DISCRETIZATION ERRORS AND CONTROL IN V.D.A

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Prediction and Errors

- Ingredients of Numerical Forecast
  - Mathematical Model
  - Observations
  - Statistics
- Errors
  - Model : physics, parametrization
  - Numerics :discretization, algorithms
  - Projection from the space of the state variable toward the space of observation
  - In observation : measurements, sampling.
- ***Errors are not always independant***

Questions :

- What is the impact of errors on the prediction?
- How to identify and control errors?
- Is it necessary to improve (better numerical discretization) the model to improve the prediction?

Variational Data Assimilation

- Model

$$\begin{cases} \dfrac{d\mathbf{u}}{dt} = F(x, \mathbf{u}), \quad t \in [0, T], \\ \mathbf{u}(0) = U, \end{cases}$$

- Cost Function

$$J(U) = \frac{\alpha}{2} \|U - U_0\|^2_{\mathcal{H}} + \frac{1}{2} \int_0^T \|H\mathbf{u} - X_{obs}\|^2_{\mathcal{H}_{obs}} \, dt,$$

- Adjoint

$$\begin{cases} \dfrac{dP}{dt} + \left[\dfrac{\partial F}{\partial \mathbf{u}}\right]^T \cdot P = H^T \left(HX - X_{obs}\right), \\ P(T) = 0. \end{cases}$$

- Gradient

$$\nabla J(U) = -P(0) + \alpha (U - U_0).$$

Error Control

- Introducing an error in the model

$$\begin{cases} \frac{d\mathbf{u}}{dt} = F(x, \mathbf{u}) + E(x, t), \\ \mathbf{u}(0) = U, \end{cases}$$

- In the cost function

$$\begin{aligned} J(U, E) &= \frac{\alpha}{2}\|U - U_0\|^2 + \frac{1}{2}\int_0^T \|H\mathbf{u} - X_{obs}\|^2 dt \\ &+ \frac{\beta}{2}\int_0^T (E, E) dt. \end{aligned}$$

- Adjoint

$$\begin{cases} \frac{dP}{dt} + \left[\frac{\partial F}{\partial \mathbf{u}}\right]^T \cdot P = H^T (H\mathbf{u} - X_{obs}), \\ P(T) = 0, \end{cases}$$

Gradient

$$
\begin{cases}
\nabla_U J(U,E) = \alpha(U - U_0) - P(0), \\
(\nabla_E J, \delta E) = \int_0^T \int_\Omega \left(- \left[\frac{\partial (F(x, \mathbf{u}(t)) + E)}{\partial E} \right]^T P(t) \right. \\
\qquad \left. + \beta E(x,t), \delta E \right) dx dt.
\end{cases}
$$

Discretization of the error

- Discretization in time and space :

$$\begin{cases} \frac{d\mathbf{u}}{dt} = F(\mathbf{u}) + \sum_{i,j} \varepsilon_{ij}\phi_j(t)X_i \\ \mathbf{u}(0) = U, \end{cases}$$

$$\begin{aligned} J(U,\Gamma) &= \frac{\alpha}{2}\|U - U_0\|^2_{\mathcal{H}} + \frac{1}{2}\int_0^T \|H\mathbf{u} - X_{obs}\|^2_{\mathcal{H}_{obs}} dt \\ &+ \frac{\beta}{2}\sum \varepsilon_{ij}^2, \end{aligned}$$

$$\nabla_{\varepsilon_{ij}} J(U,\Gamma) = \beta\varepsilon_{ij} - \int_0^T \phi_j(t)\,(X_i, P)\,dt.$$

$$\nabla_U J(U,\Gamma) = \alpha(U - U_0) - P(0)$$

Choices of the spaces

- Discretization Error

$$E \approx \sum_{p} \varepsilon(p) \nabla^{p} \mathbf{u}.$$

- With a first order scheme the discretization will depend on second order, the error could be represented in a basis of eigenvector of the Laplacian without information the eigenvectors of the covariances matrix can be considered:

$$\begin{cases} \frac{d\mathbf{u}}{dt} = F(x, \mathbf{u}) + \sum_{i,j} \varepsilon^1_{ij} \phi^1_j(t) X^1_i + \sum_{i,j} \varepsilon^2_{ij} \phi^2_j(t) X^2_i + \cdots \\ \mathbf{u}(0) = U, \end{cases}$$

VDA with Burger's equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{2}\dfrac{\partial(u^2 + au)}{\partial x} - \mu\left[\dfrac{\partial^2 u}{\partial x^2}\right] = f(x), \\ \qquad\qquad (x,t) \in (-1,1) \times (0,\infty), \\ \text{Boundary Condition: } u(-1,t) = u(+1,t) = 0 \ \forall t \in (0,\infty), \\ \text{Initial Condition: } u(x,0) = U, \\ J(U) = \inf_v J(v). \end{cases}$$

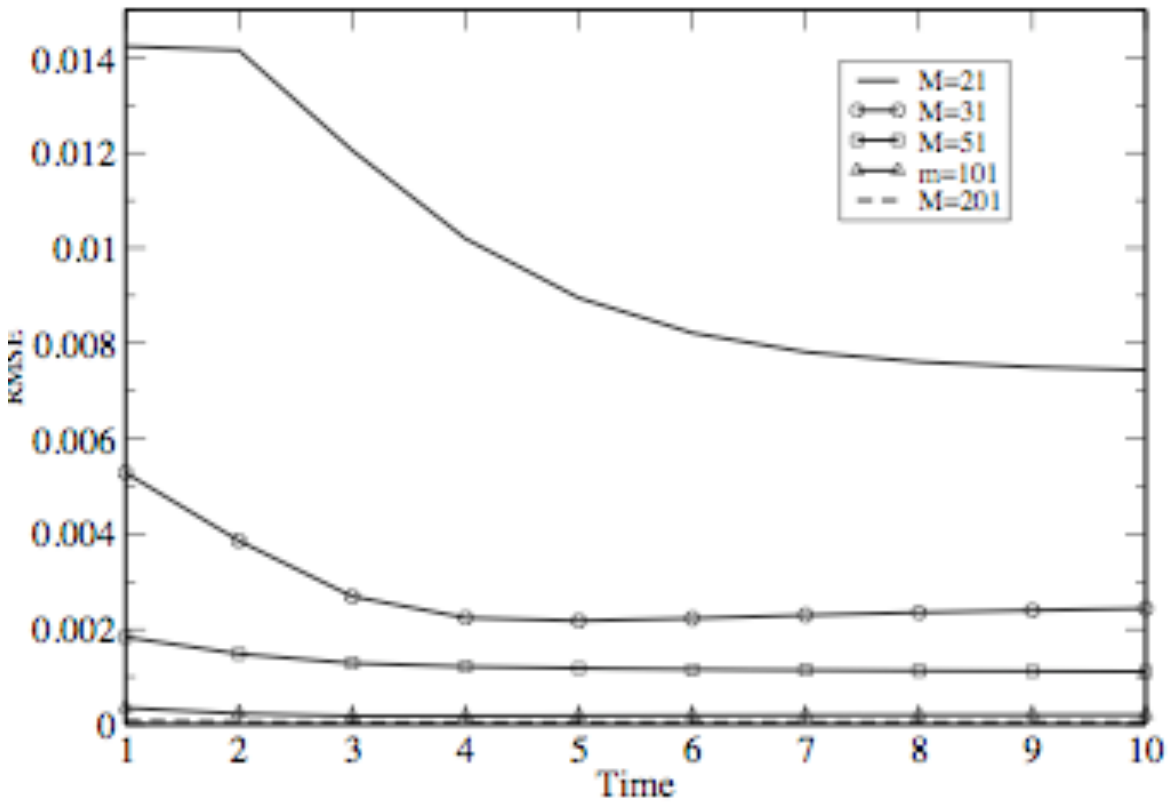
$$J(U) = \frac{1}{2}W_1 \sum_{i=2}^{M-1} (U^i - U_0^i)^2 + \frac{1}{2}W_2 \sum_{s=1}^{2}\sum_{k=1}^{K_s} (H(u(t_s,x)))^k - X_s^k)^2.$$

Burger's (2)

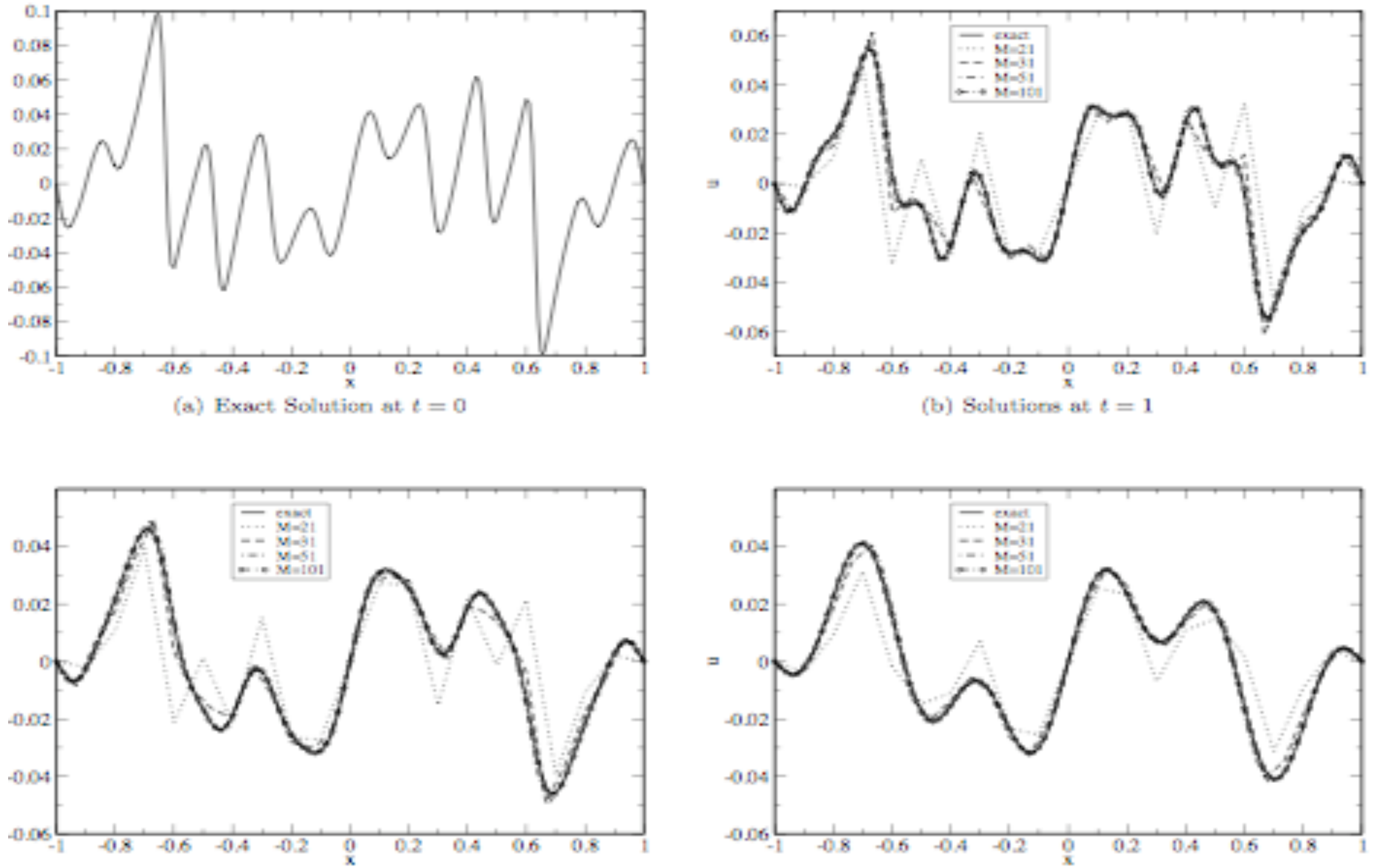
- F is choosen in such a way that the exact solution is known.
- Background term=exact solution+gaussian perturbation.
- Observations generated by the exact solution on a given regular grid.
- H is an interpolation operator.
- Consequently there are only two sources of error : discretization and interpolation.

Discretization Error

$$\text{RMSE} = \sqrt{\frac{1}{M}\sum_{i=1}^{M}(u_{ex}(i \triangle x, n \triangle t) - u_i^n)^2},$$



Exact and discretized solutions



(a) Exact Solution at $t = 0$

(b) Solutions at $t = 1$

(c) Solutions at $t = 2$

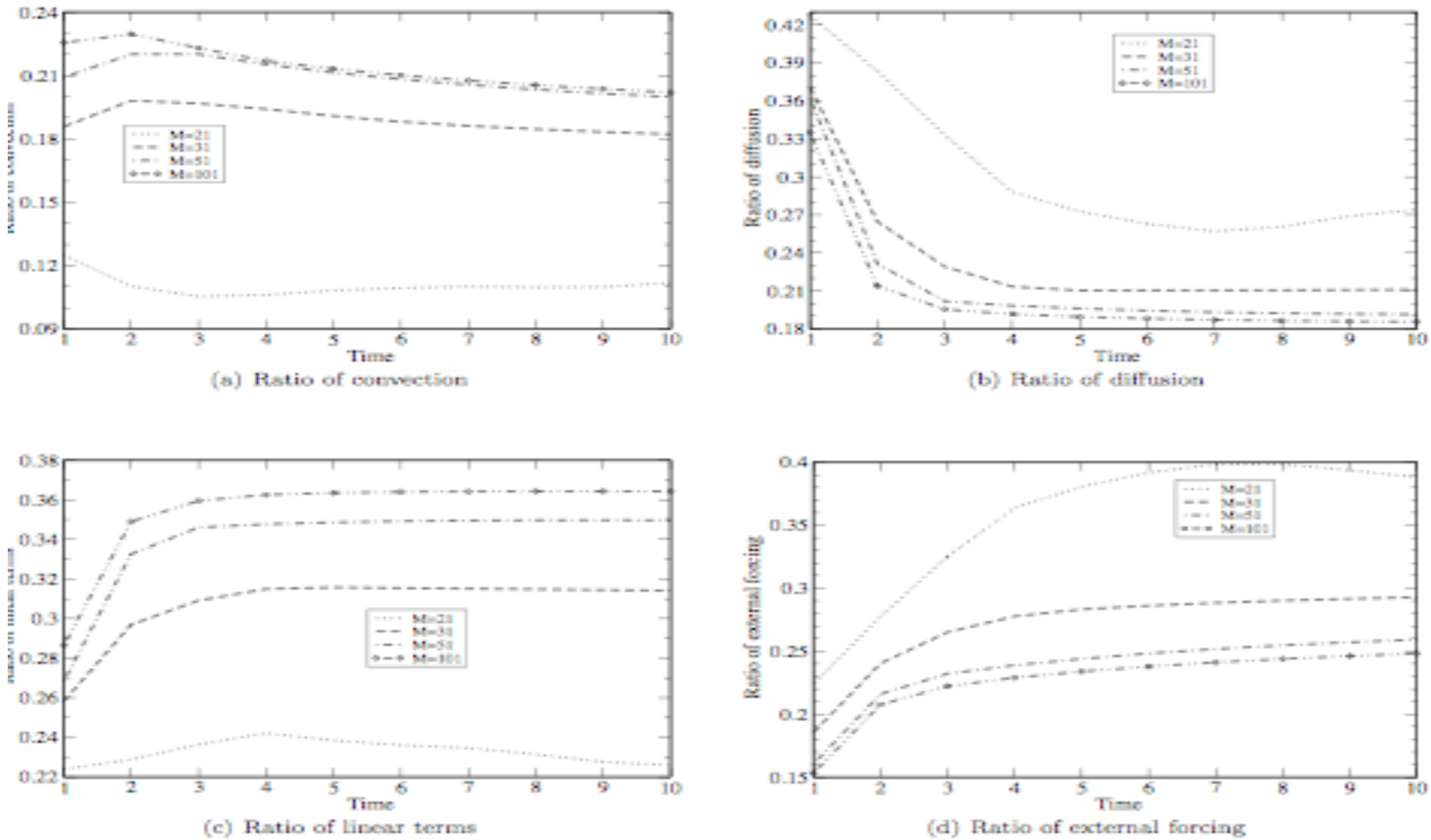
(d) Solutions at $t = 5$

Convergence of D.A.

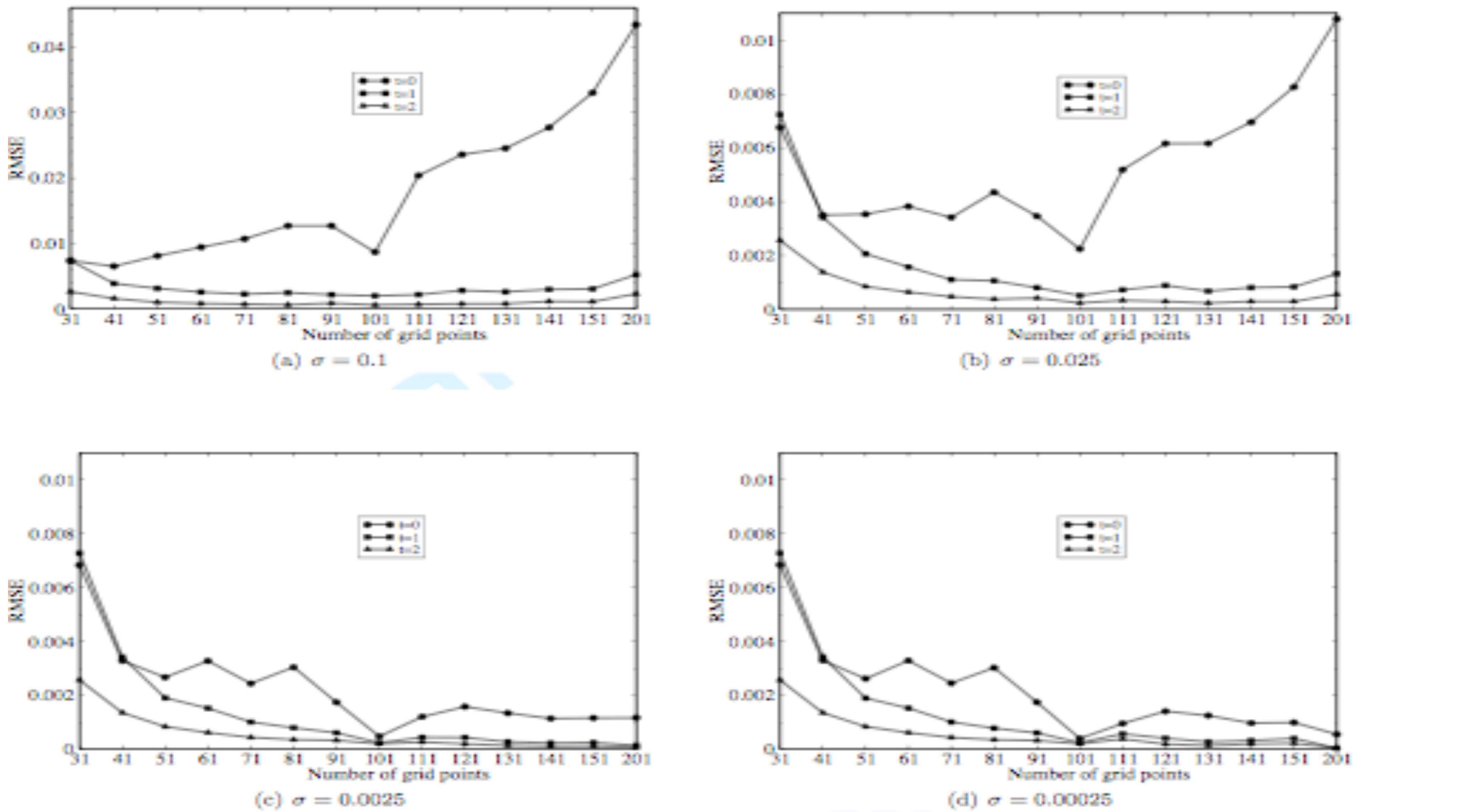
Table 1. Convergence of both the cost function and the L^2-norm of its gradient in the case $M = 101$ and $\sigma = 0.025$.

iteration	J	$\|\nabla J\|$
0	0.35078592×10^1	0.17191152×10^3
1	0.35058595×10^1	0.17180853×10^3
2	$0.76635219 \times 10^{-1}$	0.30238631
3	$0.76601067 \times 10^{-1}$	0.30068101
4	$0.76298041 \times 10^{-1}$	0.28555660
5	$0.70462862 \times 10^{-1}$	$0.12666941 \times 10^{-2}$
6	$0.70451991 \times 10^{-1}$	$0.54982245 \times 10^{-3}$
7	$0.70446412 \times 10^{-1}$	$0.47975452 \times 10^{-4}$
8	$0.70446346 \times 10^{-1}$	$0.44660779 \times 10^{-4}$
9	$0.70445638 \times 10^{-1}$	$0.10635374 \times 10^{-4}$
10	$0.70445471 \times 10^{-1}$	$0.20295671 \times 10^{-5}$
11	$0.70445438 \times 10^{-1}$	$0.67729479 \times 10^{-7}$
12	$0.70445438 \times 10^{-1}$	$0.56747292 \times 10^{-8}$
13	$0.70445437 \times 10^{-1}$	$0.23630726 \times 10^{-10}$
14	$0.70445437 \times 10^{-1}$	$0.79286146 \times 10^{-12}$
15	$0.70445437 \times 10^{-1}$	$0.23419518 \times 10^{-13}$

Evolution of various terms according to the discretization:convection,diffusion, linear term and forcing term



Influence of model resolution on D.A.



igure 4. RMSE of numerical solutions initialized with optimal initial condition at times $t = 0$, 1, and 2. The background field is the xact initial condition perturbed by Gaussian noise with mean zero and variance (a) $\sigma = 0.1$, (b) $\sigma = 0.025$, (c) $\sigma = 0.0025$, and (d) $\sigma = 0.00025$.

Influence of Observation Resolution on D.A.(1)

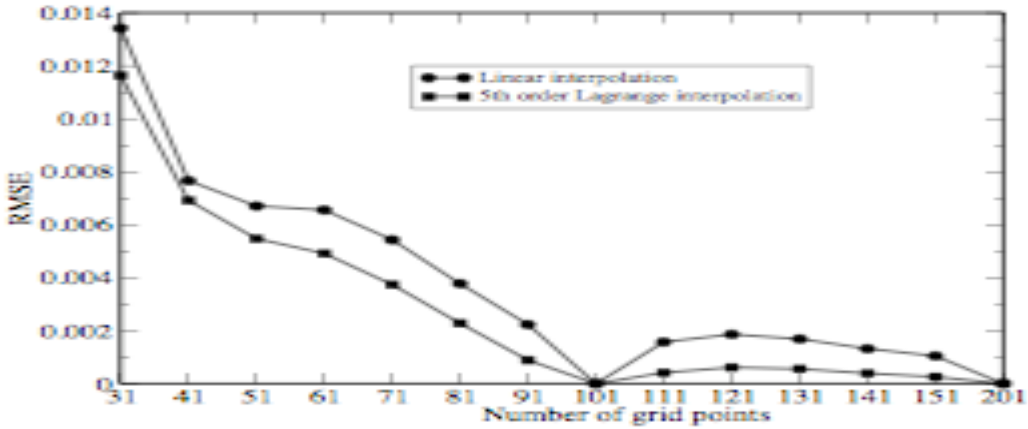
Table 2. RMSE of the numerical solution on 51-point grid.

obs number	49	99	199
$t = 0$	0.1597×10^{-2}	0.2952×10^{-2}	0.3144×10^{-2}
$t = 1$	0.1282×10^{-2}	0.1286×10^{-2}	0.1323×10^{-2}
$t = 0$	0.1277×10^{-2}	0.2835×10^{-2}	0.3110×10^{-2}
$t = 1$	0.1244×10^{-2}	0.1263×10^{-2}	0.1314×10^{-2}

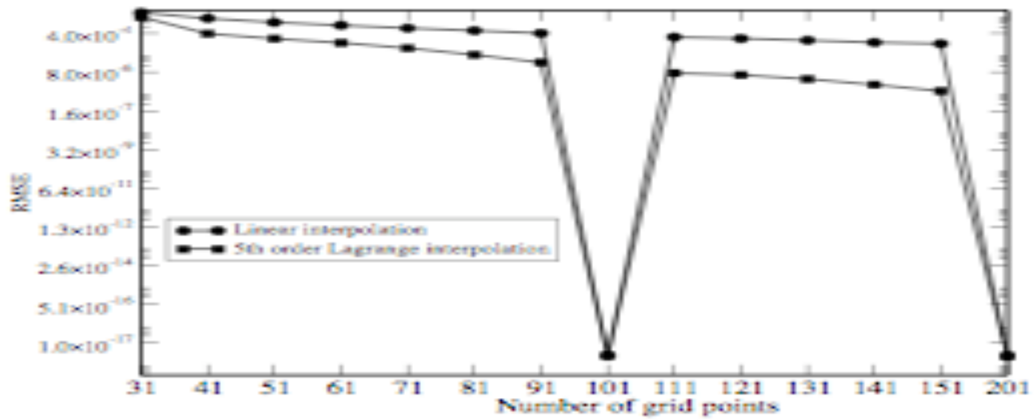
Table 3. RMSE of the numerical solution on 101-point grid.

obs number	49	99	199
$t = 0$	0.4467×10^{-2}	0.2817×10^{-2}	0.1718×10^{-2}
$t = 1$	0.1204×10^{-2}	0.9562×10^{-3}	0.6872×10^{-3}
$t = 0$	0.1956×10^{-2}	0.4035×10^{-3}	0.2411×10^{-3}
$t = 1$	0.2838×10^{-3}	0.2428×10^{-3}	0.2552×10^{-3}

Influence of Observation Operator on D.A.(1)



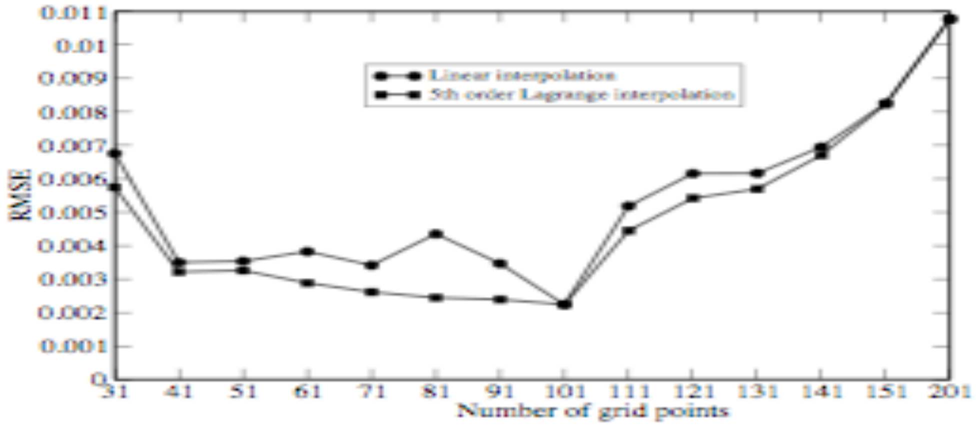
(a) $t = 0$



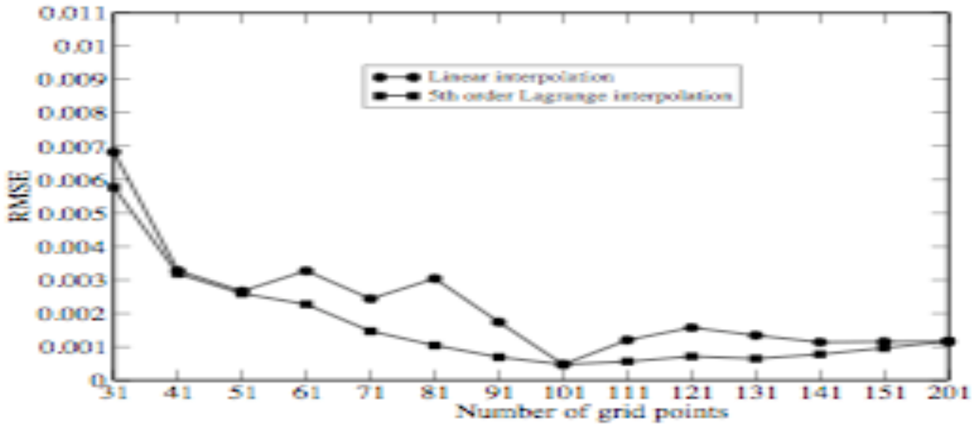
(b) $t = 1$

Figure 8. RMSE of the exact solutions interpolated on to the observation sites at (a) $t = 0$ and (b) $t = 1$.

Influence of Observation Operator on D.A.(2)



(a) $\sigma = 0.025$



(b) $\sigma = 0.0025$

e 9. RMSE of the optimal initial condition relative to the exact one with background perturbation variance (a) $\sigma = 0.025$, and $= 0.0025$.

Influence of Observation Operator on D.A.(3)

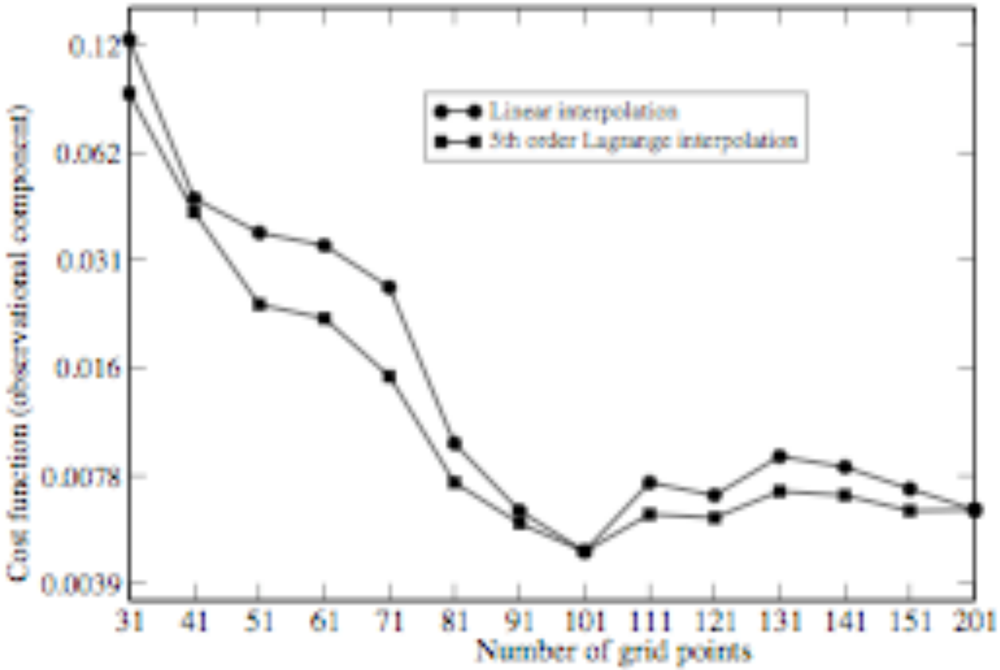
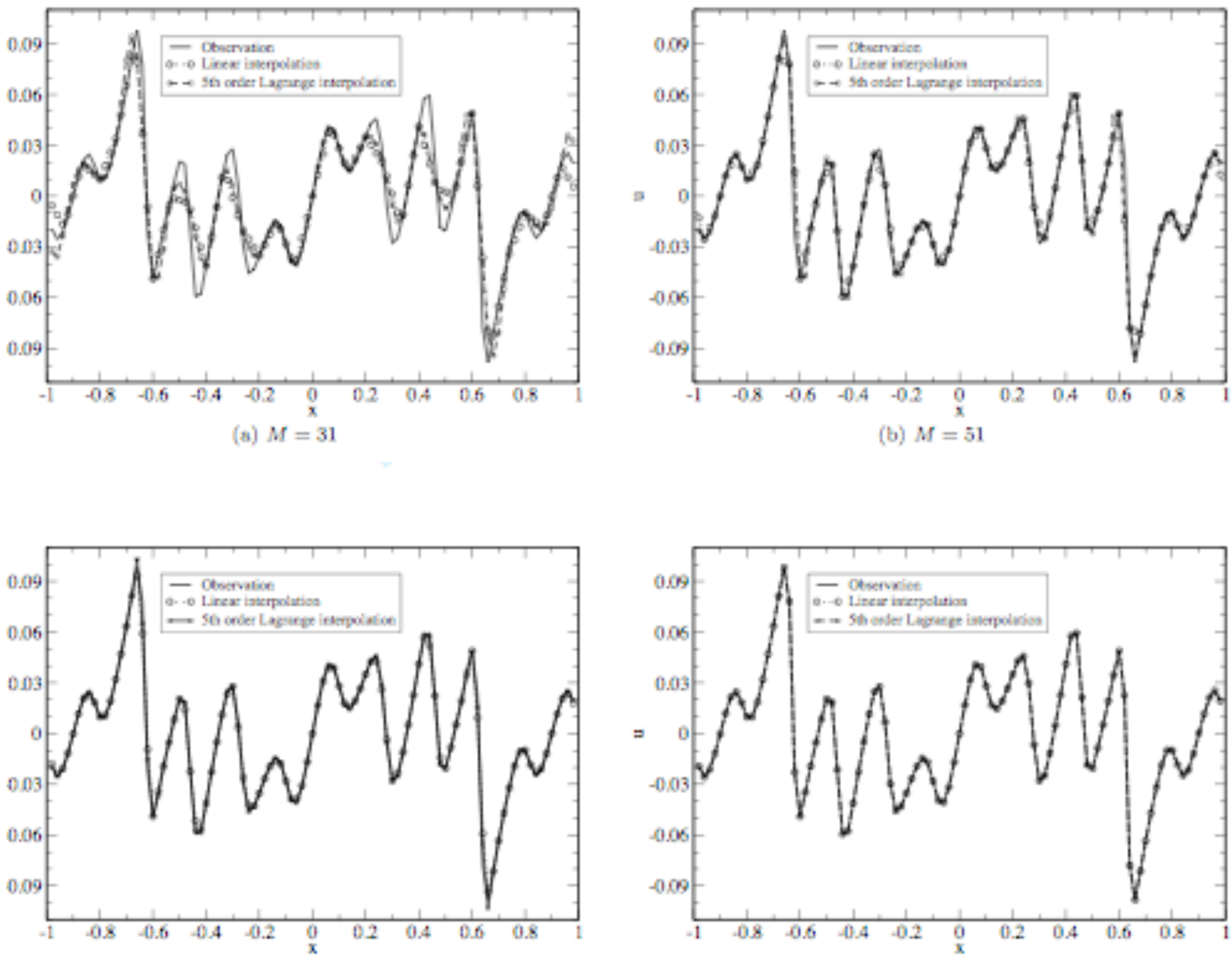


Figure 10. Observational cost function versus model resolution.



(a) $M = 31$

(b) $M = 51$

(c) $M = 81$

(d) $M = 101$

Figure 11. The observations and the exact solutions interpolated on to the observation sites.

Control of Errors (1)

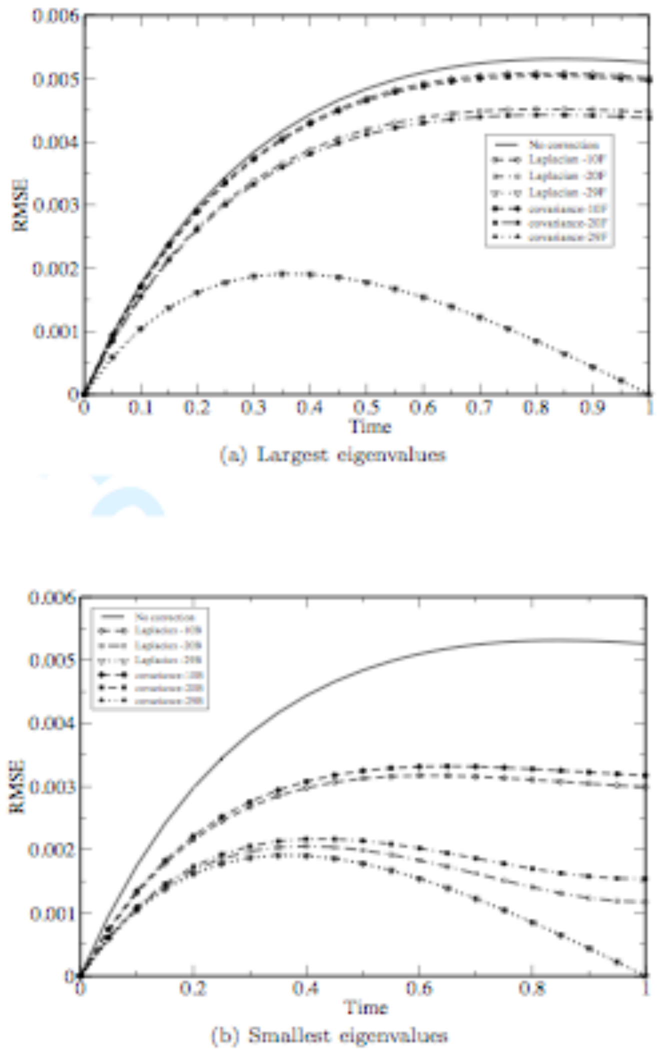
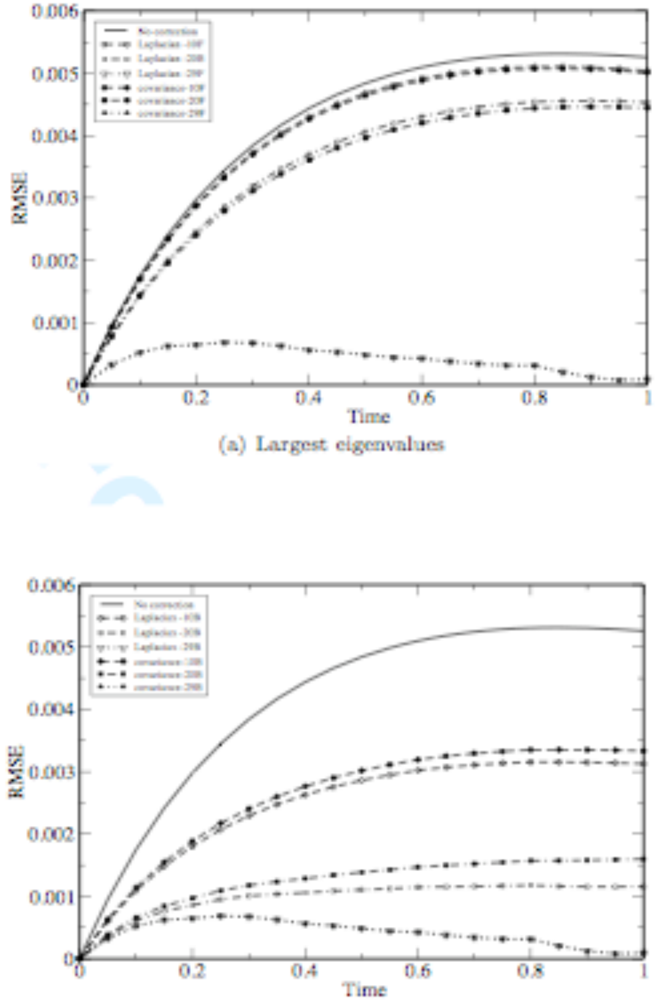


Figure 12. Comparison of the evolution of RMSE of the solutions for the modified model based on the eigenvectors of the Laplacian operator and the negative error covariance matrix. The model resolution is $M = 31$.

Control of Errors (2)



(b) Smallest eigenvalues

Figure 13. Same as figure 12 with 5 piecewise-constant subintervals of the time domain (0, 1).

Conclusions (1)

- With coarse resolution discretization are dominant. The importance of background is small
- For an increased resolution, the optimal solutio is a balance between background and observation. There exists an optimal resolution of the model.
- When the resolution is increased further beyong that of the observation the background eror is dominant.

Conclusions (2)

- Prediction can be improved by the control of errors. Without a priori knowledge on the source of errors, the eigenvectors of the covariance matrix are appropriate.
- Increasing resolution does not necessarily translate into improved assimilation. If the model solution is not sufficiently resolved high density of observationdegrades the data assimilation and prediction.
- Extension of the study to error on observations must be carried outon the Optimality System (cf. V. Shutyaev, I. Gejadze and FXLD)