Multiscale Geometric Analysis: Thoughts and Applications (a summary)

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Assimage 2005,Chamrousse, February 2005

Classical Multiscale Analysis

- Wavelets: Enormous impact
  - Theory
  - Applications
  - Many success stories
- Deep understanding of the fact that wavelets are not good for all purposes
- Consequent constructions of new systems lying beyond wavelets

Overview

- Other multiscale constructions
- Problems classical multiscale ideas do not address effectively

Fourier Analysis

- Diagonal representation of shift-invariant linear transformations; e.g. solution operator to the heat equation

  $$\partial_t u = a^2 \Delta u$$

  Fourier, *Théorie Analytique de la Chaleur*, 1812.

- Truncated Fourier series provide very good approximations of smooth functions

  $$\|f - S_n(f)\|_{L_2} \leq C \cdot n^{-k},$$

  if $f \in C^k$.

Limitations of Fourier Analysis

- Does not provide any sparse decomposition of differential equations with variable coefficients (sinusoids are no longer eigenfunctions)
- Provides poor representations of discontinuous objects (Gibbs phenomenon)

Wavelet Analysis

- Almost eigenfunctions of differential operators

$$(Lf)(x) = a(x)\partial_x f(x)$$

- Sparse representations of piecewise smooth functions

Wavelets and Piecewise Smooth Objects

- 1-dimensional example:

$$g(t) = 1_{\{t>t_0\}} e^{-(t-t_0)^2}.$$

- Fourier series:

$$g(t) = \sum c_k e^{ikt}$$

Fourier coefficients have slow decay:

$$|c|_{(n)} \geq c \cdot 1/n.$$

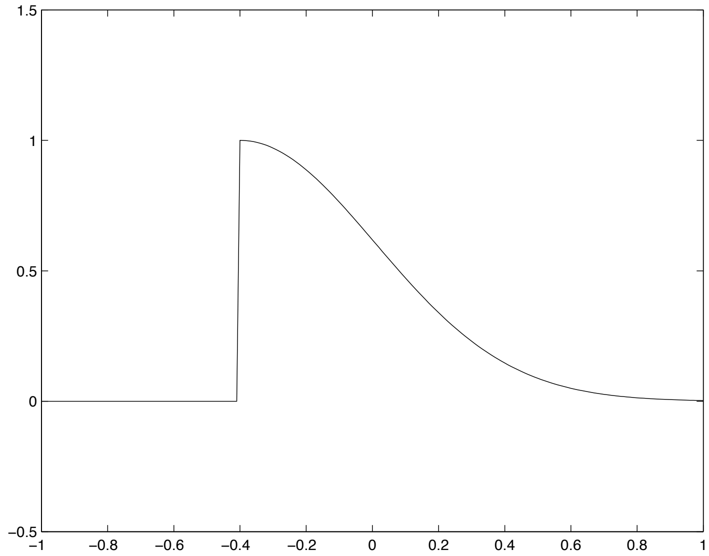
- Wavelet series:

$$g(t) = \sum \theta_\lambda \psi_\lambda(t)$$

Wavelet coefficients have fast decay:

$$|\theta|_{(n)} \leq c \cdot (1/n)^r \quad \text{for any } r > 0.$$

as if the object were non-singular



Donoho's-Candès Viewpoint

- Sparse representations of point-singularities
- Sparse representation of certain matrices
- *Simultaneously*
- Applications
  - Approximation theory
  - Data compression
  - Statistical estimation
  - Scientific computing
- More importantly: new mathematical architecture where information is organized by scale and location

Images

- Limitations of existing image representations
- Curvelets: geometry and tilings in Phase-Space
- Representation of functions, signals
- Potential applications

Three Anomalies

- Inefficiency of Existing Image Representations
- Limitations of Existing Pyramid Schemes
- Limitations of Existing Scaling Concepts

I: Inefficient Image Representations

Edge Model: Object $f(x_1, x_2)$ with discontinuity along generic C^2 smooth curve; smooth elsewhere.

Fourier is awful

Best m-term trigonometric approximation $\tilde{f}_m$

$$\|f - \tilde{f}_m\|_2^2 \asymp m^{-1/2}, \quad m \to \infty$$

Wavelets are bad

Best m-term approximation by wavelets:

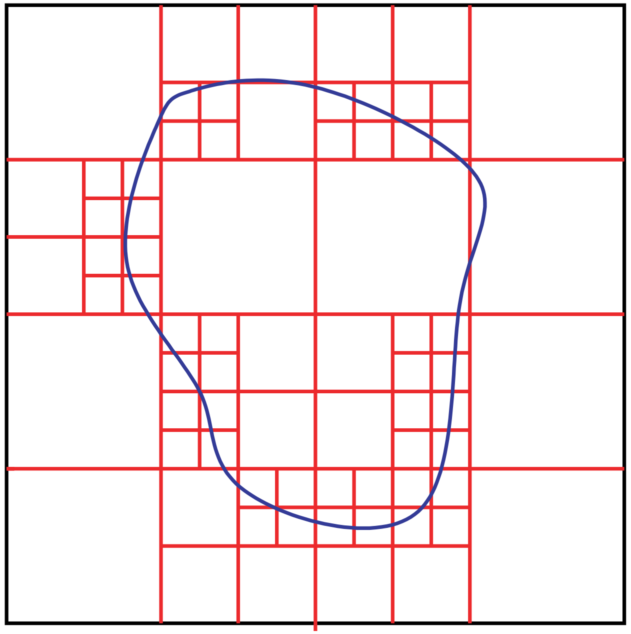
$$\|f - \tilde{f}_m\|_2^2 \asymp m^{-1}, \quad m \to \infty$$

Optimal Behavior

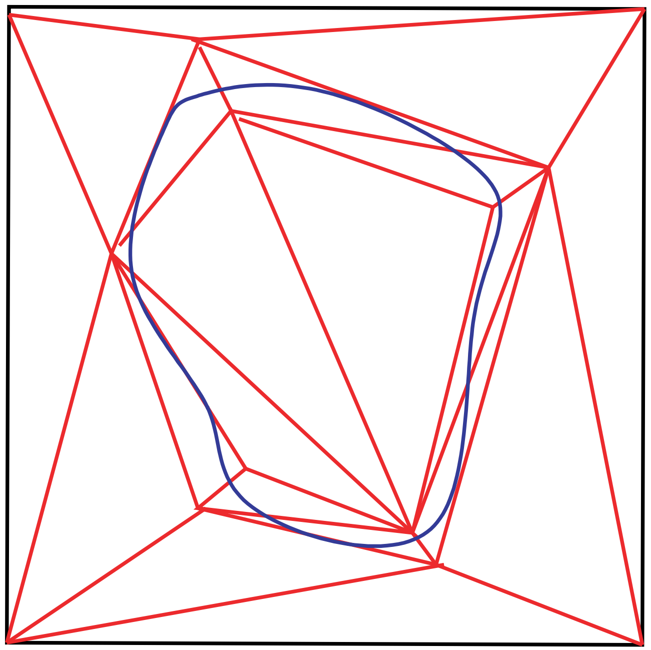
There is a 'dictionary' of 'atoms' with best m-term approximant $\tilde{f}_m$

$$\|f - \tilde{f}_m\|_2^2 \asymp m^{-2}, \quad m \to \infty$$

- No basis can do better than this.
- No depth-search limited dictionary can do better.
- No pre-existing basis does anything near this well.



(a) Wavelets



(b) Triangulations

Limitations of Existing Scaling Concepts

Traditional Scaling

$$f_a(x_1, x_2) = f(ax_1, ax_2), \qquad a > 0.$$

Curves exhibit different kinds of scaling

- Anisotropic
- Locally Adaptive

If $f(x_1, x_2) = 1_{\{y \geq x^2\}}$ then

$$f_a(x_1, x_2) = f(a \cdot x_1, a^2 x_2)$$

In Harmonic Analysis this is called Parabolic Scaling.

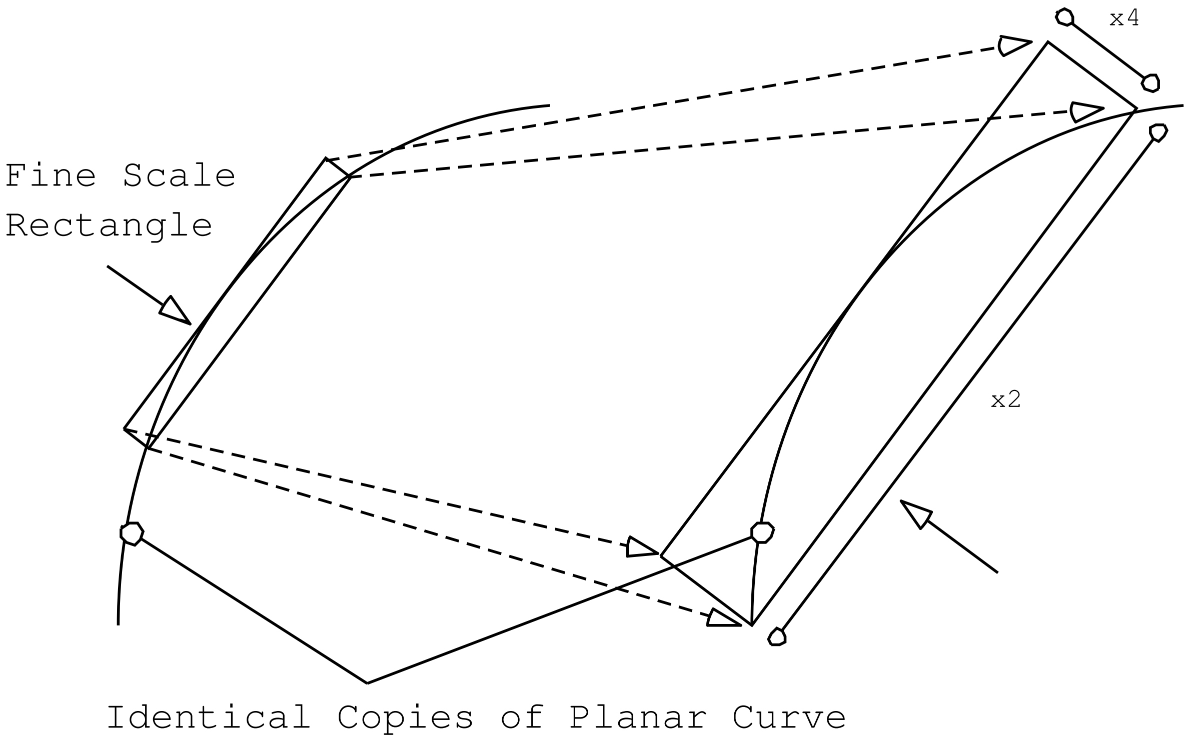


Figure 1: Curves are Invariant under Anisotropic Scaling

Curvelets

Candès and Guo, 2002.

New multiscale pyramid:

- Multiscale
- Multi-orientations
- *Parabolic (anisotropy) scaling*

$$\boldsymbol{width \approx length^2}$$

Earlier construction, Candès and Donoho (2000)

Philosophy (Slightly Inaccurate)

- Start with a waveform $\varphi(x) = \varphi(x_1, x_2)$.
  - oscillatory in x_1
  - lowpass in x_2
- *Parabolic* rescaling

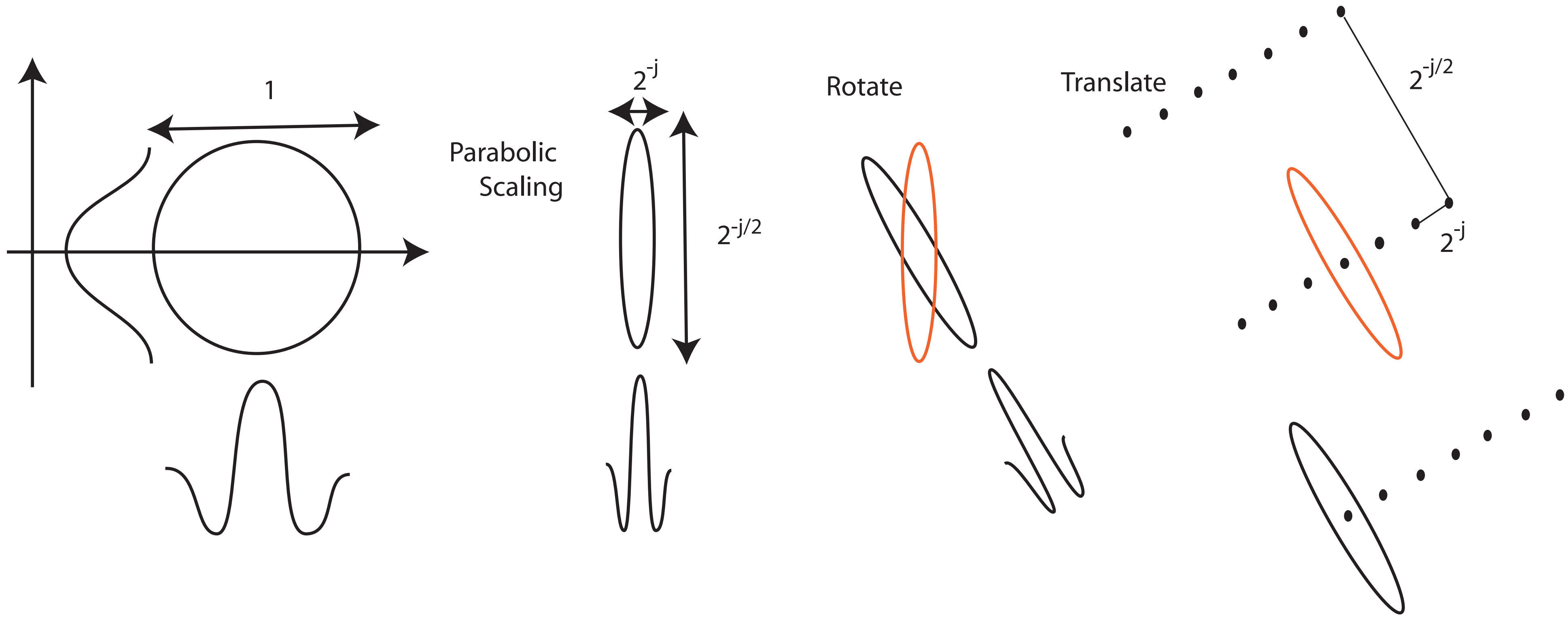
$$|D_j|\varphi(D_j x) = 2^{3j/4}\varphi(2^j x_1, 2^{j/2} x_2), \quad D_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{j/2} \end{pmatrix}, \; j \geq 0$$

- Rotation (scale dependent)

$$2^{3j/4}\varphi(D_j R_{\theta_{j\ell}} x), \quad \theta_{j\ell} = 2\pi \cdot \ell 2^{-\lfloor j/2 \rfloor}$$

- Translation (oriented Cartesian grid with spacing $2^{-j} \times 2^{-j/2}$);

$$2^{3j/4}\varphi(D_j R_{\theta_{j\ell}} x - k), \quad k \in \mathbf{Z}^2$$


1
Parabolic
Scaling
2-j
2-j/2
Rotate
Translate
2-j/2
2-j

Curvelet: Space-side Viewpoint

In the frequency domain

$$\hat{\varphi}_{j,0,k}(\xi) = \frac{2^{-3j/4}}{2\pi} W_{j,0}(\xi) e^{i\langle k,\xi\rangle}, \quad k \in \Lambda_j$$

In the spatial domain

$$\varphi_{j,0,k}(x) = 2^{3j/4}\varphi_j(x-k), \quad W_{j,0} = 2\pi\hat{\varphi}_j$$

and more generally

$$\varphi_{j,\ell,k}(x) = 2^{3j/4}\varphi_j(R_{\theta_{j,\ell}}(x - R^{-1}_{\theta_{j,\ell}}k)),$$

All curvelets at a given scale are obtained by translating and rotating a single ‘mother curvelet.’

Further Properties

- Tight frame

$$f = \sum_{j,\ell,k} \langle f, \varphi_{j,\ell,k} \rangle \varphi_{j,\ell,k} \qquad ||f||_2^2 = \sum_{j,\ell,k} \langle f, \varphi_{j,\ell,k} \rangle^2$$

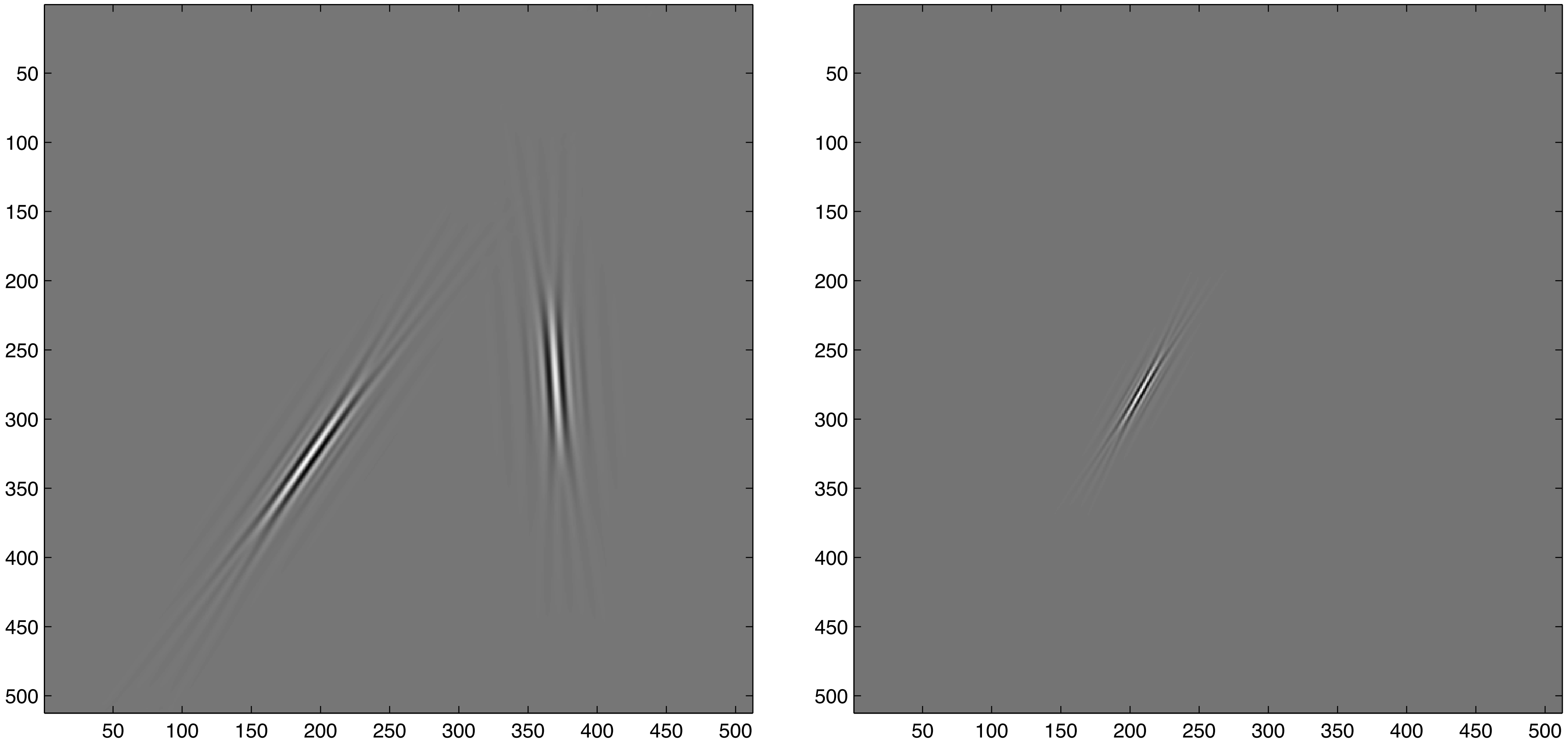
- *Geometric* Pyramid structure
  - Dyadic scale
  - Dyadic location
  - Direction
- New tiling of phase space

- Needle-shaped
- Scaling laws
  - $\boldsymbol{length \sim 2^{-j/2}}$
  - $\boldsymbol{width \sim 2^{-j}}$

$$\boldsymbol{width \sim length^2}$$

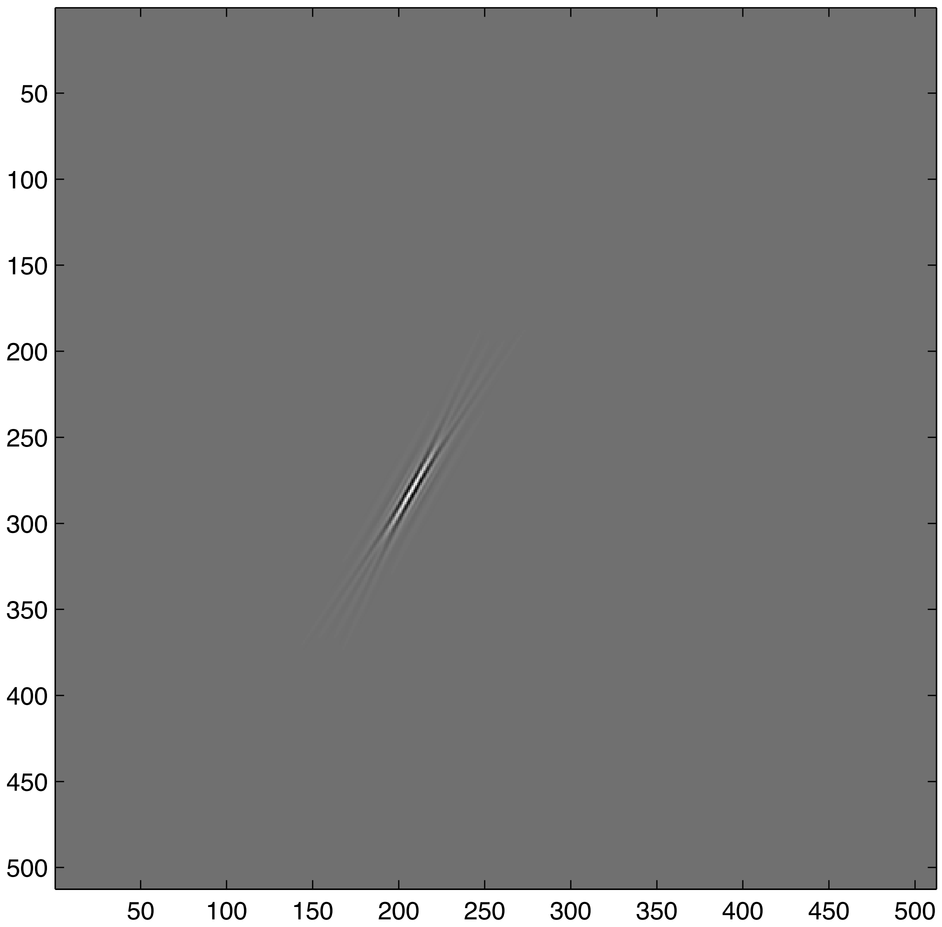
  - $\#$Directions $= \boldsymbol{2^{\lfloor j/2 \rfloor}}$
  - Doubles angular resolution at every other scale
- Unprecedented combination

Digital Curvelets

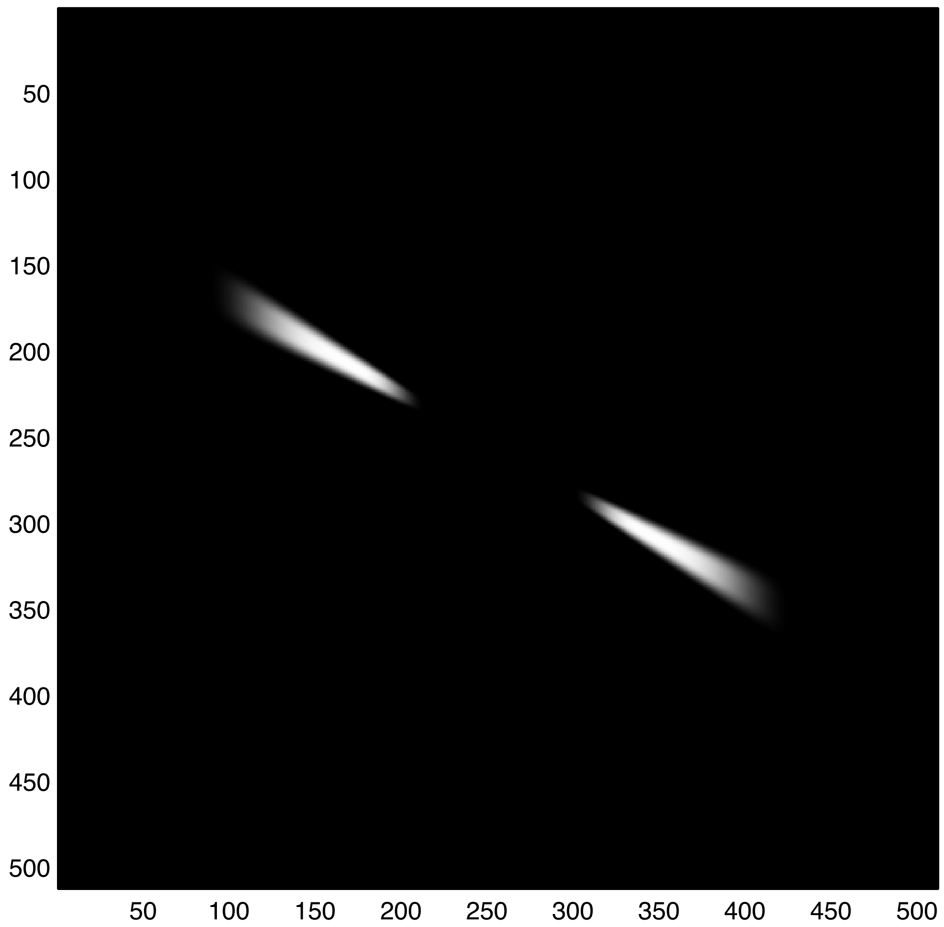


Source: Digital Curvelet Transform, Candès and Donoho (2004).

Digital Curvelets: Frequency Localization



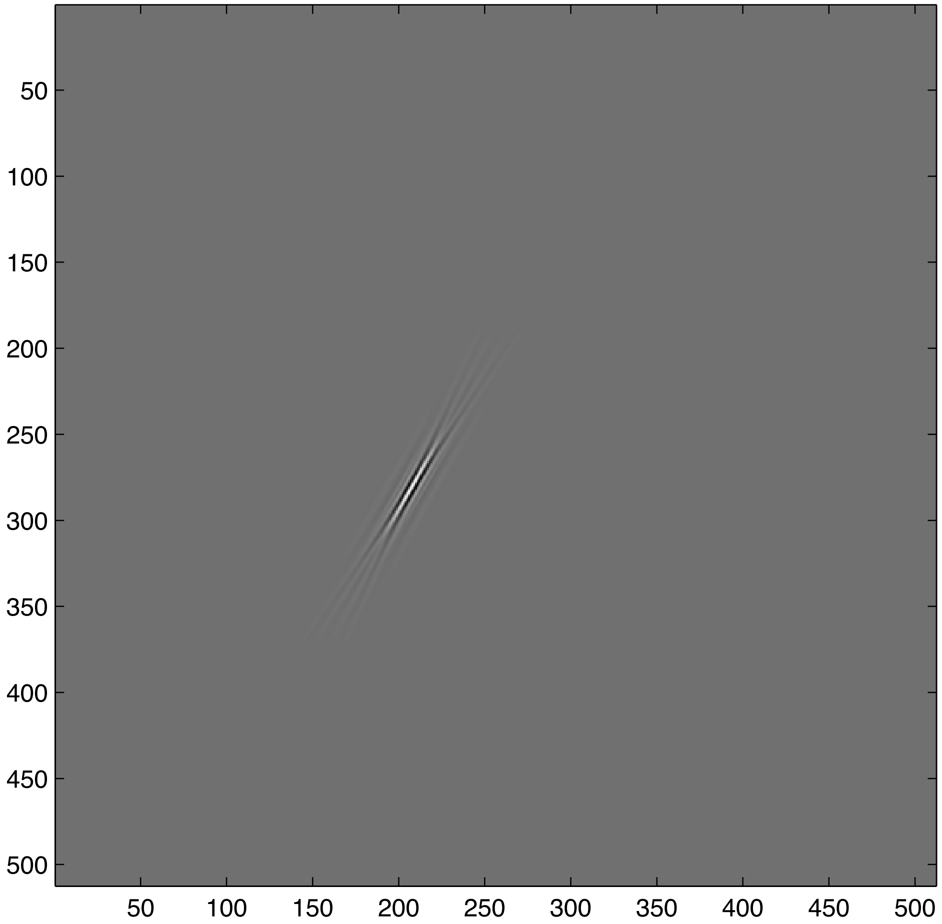
Time domain



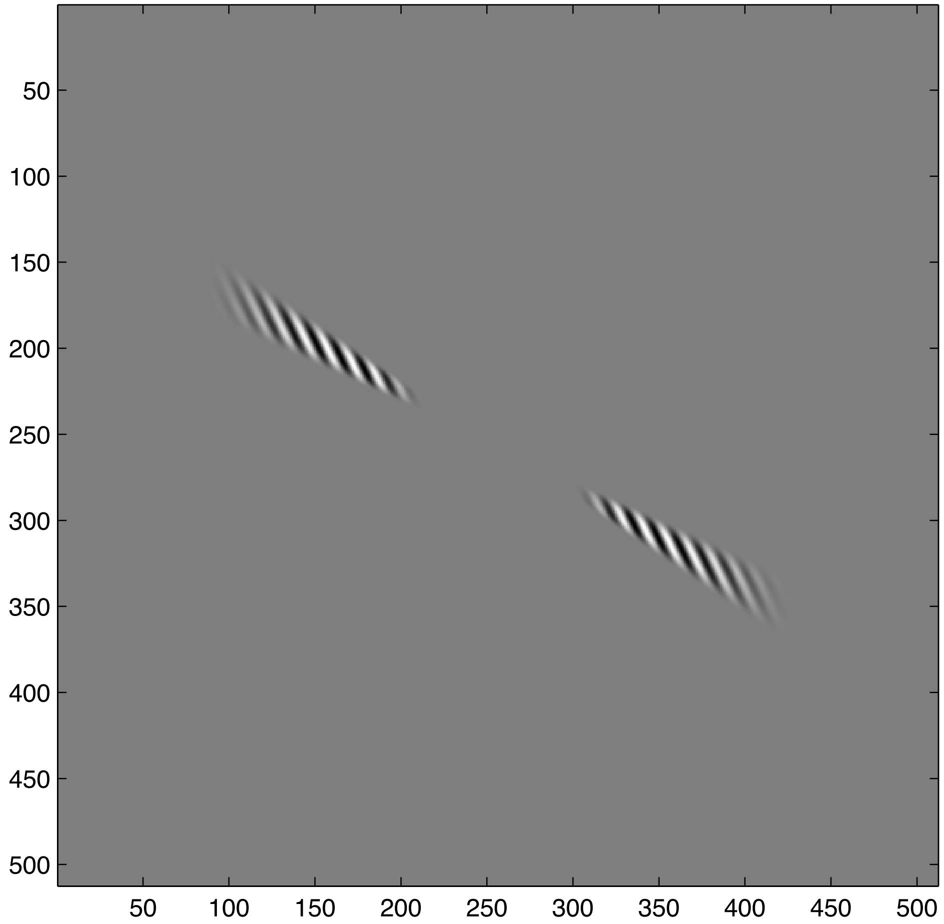
Frequency domain (modulus)

Source: Digital Curvelet Transform, Candès and Donoho (2004).

Digital Curvelets: : Frequency Localization



Time domain



Frequency domain (real part)

Source: Digital Curvelet Transform, Candès and Donoho (2004).

Curvelets and Edges

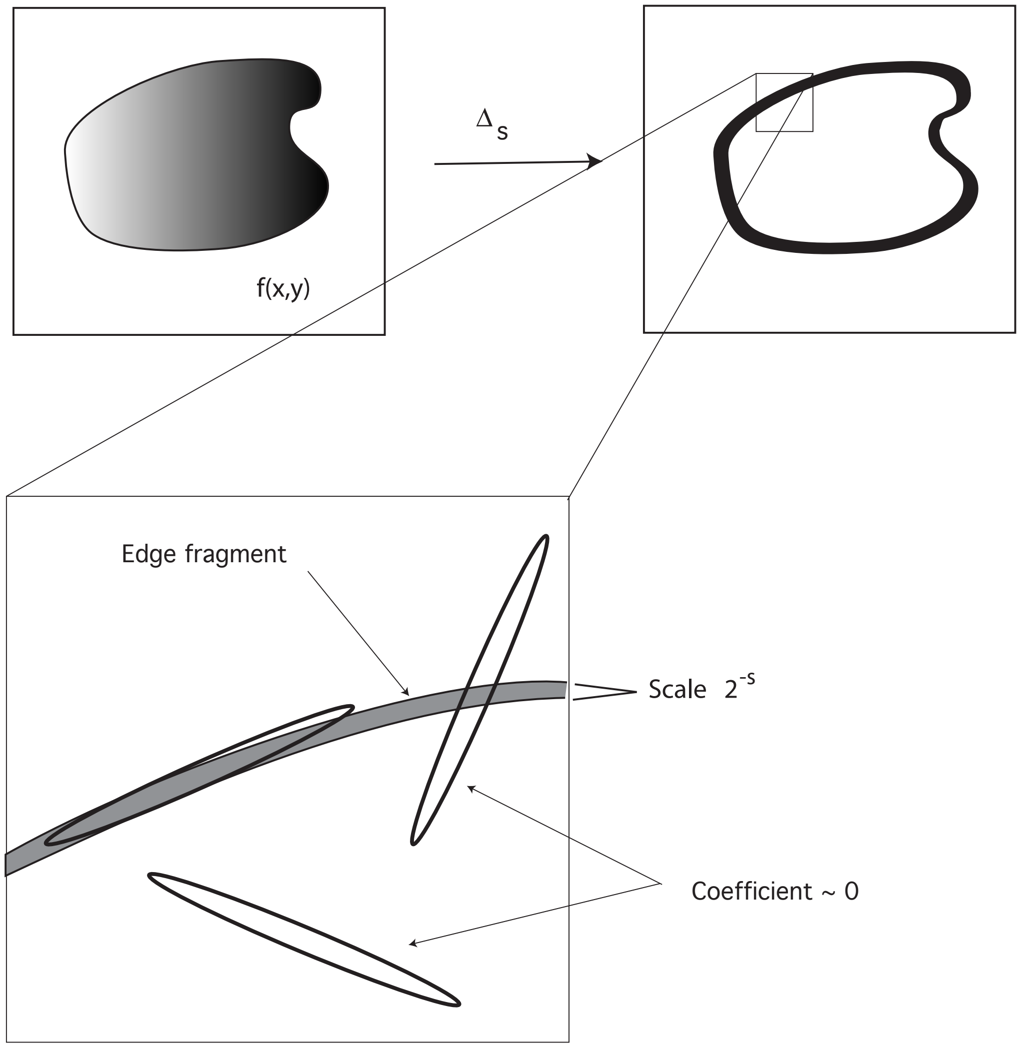
Optimality

- Suppose f is smooth except for discontinuity on C^2 curve
- Curvelet m-term approximations, naive thresholding

$$\|f - f_m^{curve}\|_2^2 \leq C\, m^{-2} (\log m)^3$$

- Near-optimal rate of m-term approximation (wavelets $\sim m^{-1}$).

Idea of the Proof



Decomposition of a Subband

Curvelets and Warpings

C^2 change of coordinates preserves sparsity (Candès 2002).

- Let $\varphi : \mathbf{R}^2 \to \mathbf{R}^2$ be a one to one C^2 function such that $\|J_\varphi\|_\infty$ is bounded away from zero and infinity.
- Curvelet expansion

$$f = \sum_\mu \theta_\mu(f)\gamma_\mu, \quad \theta_\mu(f) = \langle f, \gamma_\mu \rangle$$

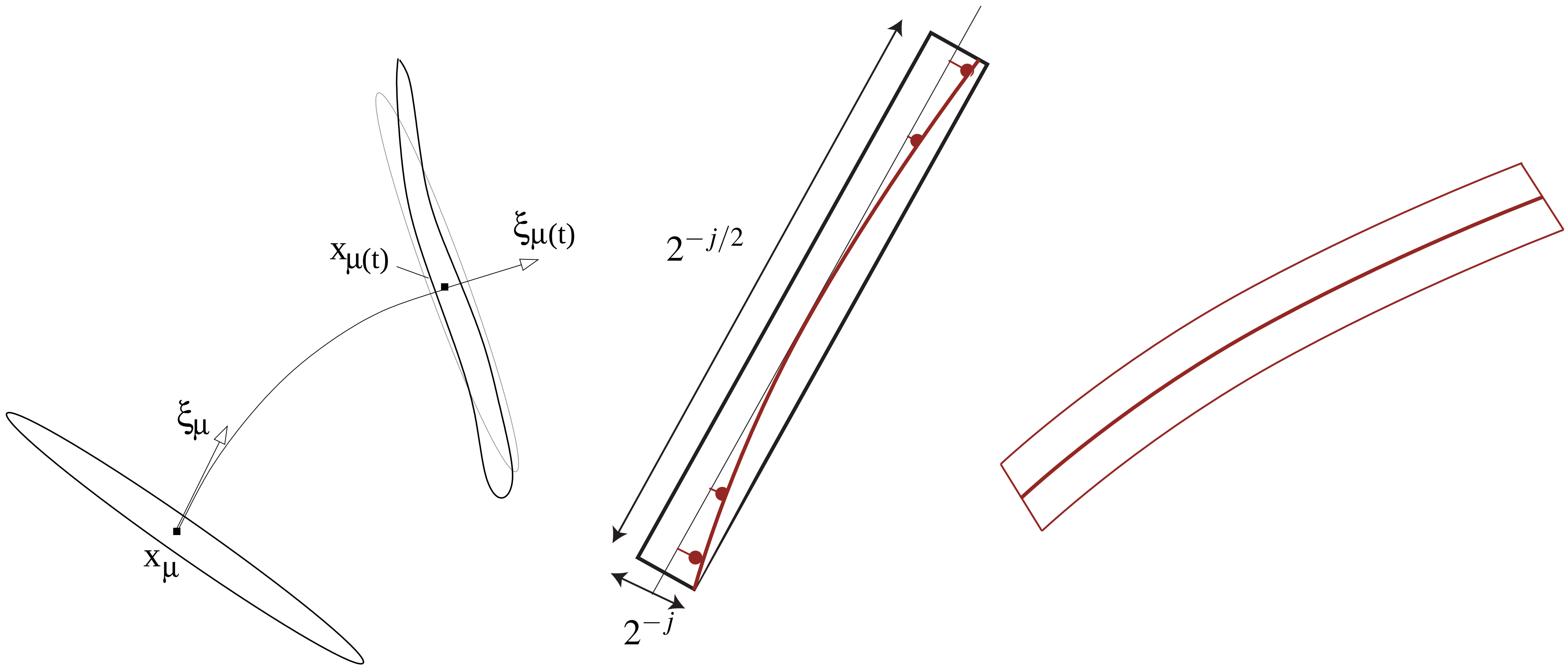
- Likewise,

$$f \circ \varphi = \sum_\mu \theta_\mu(f \circ \varphi)\gamma_\mu$$

The coefficient sequences of f and $f \circ \varphi$ are equally sparse.

Theorem 1 *Then, for each $p > 2/3$, we have*

$$\|\theta(f \circ \varphi)\|_{\ell_p} \leq C_p \cdot \|\theta(f)\|_{\ell_p}.$$



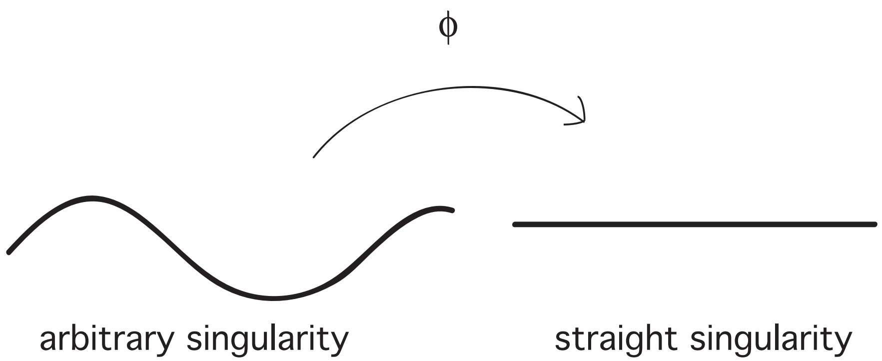
Curvelets are nearly invariant through a smooth change of coordinates

Curvelets and Curved Singularities

- f smooth except along a C^2 curve
- f_n, n-term approximation obtained by naive thresholding

$$\|f - f_n\|_{L_2}^2 \leq C \cdot (\log n)^3 \cdot n^{-2}$$

- Why?
  1. True for a straight edge
  2. Deformation preserves sparsity
- Optimal



Importance of Parabolic Scaling

- Consider arbitrary scaling (anisotropy incresases as α decreases)

$$width \sim 2^{-j}, \; length \sim 2^{-j\alpha}, \quad 0 \leq \alpha \leq 1.$$

  - ridgelets $\alpha = 0$ (very anisotropic),
  - curvelets $\alpha = 1/2$ (parabolic anisotropy),
  - wavelets $\alpha = 1$ (roughly isotropic).

- For wave-like behavior, need

$$width \leq length^2$$

- For particle-like behavior, need

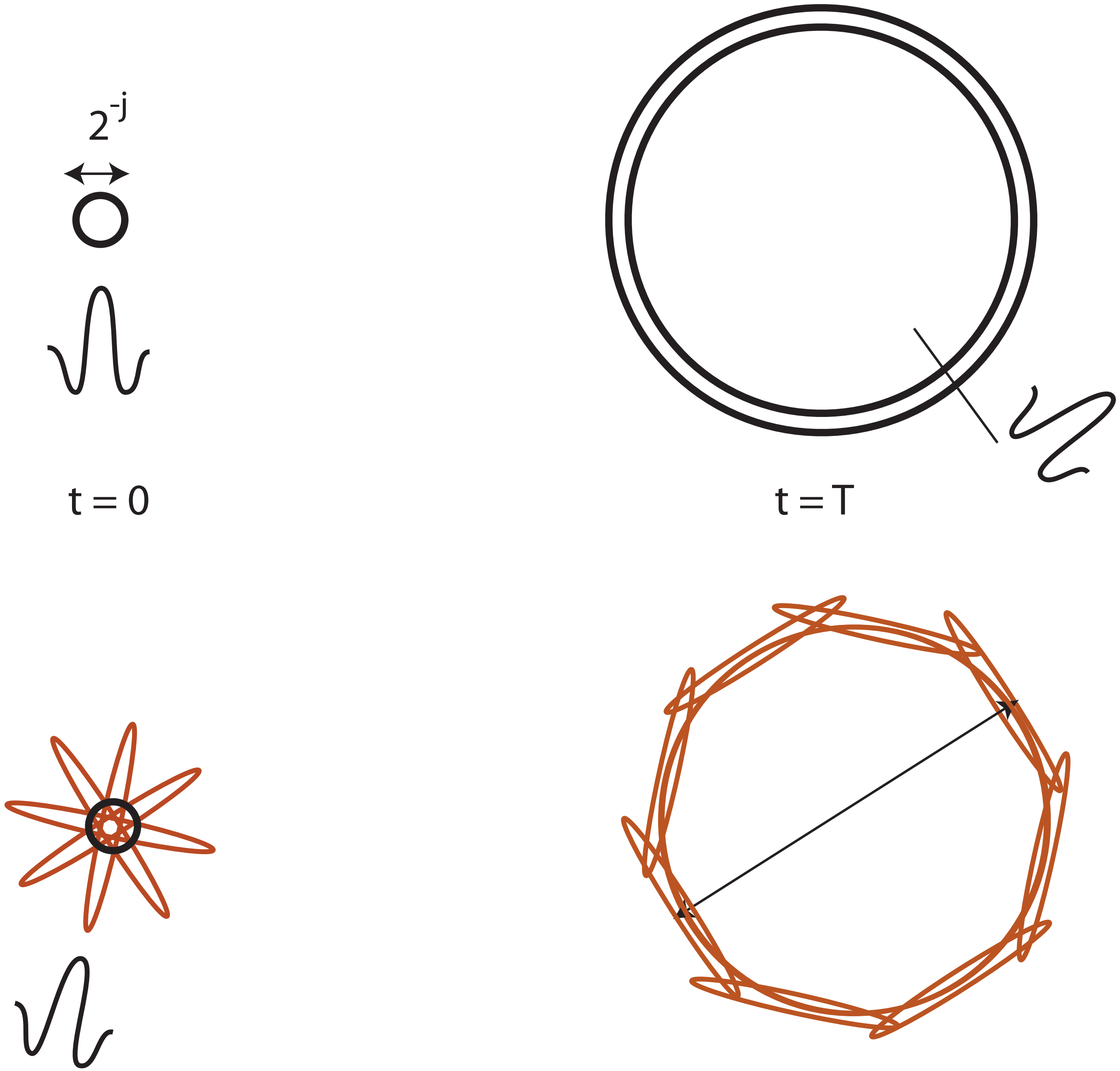
$$width \geq length^2$$

- For both (simultaneously), need

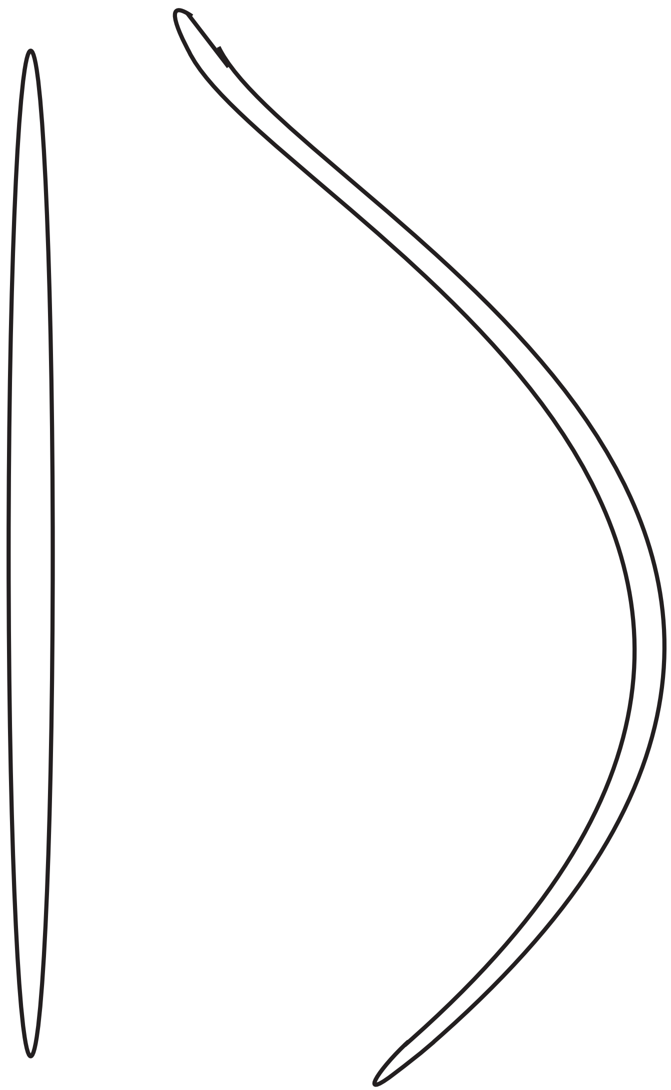
$$width \sim length^2$$

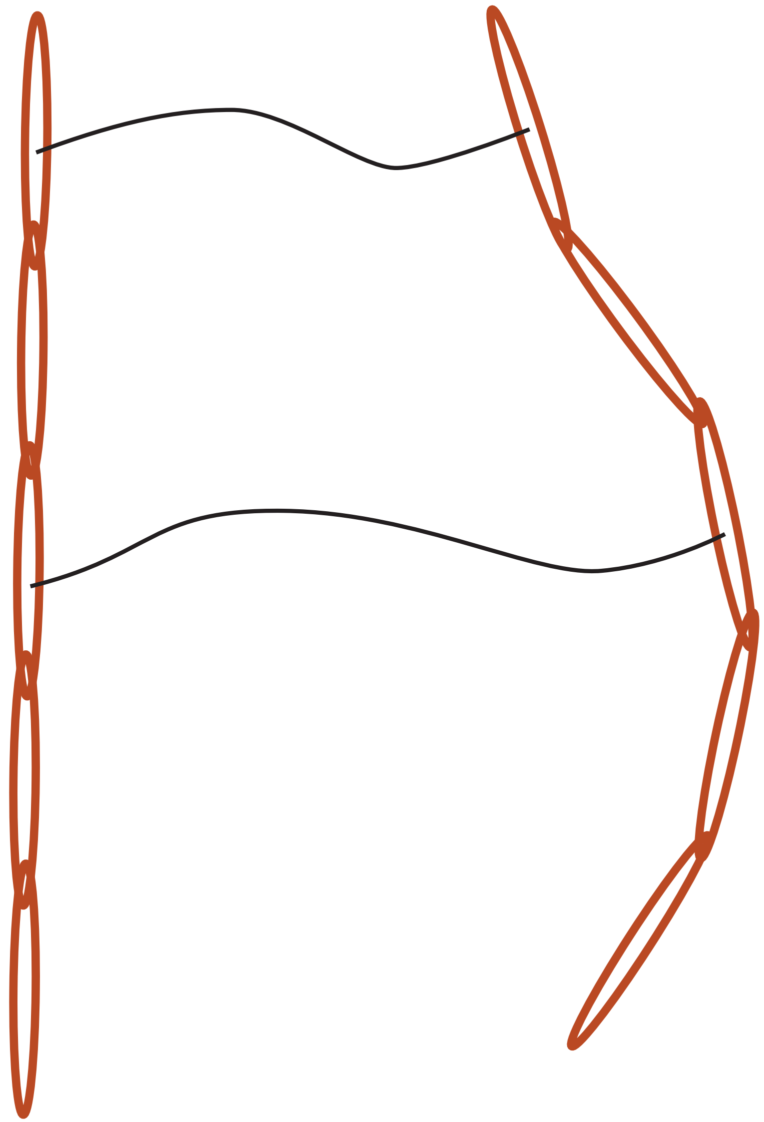
- *Only working scaling:* $\alpha = 1/2$

Examples I

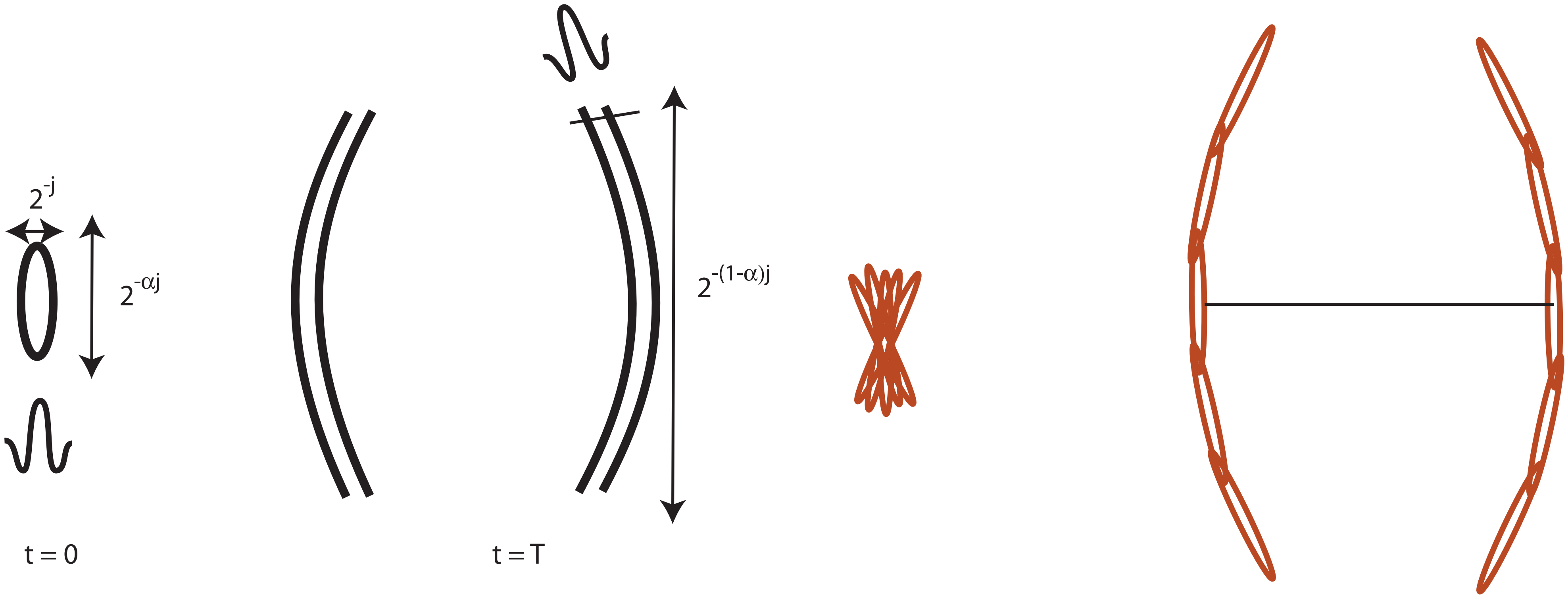


Examples II





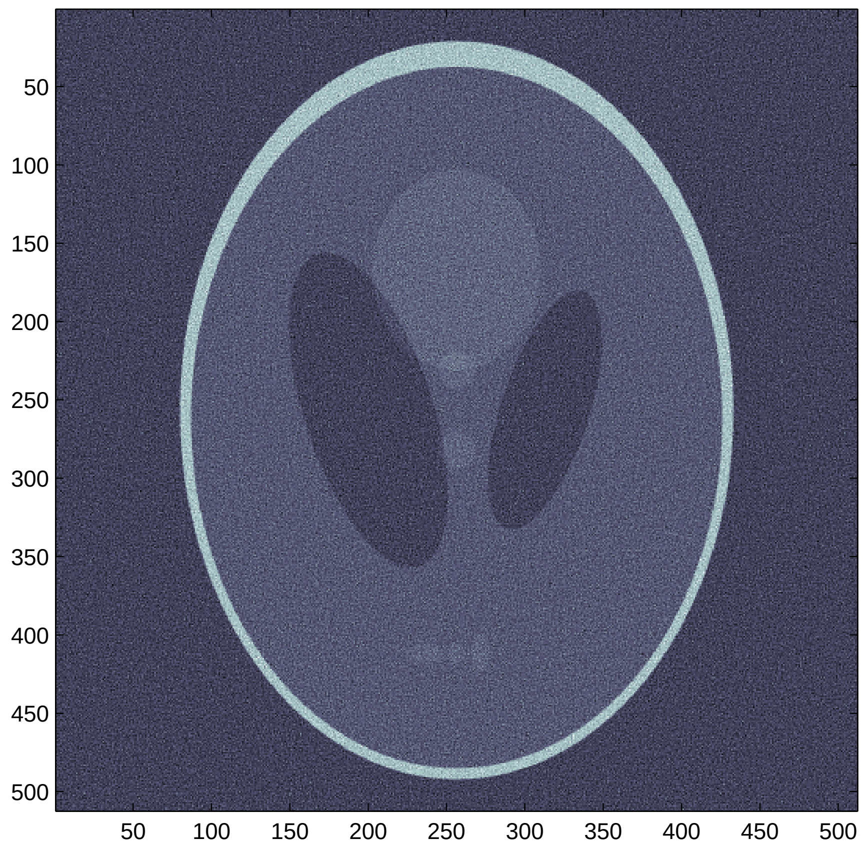
Examples III



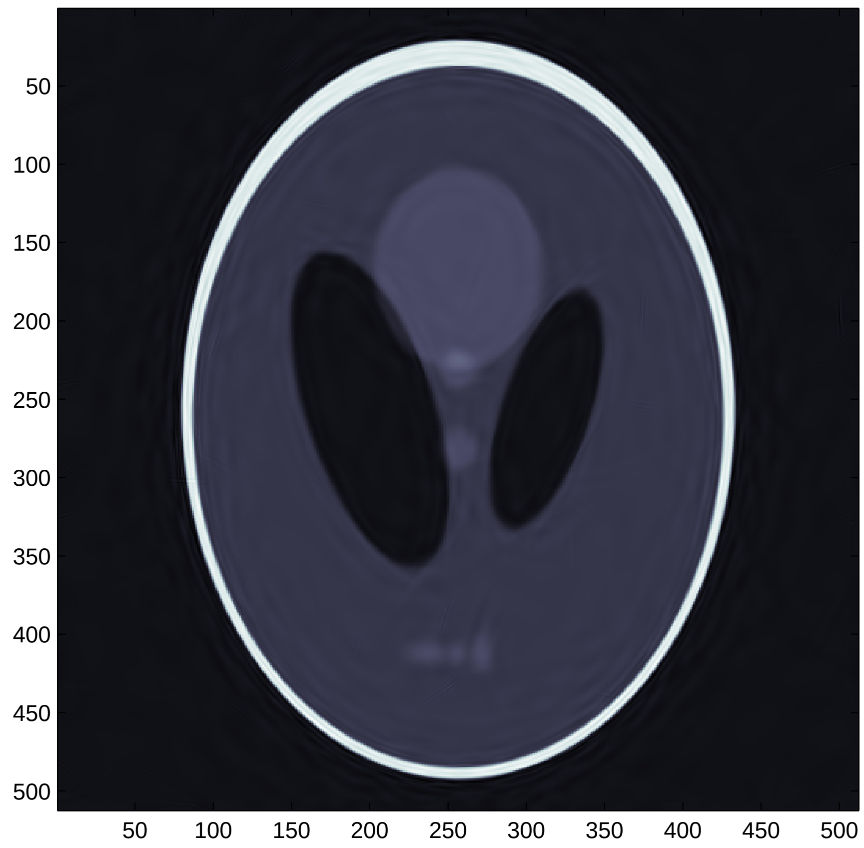
Summary

- New geometric multiscale ideas
- Key insight: geometry of Phase-Space
- New mathematical architecture
- Addresses new range of problems effectively
- Promising potential

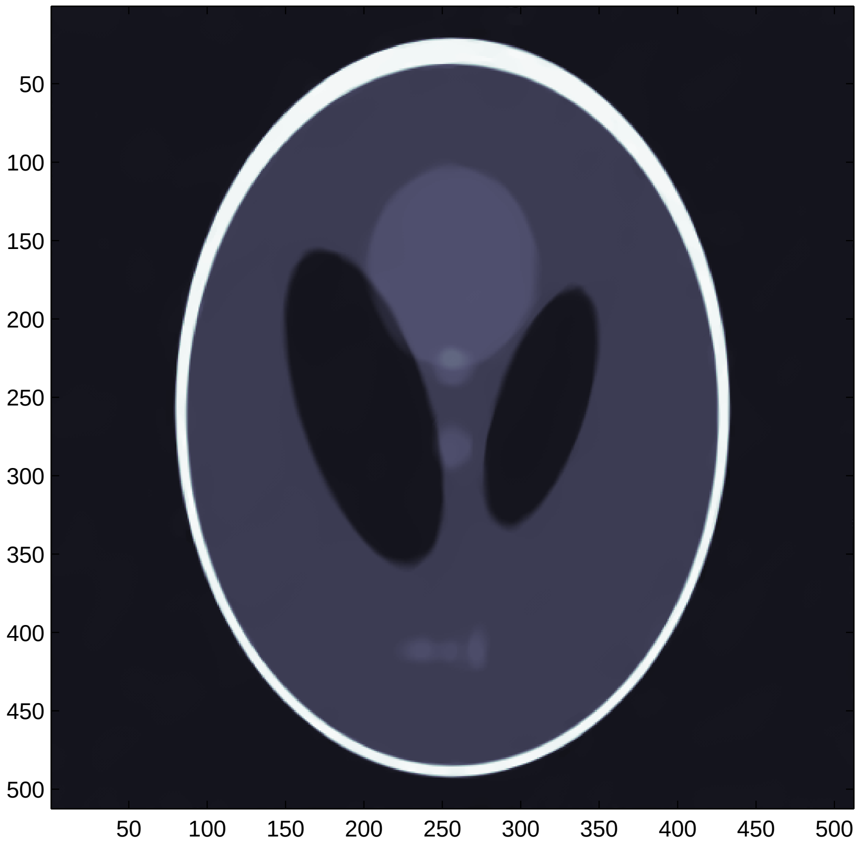
Numerical Experiments



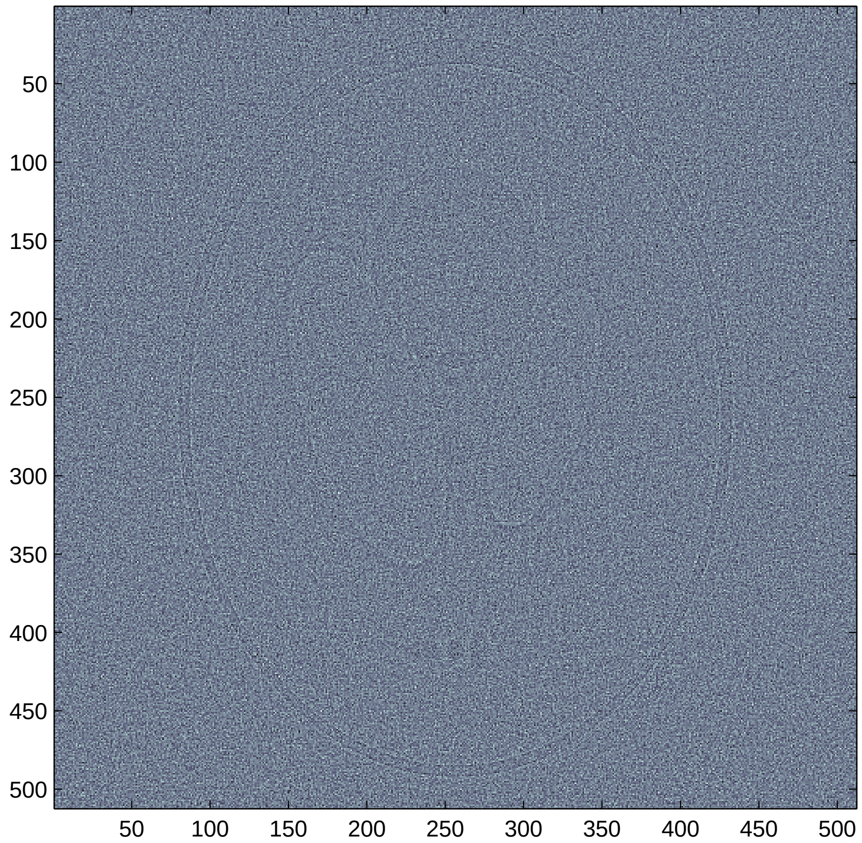
(a) Noisy Phantom



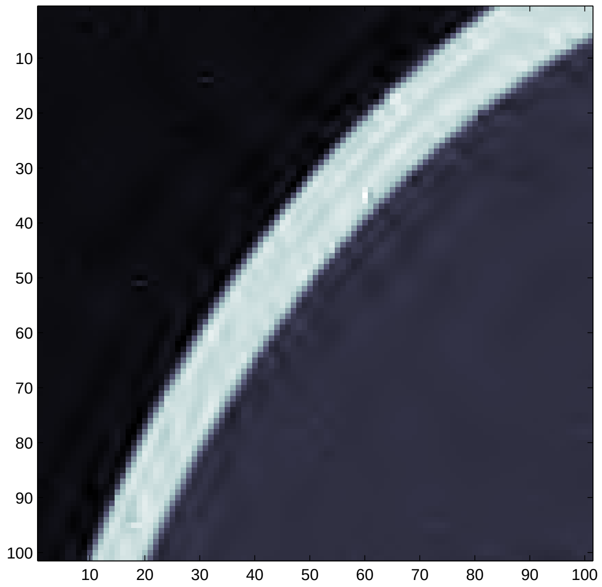
(b) Curvelets



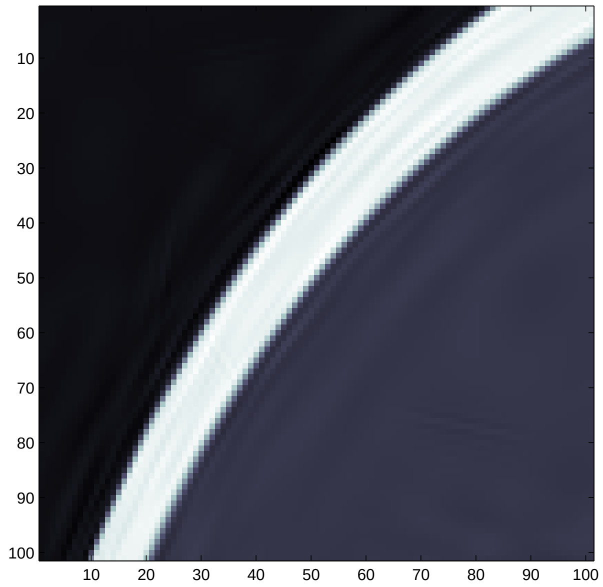
(c) Curvelets and TV



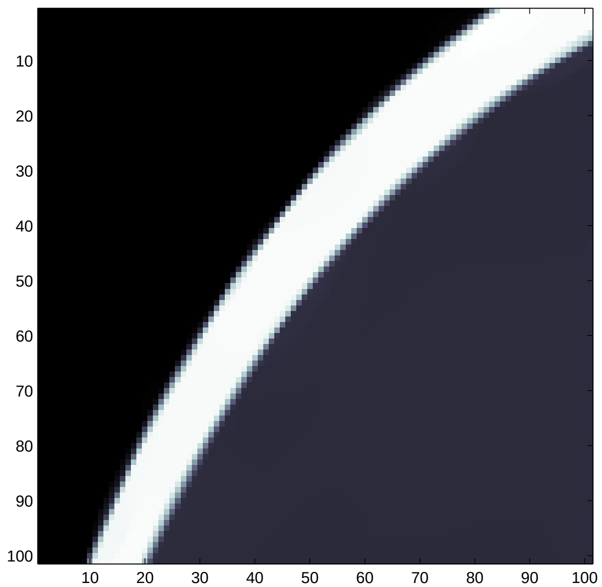
(d) Curvelets and TV: Residuals

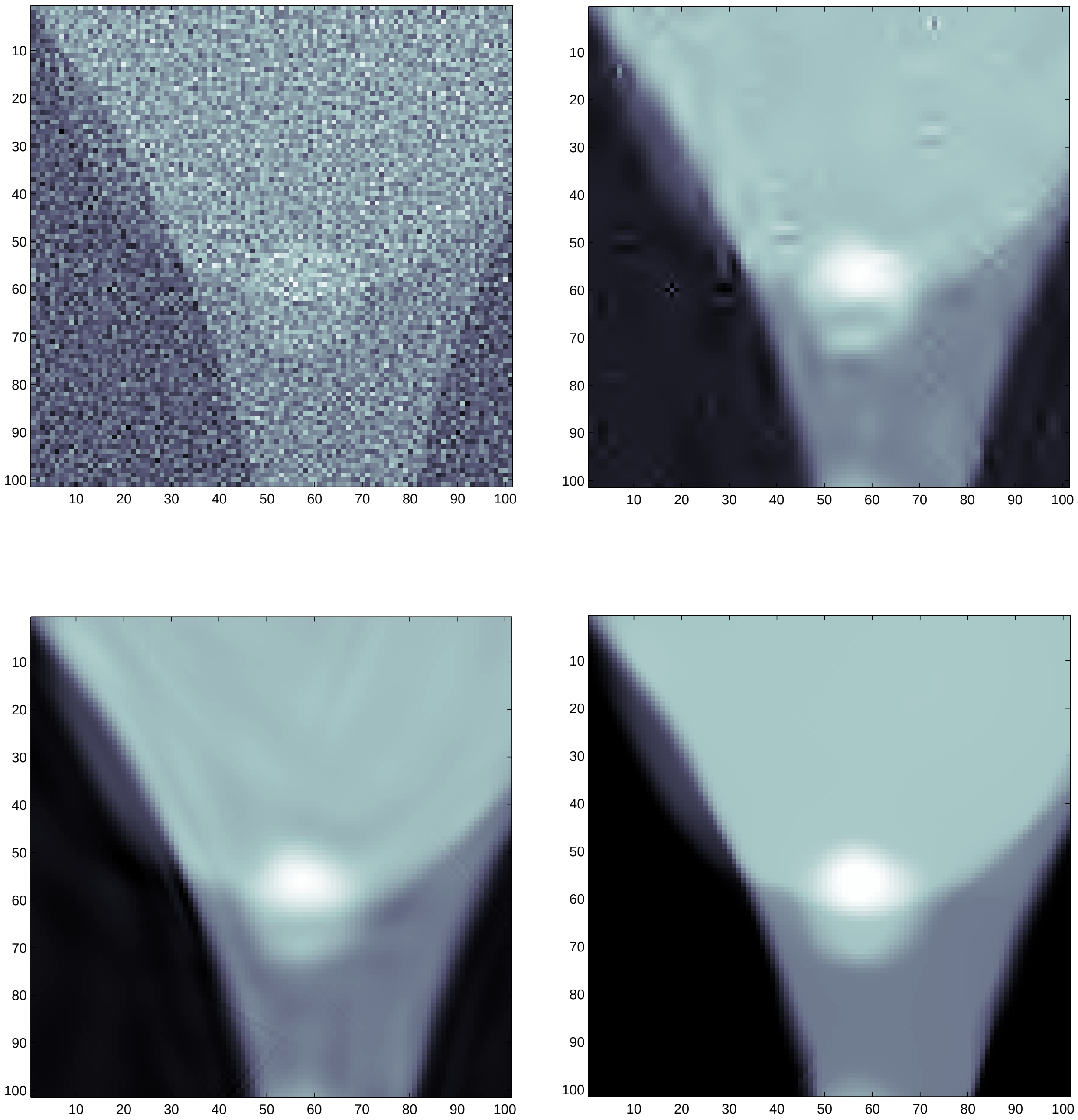


(e) Wavelets



(f) Curvelets





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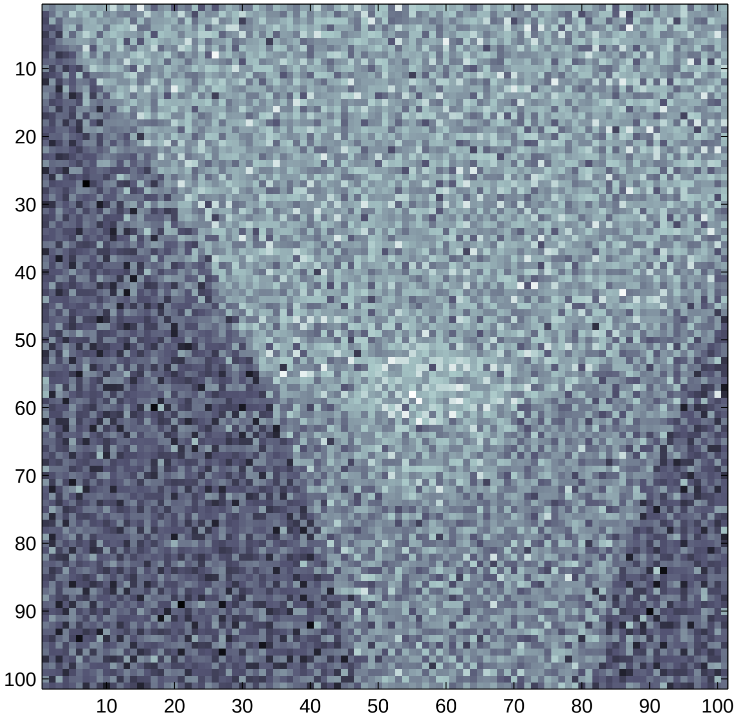
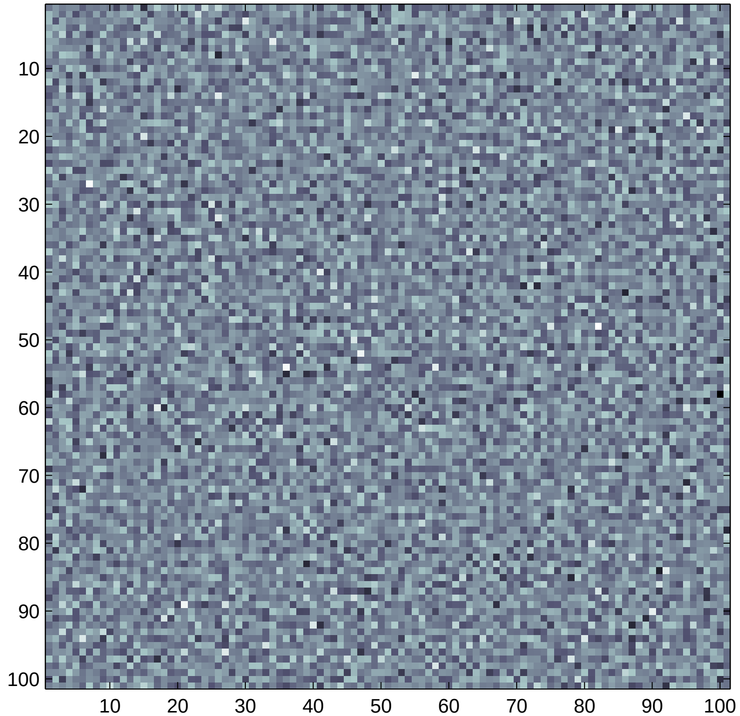
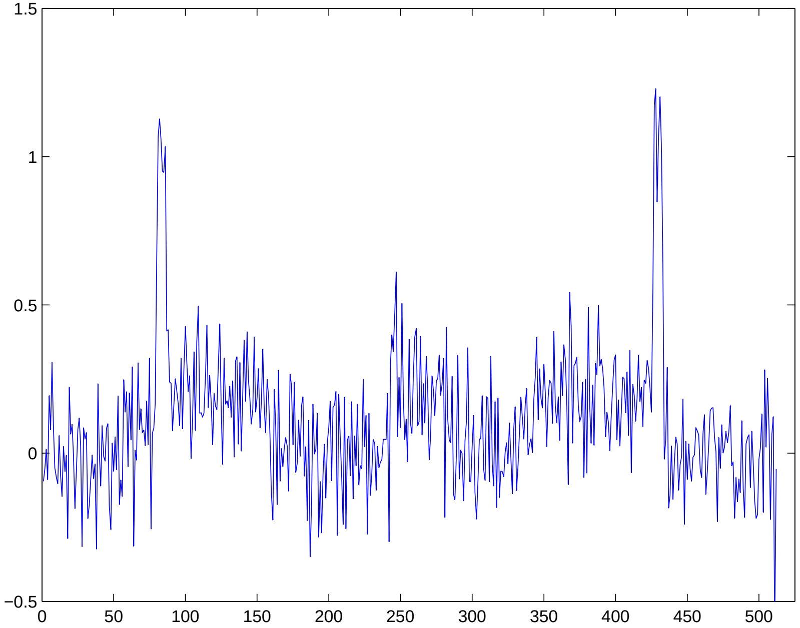


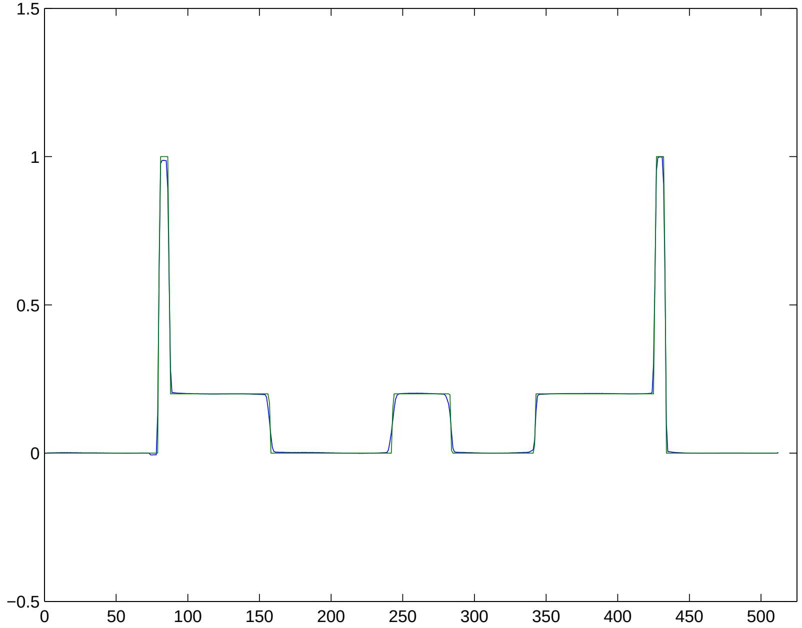
(l) Noisy



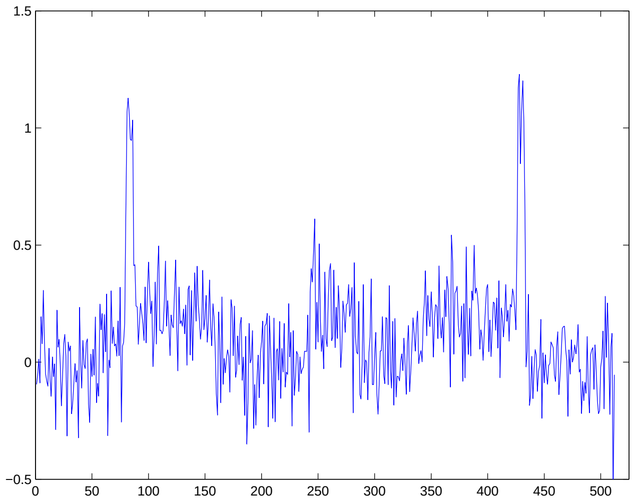
(m) Curvelets & TV: Residuals



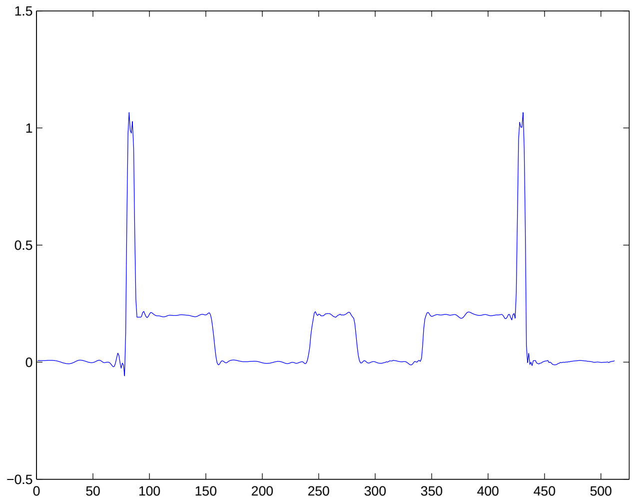
(n) Noisy Scanline



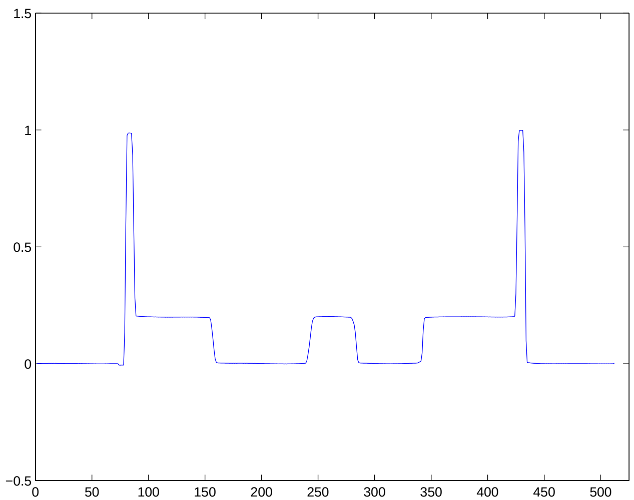
(o) True Scanline and Curvelets and TV Reconstruction



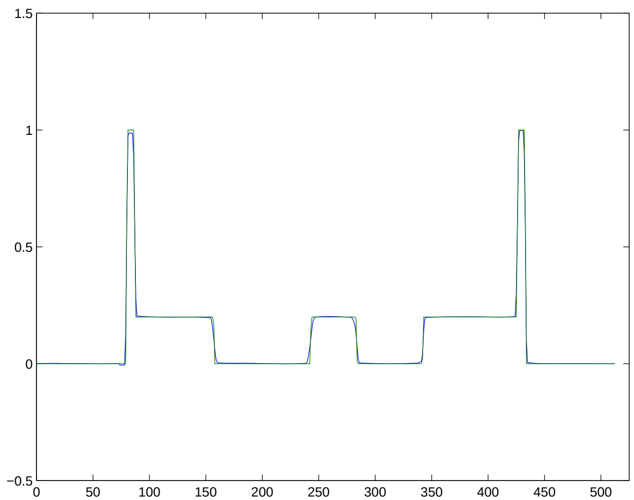
(a) Noisy Scanline



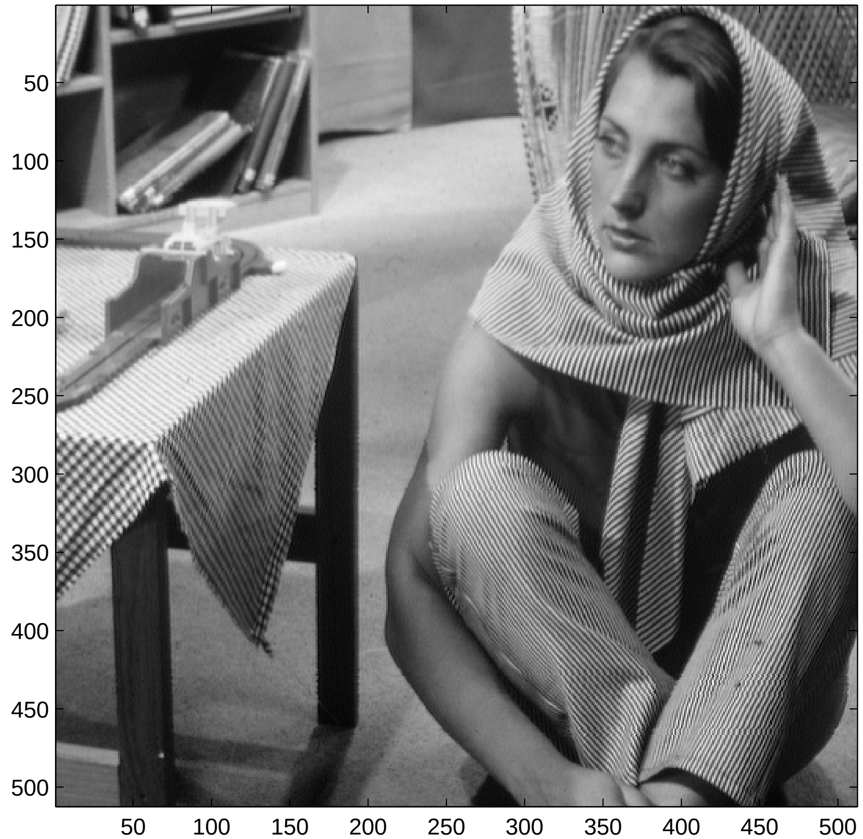
(b) Curvelets



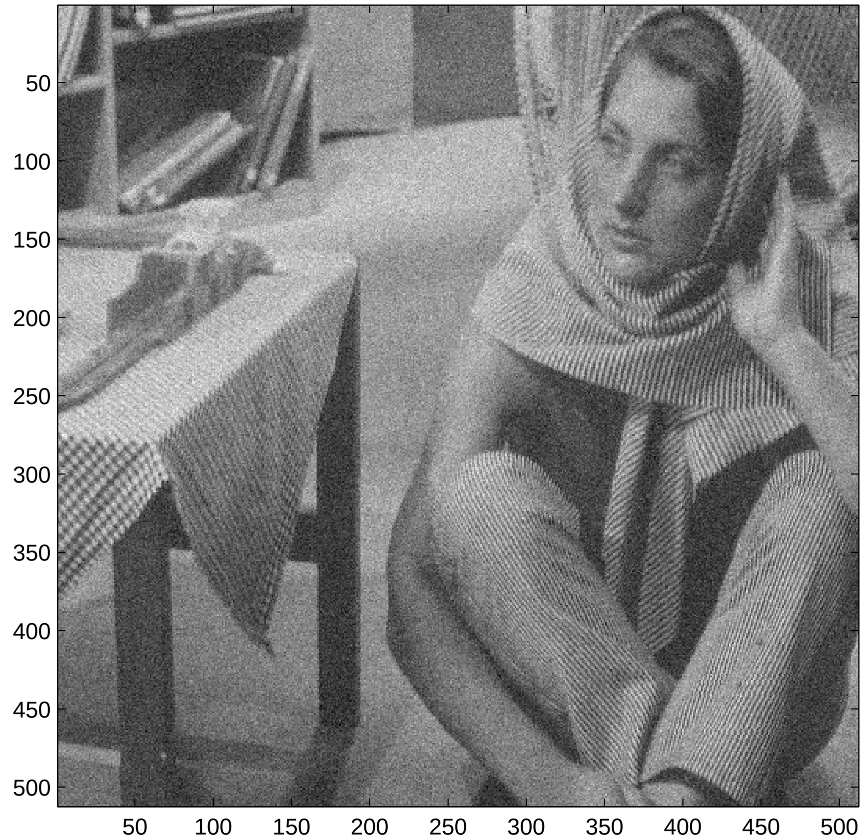
(c) Curvelets and TV



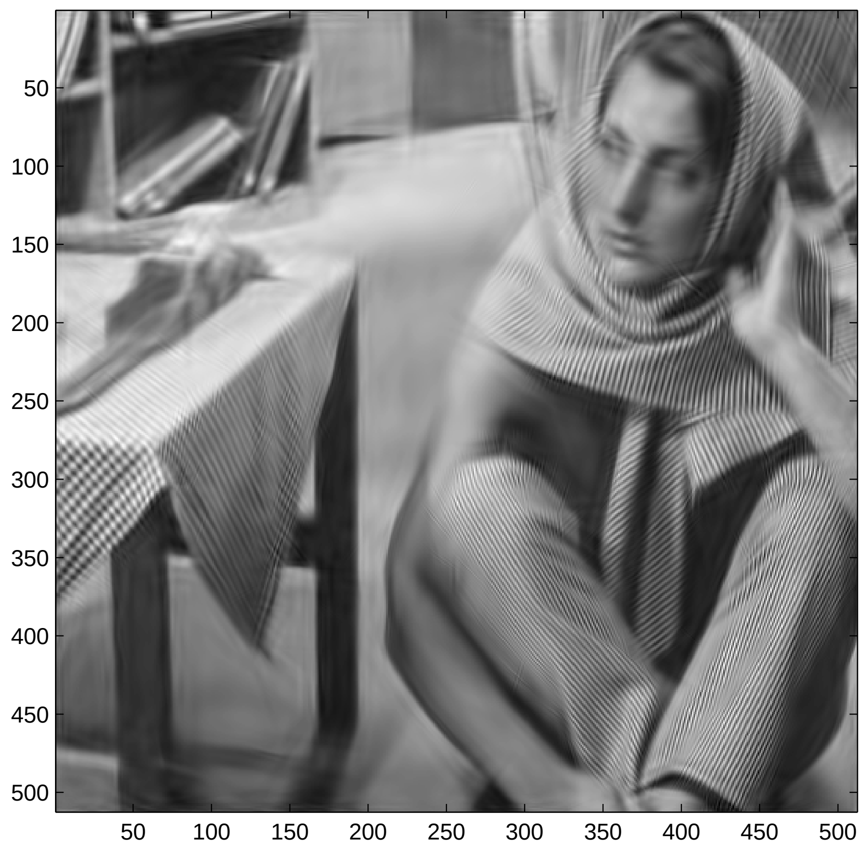
(d) True Scanline and Curvelets and TV Reconstruction



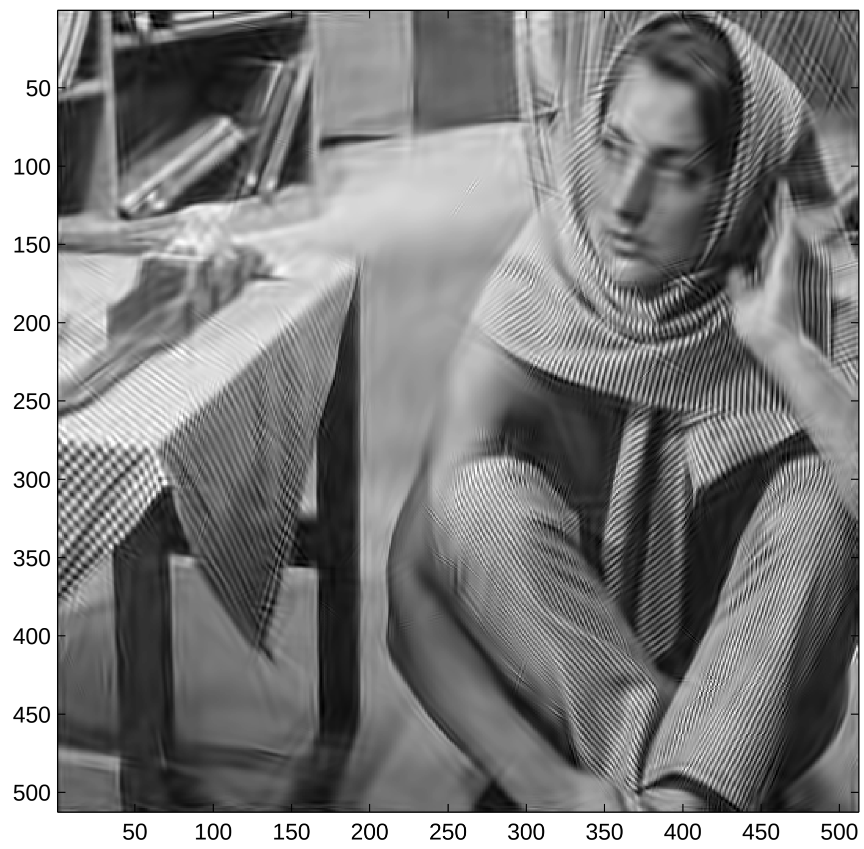
(p) Original



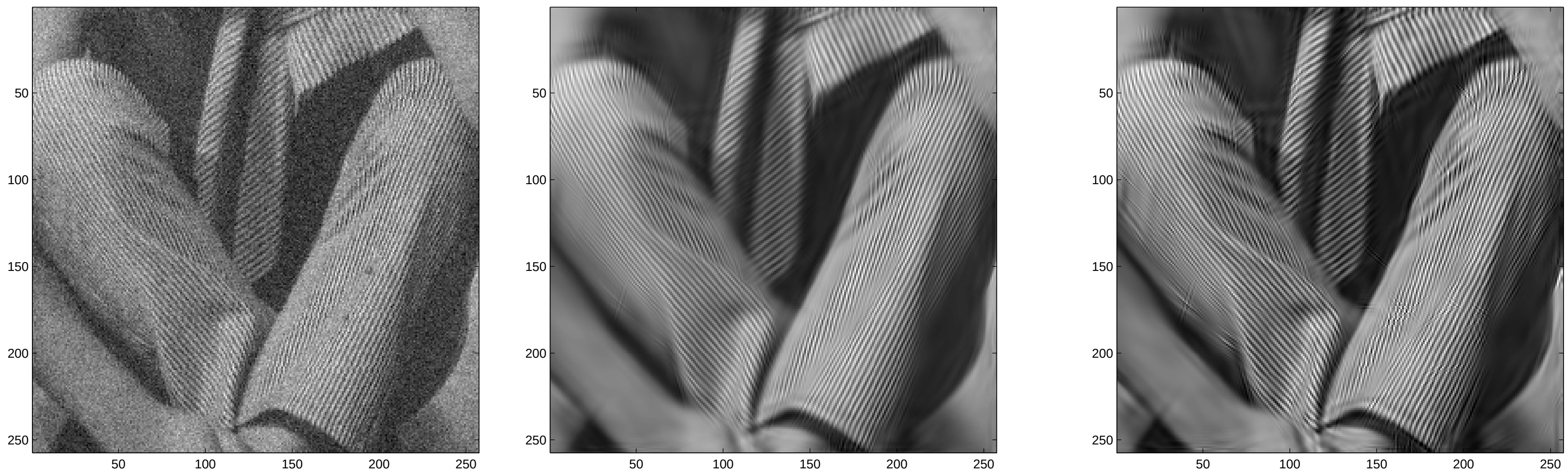
(q) Noisy



(r) Curvelets



(s) Curvelets and TV



(t) Noisy Detail (u) Curvelets (v) Curvelets and TV