Numerical solution of the variational assimilation problem using on-line SST data in the World Ocean

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1. Mathematical formulation of the problem

$$\frac{d\vec{u}}{dt} + \begin{bmatrix} 0 & -f \\ f & 0 \end{bmatrix} \vec{u} - g \cdot grad\xi + A_u\vec{u} + (A_k)^2\vec{u} = \vec{f} - \frac{1}{\rho_0} gradP_a - \frac{g}{\rho_0} grad \int\limits_0^z \rho_1(T,S)dz',$$

$$\frac{\partial \xi}{\partial t} - m\frac{\partial}{\partial x}(\int\limits_0^H \Theta(z)udz) - m\frac{\partial}{\partial y}(\int\limits_0^H \Theta(z)\frac{n}{m}vdz) = f_3,$$

$$\frac{dT}{dt} + A_T T = f_T, \ \frac{dS}{dt} + A_S S = f_S,$$

where

$$\bar{f} = g \cdot gradG, \ \Theta(z) \equiv \frac{r(z)}{R}, \quad r = R - z, \quad 0 < z < H.$$

(V.I. Agoshkov, A.V.Gusev, N.A. Diansky, R.B.Oleinikov, 2007)

Boundary conditions on the surface

$$
\begin{cases}
\left(\int\limits_0^H \Theta \vec{u} dz\right) \vec{n} + \beta_0 m_{op} \sqrt{gH}\, \xi = m_{op} \sqrt{gH}\, d_s \text{ on } \partial\Omega, \\
U_n^{(-)} u - \nu \dfrac{\partial u}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k u = \tau_x^{(a)} \big/ \rho_0, \; U_n^{(-)} v - \nu \dfrac{\partial v}{\partial z} - k_{33} \dfrac{\partial}{\partial z} A_k v = \tau_y^{(a)} \big/ \rho_0, \\
A_k u = 0, \quad A_k v = 0, \\
U_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + U_n^{(-)} d_T, \\
U_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + U_n^{(-)} d_S.
\end{cases}
$$

With the function $\phi = (u, v, \xi, T, S)$ known, we calculate

$$w(x,y,z,t) = \frac{1}{r}(m\frac{\partial}{\partial x}(\int_z^H rudz') + m\frac{\partial}{\partial y}(\frac{n}{m}\int_z^H rvdz')), (x,y,z,t) \in D \times (0, \bar{t}),$$

$$P(x,y,z,t) = P_a(x,y,t) + \rho_0 g(z-\xi) + \int_0^z g\rho_1(T,S)dz'.$$

Note, that for $U_n \equiv \underline{U} \cdot \underline{N}$ (here $U = (u, v, w)$) we always have

$$U_n = 0 \text{ on } \Gamma_{c,w} \cup \Gamma_H.$$

Problem **I**. The model approximation by splitting method.

Step 1. We consider the system:

$$
\begin{cases}
T_t + (\bar{U}, \mathbf{Grad})T - \mathbf{Div}(\hat{a}_T \cdot \mathbf{Grad}\ T) = f_T \text{ in } D \times (t_{j-1}, t_j), \\
T = T_{j-1} \text{ for } t = t_{j-1} \text{ in } D, \\
\bar{U}_n^{(-)} T - \nu_T \dfrac{\partial T}{\partial z} + \gamma_T (T - T_a) = Q_T + \bar{U}_n^{(-)} d_T \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
\dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)} T + \dfrac{\partial T}{\partial N_T} = \bar{U}_n^{(-)} d_T + Q_T \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\dfrac{\partial T}{\partial N_T} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\
T_j \equiv T \text{ on } D \times (t_{j-1}, t_j).
\end{cases}
$$

Step 2.

$$
\begin{cases}
S_t + (\bar{U}, \mathbf{Grad})S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_{j-1}, t_j), \\[2ex]
S = S_{j-1} \text{ at } t = t_{j-1} \text{ in } D, \\[2ex]
\bar{U}_n^{(-)} S - \nu_S \dfrac{\partial S}{\partial z} + \gamma_S (S - S_a) = Q_S + \bar{U}_n^{(-)} d_S \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\
\bar{U}_n^{(-)} S + \dfrac{\partial S}{\partial N_S} = \bar{U}_n^{(-)} d_S + Q_S \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\
\dfrac{\partial S}{\partial N_S} = 0 \text{ on } \Gamma_H \times (t_{j-1}, t_j), \\[2ex]
S_j \equiv S \text{ on } D \times (t_{j-1}, t_j).
\end{cases}
$$

Step 3.

$$
\begin{cases}
\underline{u}_t^{(1)} + \begin{bmatrix} 0 & -\ell \\ \ell & 0 \end{bmatrix} \underline{u}^{(1)} - g \cdot \mathbf{grad}\xi = g \cdot \mathbf{grad}G - \dfrac{1}{\rho_0}\mathbf{grad}\left(P_a + g\displaystyle\int_0^z \rho_1(\bar{T},\bar{S})dz'\right) \\
\text{ in } D \times (t_{j-1}, t_j), \\
\xi_t - \mathbf{div}\left(\displaystyle\int_0^H \Theta\underline{u}^{(1)} dz\right) = f_3 \text{ in } \Omega \times (t_{j-1}, t_j), \\
\underline{u}^{(1)} = \underline{u}_{j-1},\ \xi = \xi_{j-1} \text{ at } t = t_{j-1}, \\
\left(\displaystyle\int_0^H \Theta\underline{u}^{(1)} dz\right) \cdot n + \beta_0 m_{op}\sqrt{gH}\xi = m_{op}\sqrt{gH}d_s \text{ on } \partial\Omega \times (t_{j-1}, t_j), \\
\underline{u}_j^{(1)} \equiv \underline{u}^{(1)}(t_j) \text{ in } D
\end{cases}
$$

$$
\begin{cases}
\underline{u}_t^{(2)} + \begin{bmatrix} 0 & -f_1(\bar{u}) \\ f_1(\bar{u}) & 0 \end{bmatrix} \underline{u}^{(2)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\
\underline{u}^{(2)} = \underline{u}_j^{(1)} \text{ при } t = t_{j-1} \text{ in } D, \\
\underline{u}_j^{(2)} \equiv \underline{u}^{(2)}(t_j) \text{ in } D,
\end{cases}
$$

Step 3. (continued)

$$\begin{cases} \underline{u}_t^{(3)} + (\bar{U}, \mathbf{Grad})\underline{u}^{(3)} - \mathbf{Div}(\hat{a}_u \cdot \mathbf{Grad}\ )\underline{u}^{(3)} + (A_k)^2\underline{u}^{(3)} = 0 \text{ in } D \times (t_{j-1}, t_j), \\ \underline{u}^{(3)} = \underline{u}^{(2)} \text{ at } t = t_{j-1} \text{ in } D, \\ \bar{U}_n^{(-)}\underline{u}^{(3)} - \nu_u \dfrac{\partial \underline{u}^{(3)}}{\partial z} - k_{33}\dfrac{\partial}{\partial z}(A_k\underline{u}^{(3)}) = \dfrac{\underline{\tau}^{(a)}}{\rho_0}, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_S \times (t_{j-1}, t_j), \\ U_n^{(3)} = 0, \dfrac{\partial U^{(3)}}{\partial N_u} \cdot \bar{\underline{\tau}}_w + \left(\dfrac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \underline{\tau}_w = 0, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,c} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)}(\tilde{U}^{(3)} \cdot \underline{N}) + \dfrac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{N} + \left(\dfrac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \bar{N} = \bar{U}_n^{(-)}d, A_k\underline{u}^{(3)} = 0 \text{on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \bar{U}_n^{(-)}(\tilde{U}^{(3)} \cdot \bar{\tau}_w) + \dfrac{\partial \tilde{U}^{(3)}}{\partial N_u} \cdot \bar{\tau}_w + \left(\dfrac{\partial}{\partial N_k} A_k\underline{u}^{(3)}\right) \cdot \underline{\tau}_w = 0, A_k\underline{u}^{(3)} = 0 \text{ on } \Gamma_{w,op} \times (t_{j-1}, t_j), \\ \dfrac{\partial \underline{u}^{(3)}}{\partial N_u} = \dfrac{\tau^{(b)}}{\rho_0} \text{ on } \Gamma_H \times (t_{j-1}, t_j), \end{cases}$$

where

$$\underline{u}^{(3)} = (u^{(3)}, v^{(3)}),\ \tau^{(a)} = (\tau_x^{(a)}, \tau_y^{(a)}),$$
$$U^{(3)} = (u^{(3)}, w^{(3)}(u^{(3)}, v^{(3)})),\ \tilde{U}^{(3)} = (u^{(3)}, 0),$$
$$\tau^{(b)} = (\tau_x^{(b)}, \tau_y^{(b)}).$$

Splitting methods (G.I. Marchuk) are used to approximate subproblems on Steps 1-3 (Diansky N.A., Gusev A.V.)

Step 1:

$$(T_1)_t + L_1 T_1 = \mathcal{F}_1, \quad t \in (t_{j-1}, t_j),$$

$$T_1 = T_{j-1} \quad \text{at} \quad t = t_{j-1}$$

$$(T_2)_t + L_2 T_2 = \mathcal{F}_2 + BQ_T, \quad t \in (t_{j-1}, t_j),$$

$$T_2(t_{j-1}) = T_1(t_j).$$

$$T_2(t_j) \equiv T_j \cong T \quad \text{at} \;\; t = t_j.$$

3. Inverse problem

Let us assume, that the unique function which is obtained by observation data processing is the function T_{obs} on subdomain $\Omega_0^{(j)}$ of Ω at $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$. Let by phisical meaning the function $T_{obs} = T_{obs}^{(j)}$ is an approximation to STT data on $\Omega_0^{(j)}$, i.e to $T\Big|_{z=0}$. We permit that the function $T_{obs}^{(j)}$ is known only on the part of $\Omega \times (0, \bar{t})$, i.e. $\Omega_0^{(j)}$ at $t \in (t_{j-1}, t_j)$ and we define a support of this function as $m_0^{(j)}$. Beyond of this area we suppose function $T_{obs}^{(j)}$ is trivial.

Let the function of ocean surface heat flux Q is an "additional unknown function" on $\{\Omega_0^{(j)}\}$ (assuming that Q is known on $\{\Omega \backslash \Omega_0^{(j)}\}$) and we state the following inverse problem: *find the solution* ϕ *of the Problem* I *and function* Q, *such that,* $m_0^{(j)}(T - T_{obs}^{(j)}) = 0$.

To study this inverse problem theoretically we apply general methodology for solving data assimilation problems (Agoshkov V., 2003) and classical results of the inverse problem theory (A.N. Tikhonov, M.M. Lavrentiev, V.K. Ivanov, V.V. Vasin, V.G. Romanov, Yu.E. Anikonov, S.I. Kabanikhin).

SST data assimilation problem

We consider the cost-function in the form:

$$J_\alpha \equiv J_\alpha(Q,\phi) = \frac{1}{2}\int\limits_0^{\bar{t}} \int\limits_{\Omega_0(t)} \alpha|Q - Q^{(0)}|^2 d\Omega dt + J_0(\phi) = \sum_{j=1}^{J} J_{\alpha,j}$$

$$J_0(\phi) = \frac{1}{2}\int\limits_0^{\bar{t}} \int\limits_{\Omega_0(t)} \alpha|T - T_{obs}|^2 d\Omega dt$$

$$J_{\alpha,j} = \frac{1}{2}\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} \alpha|Q - Q^{(0)}|^2 d\Omega dt + \frac{1}{2}\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} m_0^{(j)}|T - T_{obs}^{(j)}|^2 d\Omega dt$$

Here $\alpha \equiv \alpha(\lambda,\theta,t)$ is a regularization function(is it possible, that $\alpha(\lambda,\theta,t) =$const$\geq 0$) and it may be a dimensional quantity; $Q^{(0)} \equiv Q^{(0)}(\lambda,\theta,t)$ is a given function.
We can formulate the data assimilation problem : *find the solution* ϕ *of the Problem* I *and function* Q, *such that, the functional* J_α *is minimal on the set of the solutions.*

The optimality system obtained consist of successive solving the variational assimilation problem on intervals $t \in (t_{j-1}, t_j)$, $j = 1, 2, \ldots, J$ (Agoshkov V.I., 2006). The method can be discribed as follows:

STEP 1. We solve system of equations, which arise from minimization of the functional J_α on the set of the solution of the equations. This system consists of equations for T_1, T_2, Q and system of adjoint equations:

$$\begin{cases} (T_2^*)_t + L_2^* T_2^* = B^* m_0^{(1)} (T - T_{obs}^{(1)}) \quad \text{in} \;\; D \times (t_0, t_1), \\ T_2^* = 0 \quad \text{for} \;\; t = t_1, \end{cases}$$

$$\begin{cases} (T_1^*)_t + L_1^* T_1^* = 0 \quad \text{in} \;\; D \times (t_0, t_1), \\ T_1^* = T_2^*(t_0) \quad \text{for} \;\; t = t_1 \end{cases}$$

$$\alpha(Q - Q^{(0)}) + T_2^* = 0 \quad \text{on} \;\; \Omega_0^{(1)} \times (t_0, t_1).$$

Functions T_2, $Q(t_1)$ are accepted as approximations to functions T, Q of the full solution for the Problem I at $t > t_1$, and $T_2(t_1) \cong T(t_1)$ is taken as an initial condition to solve the problem on the interval (t_1, t_2).

STEP 2. Solve problem for S:

$$S_t + (\bar{U}, \mathbf{Grad}) S - \mathbf{Div}(\hat{a}_S \cdot \mathbf{Grad}\ S) = f_S \text{ in } D \times (t_0, t_1)$$

with corresponding boundary and initial conditions. After that the function S is accepted as an approximate solution, and the function $S(t_1)$ is taken as an initial condition for the problem for the interval (t_1, t_2).

STEP 3. Solve equations of the velocity module.

Iteration process

Given $Q^{(k)}$ one solve all subproblems from step 1, adjoint problem for this step and define new correction $Q^{(k+1)}$

$$Q^{(k+1)} = Q^{(k)} - \gamma_k^{(j)}(\alpha(Q^{(k)} - Q^{(0)}) + T_2^*) \quad \text{on} \quad \Omega_0^{(j)} \times (t_{j-1}, t_j).$$

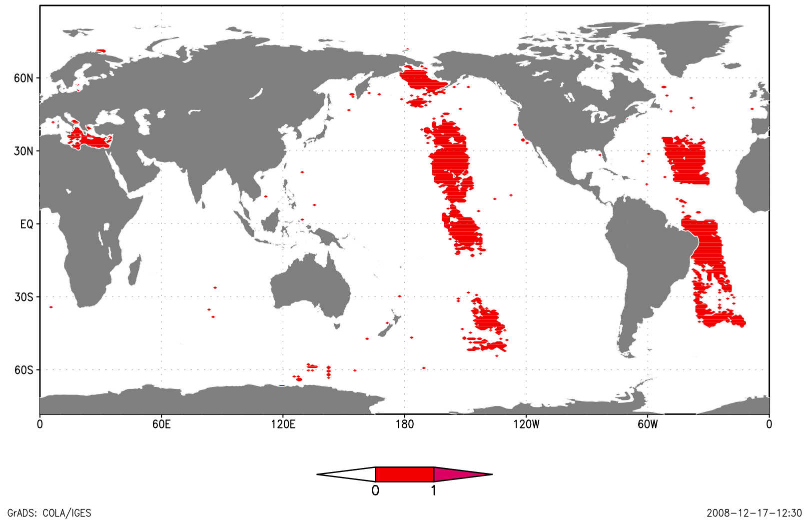
Parameters $\{\gamma_k\}$ can be calculated at $\alpha \approx +0$, by the property of dense solvability, as:

$$\gamma_k^{(j)} = \frac{1}{2} \frac{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (T - T_{obs}^{(j)})^2 \Big|_{\sigma=0} d\Omega dt}{\int\limits_{t_{j-1}}^{t_j} \int\limits_{\Omega_0^{(j)}} (T_2^*)^2 \Big|_{\sigma=0} d\Omega dt}.$$

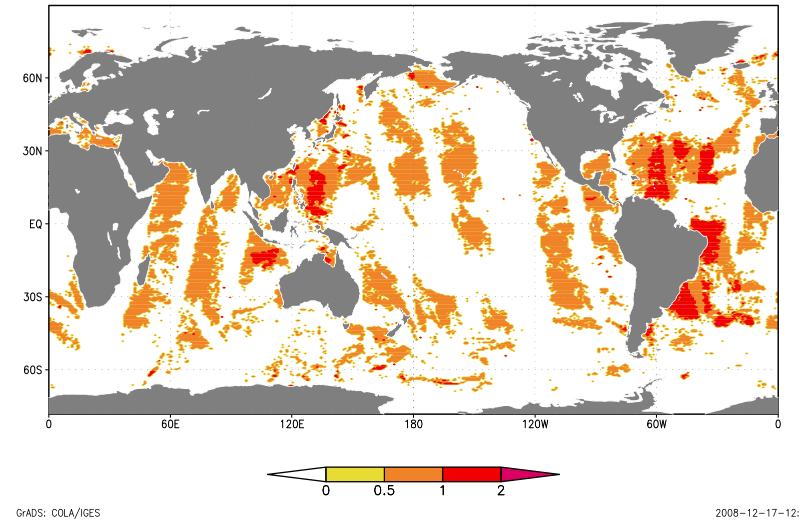
5. Numerical experiments

The object of simulation is the World Ocean. We can describe the parameters of the area studied and its geographical coordinates are: the grid 360x337x40 (latitude×longitude×depth); the first mesh point is the point with coordinates 22.5 E and 78.25 S. The grid steps with respect to x and y are constant and equal 1.0 and 0.5 degrees, respectively. The time step is equal to $\Delta t = 1$ hour.
The data of SST, which was obtained from Geophysical Center of RAS, were used for the construction of the function T_{obs}.
The mean flux for January $Q^{(0)}$ was taken from the database of NCEP (National Centers for Environmental Prediction).
The observation data assimilation module to assimilate T_{obs} was included into the thermohydrodynamics model of the World Ocean. The longest time period taken in experiments was 3 months (start from January 2004).

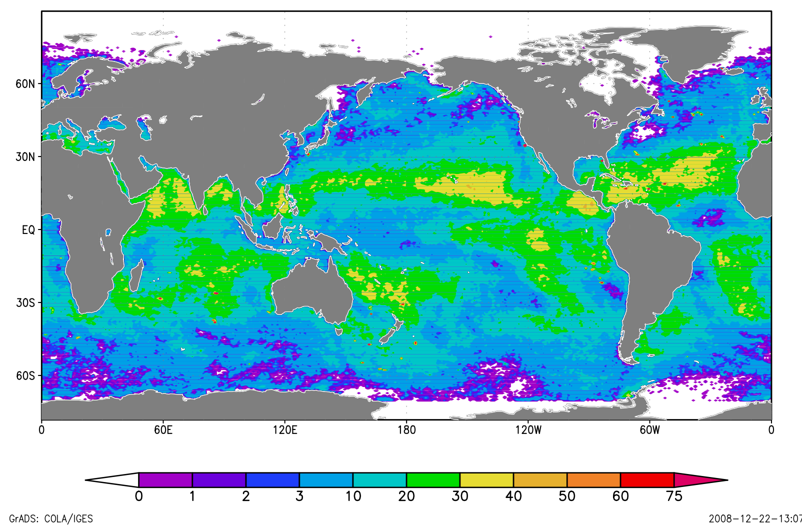
Observation data mask by some period



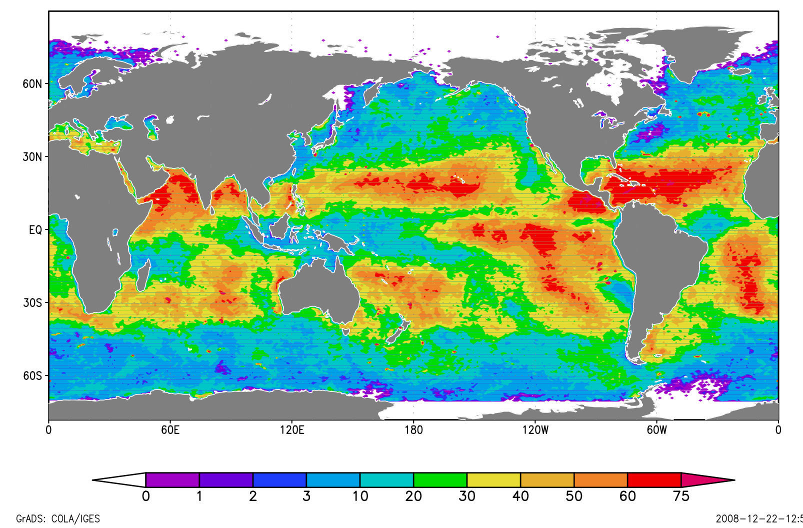
(a) 1 hour



(b) 6 hours

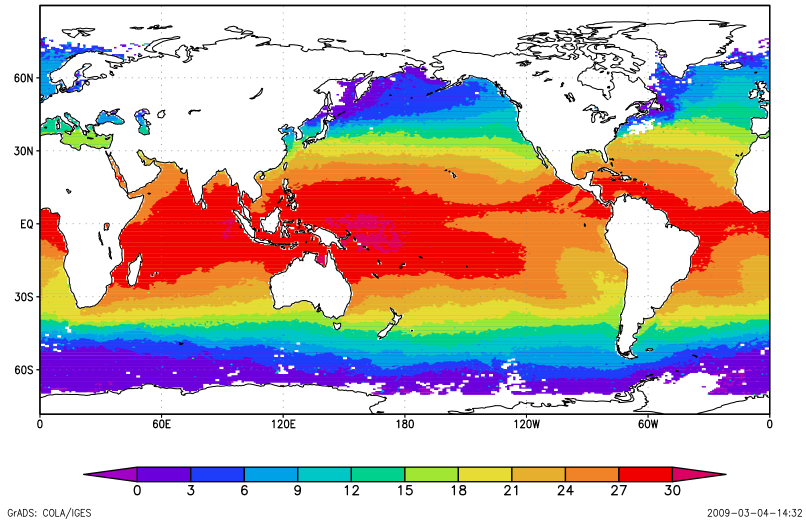


(c) 15 days

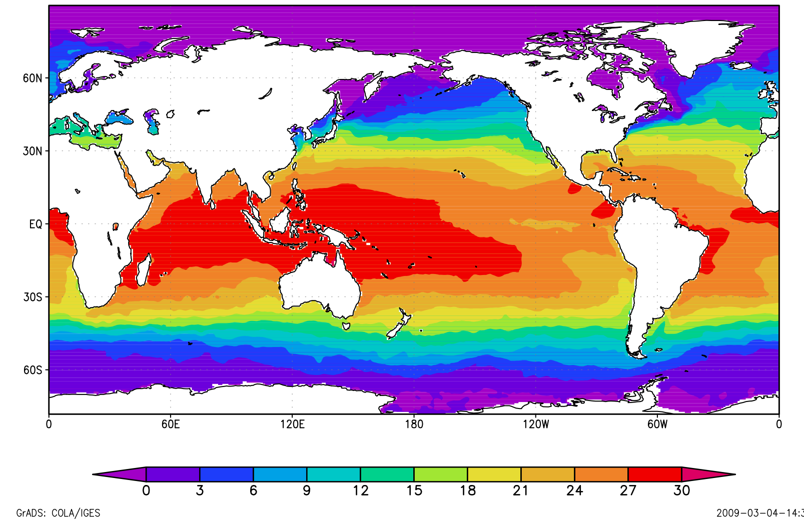


(d) 1 month

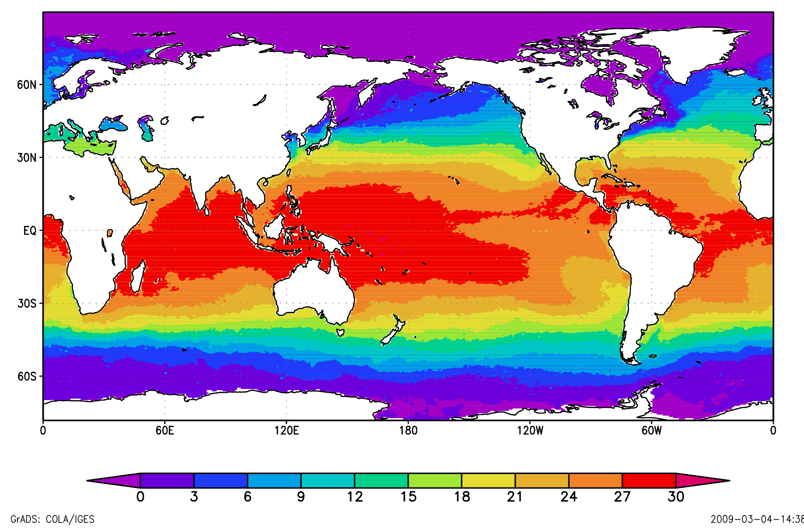
SST after 1 month of simulations (start from 1 of January)



(a) The observation data

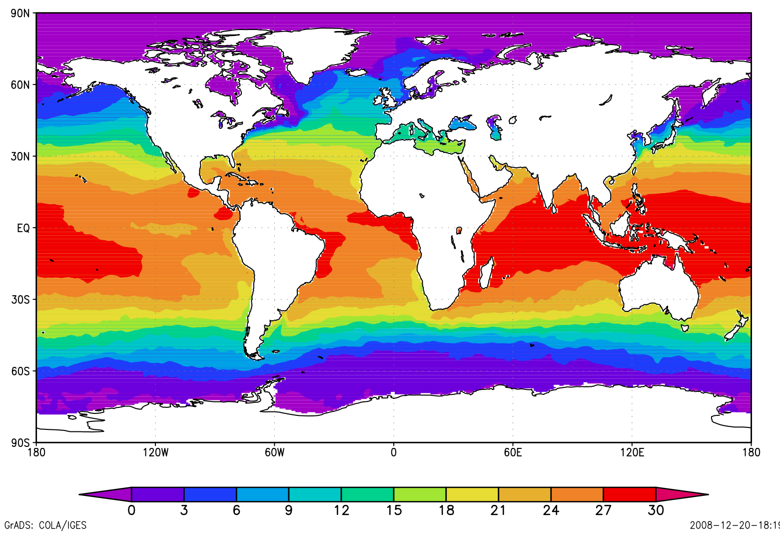


(b) SST obtaned without assimilation

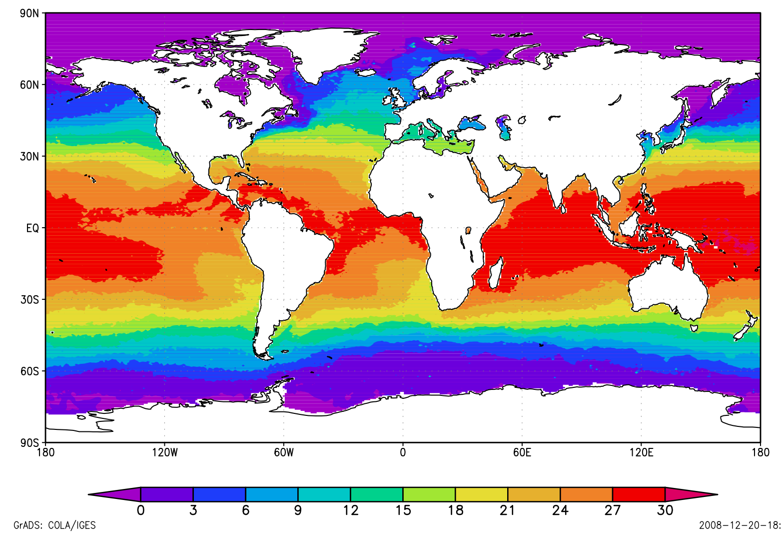


(c) SST calculated with assimilation $\alpha = 10^{-6}$

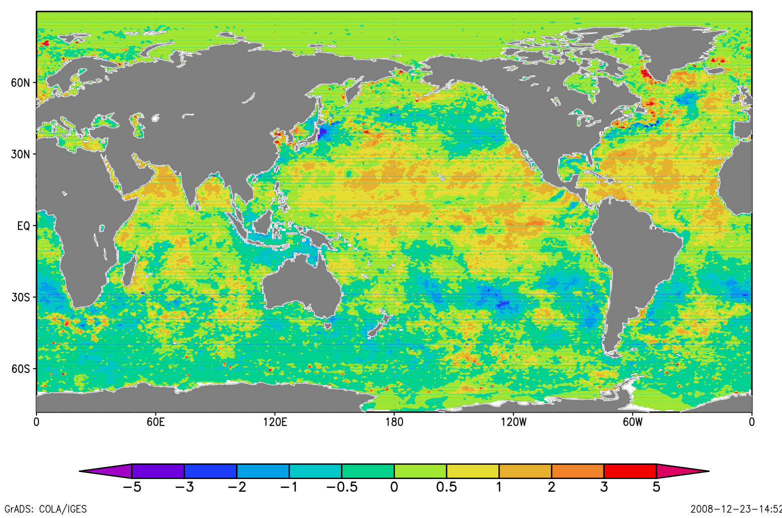
Difference on SST. Time period of assimilation is 1 month



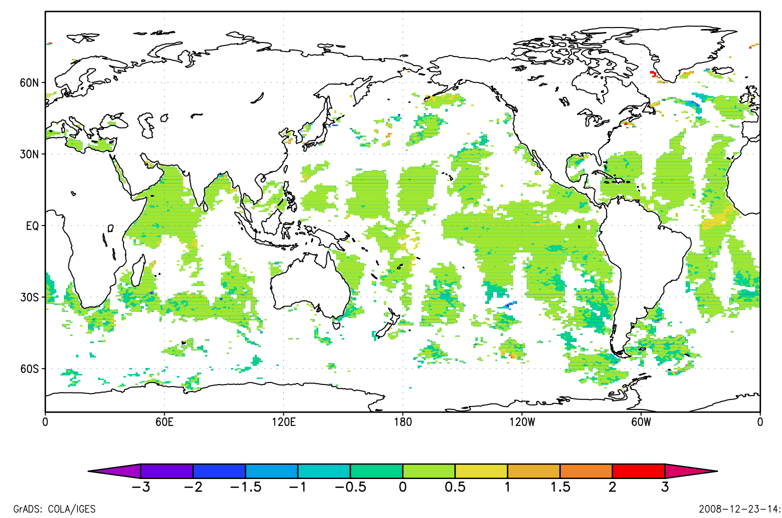
(a) T_{model}



(b) T_{assim}.

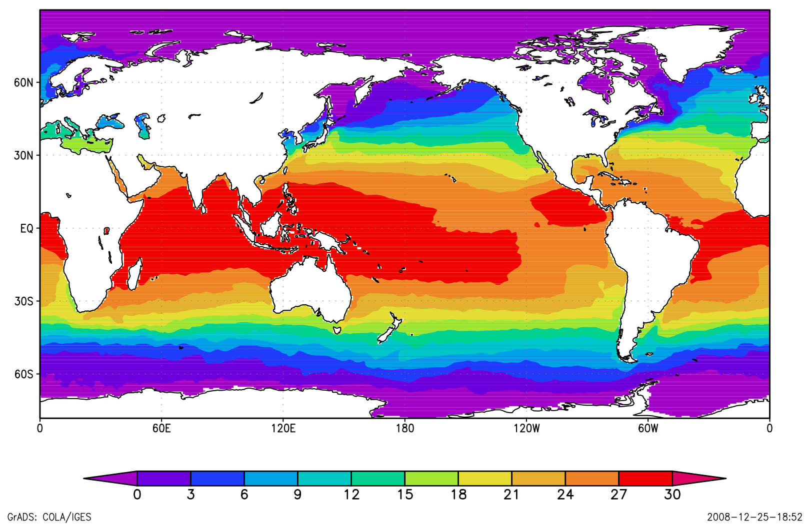


(c) $T_{\text{assim}} - T_{\text{model}}$.

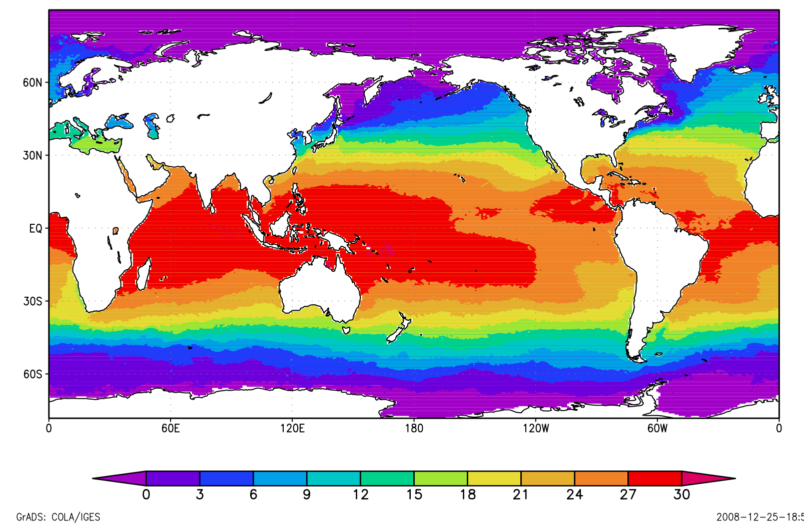


(d) $T_{\text{assim}} - T_{\text{obs}}$.

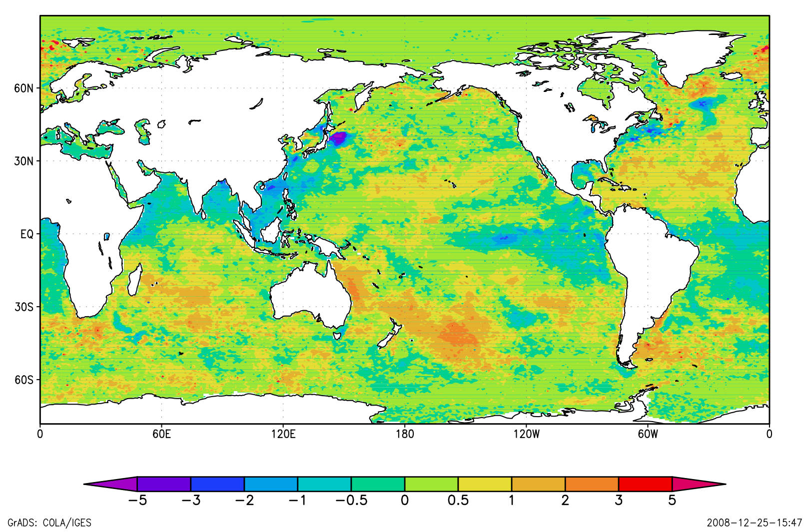
Difference on SST. Time period of assimilation is 3 months



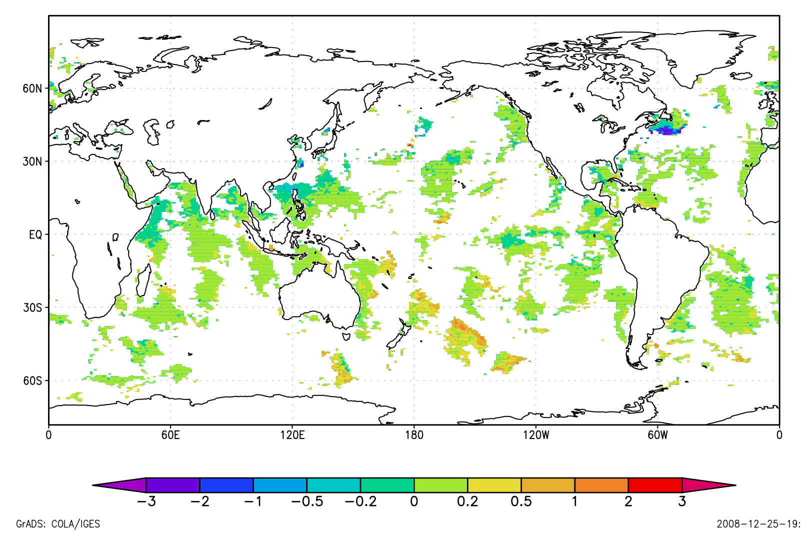
(a) T_{model}



(b) T_{assim}.

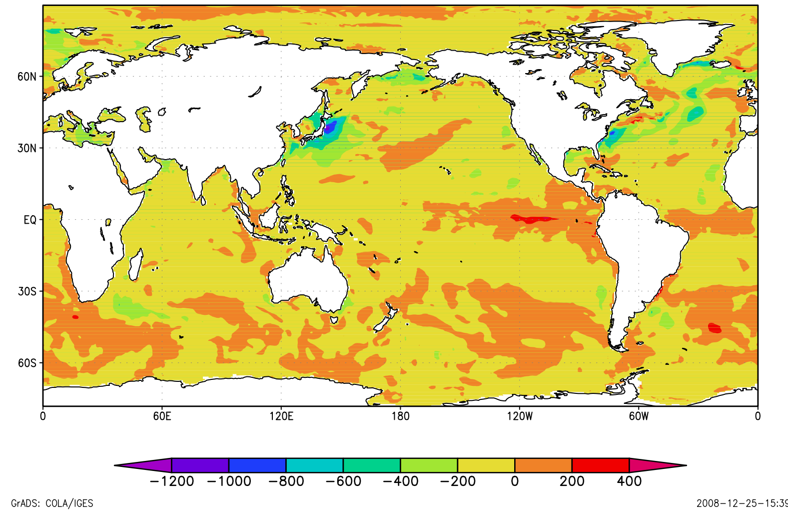


(c) $T_{\mathrm{assim}} - T_{\mathrm{model}}$.

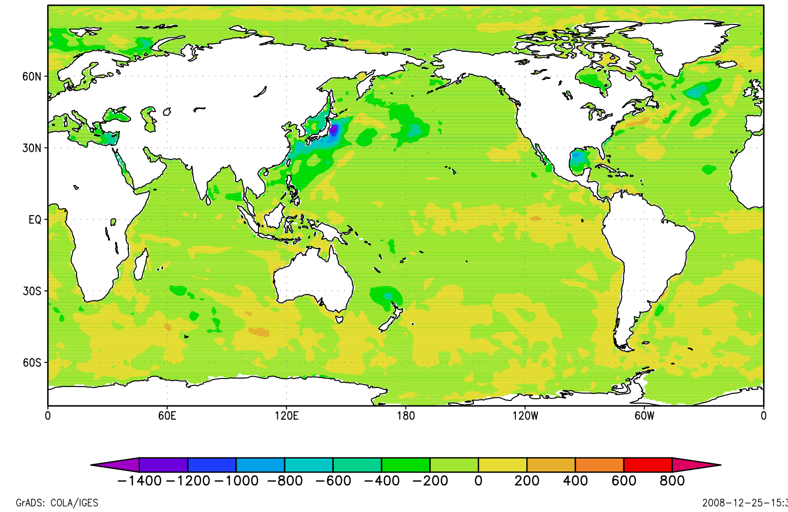


(d) $T_{\mathrm{assim}} - T_{\mathrm{obs}}$.

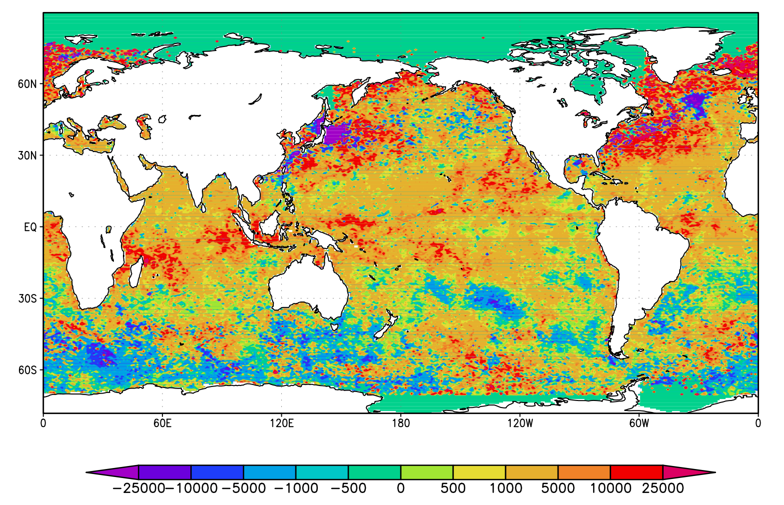
Fluxes



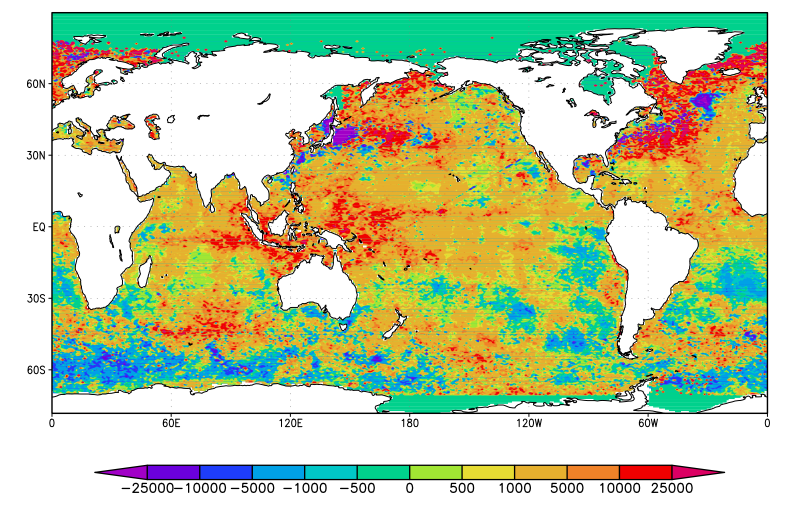
(a) Flux used in model (without assimilation, after 30 days)



(b) Flux used in model (without assimilation, after 45 days)



(c) Flux (with assimilation, 30 days of calculation)



(d) Flux (with assimilation, 45 days of calculation)

Conclusion

- The inverse and corresponding variational data assimilation problems of finding the flux on the World Ocean and SST using the observation of on-line SST data were formulated and studied.
- The numerical experiments confirm the theoretical results and advisability of using the assimilation procedure in 3D ocean and sea circulation model.
- Algorithms of the numerical solution of problems can be applied also to the corresponding problems in the dynamics of ocean and seas.