Study and solution of the variational data assimilation problems for the ocean nonlinear hydrothermodynamics model

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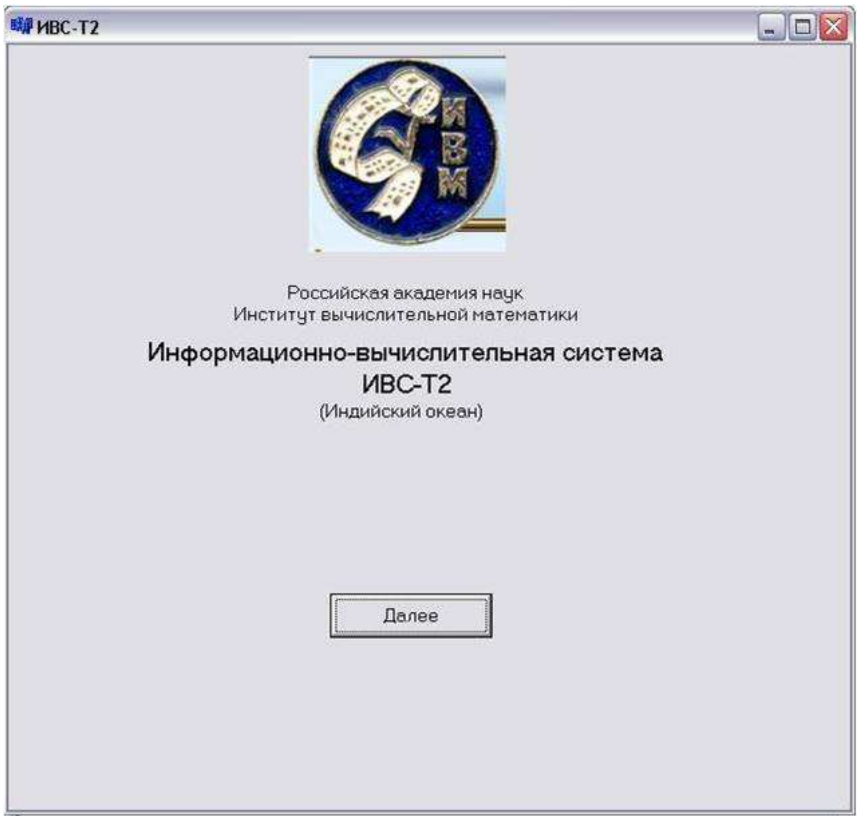
Introduction

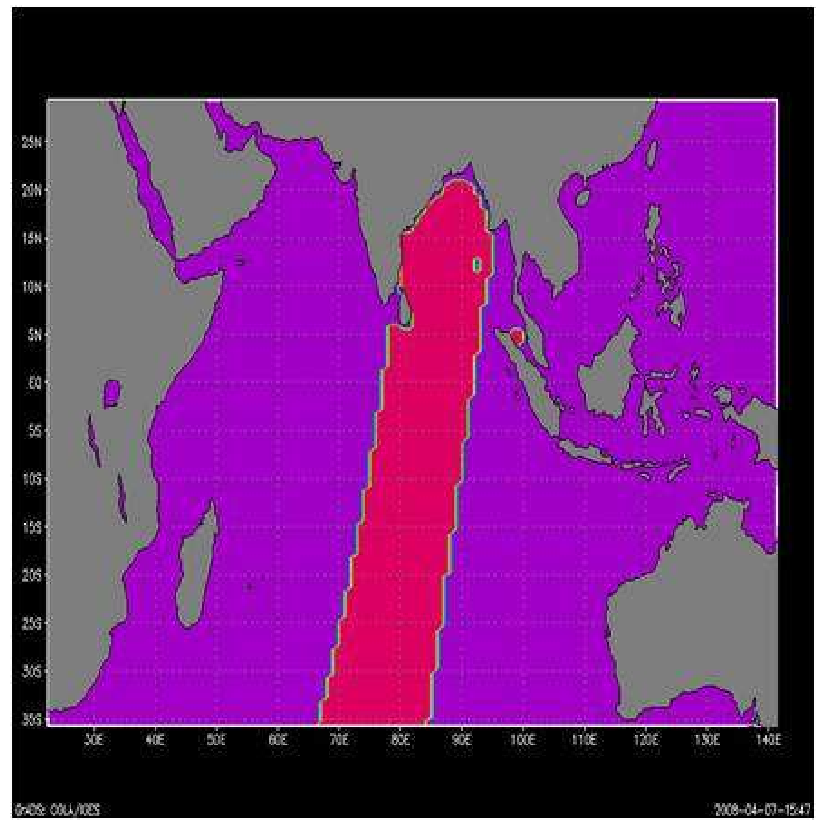
Some results of INM Group developed in 2009

- Agoshkov V.I. Study and solution of the variational data assimilation problems for the ocean nonlinear hydrothermodynamics model (see this presentation)
- Agoshkov V.I., Lebedev S.A., Parmuzin E.I. Numerical solutions of the variational assimilation problem using on-line SST data in the World Ocean (see the presentation of Parmuzin E.I.)
- Agoshkov V.I., Assovsky M.V., Botvinvsky E.A. Variational data assimilation problem for the study of the adequacy of a tidal dynamic model (see the presentation of Botvinovsky E.A.)

- Agoshkov V.I., Parmuzin E.I. et al. Development of the variational data assimilation system INM-T2 for assimilating on-line SST data.

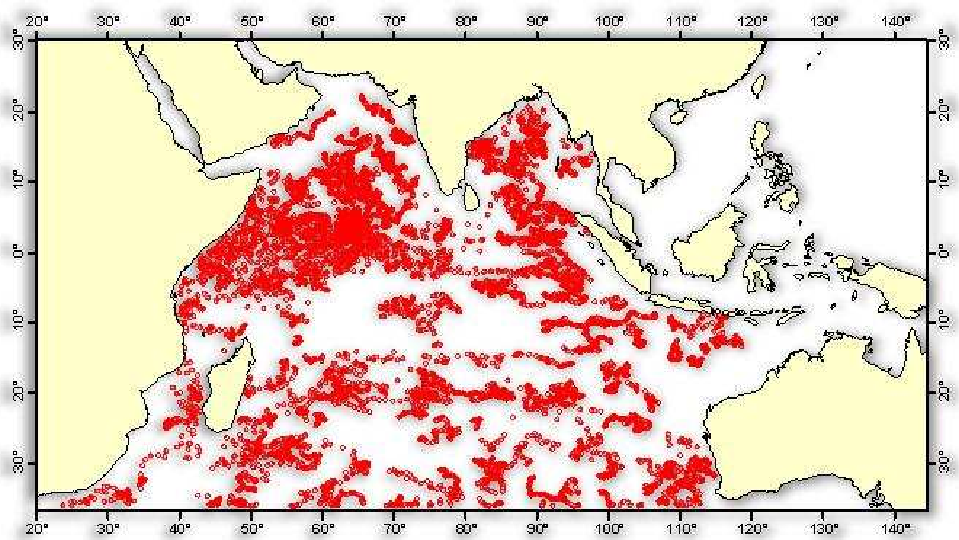


INM-T2. Welcome page

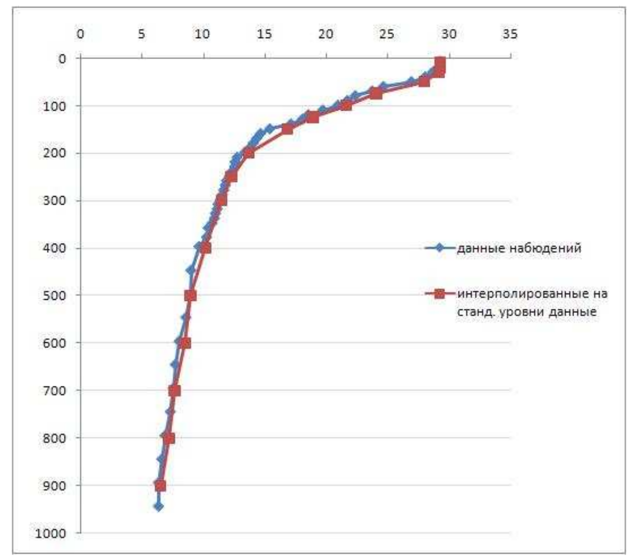


Observation data

- Agoshkov V.I., Parmuzin E.I. and Zakharova N.B. Development and study of numerical algorithms for assimilating the data on vertical sea temperature profiles with the aim to identify the vertical turbulent heat exchange coefficients



ARGO buys 2004 in Indian Ocean



Temperature profiles from ARGO

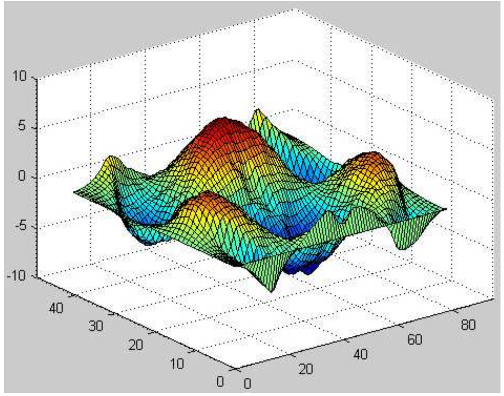
- Agoshkov V.I. Study of "image"assimilation problems for a class of linear mathematical problems

$$L\phi = f + Bu$$

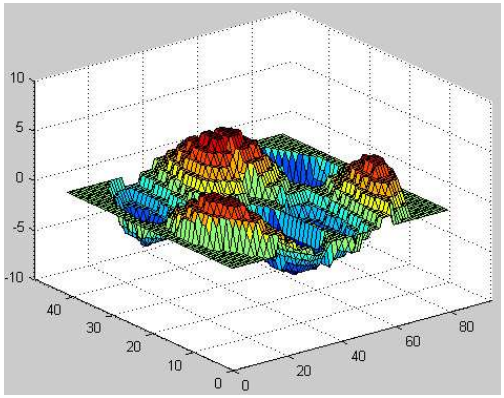
$$C\phi = K\phi_I + G$$

(Operator form statement of the image assimilation problems for linear models. Study of the solvability and uniqueness conditions. General iterative algorithms for solving the problems. Applications to "particular"models).

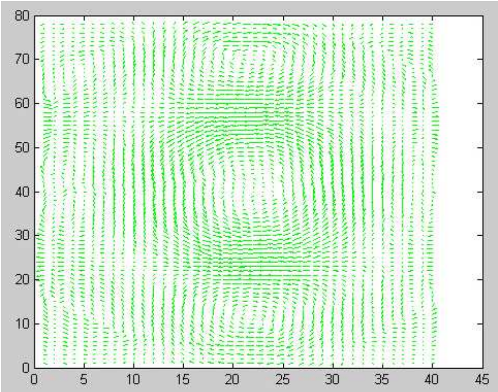
- Agoshkov V.I., Assovsky M.V., Kostrykin S.V. and Semenenko A.Yu. Study and solution of some inverse problems for equations of Stommel, Mank, Shtokman using "image"assimilation procedures.



2nd velocity comp. (model)



2nd velocity comp. (exp)



wind stress

(Statement of problems of the reconstruction of vortexes, wind stresses, velocities in the "rectangular uniform sea". Study of existence and uniqueness of solutions, data assimilation procedures and iterative algorithms. Numerical experiments.)

- Agoshkov V.I., Semenenko A.Yu. et al. Study of existence and uniqueness of solutions for the problem of reconstructions the sea surface currents based on the assimilations of images, SST and SSS.

(In process...)

1. Mathematical model of the ocean hydrothermodynamics

$$\frac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g \cdot grad\xi + A_u \vec{u} + (A_k)^2 \vec{u} = \vec{f} - \frac{1}{\rho_0} gradP_a - \frac{g}{\rho_0} grad \int_0^z \rho_1(T,S)dz',$$

$$\frac{\partial \xi}{\partial t} - m\frac{\partial}{\partial x}(\int_0^H \Theta(z)udz) - m\frac{\partial}{\partial y}(\int_0^H \Theta(z)\frac{n}{m}vdz) = f_3,$$

$$\frac{dT}{dt} + A_T T = f_T, \ \frac{dS}{dt} + A_S S = f_S,$$

where $\vec{u} = (u,v)$ and $\bar{f} = g \cdot gradG$, $\Theta(z) \equiv \frac{r(z)}{R}$, $\quad r = R - z, \quad 0 < z < H, x \equiv \lambda, y \equiv \theta, n \equiv 1/r, \ m \equiv 1/(r\cos\theta)$. The functions G, f_3, ξ_0 $\equiv \xi$ at $t = 0$ will be "additional unknowns which must be calculated too.

Boundary conditions on the "sea surface" $\Gamma_S \equiv \Omega$ at $z = 0$:

$$
\begin{cases}
\left(\int\limits_0^H \Theta \vec{u} dz\right) \vec{n} + \beta_0 m_{op} \sqrt{gH}\, \xi = m_{op} \sqrt{gH}\, d_s \text{ on } \partial\Omega, \\
U_n^{(-)} u - \nu \dfrac{\partial u}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k u = \tau_x^{(a)} \big/ \rho_0,\ U_n^{(-)} v - \nu \dfrac{\partial v}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k v = \tau_y^{(a)} \big/ \rho_0, \\
A_k u = 0, \quad A_k v = 0, \\
U_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + U_n^{(-)} d_T, \\
U_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + U_n^{(-)} d_S,
\end{cases}
$$

where
$U_n = \vec{U} \cdot \vec{N}, \vec{U} = (u, v, w) \equiv (\vec{u}, w), \vec{N} = (n_1, n_2, n_3) \equiv (\vec{n}, n_3), U_n^{(-)} = (|U_n| - U_n)/2.$
The boundary function d_T, d_S or Q_T, Q_S can be unknown also.

2. Approximation by splitting method: Problem I

Step 1. We consider the system:

$$
\begin{cases}
T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_{j-1}, t_j), \\
T = T_{j-1} \text{ for } t = t_{j-1} \text{ in } D, \\
\bar{U}_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
\dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)} T + \dfrac{\partial T}{\partial N_T} = \bar{U}_n^{(-)} d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\
T_j \equiv T \text{ on } D \times (t_{j-1}, t_j),
\end{cases}
$$

where $\Gamma_w = \Gamma_{w,c} \cup \Gamma_{w,op}$ - the "vertical lateral boundary", Γ_H - "the ocean bottom".

Step 2.

$$
\begin{cases}
S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_{j-1}, t_j), \\
S = S_{j-1} \text{ at } t = t_{j-1} \text{ in } D, \\
\bar{U}_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)} S + \dfrac{\partial S}{\partial N_S} = \bar{U}_n^{(-)} d_S + Q_S \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\
S_j \equiv S \text{ on } D \times (t_{j-1}, t_j).
\end{cases}
$$

Step 3.1

$$\begin{cases} \underline{u}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}^{(1)} - g \cdot \mathbf{grad}\xi = g \cdot \mathbf{grad} G - \frac{1}{\rho_0} \mathbf{grad} \left(P_a + g \int\limits_0^z \rho_1(\bar{T}, \bar{S}) dz' \right) \\ \text{ in } D \times (t_{j-1}, t_j), \\ \xi_t - \mathbf{div} \left(\int\limits_0^H \Theta \underline{u}^{(1)} dz \right) = f_3 \text{ in } \Omega \times (t_{j-1}, t_j), \\ \underline{u}^{(1)} = \underline{u}_{j-1}, \ \xi = \xi_{j-1} \text{ at } t = t_{j-1}, \\ \left(\int\limits_0^H \Theta \underline{u}^{(1)} dz \right) \cdot n + \beta_0 m_{op} \sqrt{gH} \xi = m_{op} \sqrt{gH} d_s \text{ on } \partial\Omega \times (t_{j-1}, t_j), \\ \underline{u}_j^{(1)} \equiv \underline{u}^{(1)}(t_j) \text{ in } D \end{cases}$$

If we write down $\underline{u}^{(1)}$ in the following form: $\underline{u}^{(1)} = \underline{U}^{(1)}(\lambda, \theta, t) + \underline{u}'(\lambda, \theta, z, t)$ where

$$\underline{U}^{(1)} = \frac{1}{H_1} \int_0^H \Theta \underline{u}^{(1)} dz, \qquad H_1 = \int_0^H \Theta dz,$$

then Step 3.1 is reduced to two subproblems for the functions $\underline{U}^{(1)}, \underline{u}_1'$.

Step 3.1

First of them is "**The ocean tide theory problem**":

$$\begin{cases} \underline{U}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{U}^{(1)} - g\,\mathrm{grad}\,\xi = g\,\mathrm{grad}\,G - \underline{I} \quad \text{in} \;\; D \times (t_{j-1}, t_j) \\ \xi_t - \mathrm{div}(H_1 \underline{U}^{(1)}) = f_3 \quad \text{in} \;\; \Omega \times (t_{j-1}, t_j) \\ \underline{U}^{(1)}(t_{j-1}) = \frac{1}{H_1} \int_0^H \Theta \underline{u}_{j-1} dz, \qquad \xi(t_{j-1}) = \xi_{j-1} \quad \text{in } \Omega \\ \\ (H_1 \underline{U}^{(1)}) \cdot \underline{n} + \beta_0 m_{\mathrm{op}} \sqrt{gH} \xi = m_{\mathrm{op}} \sqrt{gH} d_s \end{cases}$$

where

$$\underline{I} = (I_\lambda, I_\theta) = \frac{1}{\rho_0} \Big(\mathrm{grad}\, P_a + g \frac{1}{H_1} \int_0^H \Theta dz \int_0^z \mathrm{grad}\, \rho_1(\bar{T}, \bar{S}) dz' \Big).$$

The second subproblem is :

$$\left\{ \begin{array}{l} (\underline{u}'_1)_t + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}'_1 = \frac{g}{\rho_0} \Big(\frac{1}{H_1} \int_0^H \Theta dz \int_0^z \operatorname{grad} \rho_1 (\bar{T}, \bar{S}) dz' \\ - \displaystyle\int_0^z \operatorname{grad} \rho_1 (\bar{T}, \bar{S}) dz' \Big) \\ \underline{u}'_1(t_{j-1}) = \underline{u}_{j-1} - \dfrac{1}{H_1} \displaystyle\int_0^H \Theta u_{j-1} dz \end{array} \right.$$

Step 3.2

$$\begin{cases} \underline{u}_t^{(2)} + \begin{bmatrix} 0 & -f_1(\bar{u}) \\ f_1(\bar{u}) & 0 \end{bmatrix} \underline{u}^{(2)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\ \underline{u}^{(2)} = \underline{u}_j^{(1)} \text{ при } t = t_{j-1} \text{ in } D, \\ \underline{u}_j^{(2)} \equiv \underline{u}^{(2)}(t_j) \text{ in } D, \end{cases}$$

Step 3.3

$$\begin{cases} \underline{u}_t^{(3)} + (\bar{U}, \mathbf{Grad})\underline{u}^{(3)} - \mathbf{Div}(\hat{a}_u \cdot \mathbf{Grad}\ )\underline{u}^{(3)} + (A_k)^2 \underline{u}^{(3)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\ \underline{u}^{(3)} = \underline{u}^{(2)} \text{ at } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)} \underline{u}^{(3)} - \nu_u \frac{\partial \underline{u}^{(3)}}{\partial z} - k_{33} \frac{\partial}{\partial z}(A_k \underline{u}^{(3)}) = \frac{\underline{\tau}^{(a)}}{\rho_0}, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ U_n^{(3)} = 0, \frac{\partial U^{(3)}}{\partial N_u} \cdot \bar{\underline{\tau}}_w + \left(\frac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \underline{\tau}_w = 0, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)} (\tilde{U}^{(3)} \cdot \underline{N}) + \frac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{N} + \left(\frac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \bar{N} = \bar{U}_n^{(-)} d, A_k \underline{u}^{(3)} = 0 \text{on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)} (\tilde{U}^{(3)} \cdot \bar{\tau}_w) + \frac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{\tau}_w + \left(\frac{\partial}{\partial N_k} A_k \underline{u}^{(3)} \right) \cdot \underline{\tau}_w = 0, A_k \underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \frac{\partial \underline{u}^{(3)}}{\partial N_u} = \frac{\tau^{(b)}}{\rho_0} \text{ on } \Gamma_H \times (t_{j-1}, t_j), \end{cases}$$

where

$$\underline{u}^{(3)} = (u^{(3)}, v^{(3)}),\ \tau^{(a)} = (\tau_x^{(a)}, \tau_y^{(a)}),$$
$$U^{(3)} = (u^{(3)}, w^{(3)}(u^{(3)}, v^{(3)})),\ \tilde{U}^{(3)} = (u^{(3)}, 0), \quad \tau^{(b)} = (\tau_x^{(b)}, \tau_y^{(b)}).$$

3.1. Inverse and variational data assimilation problems

Problem Inv 1

Let us assume, that the unique function which is obtained by observation data processing is the function ξ_{obs} on $\bar{\Omega} \equiv \Omega \cup \partial\Omega$ at $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. Let by physical meaning this function is an approximation to sea level function ξ on Ω, i.e on the boundary, when $z = 0$. We permit that the function ξ_{obs} is known only on the part of $\Omega \times (0, \bar{t})$ and we define a support of this function as m_0. Beyond of this area we suppose function ξ_{obs} is trivial.

Let the functions G, f_3, ξ_0 are "additional unknown functions" and we state the following inverse problem - **Problem Inv 1**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem* I *and functions* G, f_3, ξ_0, *such that,* $m_0(\xi - \xi_{obs}) = 0$.

To study this inverse problem we apply general methodology for solving data assimilation problems (Agoshkov V., 2003) and classical results of the inverse problem theory (A.N. Tikhonov, M.M. Lavrentiev, V.K. Ivanov, V.V. Vasin, V.G. Romanov, M.V. Klibanov,Yu.E. Anikonov,S.I. Kabanikhin).

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(\xi_0, G, f_3, \Phi) = \frac{1}{2}\Big\{\alpha_0 \bar{t} \left\|\xi_0 - \xi^{(0)}\right\|^2_{L_2(g;\Omega)} + \alpha_f \left\|f_3 - f_3^{(0)}\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))}$$
$$+\alpha_G \left\|G - G^{(0)}\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))}\Big\} + \Im_0(\Phi) = \sum_{j=1}^{J} \int_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(\xi) = \frac{1}{2} \left\|m_0(\xi - \xi_{\text{obs}})\right\|^2_{L_2(0,\bar{t};L_2(g;\Omega))}$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\Big\{\alpha_0 \Delta t_j \left\|\xi_0 - \xi^{(0)}\right\|^2_{L_2(g;\Omega)} + \alpha_f \left\|f_3 - f_3^{(0)}\right\|^2_{L_2(g;\Omega)}(t)$$
$$+\alpha_G \left\|G - G^{(0)}\right\|^2_{L_2(g;\Omega)}(t) + \left\|m_0(\xi - \xi_{\text{obs}})\right\|^2_{L_2(g;\Omega)}(t)\Big\}.$$

Here $\alpha \equiv (\alpha_0, \alpha_f, \alpha_G)$, $\alpha_0 \geq 0$, $\alpha_f \geq 0$, $\alpha_G \geq 0$ are regularization parameters that may be dimensional values. Furthermore, it is possible to specify α_f, α_G depending on $\alpha_0 \geq 0$, (for instance, $\alpha_G = \alpha_0$, $\alpha_f = \alpha_0 \bar{t}^2$, etc.).
We can formulate the data assimilation problem - **Problem A 1**: *find the solution* ϕ *of the Problem* I *and function* G, f_3, ξ_0, *such that, the cost functional is minimal on the set of the solutions.*

Let us consider the problem on the first time step (t_0, t_1). Then the optimality conditions are:

$$\begin{cases} t_1\alpha_0(\xi_0 - \xi^{(0)}) + \xi^*(t_0) = 0 \quad \text{in } \Omega \\ \alpha_f(f_3 - f_3^{(0)}) + \xi^* = 0 \quad \text{in } \Omega \times (t_0, t_1) \\ \alpha_G(G - G^{(0)}) - \operatorname{div}\Big(\int_0^{\mathrm{H}} \Theta \underline{u}_1^* dz\Big) = 0 \quad \text{in } \Omega \times (t_0, t_1), \end{cases}$$

where $\xi^*, \underline{u}_1^*$ are the solution of the adjoint problem:

$$\begin{cases} -(\underline{u}_1^*)_t - \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}_1^* + g \operatorname{grad} \xi^* = 0 \quad \text{in } D \times (t_0, t_1) \\ -\xi_t^* + \operatorname{div}\Big(\int_0^H \Theta \underline{u}_1^* dz\Big) = m_0(\xi - \xi_{\mathrm{obs}}) \quad \text{in } \Omega \times (t_0, t_1) \\ -\Big(\int_0^H \Theta \underline{u}_1^* dz\Big) \cdot \underline{n} + \beta_0 m_{\mathrm{op}} \sqrt{gH} \xi^* = 0 \quad \text{on } \partial\Omega \times (t_0, t_1) \\ \xi^* = 0, \qquad \underline{u}_1^* = 0 \quad \text{at } t = t_1 \end{cases}$$

(or here $\xi^* = m_0(\xi - \xi_{\mathrm{obs}})(t_1)$ at $t = t_1$).

Definition : Problem Inv 1 is densely solvable if for any $\epsilon > 0$ there is a solution ϕ of the Problem I such that $\Im_0(\Phi) < \epsilon$.

Proposition 1. If $\text{supp}(\xi_{\text{obs}}) = \bar{\Omega} \times [t_0, t_1]$ and $(G, f_3)_{L_2(g;\Omega)} = 0 \ \forall t$ then Problem Inv1 is uniquely and densely solvable. The solution of Problem A1 can be taken as an approximate solution of Problem Inv1 for sufficiently small α.

Proposition 2. If $\text{mes}(\partial\Omega \cap \Gamma_{w,\text{op}}) > 0$ and the function G is sought additionally only then Problem Inv 1 is densely solvable.

Proposition 3. If $\text{mes}(\text{supp}(\xi_{\text{obs}})) > 0$ and the function f_3 is sought additionally only then Problem Inv 1 is densely solvable.

3.2. Inverse and variational data assimilation problems

Problem Inv 2

Assume that the sea surface temperature (SST), observed on a subset $\Omega^{(j)}$ of Ω, is denoted by $T_{obs} \equiv T_{obs}^{(j)}$ when $t \in (t_{j-1}, t_j), m_0^{(j)}$ is the characteristic function of this subset $(j = 1, 2, ..., J)$. Considering the boundary condition for T at $z = 0$ we write it in the following form:

$$-\nu_T \frac{\partial T}{\partial z} = Q \; at \; z = 0 \; on \; \Omega^{(j)} \times (t_{j-1}, t_j),$$

$$\bar{U}_n^{(-)} T - \nu_T \frac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \; at \;\; z = 0 \; on \; (\Omega \backslash \Omega^{(j)}) \times (t_{j-1}, t_j),$$

where the function $Q \equiv Q^{(j)} (j = 1, 2, ..., J)$.

Let the functions $Q^{(j)}$ are "additional unknown functions" and we state the following inverse problem - **Problem Inv 2**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem* I *and functions* $Q^{(j)}$, *such that,* $m_0^{(j)} (T - T_{obs}^{(j)}) = 0$ on $\Omega, j = 1, 2, ..., J$.

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(Q,\Phi) = \frac{1}{2}\int\limits_0^{\bar t}\int\limits_\Omega \alpha|Q - Q^{(0)}|^2 d\Omega dt + \Im_0(\Phi) = \sum_{j=1}^{J}\int\limits_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(Q) = \frac{1}{2}\int\limits_0^{\bar t}\int\limits_{\Omega_0(t)} m_0|T - T_{obs}|^2 d\Omega dt,$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\int\limits_{t_{j-1}}^{t_j}\int\limits_{\Omega^{(j)}} \alpha|Q - Q^{(0)}|^2 d\Omega dt + \frac{1}{2}\int\limits_{t_{j-1}}^{t_j}\int\limits_{\Omega^{(j)}} m_0^{(j)}|T - T_{obs}^{(j)}|^2 d\Omega dt.$$

Here $\alpha \geq 0$, is a "regularization"or "penalty"function, that may be constant.
Data assimilation problem - **Problem A 2**: *find the solution* ϕ *of the Problem* I *and functions* $\{Q^{(j)}\}$, *such that, the cost functional is minimal on the set of the solutions.*
It is possible to prove that *the Problem A2 is uniquely and densely solvable* . This problem has been studied and numerically solved using "adjoint equation and optimal control approaches"by Agoshkov V.I., Parmuzin E.I. and Shutyaev V.P.[2008], Agoshkov V.I., Lebedev S.A.and Parmuzin E.I.[2009].

3.3. Inverse and variational data assimilation problems

Problem Inv 3

Assume that the sea surface salinity (SSS), observed on a subset $\Omega^{(j)}$ of Ω, is denoted by $S_{obs} \equiv S_{obs}^{(j)}$ when $t \in (t_{j-1}, t_j), m_0^{(j)}$ is the characteristic function of this subset $(j = 1, 2, ..., J)$. Considering the boundary condition for S at $z = 0$ we write as:

$$-\nu_S \frac{\partial S}{\partial z} = Q \; at \; z = 0 \; on \; \Omega^{(j)} \times (t_{j-1}, t_j),$$

$$\bar{U}_n^{(-)} S - \nu_S \frac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \; at \;\; z = 0 \; on \; (\Omega \backslash \Omega^{(j)}) \times (t_{j-1}, t_j),$$

where the function $Q \equiv Q^{(j)} (j = 1, 2, ..., J)$.
Let the functions $Q^{(j)}$ are "additional unknown functions" and we state the following inverse problem - **Problem Inv 3**: *find the solution* $\phi = (u, v, \xi, T, S)$ *of the Problem*I *and functions* $Q^{(j)}$, *such that,* $m_0^{(j)} (S - S_{obs}^{(j)}) = 0$ on $\Omega, j = 1, 2, ..., J$.

Introduce the cost functional $\Im_\alpha$ of the form:

$$\Im_\alpha \equiv \Im_\alpha(Q,\Phi) = \frac{1}{2}\int_0^{\bar{t}}\int_\Omega \alpha|Q - Q^{(0)}|^2 d\Omega dt + \Im_0(\Phi) = \sum_{j=1}^{J}\int_{t_{j-1}}^{t_j} \Im_\alpha^{(j)} dt,$$

where

$$\Im_0(\Phi) \equiv \Im_0(Q) = \frac{1}{2}\int_0^{\bar{t}}\int_{\Omega_0(t)} m_0|S - S_{obs}|^2 d\Omega dt,$$

$$\Im_\alpha^{(j)} = \frac{1}{2}\int_{t_{j-1}}^{t_j}\int_{\Omega^{(j)}} \alpha|Q - Q^{(0)}|^2 d\Omega dt + \frac{1}{2}\int_{t_{j-1}}^{t_j}\int_{\Omega^{(j)}} m_0^{(j)}|S - S_{obs}^{(j)}|^2 d\Omega dt.$$

Here $\alpha \geq 0$, is a "regularization"or "penalty"function ,that may be constant.
Data assimilation problem - **Problem A 3**: *find the solution* ϕ *of the Problem* I *and functions* $\{Q^{(j)}\}$*, such that, the cost functional is minimal on the set of the solutions.*
It is possible to prove that *the Problem A3 is uniquely and densely solvable* . This problem for the case $\Omega^{(j)} \equiv \Omega (j = 1, 2, ..., J)$ has been studied and numerically solved by Agoshkov V.I., Parmuzin E.I. and Shutyaev V.P.[2008].

4.Iterative process for solving Problem A 1

For numerical implementation of the algorithm of solving the whole problem in (t_0, t_1) it is sufficient to solve two initial-boundary problems for parabolic equations (after that T and S will be defined in $D \times (t_0, t_1)$) and carry out Step 3 including the data assimilation procedure. A numerical solution of the problem at Step 3 can be obtained by the following iterative algorithm: if $f_3^{(k)}, G^{(k)}, \xi_0^{(k)}$ are defined, we solve the subproblems from the Step 3 for $\xi_0 = \xi_0^{(k)}$, $f_3 = f_3^{(k)}$, $G = G^{(k)}$ and then solve adjoint problem and compute the new approximation $f_3^{(k+1)}, G^{(k+1)}, \xi_0^{(k+1)}$.

$$
\left\{
\begin{array}{l}
\xi_0^{(k+1)} = \xi_0^{(k)} - \gamma_k(\alpha_0(\xi_0^{(k)} - \xi^{(0)}) + \xi^*(t_0)) \quad \text{in } \Omega \\
f_3^{(k+1)} = f_3^{(k)} - \gamma_k(\alpha_f(f_3^{(k)} - f_3^{(0)}) + \xi^*) \quad \text{in } \Omega \times (t_0, t_1) \\
G^{(k+1)} = G^{(k)} - \gamma_k\Big(\alpha_G(G^{(k)} - G^{(0)}) - \mathrm{div}\Big(\int_0^H \Theta u_1^* dz\Big)\Big) \\
\qquad \text{in } \Omega \times (t_0, t_1) \quad \text{provided that } \displaystyle\int_\Omega \mathrm{G}^{(i)} d\Omega = 0 \quad \forall i.
\end{array}
\right.
$$

In the case of an appropriate selection of parameters $\{\gamma_k\}$ the iterative process converges. After the criterion of stopping the iteration process is satisfied, it is necessary to used the computed $f_3^{(k+1)}, G^{(k+1)}, \xi_0^{(k+1)}$ to solve other subproblems from the Step 3 and to obtain an approximate solution to the whole problem in $D \times (t_0, t_1)$. After solving all problems and implementing the iteration process in (t_0, t_1), the variation assimilation problem is solved similarly in the subsequent intervals $(t_{j-1}, t_j), j = 2, 3, \ldots$. **In view of the established properties of unique and dense solvability of the considered Problem Inv1 and data assimilation problem in each time interval, we can state that the system of all approximate solutions $\{\phi_j\}$ imparts the minimal value of whole cost functional, i.e., is the solution to the considered problem for the whole interval $(0, \bar{t}\,)$.**

5. Numerical experiments

(Agoshkov V.I., Dianskii N.A., Gusev A.V.)

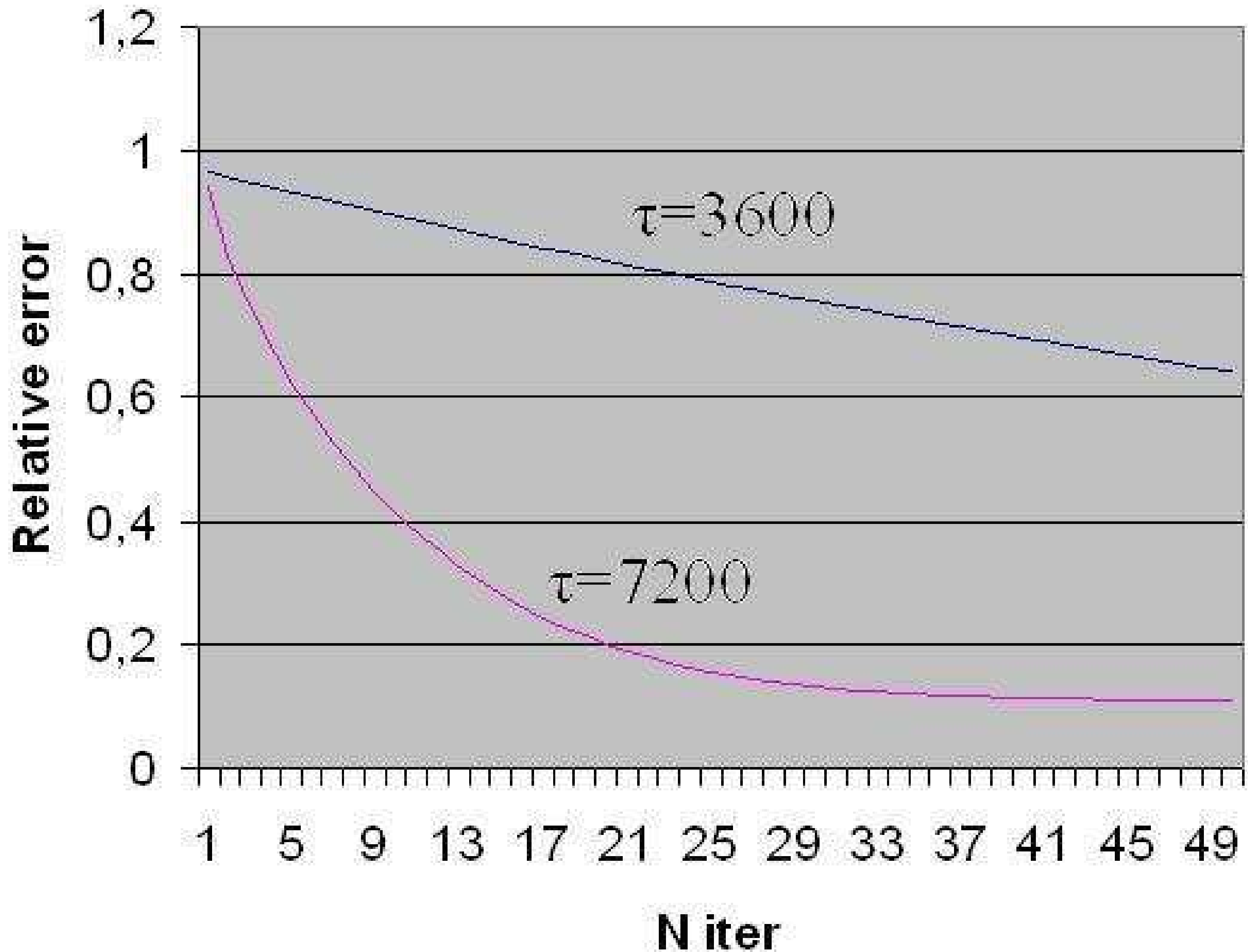
The object of simulation is the Indian Ocean. We can describe the parameters of the area studied and its geographical coordinates are: the grid 120x131x33 (latitude×longitude×depth); the first mesh point is the point with coordinates 22.5 E and 33.5 S. The grid steps with respect to x and y are constant and equal 1.0 and 0.5 degrees, respectively. The time step is equal to $\Delta t = 1$ hour.

The data of SLF, which was obtained from Geophysical Center of RAS (Lebedev S.A.), were used for the construction of the function ξ_{obs} at all time steps at each grid points.

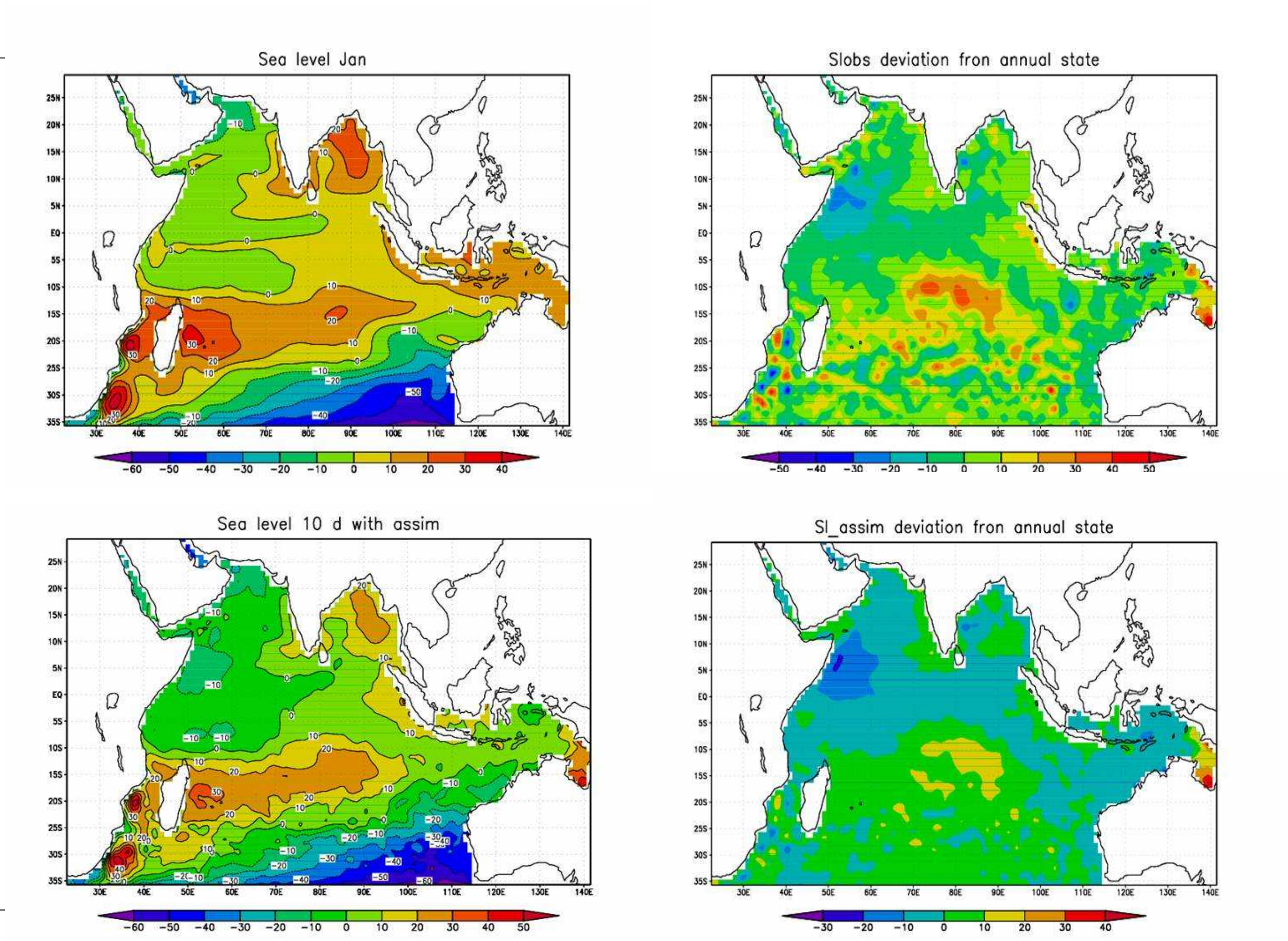
The mean value fuctions for January $G^{(0)}, f_3^{(0)}, \xi^{(0)}$ was calculated using the database of NCEP (National Centers for Environmental Prediction).

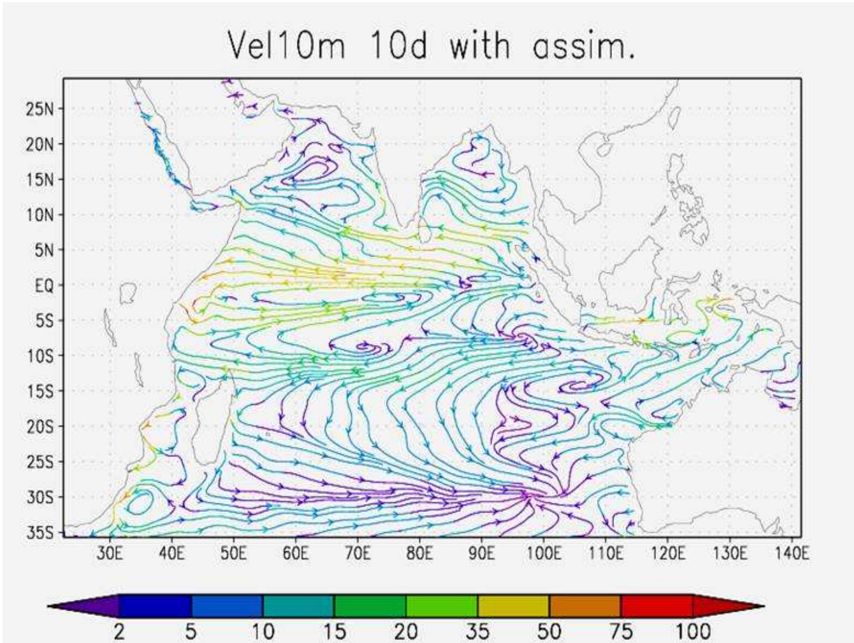
The data assimilation module to assimilate ξ_{obs} was included into the thermohydrodynamics model of the Indian Ocean. The time period taken in experiments is 1-10 days (January 2000).

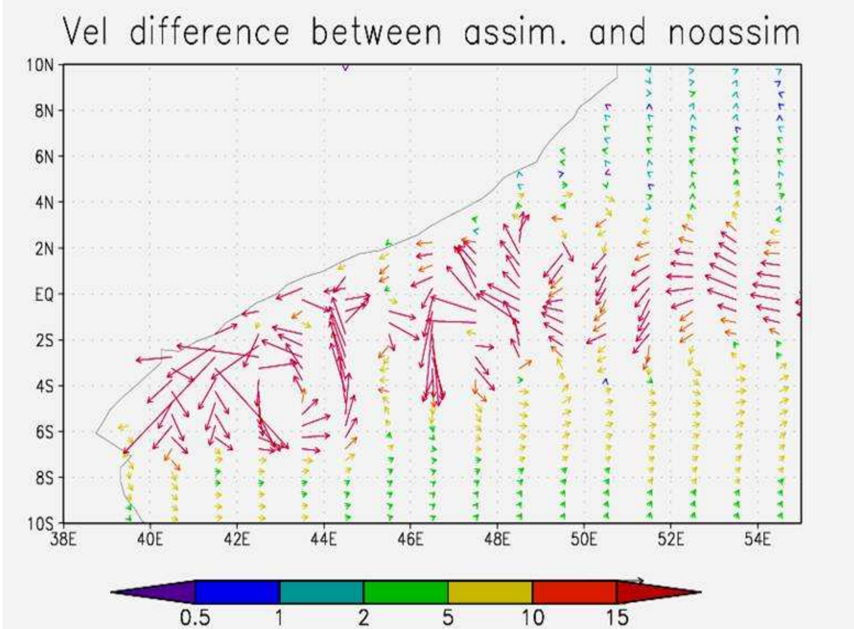
Dependence of the relative error of G, f_3 from the parameter $\tau \equiv \gamma_k \ \forall k$.

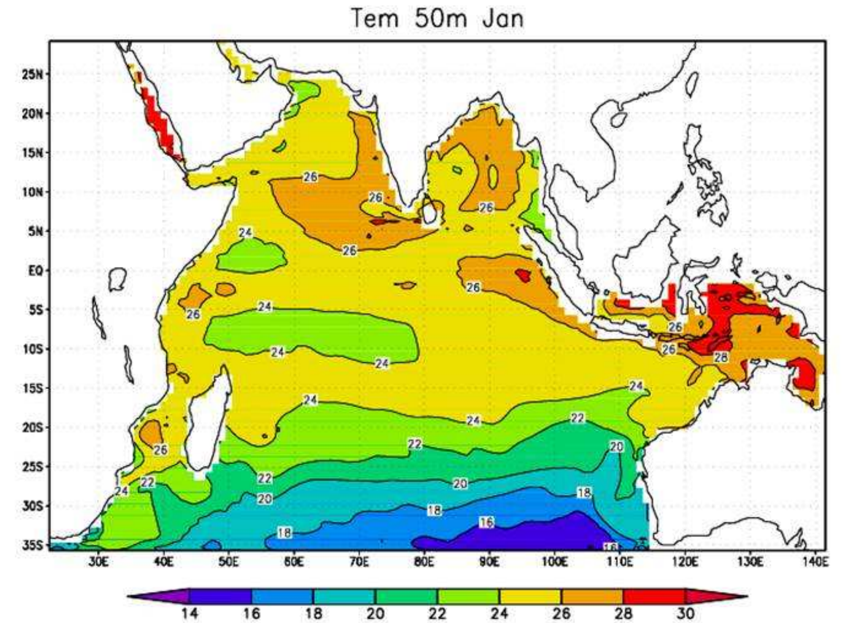


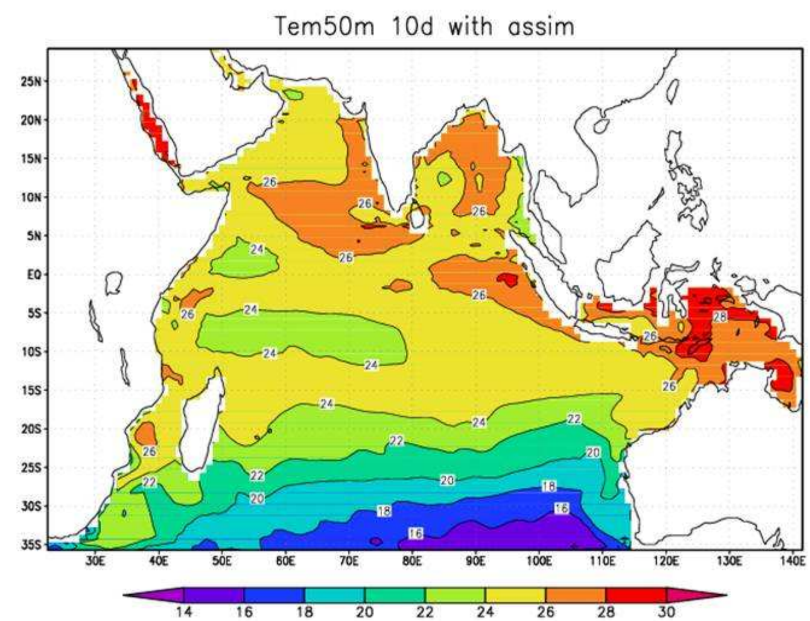
Numerical results with assimilation. The sea level function.




Vel10m 10d with assim.
2 5 10 15 20 35 50 75 100


Vel difference between assim. and noassim
0.5 1 2 5 10 15


Tem 50m Jan
14 16 18 20 22 24 26 28 30


Tem50m 10d with assim
14 16 18 20 22 24 26 28 30

Conclusions

- The inverse and corresponding variational data assimilation problems of finding the functions G, f_3, ξ_0 from the ocean dynamics mathematical models using the observation of SLF were formulated and studied.
- The numerical experiments show that assimilation of SLF have a small influence to other components of the full solution, i.e. velocity, temperature etc.
- The numerical experiments confirm the theoretical results and advisability of using the assimilation procedure in 3D ocean circulation model.
- The numerical solution of Problems A2, A3 is presented in:
  - Agoshkov V. I., Parmuzin E. I., and Shutyaev V. P. Numerical Algorithm for Variational Assimilation of Sea Surface Temperature Data. Comp. Math. and Math. Physics, 2008, Vol. 48, No. 8, pp. 1293 - 1312.
  - Agoshkov V.I., Parmuzin E.I., Shutyaev V.P. A numerical algorithm of variational data assimilation for reconstruction of salinity fluxes on the ocean surface. Russ. J. Numer. Anal. Math. Modelling, 2008, Vol. 23, No. 2, pp. 135-161.
  - Agoshkov V. I., Lebedev S. A. , and Parmuzin E. I. Numerical Solution to the Problem of Variational Assimilation of Operational Observational Data on the Ocean Surface Temperature. Izvestiya, Atmospheric and Oceanic Physics, 2009, Vol. 45, No. 1, pp. 69 - 101.