A Power Comparison for Testing Normality

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Background

Jarque–Bera (1987) pointed out that a Lagrange multiplier test is equivalent to JB test against the Pearson distributions.

Jarque–Bera

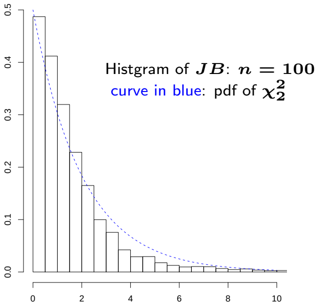
$$JB = n\left(\frac{\sqrt{b_1}^2}{6} + \frac{(b_2-3)^2}{24}\right),$$

where for a random sample $(X_1, X_2, \ldots, X_n)$, $\sqrt{b_1} = m_3/m_2^{3/2}$, $b_2 = m_4/m_2^2$, $m_j = (1/n)\sum_{i=1}^{n}(X_i - \bar{X})^j$, $j = 2, 3, 4$.

- $JB \sim \chi_2^2 \quad (n \to \infty)$ under H_0.

Motivation

- However, the χ^2 approximation does not work well.
- Unfortunately, a normalizing tr. has not been given yet.



Proposed test statistic

Nakagawa et al. (2007) proposed a modified version of the Jarque–Bera test.

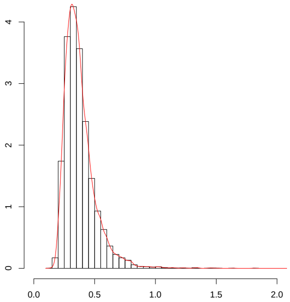
Modified Jarque–Bera test

$$JB' = \frac{\sqrt{b_1}^2}{6} + \frac{b_2^2}{24}$$

- A normalizing tr. of the null dist. for JB' has been derived.

Our goal

- When is the power of JB' test superior to that of JB?
- As H_1, the contaminated normal distributions are considered.



Power study 1

- Let $X_1, X_2, \ldots, X_n$ be i.i.d. rv with a cdf F.
- Φ: the cdf of the standard normal distribution

Omnibus testing for normality (two sided)

$$H_0\text{: } F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right) \quad (\forall x \in \mathbb{R})$$

$$H_1\text{: } F(x) \neq \Phi\left(\frac{x-\mu}{\sigma}\right) \quad (\exists x \in \mathbb{R})$$

- μ and σ may be known or unknown

- We consider JB', JB, and Shapiro–Wilk SW tests.
- As H_1, contaminated normal distributions are considered:

$$F = (1-p)N(\mu_0, \sigma_0^2) + pN(\mu, \sigma^2)$$

  - CN covers a broad range of distributions, symmetric and asymmetric ones.

Power study 2

$$H_0: X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(0,1)$$
$$H_1: X_1, X_2, \ldots, X_n \overset{\text{i.i.d.}}{\sim} (1-p)N(0,1) + pN(\mu, \sigma^2)$$

Ex.	n	μ	p	σ	
1	100	0	0.10	1, 2, 3, 4, 6	Sym.
2	100	0	0.50	1, 2, 3, 4, 6	Sym.
3	200	0	0.80	1, 2, 3, 4, 6	Sym.
4	50	3	0.50	1, 2, 3, 4, 6	Asym.

Ex.	n	σ	p	μ	
5	50	4	0.05	0, 1, 2, 3, 4	Asym.
6	50	1	0.50	0, 1, 2, 3, 4	Sym.

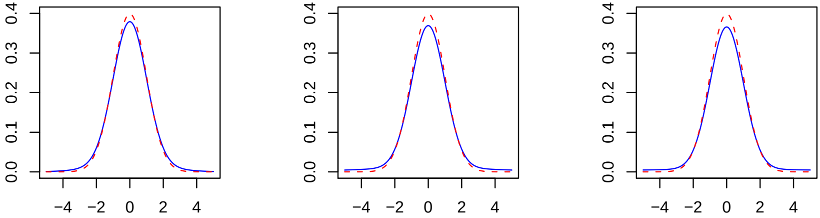
Ex.	n	μ	σ	p	
7	50	0	4	$0, 0.1, \ldots, 1$	Sym.

- significance level: $\alpha = 0.1$
- the number of replications: 10^4
- Omnibus test statistics: JB, JB', and SW

Ex.1: $n = 100, \mu = 0, p = 0.1, \sigma = 1, 2, 3, 4, 6$

$$\sqrt{\beta_1} = 0$$

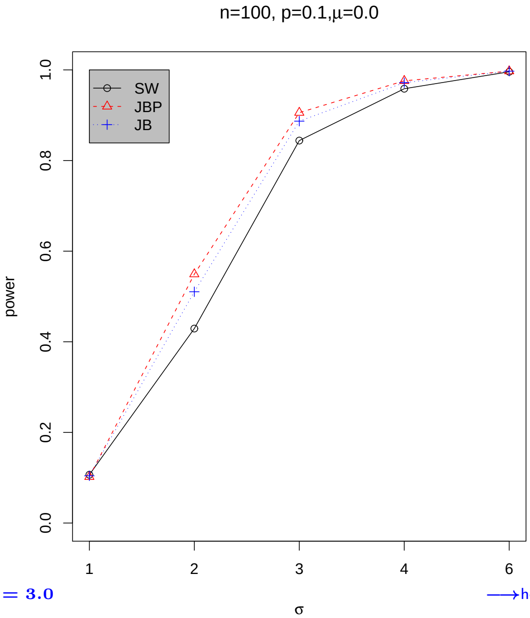
σ	1	2	3	4	6
β_2	3.00	4.44	8.33	12.72	19.33



$\sigma = 2$ $\sigma = 4$ $\sigma = 6$

- curves in red: pdf of $N(0, 1)$.
- curves in blue: pdf of $F = (1 - p)N(0, 1) + pN(\mu, \sigma^2)$.

Ex. 1: Powers of JB', JB, SW

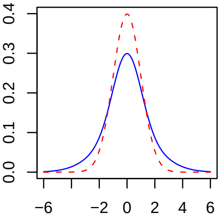


$\beta_2 = 3.0$ ⟶heavy tails

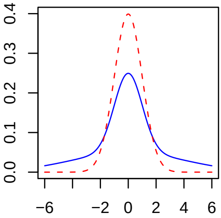
Ex. 2: $n = 100, \mu = 0, p = 0.5, \sigma = 1, 2, 3, 4, 6$

$$\sqrt{\beta_1} = 0$$

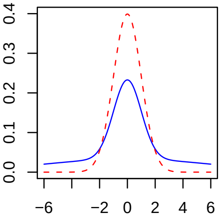
σ	1	2	3	4	6
β_2	3.00	4.08	4.92	5.34	5.68



$\sigma = 2$

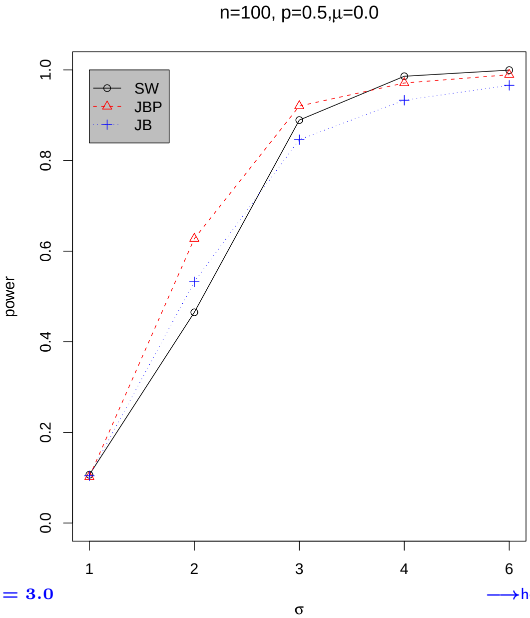


$\sigma = 4$



$\sigma = 6$

- curves in red: pdf of $N(0, 1)$.
- curves in blue: pdf of $F = (1 - p)N(0, 1) + pN(\mu, \sigma^2)$.

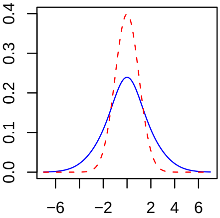


$\beta_2 = 3.0$ $\longrightarrow$ heavy tails

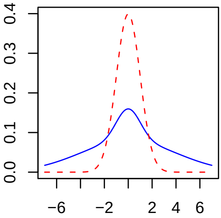
Ex. 3: $n = 200, \mu = 0, p = 0.8, \sigma = 1, 2, 3, 4, 6$

$$\sqrt{\beta_1} = 0$$

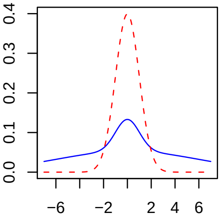
σ	1	2	3	4	6
β_2	3.00	3.37	3.56	3.64	3.70



$\sigma = 2$



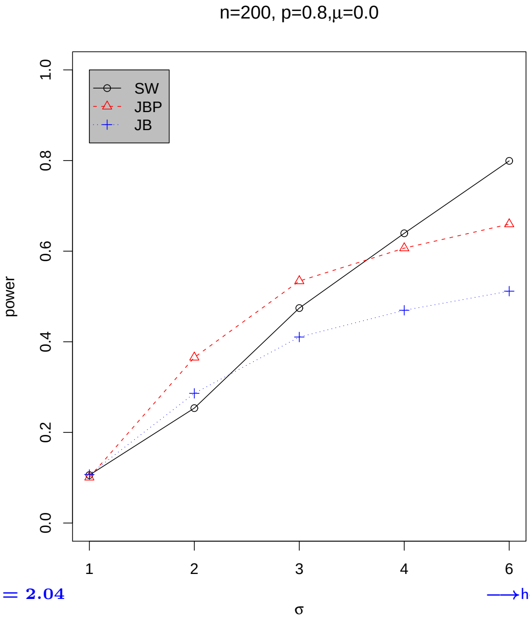
$\sigma = 4$



$\sigma = 6$

- curves in red: pdf of $N(0, 1)$.
- curves in blue: pdf of $F = (1 - p)N(0, 1) + pN(\mu, \sigma^2)$.

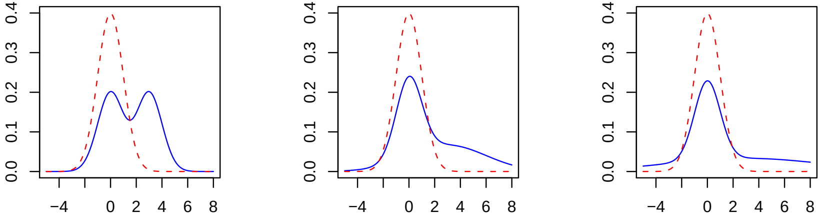
Ex. 3: Powers of JB', JB, SW



$\boldsymbol{\beta_2 = 2.04}$ ⟶heavy tails

Ex. 4: $n = 50, \mu = 3, p = 0.5, \sigma = 1, 2, 3, 4, 6$

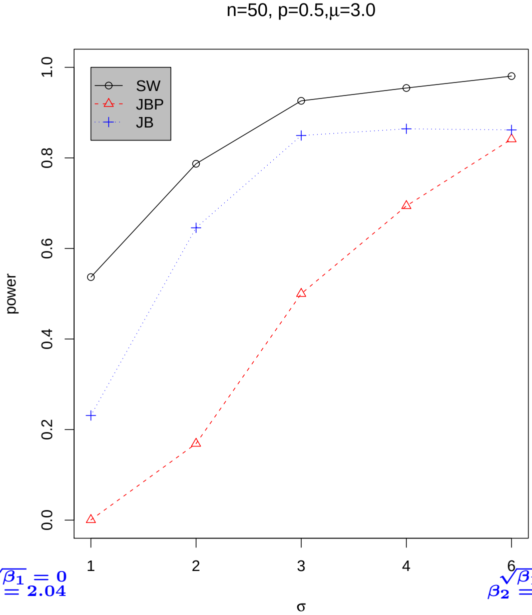
σ	1	2	3	4	6
$\sqrt{\beta_1}$	0.00	0.65	0.92	0.96	0.83
β_2	2.04	2.85	3.72	4.37	5.11



$\sigma = 2$ $\sigma = 4$ $\sigma = 6$

- curves in red: pdf of $N(0, 1)$.
- curves in blue: pdf of $F = (1 - p)N(0, 1) + pN(\mu, \sigma^2)$.

Ex. 4: Powers of JB', JB, SW

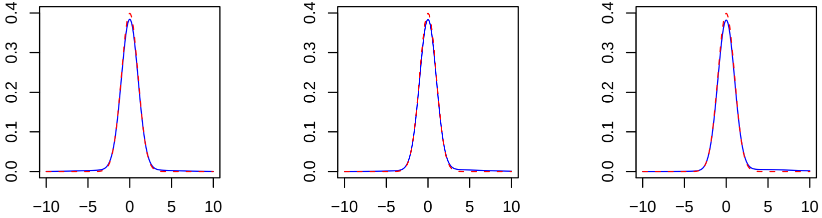


$\sqrt{\beta_1} = 0$
$\beta_2 = 2.04$

$\sqrt{\beta_1} = 0.83$
$\beta_2 = 5.11$

Ex. 5: $n = 50, \sigma = 4, p = 0.05, \mu = 0, 1, 2, 3, 4$

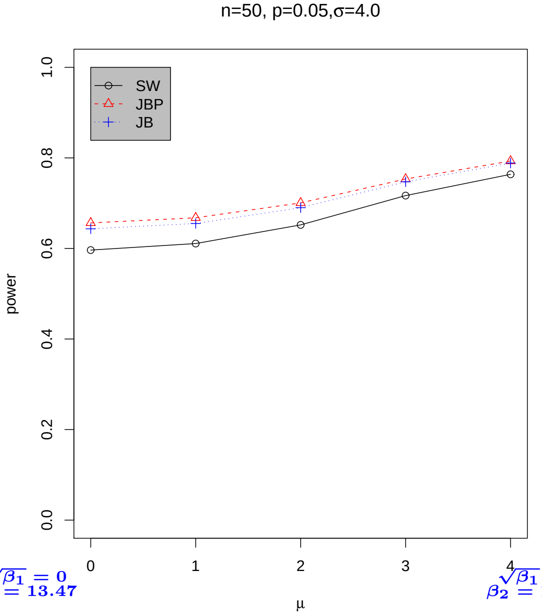
μ	0	1	2	3	4
$\sqrt{\beta_1}$	0.00	0.90	1.71	2.35	2.84
β_2	13.47	14.12	15.75	17.65	19.24



$\mu = 0$ $\mu = 2$ $\mu = 4$

- curves in red: pdf of $N(0, 1)$.
- curves in blue: pdf of $F = (1 - p)N(0, 1) + pN(\mu, \sigma^2)$.

Ex. 5: Powers of JB', JB, SW



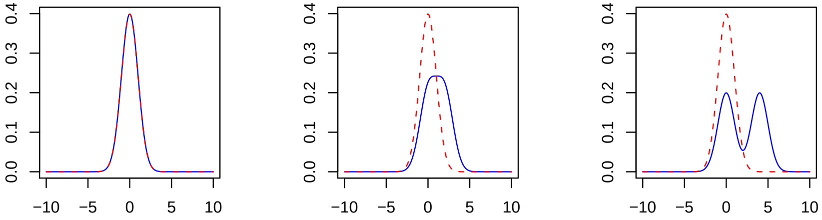
$\sqrt{\beta_1} = 0$
$\beta_2 = 13.47$

$\sqrt{\beta_1} = 2.84$
$\beta_2 = 19.24$

Ex. 6 : $n = 50, \sigma = 1.0, p = 0.5, \mu = 0, 1, 2, 3, 4$

$$\sqrt{\beta_1} = 0$$

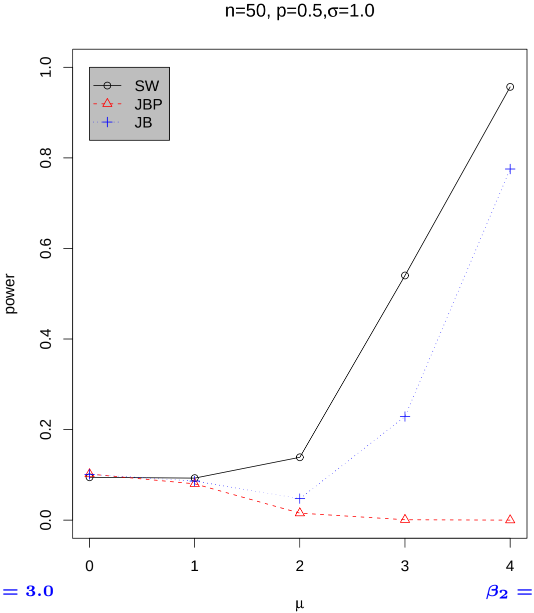
μ	0	1	2	3	4
β_2	3.00	2.92	2.50	2.04	1.72



$\mu = 0$ $\mu = 2$ $\mu = 4$

- curve in red: pdf of $N(0, 1)$.
- curves in blue: pdf of $F = (1 - p)N(0, 1) + pN(\mu, \sigma^2)$.

Ex. 6: Powers of JB', JB, SW

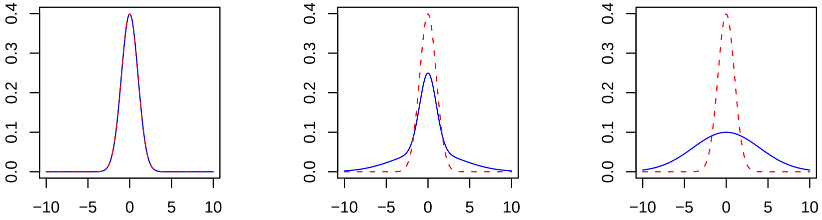


$\beta_2 = 3.0$ $\beta_2 = 1.72$

Ex. 7: $n = 50, \sigma = 4.0, \mu = 0, p = 0.0, 0.1, \ldots, 1.0$

$$\sqrt{\beta_1} = 0$$

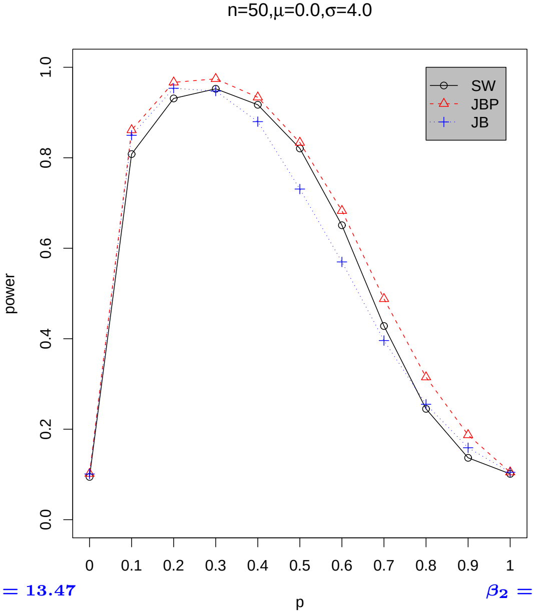
p	0.00	0.20	0.50	0.70	1.00
β_2	3.00	9.75	5.34	4.07	3.00



$p = 0.0$ $\quad$ $p = 0.5$ $\quad$ $p = 1.0$

- curves in red: pdf of $N(0, 1)$.
- curves in blue: pdf of $F = (1 - p)N(0, 1) + pN(\mu, \sigma^2)$.

Ex. 7: Powers of JB', JB, SW



$\beta_2 = 13.47$ $\beta_2 = 3.29$

Summary

Ranking:

Ex.	1	2	3	4	5	6	7
$\boldsymbol{JB}$	2	3	2	2	2	2	3
$\boldsymbol{JB'}$	1	1	1	3	1	3	1
$\boldsymbol{SW}$	3	2	1	1	3	1	2
[1]	S	S	S	A	A	S	S
[2]	HT	HT	HT	HT∼ST	HT	ST	HT

- $\boldsymbol{JB'}$ test is the best for symmetric dist. with heavy tails.
- $\boldsymbol{JB'}$ test is superior to $\boldsymbol{JB}$ test except for dist. with short tails.
- The power of $\boldsymbol{JB'}$ and $\boldsymbol{JB}$ is poor for dist. with short tails.

[1] S:Symmetric, A:Asymmetric
[2] HT:Heavy tails, ST:Short tails

Normalizing tr. of the null dist. for JB'

Under the null hypothesis H_0, let

$$A = 6 + \frac{8}{\sqrt{\beta_1}(JB')}\left[\frac{2}{\sqrt{\beta_1}(JB')} + \sqrt{1 + \frac{4}{\left(\sqrt{\beta_1}(JB')\right)^2}}\right],$$

and then, a transformed variate

$$\left(\left(1 - \frac{2}{9A}\right) - \left(\frac{1-(2/A)}{1+Z\sqrt{2/(A-4)}}\right)^{1/3}\right) / \sqrt{2/9A}$$

is asymptotically distributed as a standard normal distribution, where

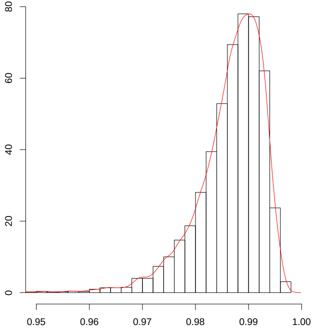
$$Z = \frac{JB' - \mu_1'(JB')}{\sqrt{\mu_2(JB')}} \quad \text{and}$$

$$\sqrt{\beta_1}(JB') = \frac{8\sqrt{6}}{n^{1/2}} + \frac{2750\sqrt{6}}{9}\frac{1}{n^{3/2}} - \frac{33968\sqrt{6}}{9}\frac{1}{n^{5/2}} + O\left(\frac{1}{n^{7/2}}\right),$$

Shapiro–Wilk test

$$SW = \frac{\left(\sum_{i=1}^{[n/2]} a_i \left(X_{(n-i+1)} - X_{(i)}\right)\right)^2}{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}$$

- $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ are the ordered statistics.



Related works

- In 1987, Jarque and Bera proposed the JB test.
  - Jarque, C. M. and Bera, A. K.A test for normality of observations and regression residuals. *Inter. Statist. Rev.*, **55**(2), 163–172, 1987.
- In 2007, Nakagawa et. al. proposed a modified version of the Jarque–Bera test JB'.
  - Nakagawa, S. and Niki, N. and Hashiguchi, H. An Omnibus test for normality. *Proc of the ninth Japan-China sympo. on statist.*, 191–194,2007.

. . . .

- In 2007, a power comparison was conducted through Monte Carlo simulation assuming the contaminated normal distributions as H_1.
  - Thadewald, T. and Buning, H. Jarque-Bera test and its competitors for testing normality-A power comparison *J. of Appl. Statist.*, **34**(1), 87–105, 2007.