A multilayer Saint-Venant system with mass exchange

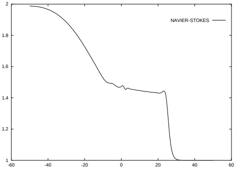
September 22, 2009

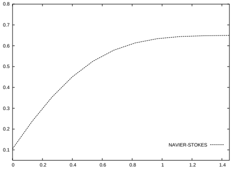
Existing models for free surface flows

- Incompressible Navier Stokes equations
  - Efficient model for a large class of flows
  - Complex implementation (moving boundaries)
  - CPU time (3d computations)
  - Robustness of the softwares for stiff flows (dam break,...)
- Shallow water equations
  - Easy implementation (fixed domain)
  - CPU time (2d computations)
  - Robustness of the software
  - Agreement with experiments... but not for all flows !
  - Constant velocity along the z-direction
  - Hydrostatic approximation

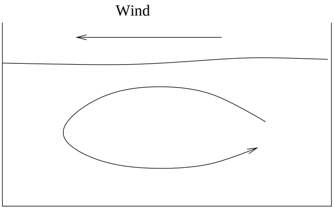
Two flows for which SW model fails

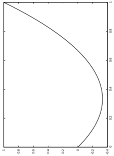
- Strong friction on the bottom $\rightsquigarrow$ Vertical dependency for u





- Wind stress in a lake $\rightsquigarrow$ Wind-induced circulation process





A multilayer SW model : The bi-fluid SW model

Introduced by Castro M., Macias J. and Pares C. (2001)

$$(BF)\begin{cases}\dfrac{\partial h_1}{\partial t}+\nabla\cdot(h_1u_1)=0,\\[2mm] \dfrac{\partial h_1u_1}{\partial t}+\nabla\cdot(h_1u_1\otimes u_1)+\nabla(\dfrac{g}{2}h_1^2) \;=\; -\dfrac{\rho_2}{\rho_1}gh_1\nabla h_2\\[6mm] \dfrac{\partial h_2}{\partial t}+\nabla\cdot(h_2u_2)=0,\\[2mm] \dfrac{\partial h_2u_2}{\partial t}+\nabla\cdot(h_2u_2\otimes u_2)+\nabla(\dfrac{g}{2}h_2^2) \;=\; -gh_2\nabla h_1\end{cases}$$

$\rightsquigarrow$ Pares et al. ('04,'07), Kurganov ('08), Abgrall-Karni ('08), Bouchut et al. ('08, '09)

From NS equations to multilayer SW models

- Derivation (following Gerbeau-Perthame...)
  - Formal asymptotic analysis of NS equ. under SW assumption
  - Vertical discretization of the fluid into N layers
  - Vertical integration of approximated NS equations by layer
- Consequences
  - Each layer has its own velocity
  - Coupling between the layers through
    - Pressure term (global coupling)
    - Mass exchange
      (local (?) coupling : interface condition for mass equation)
    - Viscous effect
      (local coupling : interface condition for momentum equation)

Hydrostatic incompressible Euler equ. with free surface

$$\left\{\begin{array}{l}\dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0,\\[2ex] \dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial uw}{\partial z} + \dfrac{\partial p}{\partial x} = 0,\\[2ex] \dfrac{\partial p}{\partial z} = -g,\end{array}\right.$$

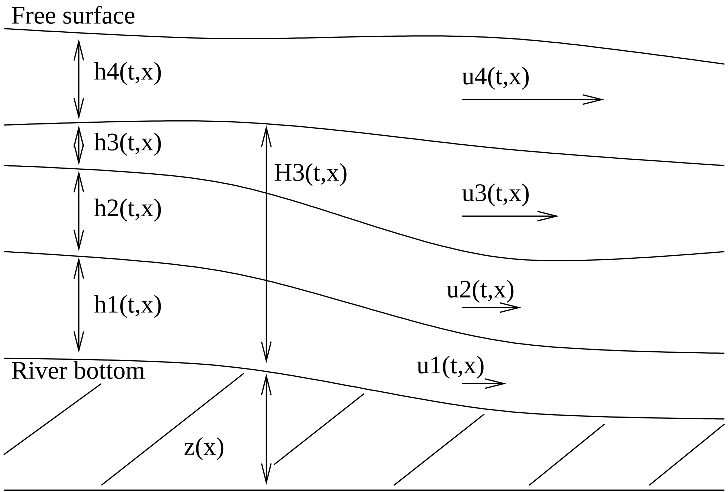
with

$$t > 0, \quad x \in \mathbb{R}, \quad z_b(x) \leq z \leq \eta(t,x),$$

and two kinematic boundary conditions (free surface and bottom)

$$\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} - w_s = 0$$

Multilayer approach



Vertical integration on a layer

$$\frac{\partial}{\partial t} h_\alpha + \frac{\partial}{\partial x} h_\alpha \bar{u}_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2},$$

$$\frac{\partial}{\partial t} h_\alpha \bar{u}_\alpha + \frac{\partial}{\partial x} h_\alpha \bar{u^2_\alpha} + g h_\alpha \frac{\partial}{\partial x} H$$
$$= -g h_\alpha \frac{\partial}{\partial x} z_b + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}$$

with the mass exchange term

$$G_{\alpha+1/2} = \frac{\partial}{\partial t} z_{\alpha+1/2} + u_{\alpha+1/2} \frac{\partial}{\partial x} z_{\alpha+1/2} - w_{\alpha+1/2}$$

Vertical integration on a layer

$$\frac{\partial}{\partial t}h_\alpha + \frac{\partial}{\partial x}h_\alpha \bar{u}_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2},$$

$$\frac{\partial}{\partial t}h_\alpha \bar{u}_\alpha + \frac{\partial}{\partial x}h_\alpha \bar{u^2_\alpha} + gh_\alpha \frac{\partial}{\partial x}H$$

$$= -gh_\alpha \frac{\partial}{\partial x}z_b + u_{\alpha+1/2}G_{\alpha+1/2} - u_{\alpha-1/2}G_{\alpha-1/2}$$

with the mass exchange term

$$G_{\alpha+1/2} = \sum_1^\alpha \frac{\partial}{\partial t}h_\beta + \frac{\partial}{\partial x}h_\beta \bar{u}_\beta$$

Vertical integration on a layer

$$\frac{\partial}{\partial t}h_\alpha + \frac{\partial}{\partial x}h_\alpha \bar{u}_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2},$$

$$\frac{\partial}{\partial t}h_\alpha \bar{u}_\alpha + \frac{\partial}{\partial x}h_\alpha \bar{u^2_\alpha} + gh_\alpha \frac{\partial}{\partial x}H$$

$$= -gh_\alpha \frac{\partial}{\partial x}z_b + u_{\alpha+1/2}G_{\alpha+1/2} - u_{\alpha-1/2}G_{\alpha-1/2}$$

- Natural conditions for lowest and uppest layers

$$G_{1/2} = G_{N+1/2} = 0$$

- Which choice for the others ?

Multilayer system without mass exchange

$$G_{\alpha+1/2} = 0 \qquad \forall\ \alpha = 0, \dots, N$$

- Energy inequality (global stability)
- Loss of hyperbolicity (interface stability)
- Non conservativity
- Kinetic interpretation (kinetic scheme)
- Numerical agreement with NS solutions for several test cases
- No physical meaning for the condition on $G_{\alpha+1/2}$
- Not adapted for wind-induced circulation processes

Wind-induced circulation process

Mass equation for each layer

$$\frac{\partial}{\partial t} h_\alpha + \frac{\partial}{\partial x} h_\alpha \bar{u}_\alpha = 0$$

Boundary conditions

$$\bar{u}_\alpha = 0 \qquad \text{(Walls)}$$

Stationnary solution

$$\bar{u}_\alpha = 0$$

$\rightsquigarrow$ Unrealistic !

Multilayer SV model with mass exchange

- No *a priori* prescription for $G_{\alpha+1/2}$
- Mass equation for one layer is not meaningfull

$$\frac{\partial}{\partial t}h_\alpha + \frac{\partial}{\partial x}h_\alpha \bar{u}_\alpha = G_{\alpha+1/2} - G_{\alpha-1/2}$$

with $G_{\alpha+1/2} = \partial_t z_{\alpha+1/2} + u_{\alpha+1/2}\partial_x z_{\alpha+1/2} - w_{\alpha+1/2}$

- We consider a single mass equation for the whole flow

$$\frac{\partial}{\partial t}H + \frac{\partial}{\partial x}\sum_1^N h_\alpha \bar{u}_\alpha = 0, \qquad h_\alpha(t,x) = \lambda_\alpha H(t,x), \qquad \sum_1^N \lambda_\alpha = 1$$

Multilayer SV model with mass exchange

- Momentum equation is now well defined

$$\lambda_\alpha \frac{\partial}{\partial t} H\bar{u}_\alpha + \lambda_\alpha \frac{\partial}{\partial x} H\bar{u_\alpha^2} + g\lambda_\alpha \frac{\partial}{\partial x} H^2 = -g\lambda_\alpha H \frac{\partial}{\partial x} z_b + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2}$$

with

$$G_{\alpha+1/2} = \sum_1^\alpha \left(\frac{\partial}{\partial x} h_\beta \bar{u}_\beta - \lambda_\beta \sum_1^N \frac{\partial}{\partial x} h_\gamma \bar{u}_\gamma \right)$$

Open question : Choice of $u_{\alpha+1/2}$

$$u_{\alpha+1/2} = \begin{cases} u_\alpha & \text{if} \quad G_{\alpha+1/2} \leq 0 \\ u_{\alpha+1} & \text{if} \quad G_{\alpha+1/2} > 0 \end{cases}$$

Properties of the new multilayer system

- System of $N+1$ equations
  (including non conservative products)
- Entropy inequality for the total energy

$$E^{mc}(x,t) = \sum_{\alpha=1}^{N} \frac{h_\alpha u_\alpha^2}{2} + \frac{g h_\alpha(\eta + z_b)}{2}$$

  $\rightsquigarrow$ ensures a global stability for the flow
- Hyperbolicity
  - Study includes the momentum exchange terms $u_{\alpha+1/2} G_{\alpha+1/2}$
  - True for the two layers system (whatever the choice of $u_{3/2}$)
  - Sometimes wrong if there is more than four layers

Kinetic interpretation

- Classical Saint-Venant system : (h, hu) is a solution if

$$M(t,x,\xi) = \frac{h(t,x)}{c}\ \chi(\frac{\xi - \mathbf{u}(t,x)}{c}), \qquad c = \sqrt{gh/2}$$

is solution of

$$\frac{\partial M_\alpha}{\partial t} + \xi \frac{\partial M_\alpha}{\partial x} = Q(t,x,\xi)$$

Kinetic interpretation

- Multilayer Saint-Venant system : (H, u^{mc}) is a solution if

$$M_\alpha(x,t,\xi) = l_\alpha \frac{H(x,t)}{c}\chi\left(\frac{\xi - u_\alpha(x,t)}{c}\right), \qquad c = \sqrt{gh/2}$$

are solutions of

$$\frac{\partial M_\alpha}{\partial t} + \xi\frac{\partial M_\alpha}{\partial x} - \\ -N_{\alpha+1/2}(x,t,\xi) + N_{\alpha-1/2}(x,t,\xi) = Q_\alpha(x,t,\xi)$$

where

$$N_{\alpha+1/2}(x,t,\xi) = G_{\alpha+1/2}(x,t)\,\delta\left(\xi - u_{\alpha+1/2}(x,t)\right),$$

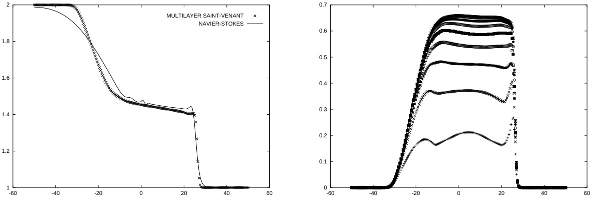
Numerical scheme

- Hyperbolic part : Kinetic scheme
  - Integration of upwind scheme for linear kinetic equations
  - No need for computation of the eigenvalues of the system
  - Stability under a modified CFL condition

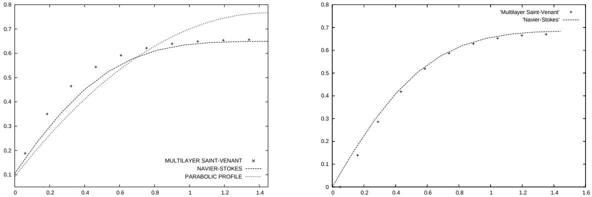
$$\Delta t^n \leq \frac{l_\alpha H_i^n \Delta x_i}{l_\alpha H_i^n \left(|u_{\alpha,i}^n| + w_M c_i^n\right) + \Delta x_i \left(\left[G_{\alpha+1/2,i}^{n+1/2}\right]_- + \left[G_{\alpha-1/2,i}^{n+1/2}\right]_+\right)}$$

- Topographic terms : Hydrostatic reconstruction
- Viscous terms : Implicit computation of momentums
  $\rightsquigarrow$ Solution of a well-posed tridiagonal $N \times N$ system

1d Dam Break



Longitudinal profiles



Vertical profiles for horizontal velocity

Wind-induced circulation : hydrostatic NS solutions

Stationnary solution far from the boundaries ($u_\alpha = cst$)

$$\partial_x p = \nu \partial_{zz} u$$

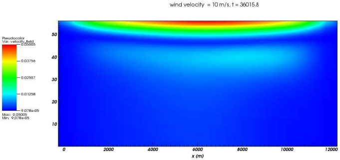
Hydrostatic assumption and boundary conditions

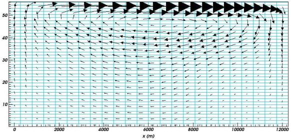
$$\nu \partial_z u = \tau \quad \text{(Surface)}, \qquad \nu \partial_z u = \kappa u \quad \text{(Bottom)}$$

Solution for large κ

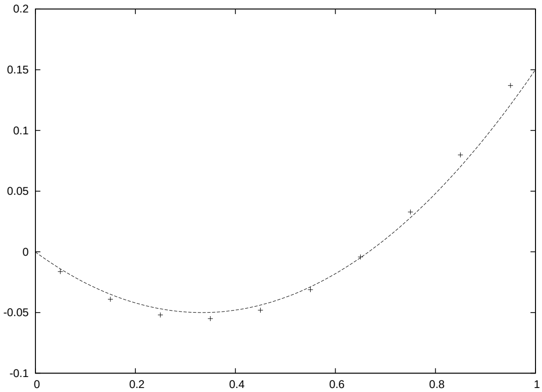
$$\partial_x h = \frac{3\tau}{2gh}, \qquad u(\xi) = 2\xi(\frac{3\xi}{2} - 1), \quad \xi = \frac{z}{h} \in [0, 1]$$

Wind-induced circulation : Multilayer solution





Wind-induced circulation : Multilayer solution



Multilayer SW vs. NS equations : CPU Time

- CPU time
  - 2D Navier-Stokes equations :
    MISTRAL software by J.F. Gerbeau and T.Lelievre
    (ALE, implicit solver...)
    $\longrightarrow$ CPU Time = 23.77 seconds (35 time steps)
  - 1D Saint-Venant system
    $\longrightarrow$ CPU Time = 0.07 second (73 time steps)
  - Multilayer Saint-Venant system
    $\longrightarrow$ CPU Time = 1.19 second (76 time steps)
- Friction $\kappa = 0.1$
- Viscosity $\mu = 0.01$
- Ten layers (Navier-Stokes and multilayer SV system)