A Mann-Whitney spatial scan statistic for continuous data

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UNIVERSITÉ MONTPELLIER 2
SCIENCES ET TECHNIQUES

Outline

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A data set to analyze

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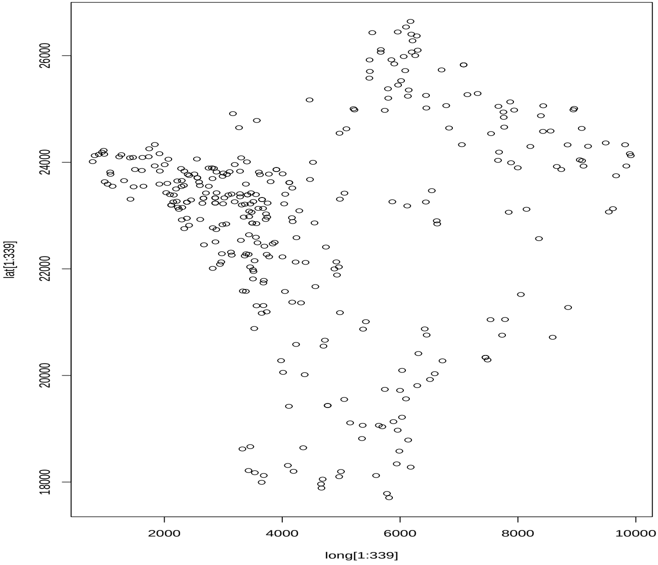

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- Marked point process : $(X_i, C_i), i = 1, \cdots, n$ where $X_i \in A \subset \mathbb{R}^d$ and $C_i \in \mathbb{R}$.

Definition :

Cluster= geographical area where the continuous variable is higher than outside.

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Question :

Are there one or more clusters ? Where ?

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Significance estimated by a Monte-Carlo procedure.

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- Elliptic clusters, with given orientation and shape (2D only).

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- Edges : $\{(i,j) : d(x_i, x_j) \leq \delta, 1 \leq i < n, i < j \leq n\}$.

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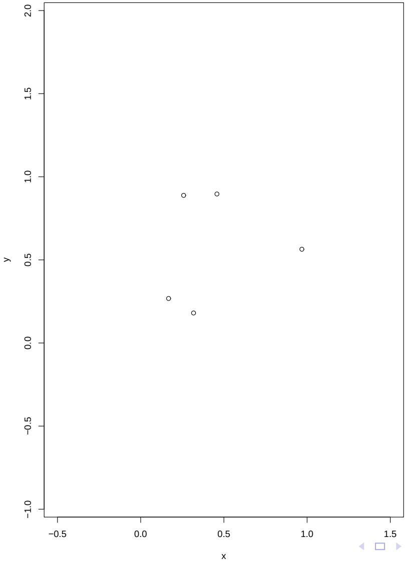

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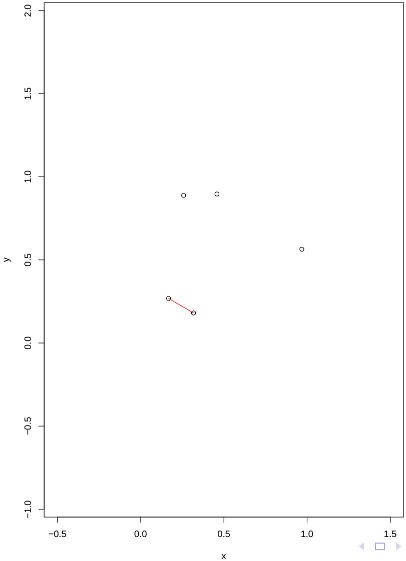

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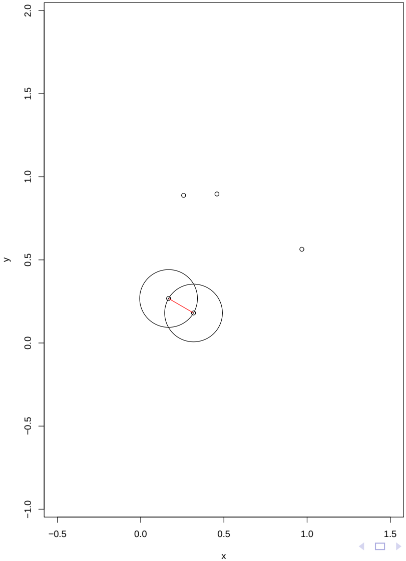

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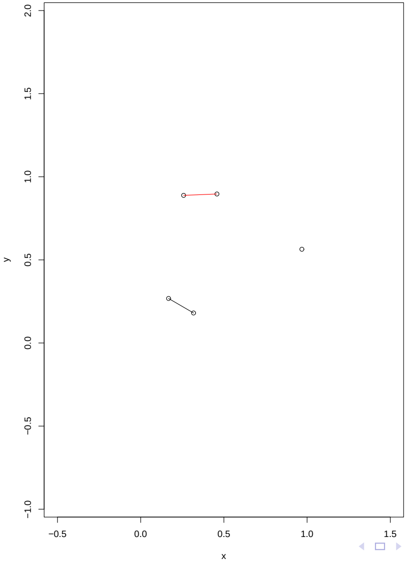

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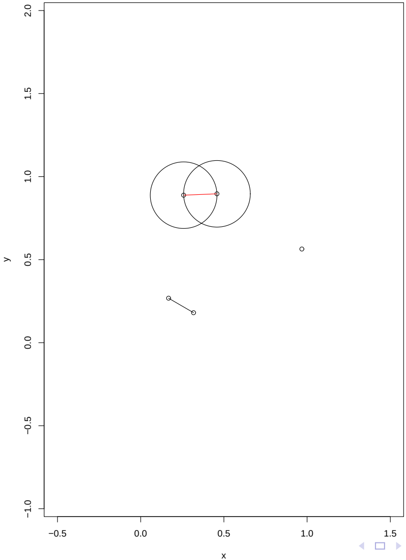

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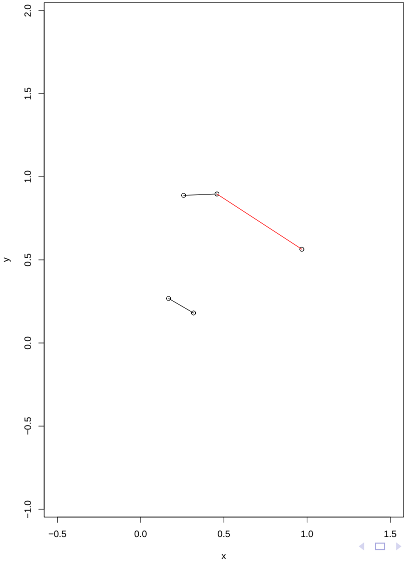

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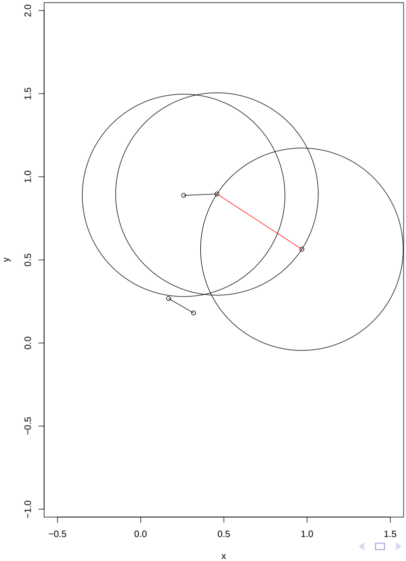

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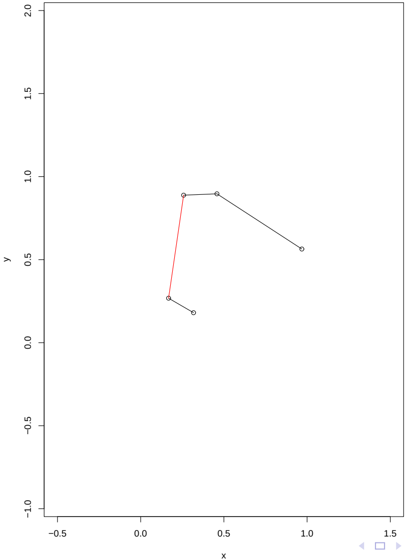

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- Only $n - 1$ areas, arbitrarily shaped.

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- $\sigma(Z)^2 = \frac{1}{n(Z)} \sum_{i:X_i \in Z} \big(C_i - \mu(Z)\big)^2$.

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Likelihood ratio :

$$I(Z) = \frac{L_{1,Z}(X_1, \cdots, X_n, C_1, \cdots, C_n)}{L_0(X_1, \cdots, X_n, C_1, \cdots, C_n)} \quad \mathbb{1}\left(\mu(Z) > \mu(\bar{Z})\right).$$

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Likelihood ratio :

$$l_{exp}(Z) = -n(Z)\log\big(\mu(Z)\big) - n(\bar{Z})\log\big(\mu(\bar{Z})\big).$$

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Likelihood ratio :

$$I_{homgau}(Z) = \frac{1}{\sigma^2_{1,Z}}.$$

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Likelihood ratio :

$$I_{hetgau}(Z) = -n(Z)\log\left(\sigma(Z)^2\right) - n(\bar{Z})\log\left(\sigma(\bar{Z})^2\right).$$

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$$I_{rank}(Z) = \frac{RS(Z) - M(Z)}{\sqrt{V(Z)}}.$$

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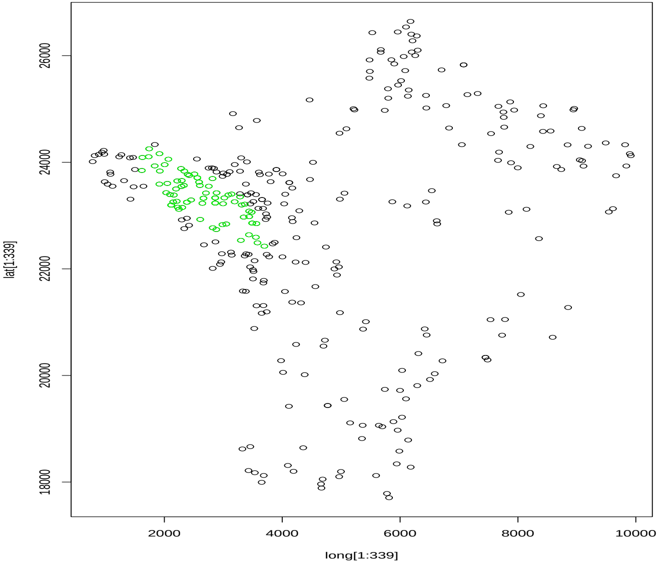

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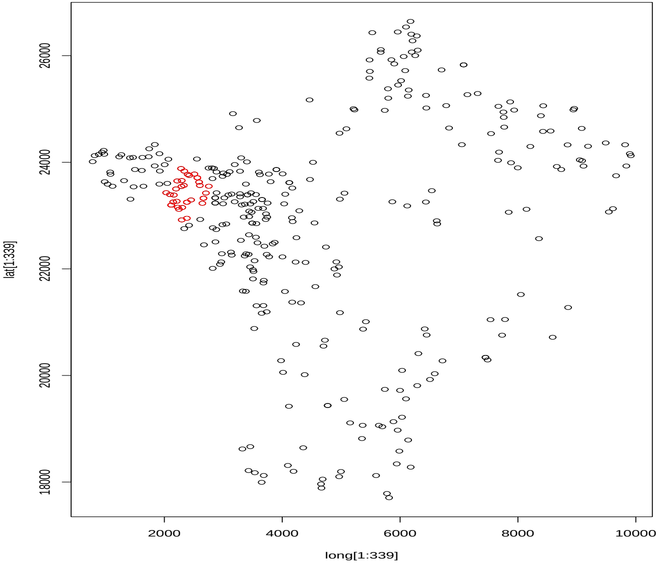

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Goal : detecting areas where galaxies are "redder".

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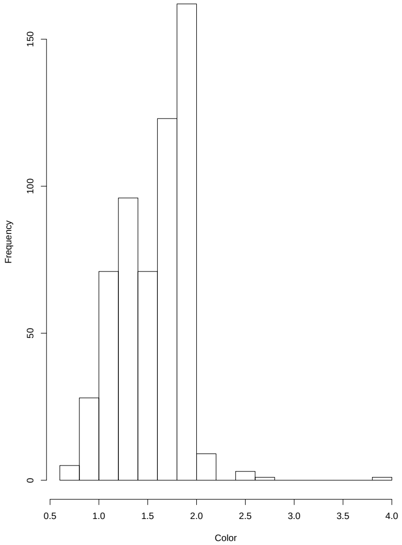

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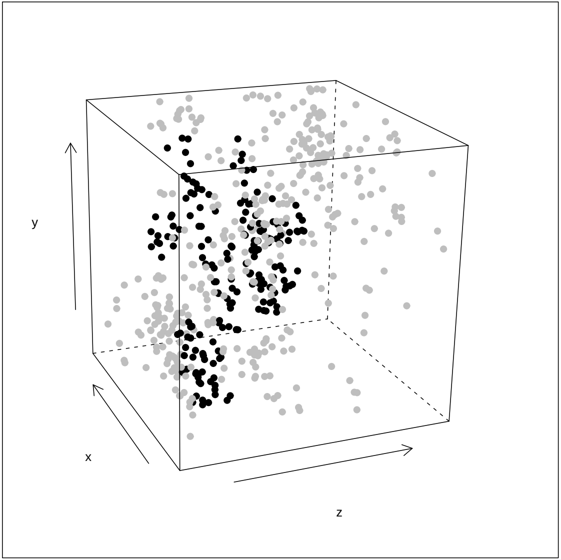

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- Simulation study to compare concentration indices.