Spline approximation of a random process with singularity

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Outline

1. Introduction. Basic Notation
2. Results
   - Optimal rate recovery
3. Numerical Experiments

Suppose a random process $X(t), t \in [0,1]$, with finite second moment is observed in a finite number of points (**sampling designs**). At any unsampled point t, we approximate the value of the process by a **composite Hermite spline**. The approximation performance on the entire interval is measured by mean errors. In this talk we deal with two problems:

- Investigating accuracy of such interpolator in mean norms
- Constructing a sequence of sampling designs with asymptotically optimal properties

Basic notation

Let $X = X(t), t \in [0,1]$, be defined on the probability space $(\Omega, \mathscr{F}, P)$. Assume that, for every t, the random variable $X(t)$ lies in the normed linear space $L^2(\Omega) = L^2(\Omega, \mathscr{F}, P)$ of zero mean random variables with finite second moment and identified equivalent elements with respect to P.
We set $||\xi|| = \left(\mathrm{E}\xi^2\right)^{1/2}$ for all $\xi \in L^2(\Omega)$ and consider the approximation by piecewise linear interpolator, based on the normed linear space $\mathscr{C}^m[0,1]$ of random processes having continuous q.m. (quadratic mean) derivatives up to order $m \geq 0$.
We define the **integrated mean norm** for any $X \in \mathscr{C}^m[0,1]$ by setting

$$||X||_p = \left(\int_0^1 ||X(t)||^p dt\right)^{1/p}, \qquad 1 \leq p < \infty,$$

and the **uniform mean norm** $||X||_\infty = \max_{[0,1]} ||X(t)||$.

Local Hölder's condition

We say that $X \in \mathcal{C}^{m,\beta}([a,b], V(\cdot))$ if $X \in \mathcal{C}^m([a,b])$ and $X^{(m)}$ is *locally Hölder continuous*, i.e., if for all $t, t+s, \in [a,b]$,

$$|| X^{(m)}(t+s) - X^{(m)}(t) || \leq V(\bar{t})^{1/2}|s|^{\beta}, 0 < \beta \leq 1, \quad (1)$$

for a positive continuous function $V(t), t \in [a,b]$, and some $\bar{t} \in [t, t+s]$.

In particular, if $V(t) = C$, $t \in [a,b]$, where C is a positive constant, then $X^{(m)}$ is *Hölder continuous*, and we denote it by $X \in \mathcal{C}^{m,\beta}([a,b], C)$

Following Berman(1974) we call process $X(t), t \in [a,b] \subseteq [0,1]$, **locally stationary** if there exists a positive continuous function $c(t)$ such that, for some $0 < \beta \leq 1$,

$$\lim_{s \to 0} \frac{||X(t+s) - X(t)||}{|s|^{\beta}} = c(t)^{1/2}, \quad \text{uniformly in } t \in [a,b].$$

We denote the class of processes which m-th q.m. satisfy the above condition over $[a,b]$ by $\mathcal{B}^{m,\beta}([a,b], c(\cdot))$.

We say that $X \in \mathcal{CB}^{m,\beta}((0,1], c(\cdot), V(\cdot))$ if there exist $0 < \beta \leq 1$ and positive continuous functions $c(t), V(t)$, $t \in (0,1]$, such that $X \in \mathcal{C}^{m,\beta}([a,b], V(\cdot))$ and $X \in \mathcal{B}^{m,\beta}([a,b], c(\cdot))$ for any $[a,b] \subset (0,1]$.

Let $X(t)$, $t \in [0,1]$, such that,

$$X \in \mathcal{C}^{l,\alpha}([0,1], M) \cap \mathcal{CB}^{m,\beta}((0,1], c(\cdot), V(\cdot)).$$

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Example:
$X(t) = B(\sqrt{t})$, $t \in [0,1]$, where $B(t)$, $t \in [0,1]$, is a fractional Brownian motion with Hurst parameter H and the covariance function $r(t,s) = (|t|^{2H} + |s|^{2H} - |t-s|^{2H})/2$

- $l = 0,\ \alpha = H/2$
- $m = 0,\ \beta = H$
- $c(t) = V(t) = (4t)^{-H}$

Suppose that for $X \in \mathcal{C}^m([0,1])$, the process and its first $r \leq m$ derivatives can be sampled at the distinct **design points** $T_n = (t_0, t_1, \ldots, t_n)$, $0 = t_0 < t_1 < \ldots < t_n = 1$. The stochastic Hermite spline of order $k = 2r + 1 \leq 2m + 1$, denoted by $H_k(X, T_n)$ is a unique solution of the interpolation problem

$$H_k^{(j)}(t_i) = X^{(j)}(t_i), \qquad i = 0, \ldots, n, \; j = 0, \ldots, r.$$

Composite Hermite spline

Define $H_{q,k}(X, T_n)$, $q \leq k$, to be a **composite Hermite spline**

$$H_{q,k}(X, T_n) := \begin{cases} H_q(X, T_n)(t), & t \in [0, t_1] \\ H_k(X, T_n)(t), & t \in [t_1, 1] \end{cases} .$$

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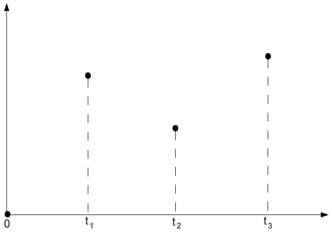

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Examples:

- $H_{1,1}$ (piecewise linear interpolator)
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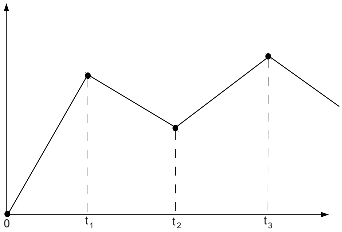

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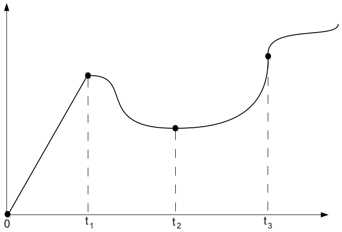
Examples:

- $H_{1,1}$ (piecewise linear interpolator)
- $H_{1,3}$



We consider **quasi regular sequences** (qRS) of sampling designs $\{T_n = T_n(h)\}$ generated by a density function $h(\cdot)$ via

$$\int_0^{t_i} h(t)dt = \frac{i}{n}, \quad i = 1, \dots, n,$$

where $h(\cdot)$ is continuous for $t \in (0, 1]$ and if $h(\cdot)$ is unbounded in $t = 0$, then $h(t) \to +\infty$ as $t \to +0$. We denote this property of $\{T_n\}$ by: $\{T_n\}$ is qRS(h).

Regularly varying function

Recall that a positive function $f(\cdot)$ is called **regularly varying** (on the right) at the origin with index ρ, if for any $\lambda > 0$,

$$\frac{f(\lambda x)}{f(x)} \to \lambda^{\rho} \quad \text{as } x \to 0+,$$

and denote this property by $f \in \mathscr{R}_{\rho}(O+)$. A natural example of such function is a power function, i.e., $f(x) = x^a \in \mathscr{R}_a(O+)$. Moreover we say that $g \in \mathscr{R}_{\rho}(r(\cdot), 0+)$ if there exists $r(x) \geq g(x), x \in [0, 1]$ such that $r \in \mathscr{R}_{\rho}(O+)$.

- (Seleznjev, Buslaev 1999)
  Optimal approximation rate for linear methods for $X \in \mathscr{C}^{l,\alpha}[0,1]$ is $n^{-(l+\alpha)}$
- (Seleznjev, 2000)
  Results on Hermite spline approximation when $X \in \mathscr{B}^{l,\alpha}([0,1], c(\cdot))$ and regular sequences of sampling designs are used

$$||X - H_k(X, T_n)|| \sim n^{-(l+\alpha)} \; as \; n \to \infty, \quad m \leq k.$$

We have a process which l-th derivative is α-Hölder on $[0, 1]$.
Can we get the approximation rate **better** than $n^{-(l+\alpha)}$?

Let us define: $H(t) = \int_0^t h(v)dv$, $G(s) = H^{-1}(s)$, and $g(s) = G'(s)$, $t, s \in [0, 1]$.

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Let $X \in \mathcal{C}^{l,\alpha}([0,1], M) \cap \mathcal{CB}^{m,\beta}((0,1], c(\cdot), V(\cdot))$. We formulate the following condition for a local Hölder function V and a sequence generating density h:

(C) *let* $g \in \mathscr{R}^+(r(\cdot), 0+)$, *where*

$$r(s) = o(s^{(m+\beta)/(l+\alpha+1/p)-1}) \text{ as } s \to 0; \tag{2}$$

if $p = \infty$, *then* $V(t)^{1/2} r(H(t))^{m+\beta} \to 0$ *as* $t \to 0$;
if $1 \leq p < \infty$ *and, additionally,* $V(G(\cdot))^{1/2} \in \mathscr{R}^+(R(\cdot), 0+)$, *then* $R(H(t))r(H(t))^{m+\beta} \in L_p[0,b]$ *for some* $b > 0$.

Theorem

Let $X \in \mathcal{C}^{l,\alpha}([0,1], M) \cap \mathcal{CB}^{m,\beta}((0,1], c(\cdot), V(\cdot))$*,* $l + \alpha \leq m + \beta$*, with the mean* $f \in C^{m,\theta}([0,1], C)$*,* $\beta < \theta \leq 1$*, be interpolated by a composite Hermite spline* $H_{q,k}(X, T_n)$*,* $l \leq q$*,* $m \leq k$*, where* T_n *is a qRS(h). Let for the density h and the local Hölder function* V*, the condition* (C) *hold. Then*

$$\lim_{n\to\infty} n^{m+\beta} ||X - H_{q,k}(X, T_n)||_p = b_{k,p}^{m,\beta} ||c^{1/2} h^{-(m+\beta)}||_p > 0. \qquad (3)$$

$X(t) = B(\sqrt{t})$, $t \in [0,1]$, where $B(t)$, $t \in [0,1]$, is a fractional Brownian motion with Hurst parameter $H = 0.8$. Then

$$X \in \mathcal{C}^{0,0.4}([0,1],1) \cap \mathcal{CB}^{0,0.8}((0,1],c(\cdot),V(\cdot)),$$

where $c(t) = V(t) = (4t)^{-0.8}$.

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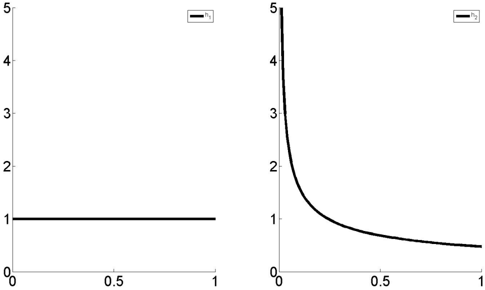
We consider the following knot densities

$$h_\lambda(t) = \frac{1}{\lambda} t^{\frac{1}{\lambda}-1}, \quad t \in (0,1], \quad \lambda > 0,$$

say, *power densities*.

We consider the **mean maximal** approximation error of the piecewise linear interpolator, and compare the efficiency of the designs generated by the power densities with parameters

- $\lambda_1 = 1$ (uniform knots distribution)
- $\lambda_2 = 2.1$



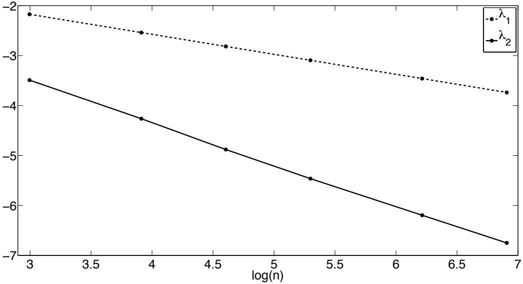


Figure: Comparison of the uniform mean errors for the uniform density $h_{\lambda_1}(\cdot)$ and $h_{\lambda_2}(\cdot)$ in a log-log scale.

The plot corresponds to the following asymptotic behavior of the approximation errors:

$$\begin{array}{lcll} e_n(h_{\lambda_1}) & \sim & C_1\, n^{-0.4}, & C_1 \simeq 0.377, \\ e_n(h_{\lambda_2}) & \sim & C_2\, n^{-0.8}, & C_2 \simeq 0.295 \text{ as } n \to \infty. \end{array}$$

For example, the minimal number of observations needed to obtain the accuracy 0.01 is approximately 8727 for the equidistant sampling density h_{λ_1}, whereas it needs only 69 knots when h_{λ_2} is used, i.e., Theorem 1 is applicable.

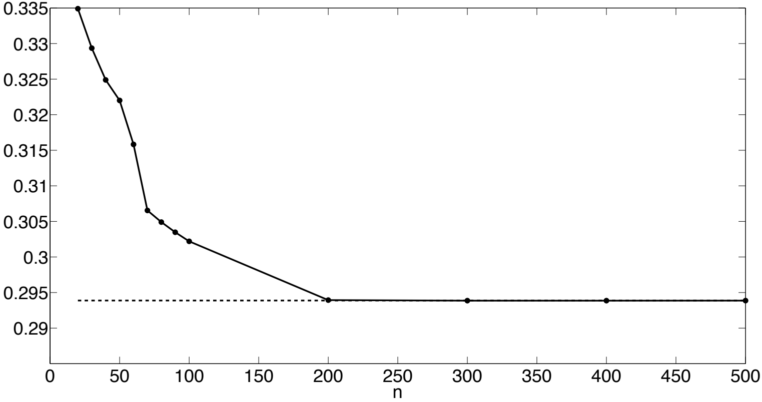


Figure: Convergence of the $n^{0.8}$ scaled uniform mean errors to the asymptotic constant for the generating density $h_2(\cdot)$.

Thank you !