Variational data assimilation problem for the study of adequacy of a tidal dynamics model

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We consider a problem on the adequacy of the model of tidal dynamics in the World Ocean. In the considered mathematical model, one of "additional sources" is a vector function F. As the criterion that the model correctly reproduces really modeled physical processes we choose the requirement that sea level function (a deviation from an equilibrium level of ocean) coincided with the satellite SSH data. This criterion can lead to a problem of minimization of a quadratic ("cost") functional for the purpose of variational assimilation of satellite data and calculation of F. After the solution of this problem we estimate the norm of F which is taken for a measure of adequacy of considered mathematical model. We prove the uniqueness and dense solvability of the minimization problem.We prove that the minimum of the "cost" functional is zero (therefore the norm of F really reflects degree of adequacy of model). Also we present special algorithms of approximation of the tidal dynamics model and the adjoint problem. The results of some numerical experiments are presented.

Statement of a problem and definitions

- We consider a system of operator equations

$$L\phi = f + Bu, \qquad (1)$$

$$C\phi = \varphi_{obs}, \qquad (2)$$

where: (1) is a mathematical model of some physical process; ϕ is a vector function characterizing the process; f is a given element of input data; $\mathcal{F} \equiv Bu$ is a "perturbation" of the equation (1) and it is given by an operator B and an "additional unknown" function u; "$C\phi$" is a set of observation ("measured") characteristics; φ_{obs} is a vector of observation data; (2) is called an "observability equation" (or a "controllability equation"). The operators L, B, C and the equations (1), (2) are considered in a system of functional spaces H, H_{obs}, H_C.

- Let us introduce:

$$\mu_C \equiv \|C\phi - \varphi_{obs}\|_{H_{obs}} - \text{"observability measure"},$$

$$S_R \equiv \{u : \|Bu\|_H \leqslant R\} \cap (\text{some restrictions on } u),\ R = const$$

- **Definitions:**
(V.A.Morozov [1987]; V.I.Agoshkov, E.A.Botvinovsky [2009])

  1. The model (1) is called **adequate** if $u \in S_R$ and $\mu_C = 0$.
  2. The model (1) is called "ε - adequate" if $u \in S_R$ exists for all $\varepsilon > 0$ such that $\mu_C \leqslant \varepsilon$.
  3. For $B \equiv 0$ (B is the "trivial operator") unperturbed model is called "adequate" ("ε - adequate") if $\mu_C = 0$ ($\mu_C \leq \varepsilon$).

- **Problem:** At given f, φ_{obs}, S_R it is needed to find solutions ϕ, u of the system (1), (2) and to compute μ_C and to check that $u \in S_R$.

Variational data assimilation problems

- A family of regularizied optimal control problems: find ϕ, u, such that

$$(3) \qquad \begin{cases} L\phi = f + Bu, \\ J_\alpha(u) = \inf\limits_{v\in D(B)} J_\alpha(v), \end{cases}$$

where

$$J_\alpha(v) = \frac{\alpha}{2}\|v\|^2_{H_c} + \frac{1}{2}||C\phi - \varphi_{obs}||^2_{H_{obs}}$$

$\alpha \geq 0, \quad \phi(v)$ is a solution of the problem (1) for $u \equiv v$;

is a *family of variational data assimilation problems* (φ_{obs} is observation data) for the mathematical model (1).
(If $\overline{D(B)} \neq H_C$ then (3) is a problem with "restrictions".)

- Let L^*, B^*, C^* be the operators adjoint to L, B, C.

Theorem. (V.I. Agoshkov [2003]; V.I.Agoshkov, E.A.Botvinovsky [2009])
If the homogeneous system

$$L^*\phi^* + C^*u^* = 0, B^*\phi^* = 0$$

has the trivial solution only, then $\forall\varepsilon > 0\ \exists\alpha > 0$ *such that a pair* ϕ_α, u_α *(which is a solution of (3)) satisfies the condition* $\mu_C = \mu_C(\alpha) \leqslant \varepsilon$.

Corollary. If the conditions of the Theorem hold true and $u_\alpha \in S_R$ the model (1) is ε-adequate. For checking ε-adequate property of the model one can use elements ϕ_α, u_α at sufficiently small $\alpha > 0$.

- An algorithm for solving the minimization problem

$$\begin{cases} L\phi = f + Bu, \\ J_\alpha(u) = \inf_{v\in D(B)} \left(\frac{\alpha}{2}\|v\|^2_{H_c} + \frac{1}{2}||C\phi - \varphi_{obs}||^2_{H_{obs}}\right) \end{cases}$$

is:

$$(4) \qquad \begin{aligned} & L\phi^{(k)} = f + Bu^{(k)}, \\ & L^* q^{(k)} = C^*(C\phi^{(k)} - \varphi_{obs}), \\ & u^{(k+1)} = u^{(k)} - \tau_k(\alpha u^{(k)} + B^* q^{(k)}), \\ & (k = 0, 1, 2, \ldots) \end{aligned}$$

The iterative process (4) is convergent under necessary constraints to $\{\tau_k\}$ (V.I.Agoshkov, [2003]).

Model of tides dynamics

- Let Ω be a part of a sphere with radius R in $\mathbb{R}^3, \partial\Omega$ is the boundary of Ω, $\lambda \in [0, 2\pi], \theta \in [0, \pi], d\Omega = R^2 sin\theta d\lambda d\theta, H(\lambda, \theta) > 0$ is the ocean depth, $T < \infty$. We consider a problem to find $\mathbf{U} \equiv (u, v), \zeta$ and "additional unknown" ψ :

$$(5) \begin{cases} H\mathbf{U}_t - \nu\Delta_1\mathbf{U} + K\mathbf{U} + gH\nabla\zeta = \mathbf{f} + \beta gH\nabla\psi, \quad \mathbf{U}\big|_{\partial\Omega} = 0, \\ \zeta_t + \mathrm{div}H\mathbf{U} = 0, \quad \mathbf{U}\big|_{t=0} = \mathbf{U}_{(0)}, \quad \zeta\big|_{t=0} = \zeta_{(0)}, \end{cases}$$

where $\mathbf{f} \equiv H\nabla\Omega^+, \Omega^+ \equiv g\zeta^+$ - "tide potential", $g = const$, $\beta = 0$ or $\beta = 1$ (at $\beta = 0$ we have the "unperturbed model"),

$$\Delta_1\mathbf{U} \equiv (\Delta u, \Delta v) + D\mathbf{U}, \quad D\mathbf{U} \equiv \frac{1}{R^2\sin^2\theta}\left(-u + 2\cos\theta\frac{\partial v}{\partial\lambda}, \ -2\cos\theta\frac{\partial u}{\partial\lambda} - v\right),$$

$$\Delta \equiv \mathrm{div}\nabla, \quad \nabla\phi \equiv \left(\frac{1}{R\sin\theta}\frac{\partial\phi}{\partial\lambda}, \ \frac{1}{R}\frac{\partial\phi}{\partial\theta}\right), \quad \mathrm{div}\mathbf{U} \equiv \frac{1}{R\sin\theta}\frac{\partial u}{\partial\lambda} + \frac{1}{R}\frac{\partial(v\sin\theta)}{\sin\theta\,\partial\theta},$$

$$K\mathbf{U} \equiv (r^*|\mathbf{U}|u - lHv, lHu + r^*|\mathbf{U}|v), \quad r^* = const, \quad r^* > 0,$$

Numerical model

(V.I.Agoshkov, E.A.Botvinovsky [2008])

- Splitting scheme at (t_{j-1}, t_j) :

(6) $$H(\mathbf{U}_j^{(1)} - \mathbf{U}_{j-1}^{(1)})/\triangle t + K_{j-1}\mathbf{U}_j^{(1)} = \mathbf{f}_j,$$

(7) $$\begin{cases} H\dfrac{\mathbf{U}_j^{(2)} - \mathbf{U}_{j-1}^{(2)}}{\triangle t} - \nu\mathbf{\Delta}_1\mathbf{U}_j^{(2)} + gH\nabla\zeta_j^{(2)} = \beta gH\nabla\psi_j, \\ \dfrac{\zeta_j^{(2)} - \zeta_{j-1}^{(2)}}{\triangle t} + \mathrm{div}H\mathbf{U}_j^{(2)} = 0, \\ \mathbf{U}_j^{(2)}\big|_{\partial\Omega} = 0, \quad \mathbf{U}_{j-1}^{(2)} = \mathbf{U}_j^{(1)}, \quad \zeta_{j-1}^{(2)} = \zeta_j^{(1)}. \end{cases}$$

- The observability equation is included in the quadratic cost functional

(8) $$J_\alpha \equiv \frac{\alpha}{2}||\psi_j||^2_{L_2(\Omega)} + \frac{1}{2}||\zeta_j^{(2)} - \zeta_j^{obs}||^2_{L_2(\Omega)} \to \inf_{\psi_j}$$

- We denote $\phi \equiv (\mathbf{U}, \zeta) \equiv (\mathbf{U}_j^{(2)}, \zeta_j^{(2)})$, $j = 1, 2, \ldots, J$ and consider the observability equation (2) as

$$\zeta = \zeta_{obs} \text{ in } \Omega, \quad (9)$$

where $\zeta_{obs} \equiv \zeta_{obs}^{(j)}, j = 1, 2, \ldots, J$ - "observation data" (altimetry, or constructed by another ("ideal") model). Then (7)-(8) can be written as:

$$L\phi = f + B\psi, \quad C\phi = \varphi_{obs} \quad (10)$$

where $f \equiv H \nabla \Omega^+, B\psi = \beta g H \nabla \psi, C\phi \equiv \zeta, \varphi_{obs} \equiv \zeta_{obs}, L\phi$ is defined by the left-hand side of (7).

- Optimality system (Euler equation):

$$\alpha\psi + B^*(L^*)^{-1}C^*(C\phi - \phi_{obs}) = 0. \tag{11}$$

- **Theorem.** *The systems (10), (11) are uniquely solvable and* $\forall\varepsilon > 0\ \exists\alpha = \alpha(\varepsilon) > 0$ *such that*

$$\mu_C \equiv \mu_C^{(j)} \equiv \|\zeta_\alpha^j - \zeta_{obs}^j\|_{H_{obs}} \leqslant \varepsilon, j = 1, 2, \ldots, J$$

where $\phi_\alpha^{(j)} \equiv (\mathbf{U}_\alpha^{(j)}, \zeta_\alpha^{(j)}), \psi^j$ *is a solution of (11) at time step* j. *And for* $\alpha = 0$ *we have the equality* $\mu_C = 0$.

For $\beta \equiv 0$, the solution has the form $\phi_0 \equiv (\mathbf{U}_0, \zeta_0), \psi \equiv 0$, where ϕ_0, ζ_0 is the solution of the unperturbed model (6), (7) at $\beta = 0$. In this case $\mu_C = \|\zeta_0^j - \zeta_{obs}^j\|_{L_2(\Omega)}$.

- The iterative process (4) for solving (11) is:

$$\begin{cases} -\nu\,\Delta_1 \mathbf{U}^{(k)} + bH\mathbf{U}^{(k)} + gH\nabla\zeta^{(k)} = \mathbf{F} + \beta gH\nabla\psi^{(k)}, \ \mathbf{U}^{(k)}\big|_{\partial\Omega} = 0, \\ g\,\mathsf{div} H\mathbf{U}^{(k)} + gb\zeta^{(k)} = G, \end{cases}$$

where $b = 1/\triangle t, \quad \mathbf{F} = bH\mathbf{U}_{j-1}^{(2)}, \quad G = gb\zeta_{j-1}^{(2)};$

$$\begin{cases} -\nu\,\Delta_1 \mathbf{Q}^{(k)} + bH\mathbf{Q}^{(k)} - gH\nabla q_3^{(k)} = 0, \ \mathbf{Q}^{(k)}\big|_{\partial\Omega} = 0, \\ -g\,\mathsf{div} H\mathbf{Q}^{(k)} + gbq_3^{(k)} = \zeta^{(k)} - \zeta_j^{obs}, \end{cases}$$

where $\mathbf{Q}^{(k)} \equiv (q_1^{(k)}, q_2^{(k)})^T;$

$$(12) \qquad \psi^{(k+1)} = \psi^{(k)} - \tau_k(\alpha\psi^{(k)} - g\,\mathsf{div} H\mathbf{Q}^{(k)}), \quad k = 0, 1, \ldots$$

Testing the iterative process

- Solutions are given by the formulas:

$$(13)\qquad \begin{aligned} u_{ex}(\lambda,\theta) &= \sin\lambda\sin^2\theta, \\ v_{ex}(\lambda,\theta) &= 0, \\ \zeta_{ex}(\lambda,\theta) &= \sin^2\theta - 2/3, \\ \Psi_{ex}(\lambda,\theta) &= \sin 2\theta \sin 2\lambda \sin\theta. \end{aligned}$$

- **Test N1.** $H \equiv 1$ (depth (m)), $R = 5$ (radius (m)), $g = 10$, $\nu = 1$, $h_\lambda = h_\theta = 1^\circ$, $\triangle t = 10$. Functional value was reduced from $J^{(1)} = 68.4$ to $J^{(20)} = 1.72 * 10^{-4}$. Relative errors in $L_2(\Omega)$ are

k	δu^k	δv^k	$\delta\zeta^k$	$\delta\Psi^k$
1	0.07281	0.06075	1.3449	1
2	0.03149	0.02704	0.6728	0.4994
5	0.009005	0.005976	0.08313	0.06181
10	0.007038	0.002827	0.002677	0.001814
20	0.007028	0.002824	$3.39 * 10^{-6}$	0.0002325

- **Test N2.** $H \equiv 4000$, $R = 6371270$, $g = 10$, $\nu = 10^8$, $h_\lambda = h_\theta = 1^\circ$, $\triangle t = 10$. Functional value wasn't being reduced: $J^{(1)} = 98.1524$, $J^{(20)} = 129.806$. Relative errors in $L_2(\Omega)$ are

k	δu^k	δv^k	$\delta \zeta^k$	$\delta \Psi^k$
1	$3.49 * 10^{-5}$	$3.35 * 10^{-5}$	$2.53 * 10^{-5}$	1
10	$3.49 * 10^{-5}$	$3.35 * 10^{-5}$	$2.53 * 10^{-5}$	1
20	$3.49 * 10^{-5}$	$3.35 * 10^{-5}$	$2.53 * 10^{-5}$	1

- **Test N3.** $\nu \to +\infty$, $H \equiv 1$, $R = 5$, $g = 10$, $h_\lambda = h_\theta = 1^\circ$, $\triangle t = 10$.

ν	δu^{20}	δv^{20}	$\delta \zeta^{20}$	$\delta \Psi^{20}$	J^1	J^{20}
1	0.0070	0.0028	$3.39 * 10^{-5}$	0.00023	68.35	$1.72 * 10^{-4}$
10	0.0088	0.0040	$3.77 * 10^{-6}$	0.00023	62.33	$1.92 * 10^{-4}$
100	0.0092	0.0042	$1.61 * 10^{-6}$	0.00044	33.71	$8.23 * 10^{-5}$
1000	0.0092	0.0043	$2.03 * 10^{-6}$	0.0037	6.08	$1.03 * 10^{-4}$
10^6	0.0092	0.0043	$1.83 * 10^{-7}$	3.7337	0.023	$9.33 * 10^{-6}$
10^8	0.0092	0.0043	$2.24 * 10^{-7}$	373.46	0.021	$1.14 * 10^{-5}$

Test N4. $R \to +\infty$, $H \equiv 1$, $\nu = 1$, $g = 10$, $h_\lambda = h_\theta = 1^\circ$, $\triangle t = 10$.

R	δu^k	δv^k	$\delta \zeta^k$	$\delta \Psi^k$	J^1	J^{20}
5	0.0070	0.0028	$3.39 * 10^{-6}$	0.0002	23.3954	$1.48845 * 10^{-10}$
10	0.0054	0.0103	$1.91 * 10^{-6}$	0.0009	91.2875	$1.88076 * 10^{-10}$
20	0.0041	0.0839	$8.80 * 10^{-5}$	0.0093	96.3419	$1.60336 * 10^{-6}$
50	0.0410	7.5331	0.2019	2.0403	1560.95	52.7667
100	1.2008	13.3968	0.5639	6.6490	2628.86	1645.51
500	0.5005	3.8048	0.9666	9.2439	539.798	144162
1000	0.2616	1.9458	0.9702	9.4770	142.084	468783
5000	0.0399	0.0337	0.0010	0.9998	5.98895	14.8578
10000	0.0199	0.0168	0.0004	0.9999	1.52071	6.91981
6371270	$3.13 * 10^{-5}$	$2.97 * 10^{-5}$	$3.76 * 10^{-9}$	0.9991	$2.71 * 10^{-5}$	$2.97 * 10^{-4}$

Numerical experiments

- Let $\beta = 0$, then $\mu_C \equiv \|\zeta_j - \zeta_{obs,j}\|_{L_2(\Omega)}$.

  Experiment 1. Ω^+ is the "analytic tide potential". Here $\zeta_j \equiv \zeta_j^{(+)}$ is the deviation from the mean level $\zeta_{mean}^{(+)}$. The function $\zeta_j^{(+)}$ is presented in fig. 1 at 14:00 of 20th January 2000.

  Experiment 2. $\Omega^+ \equiv \Omega^{(8)}$ is the sum of the eight main harmonics $(K_1, O_1, P_1, Q_1,\ M_2, S_2, N_2, K_2)$ of the tide potential, and $\zeta_j \equiv \zeta_j^{(8)}$. The function $\zeta_j^{(8)}$ is given in fig. 2 at 14:00 of 20th January 2000.

  Here we suppose $\zeta_{obs,j} \equiv \zeta_j^{(+)}$. So $\mu_C \equiv \|\zeta_j^{(8)} - \zeta_j^{(+)}\|_{L_2(\Omega)}$ is the measure of the adequacy of the model with $\Omega^{(8)}$ to the model with Ω^+ :

$$\mu_C = 0.02\|\zeta_j^{(+)}\|_{L_2(\Omega)}$$

Experiment 3. The function $\zeta_{obs}^{(8)}$ calculated from satellite altimetry (by "harmonic analysis") presented in fig.3.

$$\frac{\|\zeta^{(+)} - \zeta_{obs}^{(8)}\|_{L_2(\Omega)}}{\|\zeta_{obs}^{(8)}\|_{L_2(\Omega)}} = 0.30$$

$$\frac{\|\zeta^{(8)} - \zeta_{obs}^{(8)}\|_{L_2(\Omega)}}{\|\zeta_{obs}^{(8)}\|_{L_2(\Omega)}} = 0.31$$

The $\zeta_{obs}^{(8)}, \zeta^{(8)}, \zeta^{(+)}$ coincides in qualitative manner at all time steps. However the value of the measure of the adequacy shows that we have to take into account additional force F corresponding to the effects of the self-attraction and loading of the water.

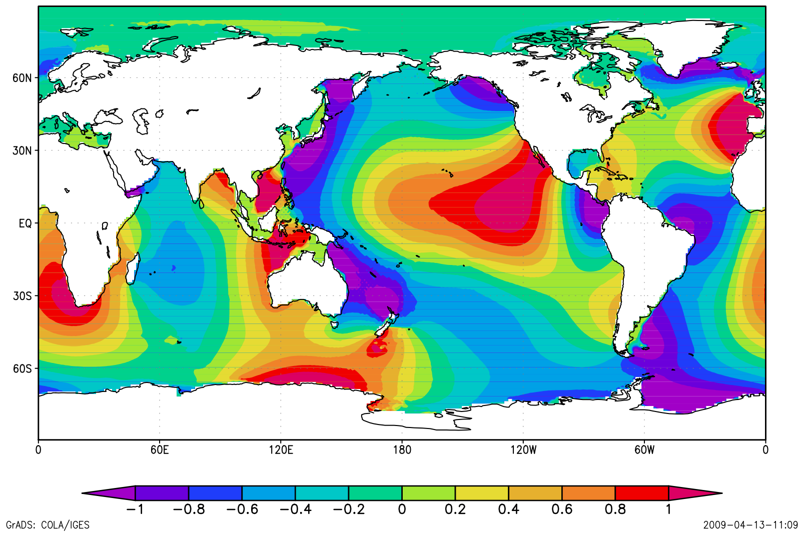

Fig. 1.

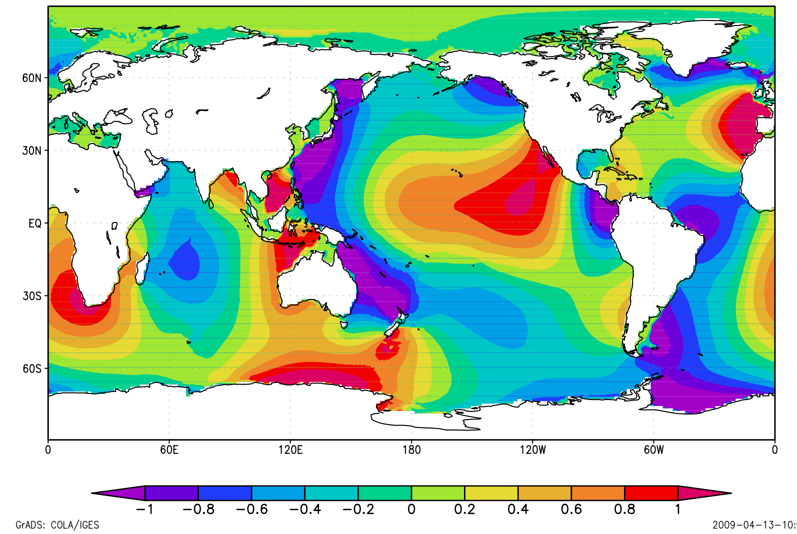

Fig. 2.

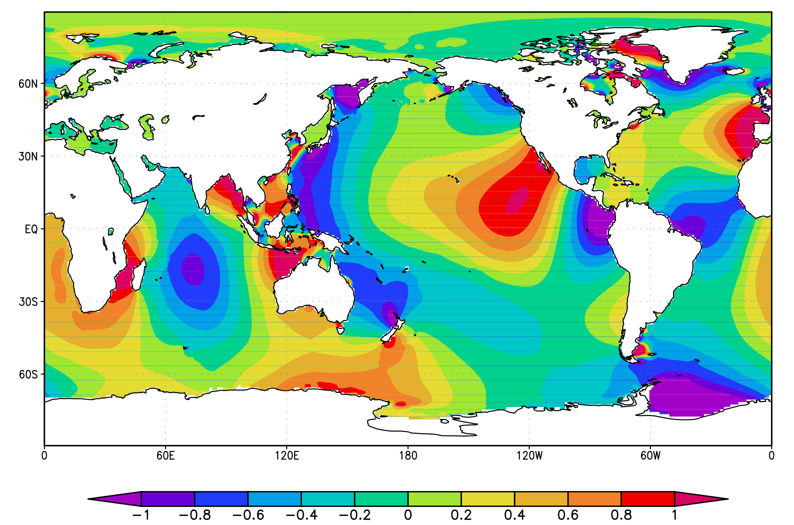

Fig. 3.

Conclusions

- The problem of the adequacy of complex mathematical models can be investigated by studying and solving the variational data assimilation problems.
- Taking into consideration the eight main harmonics the model is adequate to the model with "analytic tide potential" and it is also adequate to the observation data as a whole.
- The suggested iterative process for solving the problem of the adequacy of the considered tide dynamics model converges for a number of parameters' values. But for the detail study of the adequacy of the considered model of the tide dynamics it is needed to modify or upgrade the iterative process. Currently this is under development.

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Thank you!