

$Y_1 \sim \text{Pareto}(20{,}10)$ $\quad Y_2 \sim \text{Pareto}(40{,}10)$
$Y_3 \sim \text{Pareto}(60{,}10)$ $\quad Y_4 \sim \text{Pareto}(80{,}10)$
$Y_5 \sim \text{Pareto}(100{,}10)$ $\quad Y_6 \sim \text{Pareto}(110{,}10)$



$X_7 \sim \text{Pareto}(10{,}10)$ $\quad Y_7 \sim \text{Pareto}(1{,}5)$
$X_8 \sim \text{Pareto}(20{,}5)$ $\quad Y_8 \sim \text{Pareto}(70{,}10)$
$X_9 \sim \text{Pareto}(160{,}20)$ $\quad Y_9 \sim \text{Pareto}(70{,}10)$
$X_{10} \sim \text{Pareto}(10{,}10)$ $\quad Y_{10} \sim \text{Pareto}(20{,}15)$

The simulation mechanism

- Modified band depth introduced in López-Pintado and Romo (2009) with $J = 2$
- $B = 1000$, $\gamma = 0.5$, $1 - \alpha = 0.90$

The simulation results

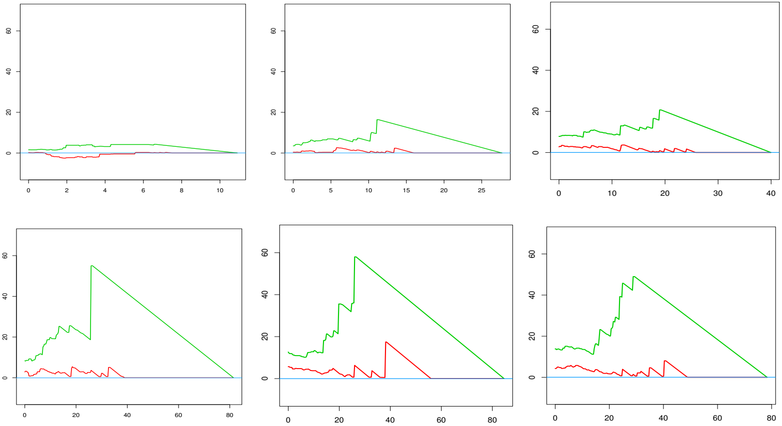


Figure: 90%-confidence band for $q_{Y_i,0.5} - q_{X,0.5}$, $i = 1, \cdots, 6$

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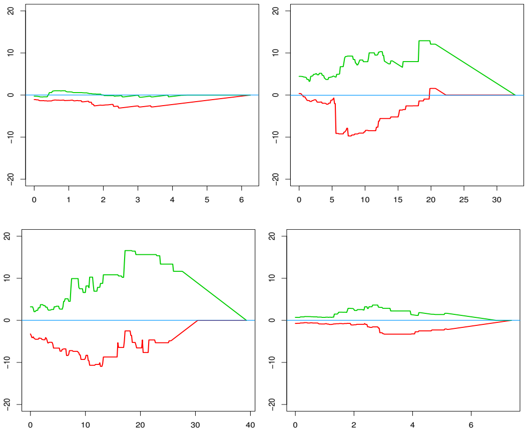

Figure: 90%-confidence band for $q_{Y_i,0.5} - q_{X,0.5}$, $i = 7, \cdots, 10$

Conclusions of the simulation study

The bands provide us with a criteria of whether two random variables are close or not with respect to a prl order or allow us to compare prl functions in an interval

LLB above the x-axis $\Rightarrow$ the random variables are ordered

ULB below the x-axis $\Rightarrow$ the random variables are ordered

LLB below the x-axis and ULB above the x-axis $\Rightarrow$ we can not say that one variable dominates the other

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Application in medicine

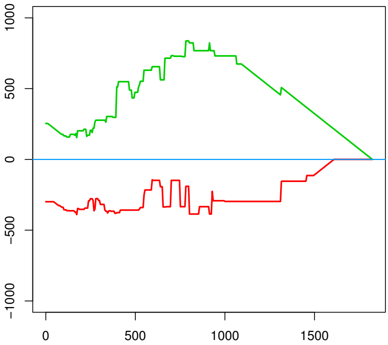

Figure: 90%-confidence bands for the difference of the merl of the patients undergoing T1 and T2

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Estimators

Let $X_1, \ldots, X_n$ be the lifetimes of the patients after a treatment. Only their right censored versions are observed, leading to the information $(\delta_1, Z_1), \ldots, (\delta_n, Z_n)$, where for $i = 1, \ldots, n$,

$$\delta_i = I_{\{X_i \leq Y_i\}} \quad \text{and} \quad Z_i = X_i \wedge Y_i = \max\{X_i, Y_i\},$$

with Y_i representing the i-th censoring random variable (I is the indicator function)

It is assumed that $Y_1, \ldots, Y_n$ are i.i.d. with $G(y) = P(Y > y) > 0$ and that G is continuous. The survival function of X can be estimated by

$$\bar{F}_{X,n}(x) = \frac{N^+(x) + 1}{n + 1} \prod_{j=1}^{n} \left(\frac{2 + N^+(Z_j)}{1 + N^+(Z_j)} \right)^{I_{\{\delta_j = 0, Z_j \leq x\}}},$$

where $N^+(x) \equiv$ number of censored and uncensored observations greater than x.
Slight variation of the Bayes estimator of Susarla and Van Ryzin (1976)