Spline approximation of a random process with singularity

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Outline

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Suppose a random process $X(t), t \in [0, 1]$, with finite second moment is observed in a finite number of points (sampling designs). At any unsampled point $t$, we approximate the value of the process by a composite Hermite spline. The approximation performance on the entire interval is measured by mean errors. In this talk we deal with two problems:

- Investigating accuracy of such interpolator in mean norms
- Constructing a sequence of sampling designs with asymptotically optimal properties
Basic notation

Let $X = X(t), t \in [0, 1]$, be defined on the probability space $(\Omega, \mathcal{F}, P)$. Assume that, for every $t$, the random variable $X(t)$ lies in the normed linear space $L^2(\Omega) = L^2(\Omega, \mathcal{F}, P)$ of zero mean random variables with finite second moment and identified equivalent elements with respect to $P$. We set $||\xi|| = (E\xi^2)^{1/2}$ for all $\xi \in L^2(\Omega)$ and consider the approximation by piecewise linear interpolator, based on the normed linear space $C^m[0, 1]$ of random processes having continuous q.m. (quadratic mean) derivatives up to order $m \geq 0$. We define the **integrated mean norm** for any $X \in C^m[0, 1]$ by setting

$$
||X||_p = \left( \int_0^1 ||X(t)||^p dt \right)^{1/p}, \quad 1 \leq p < \infty,
$$

and the **uniform mean norm** $||X||_{\infty} = \max_{[0,1]} ||X(t)||$. 
Local Hölder’s condition

We say that \( X \in C^{m,\beta}([a, b], V(\cdot)) \) if \( X \in C^m([a, b]) \) and \( X^{(m)} \) is \textit{locally Hölder continuous}, i.e., if for all \( t, t + s, \in [a, b] \),

\[
||X^{(m)}(t + s) - X^{(m)}(t)|| \leq V(\bar{t})^{1/2}|s|^\beta, \quad 0 < \beta \leq 1,
\]

for a positive continuous function \( V(t), t \in [a, b] \), and some \( \bar{t} \in [t, t + s] \).

In particular, if \( V(t) = C, t \in [a, b] \), where \( C \) is a positive constant, then \( X^{(m)} \) is \textit{Hölder continuous}, and we denote it by \( X \in C^{m,\beta}([a, b], C) \)
Local stationarity

Following Berman(1974) we call process $X(t), t \in [a, b] \subseteq [0, 1]$, **locally stationary** if there exists a positive continuous function $c(t)$ such that, for some $0 < \beta \leq 1$,

$$\lim_{s \to 0} \frac{||X(t+s) - X(t)||}{|s|^\beta} = c(t)^{1/2}, \text{ uniformly in } t \in [a, b].$$

We denote the class of processes which $m$-th q.m. satisfy the above condition over $[a, b]$ by $\mathcal{B}^{m,\beta}([a, b], c(\cdot)).$
We say that $X \in \mathcal{CB}^{m,\beta}((0, 1], c(\cdot), V(\cdot))$ if there exist $0 < \beta \leq 1$ and positive continuous functions $c(t), V(t), t \in (0, 1]$, such that $X \in \mathcal{C}^{m,\beta}([a, b], V(\cdot))$ and $X \in \mathcal{B}^{m,\beta}([a, b], c(\cdot))$ for any $[a, b] \subset (0, 1]$. 
Processes of interest

Let \( X(t), \ t \in [0, 1] \), such that,

\[
X \in \mathcal{C}^{l,\alpha}([0, 1], M) \cap \mathcal{CB}^{m,\beta}((0, 1], c(\cdot), V(\cdot)).
\]
Processes of interest

Let $X(t), t \in [0, 1]$, such that,

$$X \in \mathcal{C}^{l,\alpha}([0, 1], M) \cap \mathcal{CB}^{m,\beta}((0, 1], c(\cdot), V(\cdot)).$$

Example:

$X(t) = B(\sqrt{t}), t \in [0, 1]$, where $B(t), t \in [0, 1]$, is a fractional Brownian motion with Hurst parameter $H$ and the covariance function

$$r(t, s) = (|t|^{2H} + |s|^{2H} - |t - s|^{2H})/2$$

- $l = 0, \alpha = H/2$
- $m = 0, \beta = H$
- $c(t) = V(t) = (4t)^{-H}$
Suppose that for $X \in C^m([0,1])$, the process and its first $r \leq m$ derivatives can be sampled at the distinct design points $T_n = (t_0, t_1, \ldots, t_n)$, $0 = t_0 < t_1 < \ldots < t_n = 1$. The stochastic Hermite spline of order $k = 2r + 1 \leq 2m + 1$, denoted by $H_k(X, T_n)$, is a unique solution of the interpolation problem

$$H_k^{(j)}(t_i) = X^{(j)}(t_i), \quad i = 0, \ldots, n, \ j = 0, \ldots, r.$$
Define $H_{q,k}(X, T_n)$, $q \leq k$, to be a **composite Hermite spline**

$$H_{q,k}(X, T_n) := \begin{cases} 
H_q(X, T_n)(t), & t \in [0, t_1] \\
H_k(X, T_n)(t), & t \in [t_1, 1] 
\end{cases}.$$
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**Examples:**

- $H_{1,1}$ (piecewise linear interpolator)
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Examples:

- $H_{1,1}$ (piecewise linear interpolator)
- $H_{1,3}$
quasi Regular Sequences

We consider quasi regular sequences (qRS) of sampling designs \( \{T_n = T_n(h)\} \) generated by a density function \( h(\cdot) \) via

\[
\int_0^{t_i} h(t) dt = \frac{i}{n}, \quad i = 1, \ldots, n,
\]

where \( h(\cdot) \) is continuous for \( t \in (0, 1] \) and if \( h(\cdot) \) is unbounded in \( t = 0 \), then \( h(t) \to +\infty \) as \( t \to +0 \). We denote this property of \( \{T_n\} \) by: \( \{T_n\} \) is qRS(h).
Recall that a positive function $f(\cdot)$ is called **regularly varying** (on the right) at the origin with index $\rho$, if for any $\lambda > 0$,

$$\frac{f(\lambda x)}{f(x)} \to \lambda^\rho \quad \text{as} \quad x \to 0^+,$$

and denote this property by $f \in \mathcal{R}_\rho(O+)$. A natural example of such function is a power function, i.e., $f(x) = x^a \in \mathcal{R}_a(O+)$. Moreover we say that $g \in \mathcal{R}_\rho(r(\cdot), 0+)$ if there exists $r(x) \geq g(x), x \in [0, 1]$ such that $r \in \mathcal{R}_\rho(O+)$. 
Previous Results

- (Seleznjev, Buslaev 1999)  
  **Optimal** approximation rate for linear methods for $X \in C^{l,\alpha}[0, 1]$ is $n^{-(l+\alpha)}$

- (Seleznjev, 2000)  
  Results on Hermite spline approximation when $X \in B^{l,\alpha}([0, 1], c(\cdot))$ and regular sequences of sampling designs are used

$$||X - H_k(X, T_n)|| \sim n^{-(l+\alpha)} \text{ as } n \to \infty, \quad m \leq k.$$
We have a process which $l$-th derivative is $\alpha$-Hölder on $[0, 1]$. Can we get the approximation rate better than $n^{-(l+\alpha)}$?
Let us define: \( H(t) = \int_0^t h(v)dv \), \( G(s) = H^{-1}(s) \), and \( g(s) = G'(s) \), \( t, s \in [0, 1] \).
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Let $X \in C^{l,\alpha}([0, 1], M) \cap CB^{m,\beta}((0, 1], c(\cdot), V(\cdot))$. We formulate the following condition for a local Hölder function $V$ and a sequence generating density $h$:

\begin{align*}
(C) \text{ let } g & \in \mathcal{R}^+(r(\cdot), 0+) \text{, where} \\
& r(s) = o(s^{(m+\beta)/(l+\alpha+1/p)-1}) \text{ as } s \to 0; \\
& \text{if } p = \infty, \text{ then } V(t)^{1/2}r(H(t))^{m+\beta} \to 0 \text{ as } t \to 0; \\
& \text{if } 1 \leq p < \infty \text{ and, additionally, } V(G(\cdot))^{1/2} \in \mathcal{R}^+(R(\cdot), 0+), \text{ then} \\
& R(H(t))r(H(t))^{m+\beta} \in L_p[0, b] \text{ for some } b > 0.
\end{align*}
Optimal rate recovery

Theorem

Let $X \in \mathcal{C}^{l,\alpha}([0, 1], M) \cap \mathcal{CB}_{m,\beta}^{\alpha}((0, 1], c(\cdot), V(\cdot)), l + \alpha \leq m + \beta$, with the mean $f \in \mathcal{C}^{m,\theta}([0, 1], \mathcal{C}), \beta < \theta \leq 1$, be interpolated by a composite Hermite spline $H_{q,k}(X, T_n)$, $l \leq q, m \leq k$, where $T_n$ is a qRS($h$). Let for the density $h$ and the local Hölder function $V$, the condition (C) hold. Then

$$\lim_{n \to \infty} n^{m+\beta} \|X - H_{q,k}(X, T_n)\|_p = b_{k,p}^{m,\beta} \|c^{1/2} h^{-(m+\beta)}\|_p > 0. \quad (3)$$
Example

\[ X(t) = B(\sqrt{t}), \quad t \in [0, 1], \] where \( B(t), \quad t \in [0, 1], \) is a fractional Brownian motion with Hurst parameter \( H = 0.8. \) Then

\[ X \in C^{0,0.4}([0, 1], 1) \cap CB^{0,0.8}((0, 1], c(\cdot), V(\cdot)), \]

where \( c(t) = V(t) = (4t)^{-0.8}. \)
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where \( c(t) = V(t) = (4t)^{-0.8}. \)

We consider the following knot densities

\[ h_\lambda(t) = \frac{1}{\lambda} t^{\frac{1}{\lambda} - 1}, \quad t \in (0, 1], \quad \lambda > 0, \]

say, power densities.
We consider the **mean maximal** approximation error of the piecewise linear interpolator, and compare the efficiency of the designs generated by the power densities with parameters

- $\lambda_1 = 1$ (uniform knots distribution)
- $\lambda_2 = 2.1$
Figure: Comparison of the uniform mean errors for the uniform density \( h_{\lambda_1}(\cdot) \) and \( h_{\lambda_2}(\cdot) \) in a log-log scale.

The plot corresponds to the following asymptotic behavior of the approximation errors:

\[
\begin{align*}
  e_n(h_{\lambda_1}) & \sim C_1 n^{-0.4}, \quad C_1 \approx 0.377, \\
  e_n(h_{\lambda_2}) & \sim C_2 n^{-0.8}, \quad C_2 \approx 0.295 \text{ as } n \to \infty.
\end{align*}
\]

For example, the minimal number of observations needed to obtain the accuracy 0.01 is approximately 8727 for the equidistant sampling density \( h_{\lambda_1} \), whereas it needs only 69 knots when \( h_{\lambda_2} \) is used, i.e., Theorem 1 is applicable.
Figure: Convergence of the $n^{0.8}$ scaled uniform mean errors to the asymptotic constant for the generating density $h_2(\cdot)$. 
Thank you!