Genetics and/of basket options

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Basket derivatives

Let us consider a basket of N assets with value at time t defined by $B(t) = \sum_{i=1}^{N} a_i S_i(t)$. Then payoffs of some basket options:

- Basket call: $\{B(T) - K_B\}^+$
- Rainbow (best-of-N): $\left[\max_{1 \le i \le N} \{S_i(T)\} - K_B\right]^+$
- Atlas (Mountain range):

$$\left\{ \frac{1}{N - (N_1 + N_2)} \sum_{j=1+N_1}^{N-N_2} \frac{S_j(T)}{S_j(0)} - K_B \right\}^+$$

where $S_i(t)$ - price of the i-th basket constituent at time t, a_i - quantity of the i-th asset, K_B - exercise price (strike) of a basket option, T - time of the option's expiry, N_1,N_2 - number of best and worst performing stocks.

Research questions

1. Which pricing model is suitable for multiasset options?
2. How to estimate dependence (correlation) between assets in the basket?
3. How to estimate correlations in large dimensional baskets?

Outline

1. Motivation ✓
2. Basket dynamics in the Black-Scholes framework
3. Estimating correlation matrix
   - Historical (time series) correlation
   - Implied correlation
4. From equicorrelation to block correlation
5. Conclusion

Price dynamics of basket constituents

The price dynamic of the *i*-th stock in a basket is given by:

$$\frac{dS_i(t)}{S_i(t)} = (r - q_i)dt + \sigma_i dW_i(t) \quad (1)$$

$$\rho_{ij} dt = dW_i(t) dW_j(t) \quad (2)$$

where r - interest rate, q_i - dividend yield of a stock i, σ_i - constant volatility of the *i*-th stock, ρ_{ij} - constant correlation between the *i*-th and the *j*-th stock, W - Brownian motion.

Dynamics of the basket's value

The dynamics of the basket's value is then given by:

$$\frac{dB(t)}{B(t)} = (r - q_B)dt + \frac{\sum_{i=1}^{N} w_i S_i(t) \sigma_i dW_i(t)}{\sum_{i=1}^{N} w_i S_i(t)} = (r - q_B)dt + dZ(t) \quad (3)$$

where q_B is the dividend yield of the basket and the relative weight w_i of the i-th constituent varies over time and is given by:

$$w_i = \frac{a_i S_i(t)}{\sum_{l=1}^{N} a_l S_l(t)} \quad (4)$$

Dynamics of correlated basket constituents

Let

$$\Sigma = \begin{pmatrix} \rho_{11} & \cdots & \rho_{1N} \\ \vdots & \ddots & \vdots \\ \rho_{N1} & \cdots & \rho_{NN} \end{pmatrix}$$

the correlation matrix of a basket.
By Cholesky decomposition $\Sigma = MM^\top$ we obtain
$M = (m_{i,j})_{1\leq i\leq N, 1\leq j\leq N}$, a lower triangular matrix, a "square root" of Σ.
The process for every individual asset S_i is then defined by:

$$\frac{dS_i(t)}{S_i(t)} = (r-q)dt + \sigma_i \sum_{l=1}^{N} m_{i,l} dW_l(t) \tag{5}$$

Finally applying Itô's lemma we obtain the closed-form expression for simulation of the i-th stock process on a time interval $\Delta t = [t_1, t_2]$:

$$S_i(t_2) = S_i(t_1) \exp\left\{(r - d - \frac{\sigma_i^2}{2})\Delta t + \sigma_i \sum_{l=1}^{N} m_{i,l} \sqrt{\Delta t} g_l\right\} \quad (6)$$

where $g_l \sim \mathrm{N}(0, 1)$, i.i.d.

Historical correlation

$X_i(t) = \log S_i(t) - \log S_i(t-1)$, log returns:

$$\rho_{ij} = \frac{\sum_{k=0}^{T} \lambda^k \{X_i(t-k) - \bar{X}_i(t)\}\{X_j(t-k) - \bar{X}_j(t)\}}{\sqrt{\sum_{k=0}^{T} \lambda^k \{X_i(t-k) - \bar{X}_i(t)\}^2 \sum_{k=0}^{T} \lambda^k \{X_j(t-k) - \bar{X}_j(t)\}^2}}$$

to obtain the historical correlation matrix

$$\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{12} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1N} & \rho_{N2} & \cdots & 1 \end{bmatrix}$$

Here $\bar{X}_i(t)$ the arithmetic mean of the i-th log return calculated at time t, λ - decay parameter (RiskMetrics: $\lambda = 0.94$).

Equicorrelation matrix

Basket variance

$$\sigma^2_{Basket} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij} \tag{7}$$

replace $\begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1N} \\ \rho_{21} & 1 & \cdots & \rho_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{N1} & \rho_{N2} & \cdots & 1 \end{bmatrix}$ with $\begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{bmatrix}$,

then

$$\rho = \frac{\sigma^2_{Basket} - \sum_{i=1}^{N} w_i^2 \sigma_i^2}{2 \sum_{i=1}^{N} \sum_{j=i+1}^{N} w_i w_j \sigma_i \sigma_j} \tag{8}$$

is the average basket correlation.

Nice property: for $-\{1/(N-1)\} < \rho < 1$ - positive definite (Mardia et al, 1979).

Implied correlation

Using (8) map the implied volatility surfaces of a basket $\widehat{\sigma}_{Basket}(\kappa,\tau)$ and N constituents $\widehat{\sigma}_i(\kappa,\tau)$ to $\widehat{\rho}(\tau,\kappa)$ the **average implied correlation surface of a basket**:

$$\widehat{\rho}(\kappa,\tau) = \frac{\widehat{\sigma}^2_{Basket}(\kappa,\tau) - \sum_{i=1}^N w_i^2 \widehat{\sigma}_i^2(\kappa,\tau)}{2\sum_{i=1}^N \sum_{j=i+1}^N w_i w_j \widehat{\sigma}_i(\kappa,\tau)\widehat{\sigma}_j(\kappa,\tau)} \qquad (9)$$

Dynamic modeling of correlation surfaces

Every t we observe $(X_{t,j}, Y_{t,j})$, $1 \leq j \leq J_t$, $1 \leq t \leq T$ where

- $Y_{t,j}$ - implied correlation
- $X_{t,j}$ - two-dimensional vector of κ and τ
- T - number of observed time periods (days)
- J_t - number of observations at day t

Dynamic modeling of correlation surfaces

Including explanatory variables $X_{t,j}$ influencing the factor loadings $m_{l,j}$ rewrite (10)

$$Y_{t,j} = \sum_{l=1}^{L} Z_{t,l} m_l(X_{t,j}) + \varepsilon_{t,j} = Z_t^\top m(X_{t,j}) + \varepsilon_{t,j} \qquad (10)$$

where

- $Z_t = (Z_{t1}, \ldots, Z_{tL})^\top$ - unobservable L-dimensional process
- m - L-tuple $(m_1, ..., m_L)$ of unknown real-valued functions
- $X_{t,j}, \ldots, X_{T,J_T}$ and $\varepsilon_{t,j}, \ldots, \varepsilon_{T,J_T}$ are independent
- $\varepsilon_{t,j}$ are $i.i.d.$ with zero mean and finite second moment
- In such setting the modelling of Y_t can be simplified to modelling of $Z_t = (Z_{t,1}, ..., Z_{t,L})$, which is more feasible for $L << J$.

Dynamic modeling of correlation surfaces

$$Y_{t,j} = \sum_{l=1}^{L} Z_{t,l} \sum_{k=1}^{K} a_{l,k} \psi_k(X_{t,j}) + \varepsilon_{t,j} = Z_t^\top A \Psi_t + \varepsilon_t \tag{11}$$

where

- A - $L \times K$ coefficient matrix
- $\Psi_t = \{\psi_1(X_t), ..., \psi_R(X_t)\}^\top$ - space basis, in Park et al. (2009) a tensor product of one dimensional B-spline basis.

Choice of space basis

Estimate basis functions in a FPCA framework, motivated by Hall et. al (2006):
Find eigenfunctions corresponding to the K largest eigenvalues of the smoothed operator

$$\widehat{\psi}(u, v) = \widehat{\phi}(u, v) - \widehat{\mu}(u)\widehat{\mu}(v)$$

Choice of space basis

1. estimate $\widehat{\mu}(u)(\mu(v))$:

$$\sum_{t=1}^{T}\sum_{j=1}^{J}\{Y_{tj} - a - \sum_{c=1}^{2} b^c(u^c - X_{tj}^c)\}^2 K\left(\frac{X_{tj}-u}{h_\mu}\right)$$

2. estimate $\widehat{\phi}(u,v)$:

$$\sum_{t=1}^{T}\sum_{1\le j\ne k\le J_t}\{Y_{tj}Y_{tk} - a_0 - \sum_{c=1}^{2} b_1^c(u^c - X_{tj}^c) - \sum_{c=1}^{2} b_2^c(v^c - X_{tk}^c)\}^2$$
$$\times K\left(\frac{X_{tj}-u}{h_\phi}\right) K\left(\frac{X_{tj}-v}{h_\phi}\right)$$

3. compute $\widehat{\psi}(u,v) = \widehat{\phi}(u,v) - \widehat{\mu}(u)\widehat{\mu}(v)$ and take K eigenfunctions corresponding to the largest eigenvalues

Basis functions

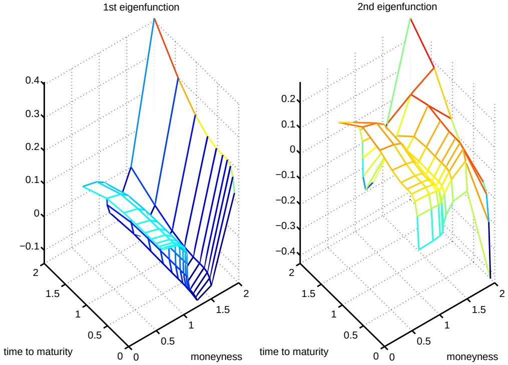
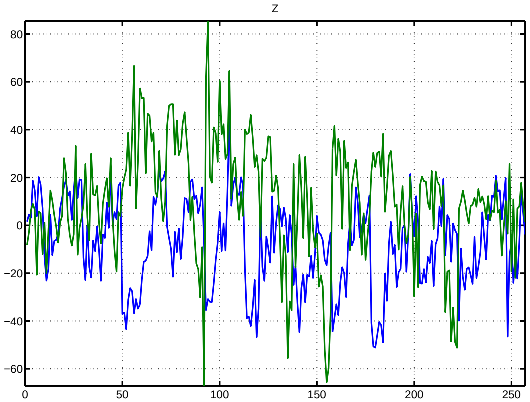


Figure 1: Eigenfunctions as basis functions estimated on 10x10 grid

Estimated time series of factors $\widehat{Z}_{t1}$, $\widehat{Z}_{t2}$



From equicorrelation to block correlation

Group assets in the basket into k blocks, then

$$\begin{bmatrix} \begin{bmatrix} 1 & \rho_1 & \cdots & \rho_1 \\ \rho_1 & 1 & \cdots & \rho_1 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1 & \rho_1 & \cdots & 1 \end{bmatrix} & \cdots & \rho_{k+1} \\ \vdots & \ddots & \vdots \\ \rho_{k+1} & \cdots & \begin{bmatrix} 1 & \rho_k & \cdots & \rho_k \\ \rho_k & 1 & \cdots & \rho_k \\ \vdots & \vdots & \ddots & \vdots \\ \rho_k & \rho_k & \cdots & 1 \end{bmatrix} \end{bmatrix}$$

Correlation matrix for 2 groups of assets (3 blocks)

$$
\begin{bmatrix}
\begin{bmatrix}
1 & \rho_1 & \cdots & \rho_1 \\
\rho_1 & 1 & \cdots & \rho_1 \\
\vdots & \vdots & \ddots & \vdots \\
\rho_1 & \rho_1 & \cdots & 1
\end{bmatrix} & \rho_3 \\
\rho_3 &
\begin{bmatrix}
1 & \rho_2 & \cdots & \rho_2 \\
\rho_2 & 1 & \cdots & \rho_2 \\
\vdots & \vdots & \ddots & \vdots \\
\rho_2 & \rho_2 & \cdots & 1
\end{bmatrix}
\end{bmatrix}
$$

Block implied correlation, 3 blocks

$$\sigma^2_{Basket}(K,\tau) = \sum_{i=1}^{N} w_i^2 \sigma_i^2(K,\tau) +$$

$$+2\sum_{i=1}^{M}\sum_{j=i+1}^{M} w_i w_j \sigma_i(K,\tau)\sigma_j(K,\tau)\rho_1(K,\tau) +$$

$$+2\sum_{i=1}^{N-M}\sum_{j=i+1}^{N-M} w_i w_j \sigma_i(K,\tau)\sigma_j(K,\tau)\rho_2(K,\tau) +$$

$$+2\sum_{i=1}^{M}\sum_{j=M+1}^{N} w_i w_j \sigma_i(K,\tau)\sigma_j(K,\tau)\rho_3(K,\tau)$$

where M - number of assets in the 1-st block.

Challenges

- Moving to high-dimensional portfolios ($N \nearrow$) with block structure of covariance matrix:
  - need well-conditioned estimate of covariance matrix (Ledoit and Wolf (2003), Bickel and Levina (2008))
  - need to define the grouping procedure and way of finding the optimal block size (Hautsch, Kyj and Oomen (2009))
- Improving correlation surface modeling:
  - need to expand the time effect in a series model Z_t as a sum of basis functions (Song Härdle and Ritov (2010))

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