Two kurtosis measures in a simulation study

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Overview

- The IF (SIF) and its role in kurtosis studies
- From inequality (Lorenz) ordering to right/left kurtosis measures
- Two kurtosis measures: SIF comparison / numerical experiments

Background

From Pearson (1905) onwards statistics textbooks have defined kurtosis operationally as **the characteristic of a distribution measured by its standardized fourth moment**. However…

Excess/defect of frequency near the mean compared to a normal curve (Pearson, 1894-1905)

Non-Gaussian type of symmetry (First Ed. ESS)

Reverse tendency toward **bimodality** (Darlington, 1970)

$$\beta_2 = E\left(\frac{X-\mu}{\sigma}\right)^4$$

Tailedness only (Ali, 1974)

Dispersion around the two values μ +/– σ (Moors, 1986)

Peakedness + tailedness (Dyson, 1943 – Finucan, 1964)

Kurtosis and the IF

An early intuition of L. Faleschini

	Faleschini (*Statistica*, 1948)	Hampel (1968) – Ruppert (1987)
Background & method	▪ Take a frequency distribution: $\{a_r, f_r\,;\ \sum f_r = P\}$ ▪ "To investigate the behaviour of β_2 when a frequency f_r is altered, we compute the **partial derivative** of β_2 wrt f_r"	▪ Consider a probability distribution F and the functional $\beta_2 = \beta_2(F)$ ▪ "How does β_2 change if we throw in an additional observation at some point x?" ▪ Contaminated distribution: $F_\varepsilon = (1-\varepsilon)F + \varepsilon\,\delta_x \qquad (0 < \varepsilon < 1)$
Main result	$\frac{\partial \beta_2}{\partial f_r} = \frac{1}{P}\left[\left(z_r^2 - \beta_2\right)^2 - \beta_2(\beta_2 - 1) - 4\gamma_1 z_r\right]$ where: $z_r = \frac{a_r - \mu}{\sigma}; \quad \gamma_1 = \frac{\mu_3}{\sigma^3}$	$IF_{\beta_2,F}(x) = \lim_{\varepsilon \downarrow 0} \frac{\beta_2(F_\varepsilon) - \beta_2(F)}{\varepsilon}$ $= (z^2 - \beta_2)^2 - \beta_2(\beta_2 - 1) - 4\gamma_1 z$ with: $z = \frac{x - \mu}{\sigma}$

Kurtosis and the IF

Faleschini's derivative – computational details

$$\beta_2 = \frac{\mu_4}{\mu_2^2}$$

$s = 2, 4$

$$\mu_s = \sum_{i=0}^{s} (-1)^i \binom{s}{i} \cdot m_{s-i} \cdot \mu^i$$

Raw moment of order $s - i$

$$m_{s-i} = \frac{1}{P} \sum_{r=1}^{n} a_r^{\,s-i} \cdot f_r$$

i-th power of μ

$$\mu = \frac{1}{P} \sum_{r=1}^{n} a_r \cdot f_r$$

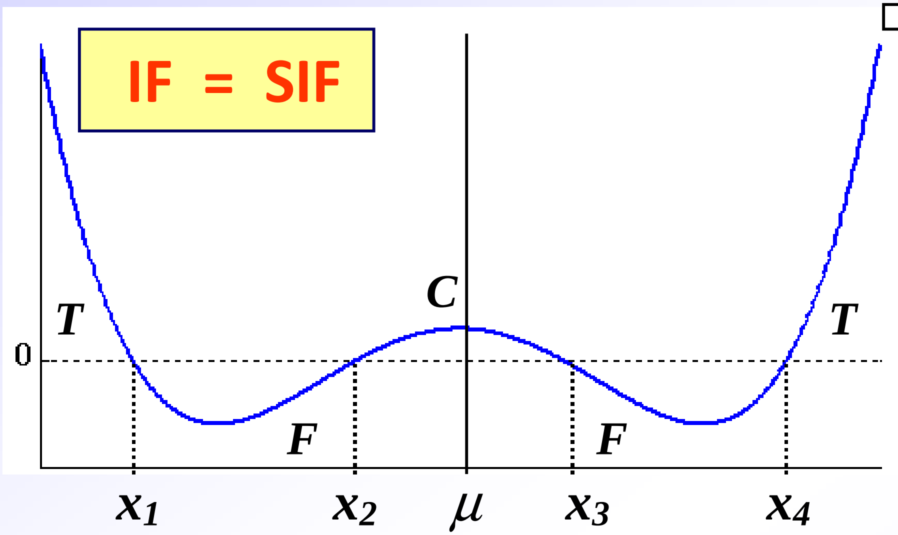
$$\frac{\partial m_{s-i}}{\partial f_r} = \frac{1}{\left(\sum f_r\right)^2} \left(a_r^{s-i} \sum f_r - \sum a_r^{s-i} \cdot f_r \right) = \frac{1}{P} \left(a_r^{s-i} - m_{s-i} \right)$$

$$\frac{\partial \mu^i}{\partial f_r} = i \mu^{i-1} \frac{1}{\left(\sum f_r\right)^2} \left(a_r \sum f_r - \sum a_r \cdot f_r \right) = \frac{1}{P} i \mu^{i-1} (a_r - \mu)$$

$$\frac{\partial \mu_s}{\partial f_r} = \sum_{i=0}^{s} (-1)^i \binom{s}{i} \frac{\partial \left(m_{s-i} \cdot \mu^i \right)}{\partial f_r} = \frac{1}{P} \left\{ (a_r - \mu)^s - \mu_s - (a_r - \mu) \cdot s \cdot \mu_{s-1} \right\}$$

Kurtosis and the IF
Kurtosis explained

In the symmetric case: $IF(x;F,\beta_2) = (z^2 - \beta_2)^2 - \beta_2(\beta_2 - 1)$



⇨ Quartic function with four real roots:

$$x_{1,2,3,4} = \mu \pm \sigma\sqrt{\beta_2 \pm \sqrt{\beta_2(\beta_2 - 1)}}$$

Unbounded

Local maximum: β_2 at $x = \mu$

Minima: $\beta_2(1 - \beta_2)$ at $x = \mu \pm \sigma\sqrt{\beta_2}$

KURTOSIS =
peakedness + tailedness
but β_2 is dominated by tailweight

This IF suggests that β_2 is likely to be overestimated by sample kurtosis for *x* distant from μ, but underestimated at intermediate values of *x*

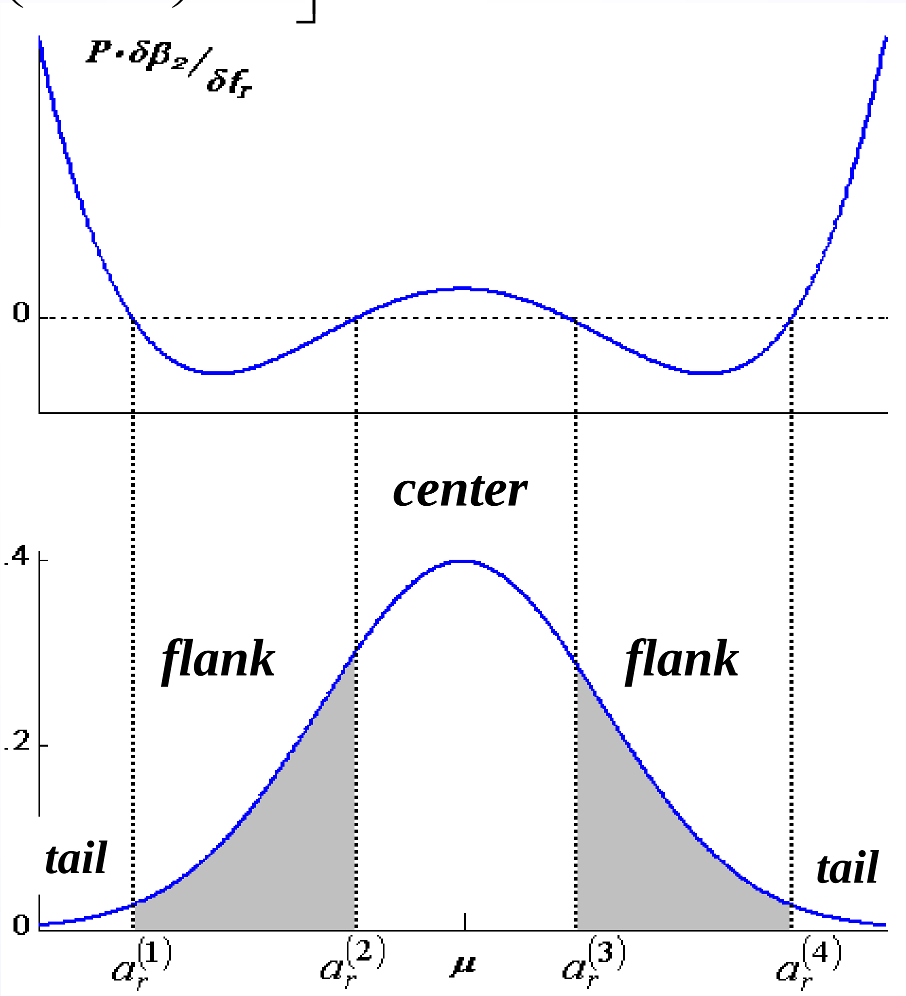
Kurtosis and the IF

The normal case and sample kurtosis

$$\frac{\partial \beta_2}{\partial f_r} = \frac{1}{P}\left[(z_r^2 - 3)^2 - 6\right] = \frac{1}{P}\left[\left(\frac{a_r - \mu}{\sigma}\right)^4 - 6\left(\frac{a_r - \mu}{\sigma}\right)^2 + 3\right]$$

<- IF = SIF

Assuming the normality of $\hat{\beta}_{2n}$ for large *n*, one can use the value $\beta_2 = 3$ to estimate the probability that *x* (a_r) lies in intervals where IF(*x*; β_2) is positive or negative.



Roots of the quartic:

$$a_r^{(1)} = \mu - 2{,}334\,\sigma \qquad a_r^{(3)} = \mu + 0{,}742\,\sigma$$

$$a_r^{(2)} = \mu - 0{,}742\,\sigma \qquad a_r^{(4)} = \mu + 2{,}334\,\sigma$$

Kurtosis and the IF
The normal case and sample kurtosis

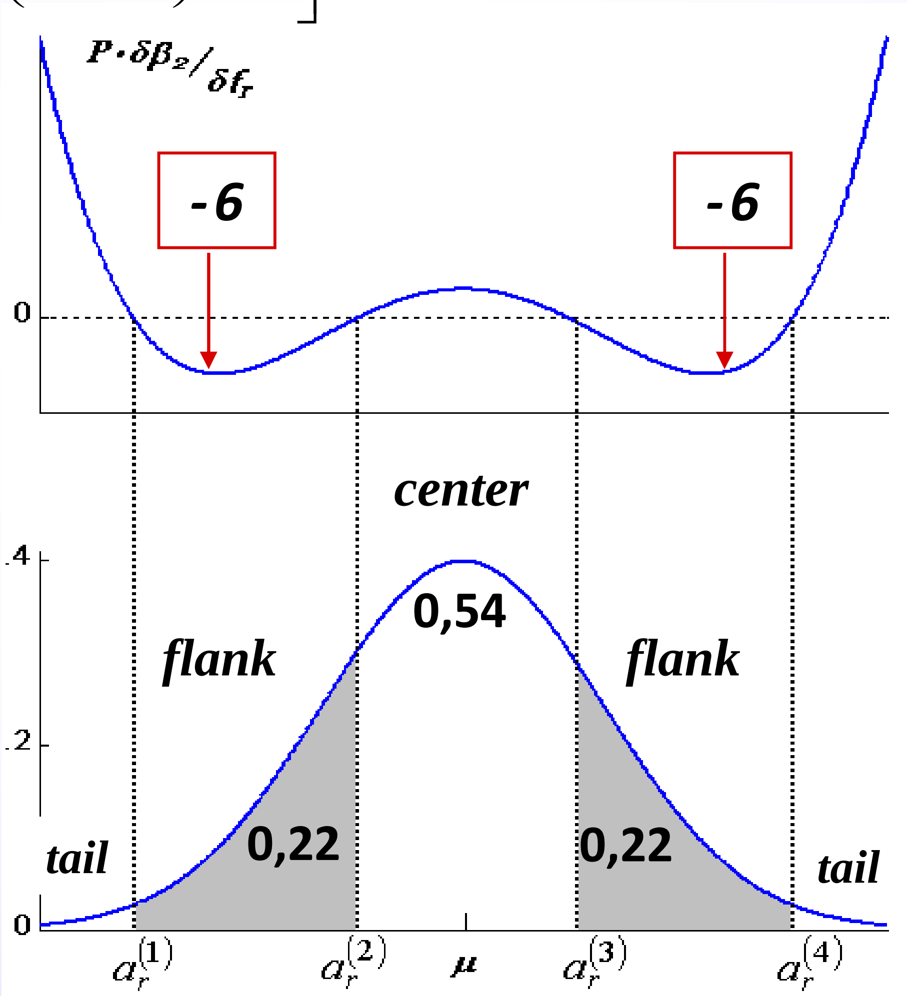
$$\frac{\partial \beta_2}{\partial f_r} = \frac{1}{P}\left[\left(z_r^2 - 3\right)^2 - 6\right] = \frac{1}{P}\left[\left(\frac{a_r - \mu}{\sigma}\right)^4 - 6\left(\frac{a_r - \mu}{\sigma}\right)^2 + 3\right]$$

<- IF = SIF

There are two intervals (***flanks***) of substantial probability (22% each) in which the IF has relatively large negative values (minimum $= -6$)

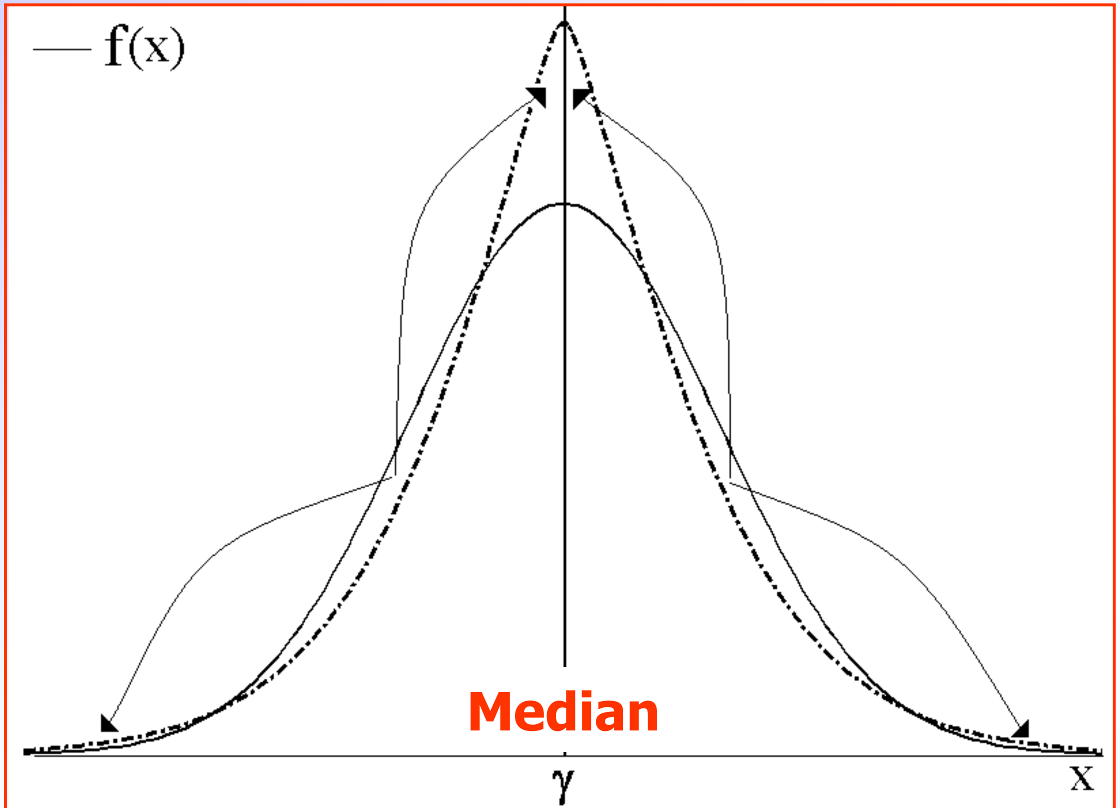
These possibly correspond to smaller values of sample kurtosis

-> Consistent with underestimation of β_2 by sample kurtosis (on average) and undercoverage of confidence intervals for β_2



Kurtosis by inequality

Zenga (ESS, 2006); Fiori (Comm. Statist., 2008)



Left deviation	**Right deviation**
$S = \gamma - X \mid X < \gamma$	$D = X - \gamma \mid X \geq \gamma$
with $E(S) = \delta_S < \infty$	with $E(D) = \delta_D < \infty$

Separate analysis of D and S:

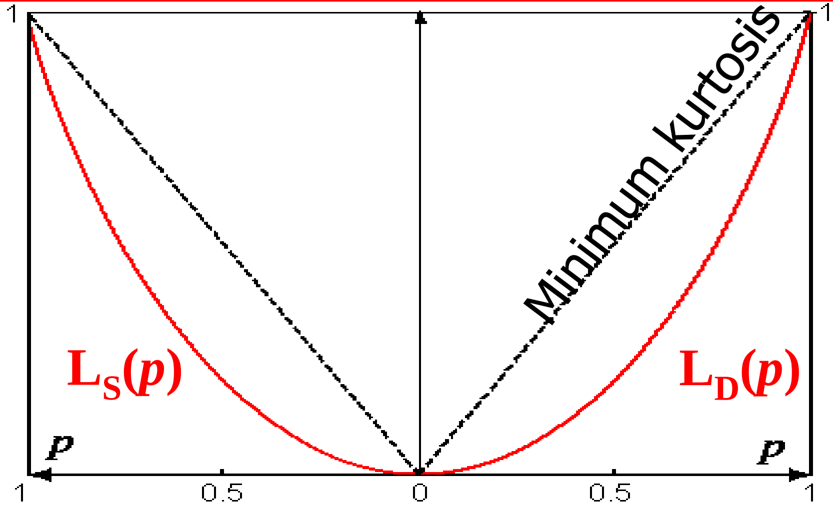
Kurtosis-increasing transformations

Pigou-Dalton transfer principle

Non-egalitarian transfers on D and S for fixed γ, δ_D, δ_S

Kurtosis by inequality

Zenga's kurtosis ordering (ESS, 2006)



Zenga's kurtosis curve

Represents kurtosis by a mirror plot of the **Lorenz curves** for **D** and **S**

Left kurtosis

Lorenz curve of S :

$$L_S(p) = \frac{1}{\delta_S} \int_0^p F_S^{-1}(t)\,dt$$

Right kurtosis

Lorenz curve of D :

$$L_D(p) = \frac{1}{\delta_D} \int_0^p F_D^{-1}(t)\,dt$$

- Unified treatment of symmetric and asymmetric distributions
- Kurtosis ordering defined via nested Lorenz curves
- Liu, Parelius and Singh (Ann. Statist., 1999) in a multivariate setting

Kurtosis by inequality

Zenga's kurtosis measures (ESS, 2006)

right **left**

$$K_1 = \frac{1}{2}[C(D) + C(S)]$$

with:

$$C(D) = 1 - \frac{\delta_D^2}{E(D^2)}$$

$$C(S) = 1 - \frac{\delta_S^2}{E(S^2)}$$

(ratios of right/left scale functionals)

right **left**

$$K_2 = \frac{1}{2}[R(D) + R(S)]$$

with:

$$R(D) = 2\int_0^1 [p - L_D(p)]dp$$

$$R(S) = 2\int_0^1 [p - L_S(p)]dp$$

(right & left Gini indexes)

Normalized measures, with values between 0 (minimum kurtosis) and 1 (maximum kurtosis)

Kurtosis measures
A look at the SIF

- Symmetric distribution ($\gamma = 0$): $F_\varepsilon = (1-\varepsilon)\,F + \varepsilon G \qquad \text{for } 0 \le \varepsilon \le 1$
- Symmetric contaminant (Ruppert, 1987): $G = 0.5\,(\delta_x + \delta_{-x})$

First measure of (right) kurtosis

$$C_D(F) = 1 - \frac{\delta_D^2\,(F)}{\mu'_{2D}(F)} \qquad \text{for } \mu'_{2D}(F) = E(D^2)$$

$$SIF\,(x; F, C_D) = \left(x - \frac{\mu'_{2D}}{\delta_D}\right)^2 - \left[\left(\frac{\mu'_{2D}}{\delta_D}\right)^2 - \mu'_{2D}\right]$$

which is positive iff

$$0 < x < \frac{\mu'_{2D}}{\delta_D}\left(1 - \sqrt{C_D}\right)$$

$$x > \frac{\mu'_{2D}}{\delta_D}\left(1 + \sqrt{C_D}\right)$$

Kurtosis measures
A look at the SIF

Second measure of (right) kurtosis

$$R_D(F) = \frac{\Delta_D(F)}{2\delta_D(F)}$$

where $\Delta(F)$ stands for the Gini mean difference:

$$\Delta(F) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|\, dF(x)\, dF(y)$$

$$SIF(x; F, R_D) = \frac{1}{\delta_D}\left[\delta_{D,x} - R_D(x+\delta_D)\right]$$

where $\delta_{D,x}(F) = E\left[|X-x| \mid X>0\right]$

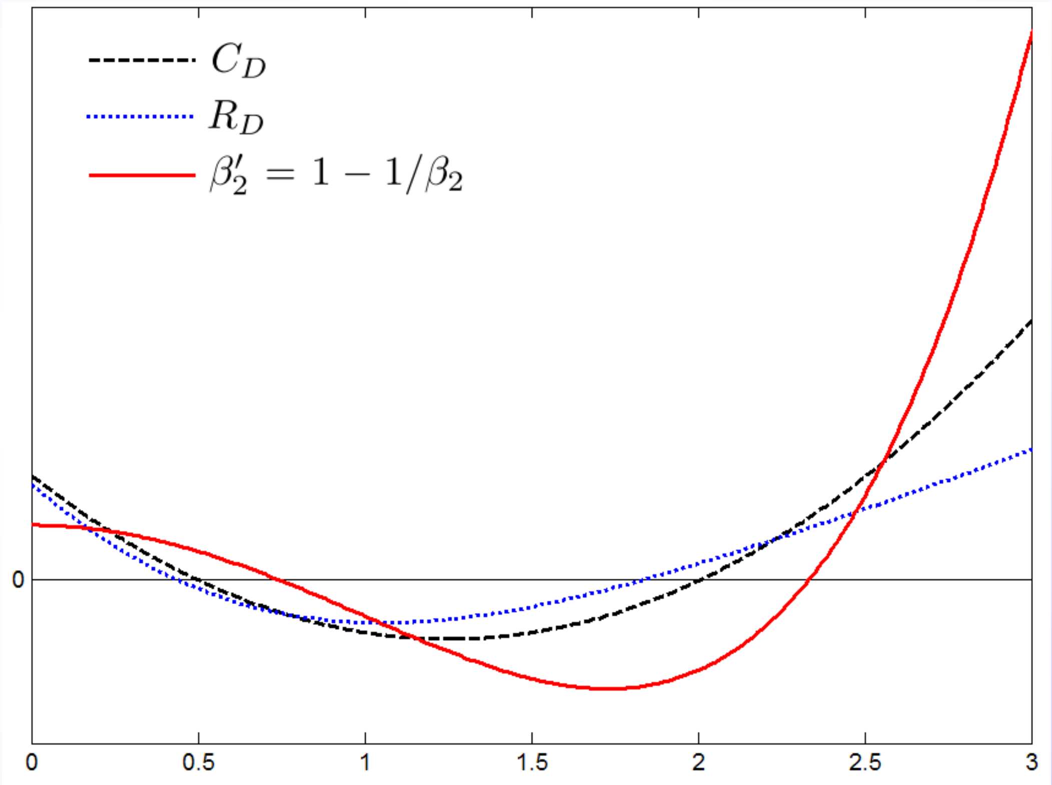
Kurtosis measures

SIF comparison at standard normal

All the measures are increased by contamination in the tails and at the center and are decreased by contamination in the shoulders/flanks.

Having unbounded SIF, they are sensitive to the location of tail outliers as well as their frequency. However, conventional kurtosis is much more sensitive (quartic SIF).

The magnitude of *min SIF* is considerably larger for conventional kurtosis.

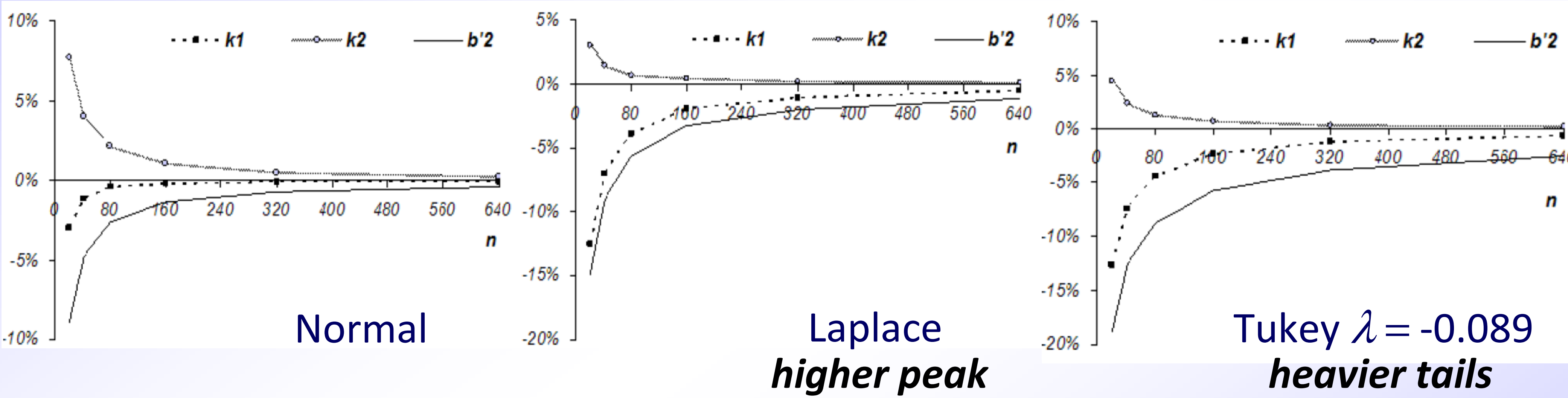


Kurtosis measures

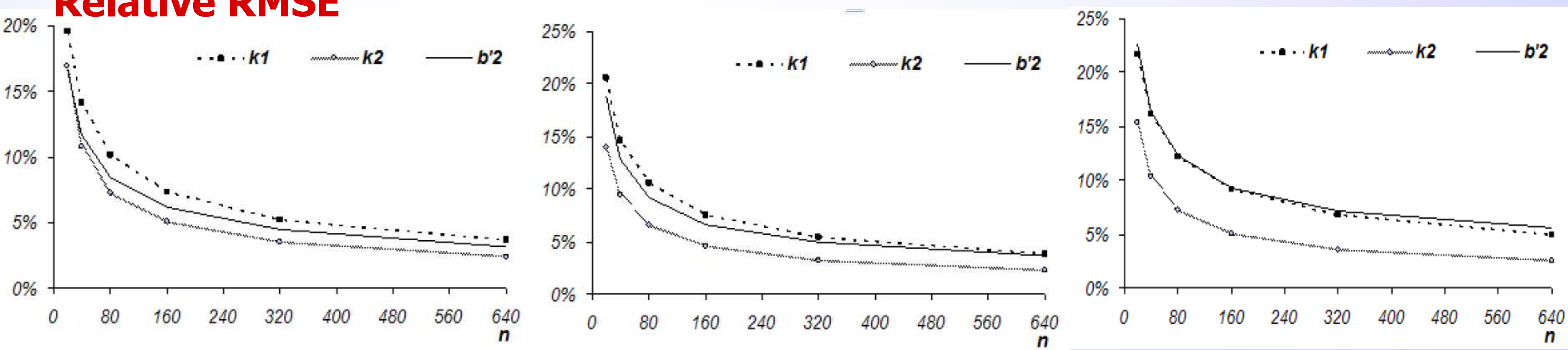
Monte Carlo experiment
(N = 20,000; n = 20 to 640)

Distribution	K_1	K_2	β_2'
Normal	0.3634	0.4142	0.6667
Laplace	0.5	0.5	0.8333
Tukey lambda	0.4614	0.4638	0.8418

Relative bias



Relative RMSE



Kurtosis measures
Small/medium sample behaviour

Bootstrap confidence intervals at 95% level (percentile method)

B = 2,000 bootstrap resamples; N = 10,000 replications

Empirical coverage

	$n = 40$			$n = 160$		
	Normal	**Laplace**	**Lambda**	**Normal**	**Laplace**	**Lambda**
K_1	0.9879	0.8420	0.8561	0.9700	0.8702	0.8710
K_2	0.9877	0.9770	0.9903	0.9760	0.9620	0.9597
β_2'	0.8825	0.5500	0.3573	0.8810	0.6520	0.4760

Average length

	$n = 40$			$n = 160$		
	Normal	**Laplace**	**Lambda**	**Normal**	**Laplace**	**Lambda**
K_1	0.2112	0.2302	0.2300	0.1036	0.1278	0.1331
K_2	0.1848	0.1901	0.1914	0.0826	0.0899	0.0914
β_2'	0.2591	0.2608	0.2756	0.1363	0.1416	0.1611

- K_2 is the measure which is likely to be estimated with the highest accuracy
- The sample performance of K_2 improves as the parent distribution becomes more peaked (Laplace)

Directions for research

- Rigorous asymptotic inference for the new measure K_2, possibly in view of practical (financial?) applications of right/left kurtosis
- Check compatibility between the inequality-based concept of kurtosis and some robust (e.g. quantile-based) measures recently proposed in literature (Groeneveld, 1998; Schmid & Trede, 2003; Brys, Hubert & Struyf, 2006; Kotz & Seier, 2007; …)

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