Analysis of Competing Risks in the Pareto Model for Progressive Censoring with binomial removals

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Outline

We study the competing risks model when the data is progressively type II censored with random removals which follows a binomial distribution. Under the assumptions of independent causes of failure and using Pareto distribution as the distribution of lifetime of each unit

- Competing Risks
- Censoring is inevitable in the lifetime study
- Progressive type II censoring with random removal
- The data from a progressively Type II censored sample is as follows:

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- It is assumed here that the causes of failures follow pareto distributions. The pareto distribution has been used commonly to model naturally occurring phenomenon in which the distributions of random variables of interest have long tails;

Notations

- We put n independent and identical units on the life test. The test is terminated when $m \leq n$, m is pre-specified, units failed.
- The lifetime of i-th unit is denoted by $X_i, i = 1, 2, \ldots, n$, and X_{ij} denotes the time of failure of the i-th unit by the cause j where $j = 1, 2$, so $X_i = min\{X_{i1}, X_{i2}\}$.

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- $F(.)$: cumulative distribution function of X_i,
- $F_j(.)$: cumulative distribution function of X_{ij},
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- The distribution of the random variable X_{ij} is Pareto with parameters (α_j, β), $j = 1, 2$ and $i = 1, 2, \ldots, n$. The pdf of X_{ij}, $j = 1, 2$, for each $i = 1, 2, \ldots, n$, is

$$f_j(x) = \frac{\alpha_j \beta^{\alpha_j}}{x^{\alpha_j + 1}} \qquad x \geq \beta, \beta > 0, \alpha_j > 0$$

- The survival function, sf, and the hazard rate function, hrf, are respectively

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- When the first failure occurs, (1) we observe two values $X_{1:m:n}$ and $\delta_1 \in \{1,2\}$ where $X_{1:m:n}$ denotes the first order statistics out of the m failed items, which in turn denotes the statistics from the whole sample; (2) R_1 of surviving unites are randomly selected and removed, where R_1 follows binomial distribution with parameters $n-m$ and p. When the i-th failure occurs, $i = 2, \ldots, m-1$: (1) we observe two values $X_{i:m:n}$ and $\delta_i \in \{1,2\}$; (2) R_i of surviving unites are randomly selected and removed, where R_i follows binomial distribution with parameters $n - m - \sum_{l=1}^{i-1} R_l$ and p.
- Finally this experiment terminates when the m-th failure occurs, and (1) we observe two values $X_{m:m:n}$ and $\delta_m \in \{1,2\}$; (2) the rest $R_m = n - m - \sum_{i=1}^{m-1} R_i$ surviving units are all removed from the test. Here, $\delta_i = j$, $j = 1, 2$

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$$L(\theta; y, \delta, R) = L_1(\theta; y, \delta|R = r)P(R, p)$$

where

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$$c\prod_{i=1}^{m}[f_1(y_i)\bar{F}_2(y_i)]^{I(\delta_i=1)}[f_2(y_i)\bar{F}_1(y_i)]^{I(\delta_i=2)}[\bar{F}(y_i)]^{r_i}$$

where $c = n(n - r_1 - 1)\cdots(n - r_1 - r_2 - \cdots - r_{m-1} - m + 1)$ and $\bar{F}(y_i) = \bar{F}_1(y_i)\bar{F}_2(y_i)$.

$$L_1(\theta; y, \delta | R = r) = c\, \alpha_1^{n_1} \alpha_2^{n_2} \left(\prod_{i=1}^{m} \frac{1}{y_i} \right) \frac{\beta^{(\alpha_1+\alpha_2)\left(\sum_{i=1}^{m} r_i+1\right)}}{\left(\prod_{i=1}^{m} y_i^{r_i+1}\right)^{\alpha_1+\alpha_2}}$$

$$= c\, \alpha_1^{n_1} \alpha_2^{n_2} \left(\prod_{i=1}^{m} \frac{1}{y_i} \right) e^{-(\alpha_1+\alpha_2)\sum_{i=1}^{m}(r_i+1)\ln \frac{y_i}{\hat{\beta}}} \qquad \beta \le y_1 \le \cdots \le y_m$$

where $n_j = \sum_{i=1}^{m} I(\delta_i = j) \qquad j = 1, 2.$

$$L(\theta; y, \delta, R) = c^* \; \alpha_1^{n_1} \alpha_2^{n_2} \left(\prod_{i=1}^{m} \frac{1}{y_i} \right) e^{-(\alpha_1+\alpha_2) \sum_{i=1}^{m} (r_i+1) \ln \frac{y_i}{\hat{\beta}}}$$

$$\times p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i} \qquad \beta \le y_1 \le \cdots \le y_m$$

where

$$c^* = \frac{(n-m)!c}{\prod_{i=1}^{m-1} r_i! (n - m - \sum_{i=1}^{m-1} r_i)!}$$

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- $$\hat{\beta} = Y_1$$

- $$\hat{\alpha}_1 = \frac{n_1}{\sum_{i=1}^{m}(R_i+1)ln\frac{y_i}{\hat{\beta}}} \quad \text{and} \quad \hat{\alpha}_2 = \frac{n_2}{\sum_{i=1}^{m}(R_i+1)ln\frac{y_i}{\hat{\beta}}} = \frac{m}{Z} - \hat{\alpha}_1$$

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To construct the UMVUE's, we consider the following transformation (see Balakrishnan and Aggarwala (2000)):

$$\begin{cases} Z_1 = nX_1, \\ Z_2 = (n - R_1 - 1)(X_2 - X_1), \\ . \\ . \\ . \\ Z_m = (n - R_1 - \cdots - R_{m-1} - m + 1)(X_m - X_{m-1}). \end{cases}$$

The Z_i's are called the spacings. It can be easily seen (Balakrishnan and Aggarwala (2000)) that Z_i's are $i.i.d.$ $exp(\alpha_1 + \alpha_2)$ random variables. Therefore, $\sum_{i=1}^{m}(R_i + 1)ln\frac{y_i}{\beta} = \sum_{i=1}^{m}(R_i + 1)X_i = \sum_{i=1}^{m} Z_i$ is distributed as a gamma$(m, \alpha_1 + \alpha_2)$ random variable. Since n_1 is a Bin$(m, \frac{\alpha_1}{\alpha_1+\alpha_2})$, and n_1 is independent of $\sum_{i=1}^{m}(R_i + 1)X_i$, we have for $m > 1$

$$E(\hat{\alpha}_1) = \frac{m\,\alpha_1}{\alpha_1 + \alpha_2} \times \frac{\alpha_1 + \alpha_2}{m-1} = \frac{m}{m-1} \times \alpha_1$$

and

$$E(\hat{\alpha}_2) = \frac{m\,\alpha_2}{\alpha_1 + \alpha_2} \times \frac{\alpha_1 + \alpha_2}{m-1} = \frac{m}{m-1} \times \alpha_2$$

Hence UMVUE's of α_1 and α_2 are given by

$$\tilde{\alpha}_1 = \frac{m-1}{m}\,\hat{\alpha}_1 \qquad \text{and}\tilde{\alpha}_2 = \frac{m-1}{m}\,\hat{\alpha}_2$$

The MLE of parameter p is:

$$\hat{p} = \frac{\sum_{i=1}^{m-1} R_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i)R_i + \sum_{i=1}^{m-1} R_i}.$$

- The percentile bootstrap (Boot-p) proposed by Efron[7],
- The bootstrap-t method (Boot-t) proposed by Hall[8].
  It is observed that in this type of situations (Kundu et al., [10]), the non-parametric bootstrap method does not work well. For a fixed set of $R = (r_1, \ldots, r_m)$, we propose the following two parametric bootstrap confidence intervals for α_1 and α_2.

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- Estimate α_1 and α_2, say $\hat{\alpha}_1$ and $\hat{\alpha}_2$, from the sample $\{(x_{1:m:n}, \delta_1, R_1), (x_{2:m:n}, \delta_2, R_2), \ldots, (x_{m:m:n}, \delta_m, R_m)\}$.
- Generate a bootstrap sample $\{X^*_{1:m:n}, \ldots, X^*_{m:m:n}\}$, using $\hat{\alpha}_1$, $\hat{\alpha}_2$, $R_1, \ldots, R_m$.
- Obtain the bootstrap estimate of α_1 and α_2, say $\hat{\alpha}^*_1$ and $\hat{\alpha}^*_2$ using the bootstrap sample.
- Repeat Step [2] NBOOT times.
- Let $\widehat{CDF}_j(x) = P(\hat{\alpha}^*_j \leq x)$ be the cumulative distribution function of $\hat{\alpha}^*_j$, $j = 1, 2$. Define $\hat{\alpha}_{j,Boot-p}(x) = \widehat{CDF}_j^{\,-1}(x)$ for a given x.
- The approximate $100(1-\gamma)\%$ confidence interval for α_j is given by

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- Repeat Steps [1] to [4] NBOOT times.
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- Let $\widehat{CDF}_j(x) = P(T_j^* \leq x)$ be the cumulative distribution function of T_j^*, $j = 1, 2$. For a given x, define
  $\hat{\alpha}_{j,Boot-t}(x) = \hat{\alpha}_j + m^{-\frac{1}{2}} \sqrt{\hat{V}(\hat{\alpha}_j^*)}\,\widehat{CDF}_j^{-1}(x)$.
- The approximate $100(1-\gamma)\%$ confidence interval for α_j is given by

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We assume α_1 and α_2 are independently distributed with gamma(a_1, b_1) and gamma(a_2, b_2) priors, respectively, and the parameter β is assumed to be known. Then the posterior density of α_1 and α_2 based on the gamma priors is given by

$$l(\alpha_1, \alpha_2|\mathbf{y}, \mathbf{R}=\mathbf{r}) = k\,\alpha_1^{n_1+a_1-1}\, e^{-\left(b_1+\sum_{i=1}^{m}(r_i+1)\ln\frac{y_i}{\beta}\right)\alpha_1} \alpha_2^{n_2+a_2-1}\, e^{-\left(b_2+\sum_{i=1}^{m}(r_i+1)\ln\frac{y_i}{\beta}\right)\alpha_2}$$

Here, k is the normalizing constant that ensures $l(\alpha_1, \alpha_2|\mathbf{y}, \mathbf{R}=\mathbf{r})$ is a proper density function. Hence the Bayes estimates of α_1 and α_2 under square loss are

$$\hat{\alpha}_{1Bayes} = \frac{n_1 + a_1}{b_1 + \sum_{i=1}^{m}(r_i+1)\ln\frac{y_i}{\beta}} \quad \text{and}$$

$$\hat{\alpha}_{2Bayes} = \frac{n_2 + a_2}{b_2 + \sum_{i=1}^{m}(r_i+1)\ln\frac{y_i}{\beta}}$$

Table: (1), Original sample from Lawless

t_i	δ_i	t_i	δ_i	t_i	δ_i	t_i	δ_i	t_i	δ_i	t_i	δ_i
11	2	708	2	1990	1	2551	1	2831	2	3504	1
35	2	958	2	2223	1	2565	*	3034	1	4329	1
49	2	1062	2	2327	2	2568	1	3059	2	6367	*
170	2	1167	1	2400	1	2694	1	3112	1	6976	1
329	2	1594	2	2451	2	2702	2	3214	1	7846	1
381	2	1925	1	2471	1	2761	2	3478	1	13403	*

Here we have $n = 36$, $n_1 = 17$, $n_2 = 16$, $n^* = 3$, $\sum_{i=1}^{n} t_i = 99245$.
Using the above data, without censoring, with $\beta = 11$ we computed the MLE of the parameters α_1, α_2, corresponding variances of these estimates and the 0.95% C.I. of α_1, α_2. Table 2 gives the results obtained.

Table: (2), Parameter's estimation using the original sample

parameter	MLE	Var	95% C.I.
α_1	0.1045	6.153×10^{-4}	[0.0967, 0.1126]
α_2	0.0984	5.517×10^{-4}	[0.0907, 0.1061]

Table: (3), Progressive Type II censoring samples

	$m = 15, p = 0.2$
R	[4, 5, 1, 2, 2, 2, 0, 0, 2, 1, 0,1, 0, 1, 0]
(X, δ)	(11,2),(170,2),(708,2),(1167,1),(2327,2),(2471,1),(2568,1),(2 (2761,2),(3034,1),(3214,1),(3504,1),(6367,$*$),(6976,1),(134
	$m = 20, p = 0.2$
R	[2, 3, 4, 2, 2, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0]
(X, δ)	(11,2),(170,2),(708,2),(1167,1),(1990,1),(2471,1),(2568, (2702,2),(2761,2),(2831,2),(3034,1),(3059,2),(3214,1),(34 (3504,1),(4329,1),(6367,$*$),(6976,1),(7846,1),(13403,$*$
	$m = 25, p = 0.2$
R	[3, 0, 1, 0, 2, 0, 2, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0,
(X, δ)	(11,2),(170,2),(329,2),(958,2),(1062,2),(1594,2),(1925,

Table: (4), Progressive Type II censoring samples

	$m = 15, p = 0.5$
R	[7, 7, 4, 0, 2, 1, 0, 0, 0, 0, 0, 0,0, 0, 0]
(X, δ)	(11,2),(1062,2),(1594,2),(2327,2),(2702,2),(3034,1),(3112,1) (3478,1),(3504,1),(4329,1),(6367,$*$),(6976,1),(7846,1),(13
	$m = 20, p = 0.5$
R	[9, 2, 3, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
(X, δ)	(11,2),(1062,2),(1990,1),(2471,1),(2565,$*$),(2568,1),(269 (2702,2),(2831,2),(3034,1),(3059,2), (3112,1),(3214,1),(3 (3504,1),(4329,1),(6367,$*$),(6976,1),(7846,1),(13403
	$m = 25, p = 0.5$
R	[3, 3, 1, 2, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, (
(X, δ)	(11,2),(329,2),(1062,2),(1594,2),(1990,1),(2327,2),(245

Table: (5), The MLE, Variances and C.I. for the data in table 3

$m = 15, p = 0.2$			
parameter	MLE	Var	95% C.I.
θ_1	0.0602	4.1547×10^{-4}	[0.0535, 0.0668]
θ_2	0.0376	2.6838×10^{-4}	[0.0323, 0.0430]
$m = 20, p = 0.2$			
parameter	MLE	Var	95% C.I.
α_1	0.0735	4.6152×10^{-4}	[0.0666, 0.0806]
α_2	0.0468	3.0065×10^{-4}	[0.0411, 0.0525]
$m = 25, p = 0.2$			
parameter	MLE	Var	95% C.I.
α_1	0.0926	5.5791×10^{-4}	[0.0849, 0.1004]
α_2	0.0556	3.3818×10^{-4}	[0.0496, 0.0616]

Table: (6), The MLE, Variances and C.I. for the data in table 3

$m = 15, p = 0.5$			
parameter	MLE	Var	95% C.I.
α_1	0.0693	4.8434×10^{-4}	[0.0621, 0.0765]
α_2	0.0308	2.2742×10^{-5}	[0.0259, 0.0357]
$m = 20, p = 0.5$			
parameter	MLE	Var	95% C.I.
α_1	0.0824	5.6124×10^{-4}	[0.0747, 0.0902]
α_2	0.0488	3.7332×10^{-4}	[0.0425, 0.0551]
$m = 25, p = 0.5$			
parameter	MLE	Var	95% C.I.
α_1	0.0836	5.1272×10^{-4}	[0.0726, 0.0909]
α_2	0.0643	3.9871×10^{-4}	[0.0577, 0.0708]