From Saint-Venant to Navier-Stokes with kinetic schemes

Coupling hydrodynamics and biology

J. Sainte-Marie[1,2]

[1]INRIA Rocquencourt, [2]Saint-Venant Laboratory (EDF R&D - MEEDDAT)

Involved persons

- M.O. Bristeau, M. Pelanti, B. Perthame
- N. Goutal, L. Martin, M.J. Salençon (EDF R&D)
- E. Audusse (Galilée inst., univ. Paris XIII & BANG)

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Shallow water systems

$$(NS)\left\{\begin{array}{l}\operatorname{div}\underline{\mathbf{u}} = 0,\\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)\underline{\mathbf{u}} + \nabla p = \mathbf{G} + \operatorname{div}\underline{\underline{\Sigma}},\end{array}\right.$$

- small parameter $\varepsilon = \frac{H_0}{L_0}$, expansion in $\mathcal{O}(\varepsilon^2)$

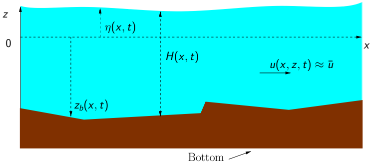
$$(SV)\left\{\begin{array}{l}\frac{\partial H}{\partial t} + \frac{\partial(H\bar{u})}{\partial x} = 0,\\ \frac{\partial(H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} - \kappa(\bar{u})\end{array}\right.$$

- Low complexity, many applications
- "Reduced" validity domain

(NS) $\quad$ ($\approx$ NS)

$\bigcup$

(SV)



Outline

1. Three research axis
   - Non-hydrostatic terms
   - Hydrodynamics/biology coupling
   - Model reduction - inverse problems
2. Methodology
   - Models (no more hyperbolic systems)
   - Analysis of proposed models
3. Applications
   - Not so shallow water systems
   - Hydrodynamics + ecology

Axis 1 : Non-hydrostatic terms

(Navier-Stokes)
$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\Sigma_{xx} + \frac{\partial}{\partial z}\Sigma_{xz}, \\ \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g + \frac{\partial}{\partial x}\Sigma_{zx} + \frac{\partial}{\partial z}\Sigma_{zz}, \end{cases}$$

When w is small or $\frac{H_0}{L_0} \ll 1$

- Hydrostatic

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g \quad \Rightarrow \quad \frac{\partial p}{\partial z} = -g$$

- Not always valid
  - Large vertical accelerations (bottom variations)
  - Varying density (Archimede' principle)

Axis 1 : Gravity waves

Non-hydrostatic

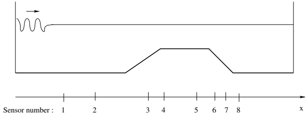
$$u(x,z,t) = \bar{u}(x,t) + \mathcal{O}(\varepsilon), \quad \frac{\partial w}{\partial t} + \frac{\partial p}{\partial z} = -g$$

$$\begin{cases} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) + \mathcal{D}(H)\frac{\partial^3 \bar{u}}{\partial^2 x \partial t} = -H\frac{\partial z_b}{\partial x} - \kappa(\bar{u}) \end{cases}$$

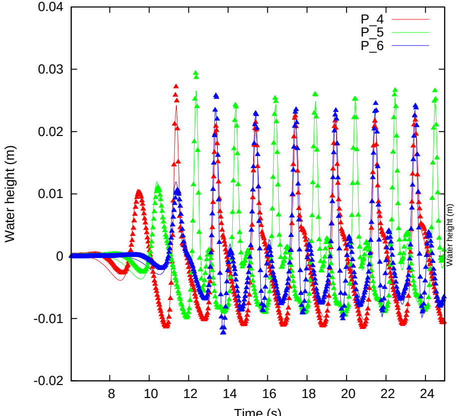
- Peregrine 1957, BBM 1972, Bristeau-JSM 2008
- Kinetic type interpretation & scheme
- Implementation in Mascaret (©EDF-MEEDDAT)
- Also done with variable section

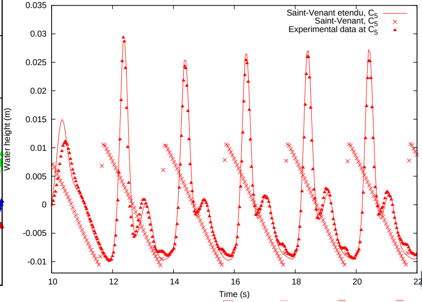
Axis 1 : Experimental validation

- Dinguemans experiments



- Animation
- Comparisons (sensors 4, 5 and 6)

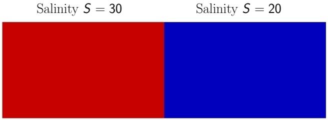




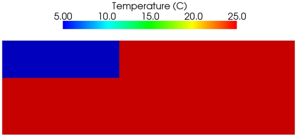
Axis 1 : Archimede' principle

- Limitations for hydrostatic systems

- Re-ordering



- Behaviour (hot/cold water)



$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0 \\ \frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho u w}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial \rho w}{\partial t} + \frac{\partial \rho u w}{\partial x} + \frac{\partial \rho w^2}{\partial z} + \frac{\partial p}{\partial z} = -\rho g \end{cases}$$

- Static equilibrium stable if $\frac{\partial \rho}{\partial z} \leq 0$
- Rayleigh-Besnard type instabilities

Axis 2 : Water + tracer (pollutant) + ...

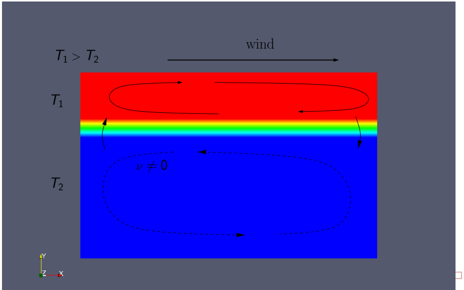
- Towards real life applications : not only water
- Varying density w.r.t. temperature or salinity
- O_2 cycle modelling and simulation
  - water/atmosphere energy exchanges
  - large geometrical domains, large time scale
  - **multiscale modelling (water $\leftrightarrow$ biology), reactions terms**
- Cyanobacteries population dynamics

APPLICATION Ecology / Water quality management

Axis 2 : Upwelling phenomena (I)

3D Euler/Navier-Stokes (hydrostatic) with variable density

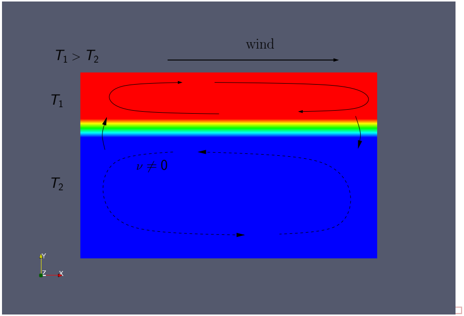
$$
\begin{cases}
\dot{\rho} + \operatorname{div}(\rho \underline{\mathbf{u}}) = 0, \\
\dot{\overline{\rho \underline{\mathbf{u}}}} + (\underline{\mathbf{u}}.\nabla)(\rho \underline{\mathbf{u}}) + \nabla p = \rho \mathbf{G}, \\
\dot{\overline{\rho T_j}} + \operatorname{div}(\rho T_j \underline{\mathbf{u}}) = \frac{\partial}{\partial z}\left(\mu_T \frac{\partial T_j}{\partial z}\right), \quad j = 1, \ldots, p \\
\rho = \rho(\{T_j\}_{j=1}^{p}) = \rho(H, \{T_j\}_{j=1}^{p})
\end{cases}
$$



Axis 2 : Upwelling phenomena (I)

3D hydrostatic Euler/Navier-Stokes (constant density)

$$\begin{cases} \operatorname{div}(\underline{\mathbf{u}}) = 0, \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)(\underline{\mathbf{u}}) + \nabla p = \mathbf{G}, \\ \dot{\overline{T_j}} + \operatorname{div}(T_j \underline{\mathbf{u}}) = 0, \quad j = 1, \ldots, p \end{cases}$$



Axis 3 : model reduction (in time)

- **Large scale problems (space & time)**
- Explicit schemes
  - simple but time step δt small
  - stability under a CFL condition
  - large number of time steps but $t \longrightarrow t + \delta t$ not costly
- Implicit schemes
  - more complex but δt larger
  - stability
  - few number of time steps but $t \longrightarrow t + \delta t$ costly
- Implicit version of kinetic schemes
  - to be done ?
  - other strategies (rigid/deformable lid)

Axis 3 : model reduction (in space)

1. Large model discretized in space (dimension n)

$$M\dot{X} = K(X) + B$$

2. Reduced model
   - ϕ_l projection of rank $l \ll n$, $X = P\tilde{I}_l X_l = \phi_l X_l$

$$({}^t\phi_l M \phi_l)\dot{X}_l = {}^t\phi_l K(\phi_l X_l) + {}^t\phi_l B$$

   - Widely used for dynamical systems, data analysis
3. Problem (X, X_l)
   - Error $\varepsilon^l = X - \phi_l X \ll 1$
   - Known results for finite elements
   - For finite volumes ?

Axis 3 : Filtering/estimation for kinetic equations

- Filtering (DAT) for hyperbolic conservation laws ?

$$\dot{X} = \frac{\partial F(X)}{\partial x} + K(Y_{mes} - HX)$$

- Integration of source terms
- Use kinetic interpretation for filtering
- Model, filter and numerical scheme

$$\frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - K \frac{\partial}{\partial \xi} (\xi(M - M_{mes})) = Q$$

where $M_{mes} = \frac{H_{mes}}{c_{mes}} \chi \left(\frac{\xi - u}{c_{mes}} \right)$ or $M_{mes} = \frac{H}{c} \chi \left(\frac{\xi - u_{mes}}{c} \right)$ or $M_{mes} = \frac{H_{mes}}{c_{mes}} \chi \left(\frac{\xi - u_{mes}}{c_{mes}} \right)$

Towards Navier-Stokes (Euler)

Averaged Euler

- Starting point

(Euler) $$\left\{\begin{array}{l}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\\ \frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uw}{\partial z}+\frac{\partial p}{\partial x}=0\\ \frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}+\frac{\partial p}{\partial z}=-g\end{array}\right.$$

- Vertical integration $i=0,1$

$$\left\{\begin{array}{l}\int_{z_b}^{\eta} z^i\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}\right)dz=0\\ \int_{z_b}^{\eta} z^i\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uw}{\partial z}+\frac{\partial p}{\partial x}\right)dz=0\\ \int_{z_b}^{\eta} z^i\left(\frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}+\frac{\partial p}{\partial z}+g\right)dz=0\end{array}\right.$$

Towards Navier-Stokes (Euler)

Averaged Euler

- Starting point

$$(\text{Euler})\quad \begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \\ \dfrac{\partial u}{\partial t} + \dfrac{\partial u^2}{\partial x} + \dfrac{\partial uw}{\partial z} + \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial w}{\partial t} + \dfrac{\partial uw}{\partial x} + \dfrac{\partial w^2}{\partial z} + \dfrac{\partial p}{\partial z} = -g \end{cases}$$

- Transport-reaction type system

$$\begin{cases} \dfrac{\partial H}{\partial t} + \dfrac{\partial}{\partial x}(H\overline{u}) = 0 \\ \dfrac{\partial}{\partial t}(H\overline{u}) + \dfrac{\partial}{\partial x}\left(H\overline{u}^2 + H\overline{p}\right) = -\, p|_b \dfrac{\partial z_b}{\partial x} \\ \dfrac{\partial}{\partial t}(H\overline{w}) + \dfrac{\partial}{\partial x}(H\overline{wu}) = p|_b - gH \\ \dfrac{\partial}{\partial t}\dfrac{1}{2}(\eta^2 - z_b^2) + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}(\eta^2 - z_b^2)\overline{u}\right) - H\overline{w} = 0 \\ \dfrac{\partial}{\partial t}\left(\dfrac{1}{2}(\eta^2 - z_b^2)\overline{u}\right) + \dfrac{\partial}{\partial x}\left(\dfrac{1}{2}(\eta^2 - z_b^2)\overline{u}^2 + \int_{z_b}^{\eta} zp\ dz\right) - H\overline{wu} = -\dfrac{1}{2}\dfrac{\partial z_b^2}{\partial x}\, p|_b \\ \dfrac{\partial}{\partial t}\int_{z_b}^{\eta} zw\ dz + \dfrac{\partial}{\partial x}\overline{u}\int_{z_b}^{\eta} zw\ dz - \int_{z_b}^{\eta} w^2\ dz - H\overline{p} = z_b\, p|_b - \dfrac{g}{2}(\eta^2 - z_b^2) \end{cases}$$

with $H\bar{w} = \int_{z_b}^{\eta} w\ dz$

Towards Navier-Stokes (Euler)

Averaged Euler

- Starting point

(Euler) $$\left\{\begin{array}{l}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\\ \frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uw}{\partial z}+\frac{\partial p}{\partial x}=0\\ \frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}+\frac{\partial p}{\partial z}=-g\end{array}\right.$$

- Kinetic interpretation

$$(B_M) \quad \frac{\partial M}{\partial t}+\xi\frac{\partial M}{\partial x}-g\frac{\partial M}{\partial \gamma}=Q_1, \qquad (1)$$

$$(B_{R,M}) \quad \frac{\partial R}{\partial t}+\xi\frac{\partial R}{\partial x}-\gamma M=Q_2, \qquad (2)$$

with the Gibbs equilibria

$$M(x,t,\xi,\gamma)=\frac{H}{c_1c_2}\chi\left(\frac{\xi-\bar{u}}{c_1}\right)\psi\left(\frac{\gamma-\overline{w}}{c_2}\right)$$
$$R(x,t,\xi,\gamma)=\frac{\eta^2-z_b^2}{2}\chi\left(\frac{\xi-\bar{u}}{c_3}\right)\delta\left(\gamma-\check{w}\right)$$

- Proof: $\int_{\mathbb{R}^2}(1,\xi,\gamma,|\xi|^2/2)\mathcal{B}_M d\xi d\gamma$, $\int_{\mathbb{R}^2}(1,\xi,\gamma)\mathcal{B}_{R,M} d\xi d\gamma$
- Also multilayer version (full Euler 3D), JSM 2009

Analysis of proposed models

Hydrostatic case

- The multilayer system & Euler hydrostatic are not always hyperbolic
- Adaptation of the results for Euler gas dynamics (Lions-Perthame-Souganidis)

Non hydrostatic case

- numerical schemes (for reaction terms)
- source terms

Kinetic schemes

- in non-hyperbolic cases
- at the discrete level

Typical applications (Etang de Berre)

- Large scale : time (1 year) & space (155 km^2)
- Water with variable density (temperature & salinity)
- Hydrodynamics and biology (O_2 cycle)
- Collaborations, partnerships
  - EDF-LNHE
  - UMLV-ENPC (Bouchut, Ern), Univ. Washington (R. LeVeque), Univ. Malaga, Univ. Catane

General outlook

- From applied mathematics to applications
  - hyperbolic type systems
  - math. anal., num. anal,
  - industrial problems
- From Saint-Venant to Navier-Stokes
  - multilayer system, variable density
  - kinetic interpretation, associated numerical scheme
  - non-hydrostatic terms
- Applied mathematics
  - analysis of the obtained models
  - inverse problems using kinetic type interpretations
- Exprerimental validations/collaborations needed
- Biology/hydrodynamics coupling
- Related problems