Method for solving the inverse problems on sea surface heat flux and on vertical turbulent heat exchange coefficient

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Introduction

INM RAS:

(Marchuk G.I., Agoshkov V., Zalesny V., Shutyaev V., Diansky N., Parmuzin E., Botvinovsky E., Lebedev S., Ipatova V., Gusev A., Kochurov A.)

- **2002-2003** – Development of the general methodology
- **2004-2005** – First works on assimilation of sea surface temperature.
- **2005-2006** – Study of problems for semidiscrete models.
- **2007** – Existence theorems for "continuous equations".
- **2007** – Methods and technology for solving inverse hydrothermodynamics problem in ocean with variational assimilation of sea surface temperature.

- **2007** – Variational data assimilation system INM-T1.
- **2008** – Study and numerical solution of the variational assimilation problem using on-line SST data.
- **2008** – Numerical methods for solving the variational assimilation problem using sea surface salinity.
- **2008** – Variational data assimilation system INM-T2.
- **2008** – Study and solution of the inverse problem on vertical turbulent heat exchange coefficient by assimilation of ”vertical profiles” of temperature.

1. Mathematical model

$$\frac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g \cdot grad\xi + A_u \vec{u} + (A_k)^2 \vec{u} = \vec{f} - \frac{1}{\rho_0} gradP_a - \frac{g}{\rho_0} grad \int_0^z \rho_1(T,S)dz',$$

$$\frac{\partial \xi}{\partial t} - m\frac{\partial}{\partial x}(\int_0^H \Theta(z)udz) - m\frac{\partial}{\partial y}(\int_0^H \Theta(z)\frac{n}{m}vdz) = f_3,$$

$$\frac{dT}{dt} + A_T T = f_T, \ \frac{dS}{dt} + A_S S = f_S,$$

where

$$\bar{f} = g \cdot gradG, \ \Theta(z) \equiv \frac{r(z)}{R}, \quad r = R - z, \quad 0 < z < H.$$

(V.I. Agoshkov, A.V.Gusev, N.A. Diansky, 2007)

Boundary conditions on the surface

$$
\begin{cases}
\left(\int\limits_0^H \Theta \vec{u} dz\right) \vec{n} + \beta_0 m_{op} \sqrt{gH}\, \xi = m_{op} \sqrt{gH}\, d_s \text{ on } \partial\Omega, \\
U_n^{(-)} u - \nu \dfrac{\partial u}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k u = \tau_x^{(a)} \Big/ \rho_0, \; U_n^{(-)} v - \nu \dfrac{\partial v}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k v = \tau_y^{(a)} \Big/ \rho_0, \\
A_k u = 0, \quad A_k v = 0, \\
U_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + U_n^{(-)} d_T, \\
U_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + U_n^{(-)} d_S.
\end{cases}
$$

With the function $\phi = (u, v, \xi, T, S)$ known, we calculate

$$w(x,y,z,t) = \frac{1}{r}(m\frac{\partial}{\partial x}(\int\limits_z^H rudz') + m\frac{\partial}{\partial y}(\frac{n}{m}\int\limits_z^H rvdz')), (x,y,z,t) \in D \times (0,\,\bar{t}),$$

$$P(x,y,z,t) = P_a(x,y,t) + \rho_0 g(z-\xi) + \int\limits_0^z g\rho_1(T,S)dz'.$$

Note, that for $U_n \equiv \underline{U} \cdot \underline{N}$ (here $U = (u,v,w)$) we always have

$$U_n = 0 \;\text{ on }\; \Gamma_{c,w} \cup \Gamma_H .$$

Problem **I**: Approximation of the model by splitting method

Step 1. We consider the system:

$$
\begin{cases}
T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_{j-1}, t_j), \\
\\
T = T_{j-1} \text{ for } t = t_{j-1} \text{ in } D, \\
\\
\bar{U}_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
\dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)} T + \dfrac{\partial T}{\partial N_T} = \bar{U}_n^{(-)} d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\
\\
T_j \equiv T \text{ on } D \times (t_{j-1}, t_j).
\end{cases}
$$

Step 2.

$$\begin{cases} S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_{j-1}, t_j), \\ S = S_{j-1} \text{ at } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ \dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)} S + \dfrac{\partial S}{\partial N_S} = \bar{U}_n^{(-)} d_S + Q_S \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\ S_j \equiv S \text{ on } D \times (t_{j-1}, t_j). \end{cases}$$

Step 3.

$$
\begin{cases}
\underline{u}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}^{(1)} - g \cdot \mathbf{grad}\xi = g \cdot \mathbf{grad}G - \dfrac{1}{\rho_0}\mathbf{grad}\left(P_a + g \displaystyle\int_0^z \rho_1(\bar{T}, \bar{S}) dz' \right) \\
\text{ in } D \times (t_{j-1}, t_j), \\
\xi_t - \mathbf{div}\left(\displaystyle\int_0^H \Theta \underline{u}^{(1)} dz \right) = f_3 \text{ in } \Omega \times (t_{j-1}, t_j), \\
\underline{u}^{(1)} = \underline{u}_{j-1},\ \xi = \xi_{j-1} \text{ at } t = t_{j-1}, \\
\left(\displaystyle\int_0^H \Theta \underline{u}^{(1)} dz \right) \cdot n + \beta_0 m_{op} \sqrt{gH} \xi = m_{op} \sqrt{gH} d_s \text{ on } \partial\Omega \times (t_{j-1}, t_j), \\
\underline{u}_j^{(1)} \equiv \underline{u}^{(1)}(t_j) \text{ in } D
\end{cases}
$$

$$
\begin{cases}
\underline{u}_t^{(2)} + \begin{bmatrix} 0 & -f_1(\bar{u}) \\ f_1(\bar{u}) & 0 \end{bmatrix} \underline{u}^{(2)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\
\underline{u}^{(2)} = \underline{u}_j^{(1)} \text{ for } t = t_{j-1} \text{ in } D, \\
\underline{u}_j^{(2)} \equiv \underline{u}^{(2)}(t_j) \text{ in } D,
\end{cases}
$$

Step 3. (continued)

$$\begin{cases} \underline{u}_t^{(3)} + (\bar{U}, \mathbf{Grad})\underline{u}^{(3)} - \mathbf{Div}(\hat{a}_u \cdot \mathbf{Grad}\ )\underline{u}^{(3)} + (A_k)^2 \underline{u}^{(3)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\ \underline{u}^{(3)} = \underline{u}^{(2)} \text{ at } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)} \underline{u}^{(3)} - \nu_u \dfrac{\partial \underline{u}^{(3)}}{\partial z} - k_{33} \dfrac{\partial}{\partial z}(A_k \underline{u}^{(3)}) = \dfrac{\underline{\tau}^{(a)}}{\rho_0}, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ U_n^{(3)} = 0, \dfrac{\partial U^{(3)}}{\partial N_u} \cdot \bar{\underline{\tau}}_w + \left(\dfrac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \underline{\tau}_w = 0, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)} (\tilde{U}^{(3)} \cdot \underline{N}) + \dfrac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{N} + \left(\dfrac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \bar{N} = \bar{U}_n^{(-)} d, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,op} \times (t_{j-1}, \\ \bar{U}_n^{(-)} (\tilde{U}^{(3)} \cdot \bar{\tau}_w) + \dfrac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{\tau}_w + \left(\dfrac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \underline{\tau}_w = 0, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j) \\ \dfrac{\partial \underline{u}^{(3)}}{\partial N_u} = \dfrac{\tau^{(b)}}{\rho_0} \text{ on } \Gamma_H \times (t_{j-1}, t_j), \end{cases}$$

where

$$\underline{u}^{(3)} = (u^{(3)}, v^{(3)}),\ \tau^{(a)} = (\tau_x^{(a)}, \tau_y^{(a)}),$$
$$U^{(3)} = (u^{(3)}, w^{(3)}(u^{(3)}, v^{(3)})),\ \tilde{U}^{(3)} = (u^{(3)}, 0),$$
$$\tau^{(b)} = (\tau_x^{(b)}, \tau_y^{(b)}).$$

Splitting methods (G.I. Marchuk) are used to approximate subproblems on Steps 1-3

Step 1:

$$(T_1)_t + L_1 T_1 = \mathcal{F}_1, \quad t \in (t_{j-1}, t_j),$$

$$T_1 = T_{j-1} \quad \text{at} \quad t = t_{j-1}$$

$$(T_2)_t + L_2 T_2 = \mathcal{F}_2 + BQ_T, \quad t \in (t_{j-1}, t_j),$$

$$T_2(t_{j-1}) = T_1(t_j).$$

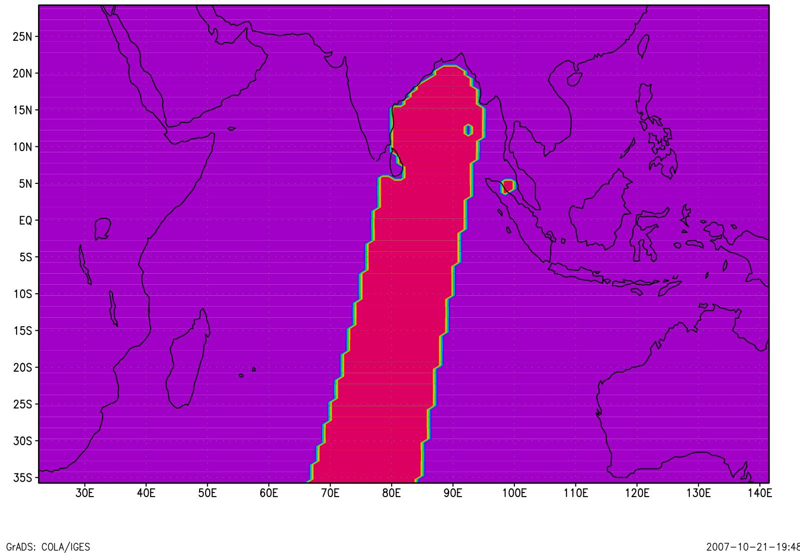
$$T_2(t_j) \equiv T_j \cong T \quad \text{at} \;\; t = t_j.$$

3. Inverse problem and assimilation of on-line SST data

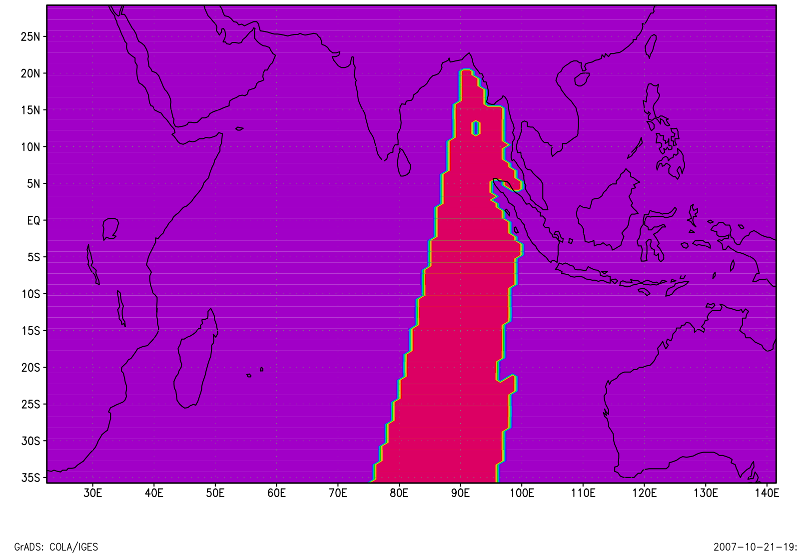
Let us assume that the unique function which is obtained by observation data processing is the function T_{obs} on $\Omega_0^{(j)}$ at $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. Let by physical meaning the function $T_{obs} = T_{obs}^{(j)}$ is an approximation to STT data. We permit that the function $T_{obs}^{(j)}$ is known only on the part of $\Omega \times (0, \bar{t})$, i.e. on $\Omega_0^{(j)}$ at $t \in (t_{j-1}, t_j)$ and we define the support of this function as $m_0^{(j)}$. Beyond of this area we suppose function $T_{obs}^{(j)}$ is trivial.

Let the function of sea (ocean) surface heat flux Q is an "additional unknown function" on $\{\Omega_0^{(j)}\}$ (assuming that Q is known on $\{\Omega \backslash \Omega_0^{(j)}\}$) and we state the following inverse problem: *find the solution* ϕ *of the Problem* I *and the function* Q *such that* $m_0^{(j)}(T - T_{obs}^{(j)}) = 0$.

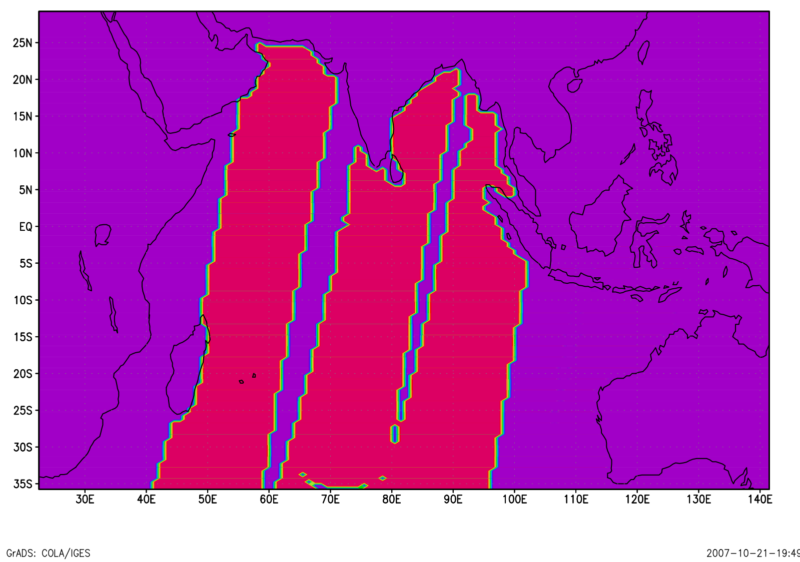
Observation data mask by hours



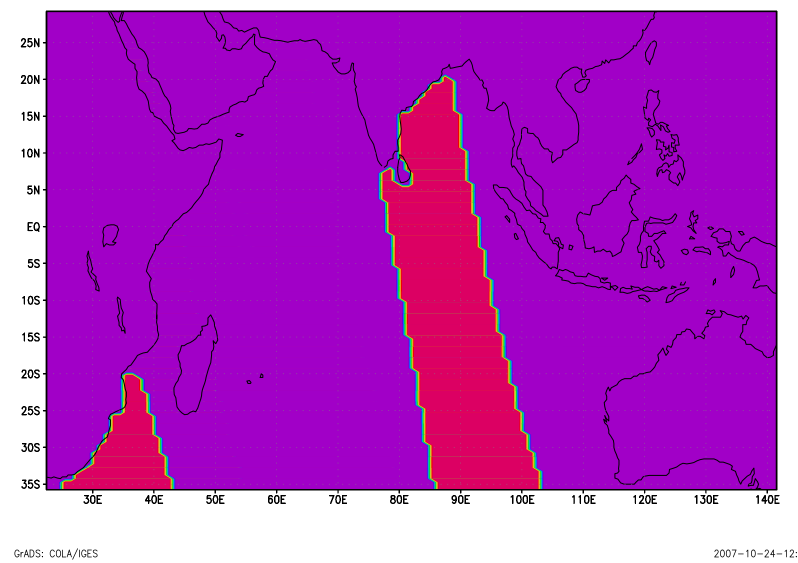
(a)



(b)



(c)



(d)

4. SST data assimilation problem

We consider the cost-function of the form:

$$J_\alpha \equiv J_\alpha(Q,\phi) = \frac{1}{2}\int_0^{\bar{t}} \int_{\Omega_0(t)} \alpha |Q - Q^{(0)}|^2 d\Omega dt + J_0(\phi) = \sum_{j=1}^{J} J_{\alpha,j} \qquad (*)$$

$$J_0(\phi) = \frac{1}{2}\int_0^{\bar{t}} \int_{\Omega_0(t)} \alpha |T - T_{obs}|^2 d\Omega dt$$

$$J_{\alpha,j} = \frac{1}{2}\int_{t_{j-1}}^{t_j} \int_{\Omega_0^{(j)}} \alpha |Q - Q^{(0)}|^2 d\Omega dt + \frac{1}{2}\int_{t_{j-1}}^{t_j} \int_{\Omega_0^{(j)}} m_0^{(j)} |T - T_{obs}^{(j)}|^2 d\Omega dt$$

Here $\alpha \equiv \alpha(\lambda,\theta,t)$ is a regularization function(is it possible, that $\alpha(\lambda,\theta,t)$ =const≥ 0) and it may be a dimensional quantity; $Q^{(0)} \equiv Q^{(0)}(\lambda,\theta,t)$ is a given function.
We can formulate the data assimilation problem : *find the solution* $\phi \equiv (u, v, \xi, T, S)$ *of the Problem* I *and the function* Q *such that the functional* J_α *is minimal on the set of the solutions.*

Theorem. Let J_α be defined by formula $(*)$ for T_{obs}, $Q^{(0)} \in L_2(\Omega_0^{(j)} \times (t_{j-1}, t_j))$, $j = 1, 2, \ldots, J$. Then the variational assimilation problem of finding a solution $\phi \equiv (u, v, \xi, T, S)$ to Problem I and a function $Q \in L_2(\Omega_0^{(j)} \times (t_{j-1}, t_j))$, such that they minimize the functional, is uniquely solvable for any $\alpha > 0$. For $\alpha = 0$ this problem is uniquely and densely solvable and, as a sequence of solutions minimizing J_0, we can choose a sequence of regularized solutions to the variational assimilation problem for J_α as $\alpha \to +0$, moreover, $\inf J_0 = 0$.

Corollary. Under the conditions of the unique and dense solvability of the variational assimilation problem on (t_{j-1}, t_j) the solution to the original assimilation problem on $(0, \bar{t})$ is reduced to the sequential solution of the corresponding problems on intervals (t_{j-1}, t_j).

The optimality system obtained consist of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$ (Agoshkov V.I., 2006). The method can be described as follows:

STEP 1. We solve the system of equations, which arise from minimization of the functional J_α on the set of the solution of the equations. This system consists of equations for T_1, T_2, Q and system of adjoint equations:

$$\begin{cases} (T_2^*)_t + L_2^* T_2^* = B^* m_0^{(1)} (T - T_{obs}^{(1)}) \quad \text{in} \;\; D \times (t_0, t_1), \\ T_2^* = 0 \quad \text{for} \;\; t = t_1, \end{cases}$$

$$\begin{cases} (T_1^*)_t + L_1^* T_1^* = 0 \quad \text{in} \;\; D \times (t_0, t_1), \\ T_1^* = T_2^*(t_0) \quad \text{for} \;\; t = t_1 \end{cases}$$

$$\alpha(Q - Q^{(0)}) + T_2^* = 0 \quad \text{on} \;\; \Omega_0^{(1)} \times (t_0, t_1).$$

Functions T_2, $Q(t_1)$ are accepted as approximations to functions T, Q of the full solution for the Problem I at $t > t_1$, and $T_2(t_1) \cong T(t_1)$ is taken as an initial condition to solve the problem on the interval (t_1, t_2).

STEP 2. Solve problem for S:

$$S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\; S) = f_S \text{ in } D \times (t_0, t_1)$$

with corresponding boundary and initial conditions. After that the function S is accepted as an approximate solution, and the function $S(t_1)$ is taken as an initial condition for the problem for the interval (t_1, t_2).

STEP 3. Solve equations of the velocity module.

5. Iterative process

Given $Q^{(k)}$, one solve all subproblems from step 1, adjoint problem for this step and define new correction $Q^{(k+1)}$ by

$$Q^{(k+1)} = Q^{(k)} - \gamma_k^{(j)}(\alpha(Q^{(k)} - Q^{(0)}) + T_2^*) \quad \text{on} \;\; \Omega_0^{(j)} \times (t_{j-1}, t_j).$$

Parameters $\{\gamma_k\}$ can be calculated at $\alpha \approx +0$, by the property of dense solvability, as:

$$\gamma_k^{(j)} = \frac{1}{2} \, \frac{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (T - T_{obs}^{(j)})^2 \Big|_{\sigma=0} d\Omega dt}{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (T_2^*)^2 \Big|_{\sigma=0} d\Omega dt}.$$

6. Inverse problem on the vertical turbulent heat exchange coefficient

Let ν_T, Q in Subproblem of Problem I (with $T_2 \equiv T$):

$$T_t + \frac{1}{2}\left(w_1 \frac{\partial T}{\partial z} + \frac{1}{r^2}\frac{\partial(r^2 w_1 T)}{\partial z}\right) - \frac{1}{r^2}\frac{\partial}{\partial z} r^2 \nu_T \frac{\partial T}{\partial z} = f_T \text{ in } D \text{ when } t \in (t_{j-1}, t_j),$$

$$T = T_1(t_j) \quad \text{at } t = t_{j-1},$$

$$-\nu_T \frac{\partial T}{\partial z} = Q \text{ at } z = 0 \text{ on } \Omega_0^{(j)} \times (t_{j-1}, t_j),$$

$$\bar{U}_n^{(-)} T - \nu_T \frac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \text{ at } z = 0 \text{ on } (\Omega \backslash \Omega_0^{(j)}) \times (t_{j-1}, t_j)$$

$$\nu_T \frac{\partial T}{\partial z} = 0 \text{ at } z = H,$$

be *additional unknowns.*

The inverse Problem O_1.

Find $\phi \equiv (u, v, \xi, T, S)$ and ν_T, Q such that ϕ is the solution of Problem I and the relations

$$T = T_{obs}^{(j)} \text{ on } \Omega_0^{(j)} \times (t_{j-1}, t_j),$$

$$T = T_{obs,1}^{(j)} \text{ in } D_0^{(j)} \times (t_{j-1}, t_j)$$

$$(j = 1, 2, \ldots, J)$$

hold true.

- Assume that:

$$[0, H(x,y)] = \bigcup_{k=1}^{N} [h_{k-1}(x,y), h_k(x,y)]$$

and introduce the space $L_\infty^{(N)}$:

$$L_\infty^{(N)} : \nu = \sum_{i=1}^{N} \nu_i(x,y,t)\chi_i(z), \quad t \in (t_{j-1}, t_j)$$

$$\forall \nu_i \in L_\infty(\Omega_0^{(j)} \times (t_{j-1}, t_j)),$$

$$\chi_i(z) = \{1 \text{ on } [h_{i-1}, h_i]; 0 \text{ on } [0, H] \backslash [h_{i-1}, h_i]\}$$

Theorem. Assume that

(a) $T_{obs,1}^{(j)} \in W_2^1(0,H)\ \forall (x,y,t) \in \Omega_0^{(j)} \times (t_{j-1}, t_j)$

(b) $\int\limits_0^H \left(\dfrac{\partial T_{obs,1}^{(j)}}{\partial z} \right)^2 dz \neq 0\ \forall (x,y,t) \in \Omega_0^{(j)} \times (t_{j-1}, t_j)$

(c) $Q \in L_2(\Omega_0^{(j)} \times (t_{j-1}, t_j)), \nu_T \in L_\infty^{(N)}$.

Then the Problem O_1 is uniquely solvable ($j = 1, 2, \ldots, J$).

One of the processes of approximate solving the Problem O_1

- We solve the Problem O for given $\nu_{T,0}$ and find T_0, Q_0.
- Introduce the "first corrections"

$$\nu_{T,1} \equiv \nu_T - \nu_{T,0}, \quad Q_1 \equiv Q - Q_0,$$

$$T_1 = T - T_0 \quad \text{in} \quad D \times (t_{j-1}, t_j)$$

and write down the approximate linear inverse problem to calculate $\nu_{T,1}$, Q_1

- We reduce the inverse problem to *the variational data assimilation procedure*.
- The variational data assimilation problem is solved by methods of the extremum theory problems.
- As results we calculate $\nu_{T,1}$, Q_1 and

$$\nu_T \cong \nu_{T,0} + \nu_{T,1}, \quad Q \cong Q_0 + Q_1$$

and approximate solutions of Problem I for each $(t_{j-1}, t_j) \quad \forall j$

Conclusion

- The inverse and corresponding variational data assimilation problems of finding the flux on the ocean and sea surface using the observation of on-line SST data were formulated and studied.
- The inverse problem on the heat flux on the sea surface and coefficient of vertical turbulent heat exchange was studied; the algorithms to solve the problem were proposed.
- The theoretical results and algorithms of the numerical solution of problems can be applied also to the corresponding problems in the dynamics of Black Sea.
- One of the problems is: one needs to construct effective methods for obtaining of on-line vertical profiles of temperature.
- The constructions of "INM-T3"(with calculation of Q, ν_T and on-line solving the corresponding variational data assimilations problems).