Image-based modelling of ocean surface circulation from satellite acquisitions

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Objective

- Estimation of surface circulation (2D motion $w(x, t)$) from an image sequence $T(x, t)$
- Data Assimilation

Figure: Satellite data acquired over the Black Sea. Circulation

- Important issue for pollutant transport and meteorology forecast
State-of-the-art

Image Processing

- Image structures as tracer of the sea surface circulation

- Optical flow methods:
  - They compute translational displacement between two observations
  - They are ill-posed (the Aperture Problem) and required spatial regularization
  - Validity of brightness constancy assumption?
  - ⇒ not physically suited for ocean circulation
State-of-the-art
Ocean circulation models and images

- Circulation: advanced 3D oceanographic models are available (Navier-Stokes, see NEMO project (http://www.nemo-ocean.net) for instance)

- But: only a thin upper layer of ocean is observable (from satellite)

- From 3D Navier-stokes and various simplifications, a 2D ocean surface circulation model is derived → shallow water equations

- Compute an optimal solution w.r.t. the model and fitting observations: use of Data Assimilation techniques

- Need of an observation model: link between state vector and observations
State-of-the-art

Proposed method

- Shallow water model requires information on temperature and upper layer thickness

- Temperature: available from Sea Surface Temperature images (NOAA-AHVRR sensors)

- Layer thickness: not available from remote sensing

⇒ We propose a method to compute surface circulation from SST image and without information on the upper layer thickness
⇒ Use of a rough model, missing information will be represented in an additional model
⇒ Solution is computed using a weak 4D-Var formulation
Model

Shallow water equations

- State vector is velocity, \( \mathbf{w} = (u, v)^T \), surface temperature \( (T_s) \) and upper layer thickness \( (\eta) \):

\[
\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} + fv - g' \frac{\partial \eta}{\partial x} + K_w \Delta u \tag{1}
\]

\[
\frac{\partial v}{\partial t} = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - fu - g' \frac{\partial \eta}{\partial y} + K_w \Delta v \tag{2}
\]

\[
\frac{\partial \eta}{\partial t} = -\frac{\partial (u\eta)}{\partial x} - \frac{\partial (v\eta)}{\partial y} - \eta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{3}
\]

\[
\frac{\partial T_s}{\partial t} = -u \frac{\partial T_s}{\partial x} - v \frac{\partial T_s}{\partial y} + K_T \Delta T_s \tag{4}
\]

- \( K_w, K_T \) are diffusive constants, \( f \) the Coriolis parameter and \( g' \) the reduced gravity (see paper for details)

- Functions \( \mathbf{w}, T_s \) and \( \eta \) are defined on a space-time domain: \( \Omega \times [0, T], \Omega \subset \mathbb{R}^2 \)
Geophysical forces (in red) are grouped

\[
\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} + f v - g' \frac{\partial \eta}{\partial x} + K_w \Delta u \tag{5}
\]

\[
\frac{\partial v}{\partial t} = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} - f u - g' \frac{\partial \eta}{\partial y} + K_w \Delta v \tag{6}
\]

\[
\frac{\partial \eta}{\partial t} = -\frac{\partial (u \eta)}{\partial x} - \frac{\partial (v \eta)}{\partial y} - \eta \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{7}
\]

\[
\frac{\partial T_s}{\partial t} = -u \frac{\partial T_s}{\partial x} - v \frac{\partial T_s}{\partial y} + K_T \Delta T_s \tag{8}
\]

in a hidden part, \( \mathbf{a} = (a_u, a_v)^T \), named “additional model”
Proposed model

- The previous system is rewritten as:

\[
\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial y} + a_u \tag{9}
\]

\[
\frac{\partial v}{\partial t} = -u \frac{\partial v}{\partial x} - v \frac{\partial v}{\partial y} + a_v \tag{10}
\]

\[
\frac{\partial T_s}{\partial t} = -u \frac{\partial T_s}{\partial x} - v \frac{\partial T_s}{\partial y} + K_T \Delta T_s \tag{11}
\]

- with

\[
a_u = f v - g' \frac{\partial \eta}{\partial x} + K_w \Delta u \tag{12}
\]

\[
a_v = -f u - g' \frac{\partial \eta}{\partial y} + K_w \Delta v \tag{13}
\]

where $\eta$ verifies Eq.(3)
Proposed model

- The additional model $\mathbf{a}$ is now considered as an unknown that we want to retrieve.
- The state vector is $\mathbf{X} = (\mathbf{w} \quad T_s)^T$, and the model ruling the evolution in time of $\mathbf{X}$ is summarized as:

$$
\frac{\partial \mathbf{X}}{\partial t}(\mathbf{x}, t) + \mathbb{M}(\mathbf{X})(\mathbf{x}, t) = \begin{pmatrix} \mathbf{a}(\mathbf{x}, t) \\ 0 \end{pmatrix} \quad \mathbf{x} \in \Omega, \ t \in (0, T]
$$

(14)

with

$$
\mathbb{M}(\mathbf{X}) = \begin{pmatrix}
\frac{u}{\partial x} + \frac{v}{\partial y} \\
\frac{u}{\partial x} + \frac{v}{\partial y} \\
\frac{u}{\partial x} + \frac{v}{\partial y} + K_T \Delta T_s
\end{pmatrix}
$$
Initialization and Observation model

- Need of an initial condition for $X(0)$:
  \[ X(x, 0) = X_b(x) + \varepsilon_B(x), \quad x \in \Omega \]  
  \(\varepsilon_B\) Gaussian with covariance matrix $B$

- Observations are SST images, $T(t_i)$, available at some given dates $t_1, \cdots, t_N$

- The observation operator $\mathbb{H}$ projects the state vector in the observation space. It is defined as:
  \[ \mathbb{H}(X) = T_s \]  

- Link between state vector and observation:
  \[ \mathbb{H}(X)(x, t_i) = T(x, t_i) + \varepsilon_R(t_i), \quad x \in \Omega, i = 1, \cdots, N \]  
  \(\varepsilon_R\) Gaussian with covariance matrix $R$
Data assimilation
Weak 4D-Var formulation

To solve:

\[
\frac{\partial \mathbf{X}}{\partial t}(\mathbf{x}, t) + \mathbb{M}({\mathbf{X}})(\mathbf{x}, t) = \begin{pmatrix} \mathbf{a}(\mathbf{x}, t) \\ 0 \end{pmatrix}, \quad \mathbf{x} \in \Omega, \ t \in (0, T] \quad (18)
\]

\[
\mathbb{H}(\mathbf{X})(\mathbf{x}, t_i) = T(\mathbf{x}, t_i) + \varepsilon_R(t_i), \quad \mathbf{x} \in \Omega, \ i = 1, \cdots, N
\]

\[
\mathbf{X}(\mathbf{x}, 0) = \mathbf{X}_b(\mathbf{x}) + \varepsilon_B(\mathbf{x}), \quad \mathbf{x} \in \Omega
\]

we minimize the cost function \( J \):

\[
J(\varepsilon_B, \mathbf{a}(t_1), \cdots, \mathbf{a}(t_N)) = \langle \varepsilon_B, B^{-1} \varepsilon_B \rangle + \gamma \sum_{i=1}^{N} \| \nabla \mathbf{a}(t_i) \|^2 + \\
\sum_{i=1}^{N} \langle \mathbb{H}(\mathbf{X})(t_i) - T(t_i), R^{-1}(\mathbb{H}(\mathbf{X})(t_i) - T(t_i)) \varepsilon_B \rangle
\]

under the constraint of Eq.(18)

\( \gamma \) term is introduced to prevent numerical instabilities
Weak 4D-Var formulation

Computation of $\nabla J$

**Theorem 1**

Let $\lambda(x, t)$ be an auxiliary variable (named adjoint variable) as solution of:

\[
\begin{align*}
\lambda(T) &= 0 \quad (20) \\
-\frac{\partial \lambda(t)}{\partial t} + \left( \frac{\partial M}{\partial X} \right)^* \lambda &= \mathbb{H}^T R^{-1} [\mathbb{H}X(t) - T(t)] \quad t = t_i \quad (21) \\
&= 0 \quad t \neq t_i \quad (22)
\end{align*}
\]

Then, gradient of $J$ is:

\[
\begin{align*}
\frac{\partial J}{\partial \varepsilon_B} &= 2 \left( B_i^{-1} \varepsilon_B + \lambda(0) \right) \quad (23) \\
\frac{\partial J}{\partial a(t_i)} &= 2 \left( -\gamma \Delta a(t_i) + \lambda(t_i) \right) \quad (24)
\end{align*}
\]
Weak 4D-Var formulation

Algorithm

1. **Forward pass**: integrate forward in time $X(t)$, compute $J$

2. **Backward pass**: integrate backward in time $\lambda(t)$, compute $\nabla J$

3. Perform a steepest descent using numerical solver and get new values for $X(0)$ and $a(t_i)$, $i = 1 \cdots N$

4. Repeat steps 1, 2, 3 up to convergence
Implementation

- Eq. (18) must be discretized:
  - In time: Euler scheme
  - In space:
    - components $u$ and $v$ are transported by a non linear advection (Burger equations): Godunov scheme
    - component $I_s$ is transport by a linear advection: a first order up-wind

- Adjoint model: operator $(\frac{\partial M}{\partial x})^*$ in Eq. (21) is formally defined as a dual operator:

$$\left\langle \phi, \left(\frac{\partial M}{\partial x}\right)^* \psi \right\rangle = \left\langle \left(\frac{\partial M}{\partial x}\right) \phi, \psi \right\rangle$$

It must be determined from the discrete model $M_{\text{dis}}$: we use an automatic differentiation software, Tapenade [Hascoët and Pascual, 2013]

- Steepest descent is performed by BFGS solver [Byrd et al., 1995]
Data assimilation setup

- Model: temperature diffusion is neglected ($K_T = 0$)
- Initial condition $\mathbf{X}_b = (\mathbf{w}_b \quad I_b)^T$:
  - no information available on initial velocity, we set $\mathbf{w}_b = \mathbf{0}$
  - $I_b$ is initialized to the first available observation $I(t_1)$
- Covariance matrix $B$: without information on initial velocity we choose $\varepsilon_B = (0 \quad 0 \quad \varepsilon_{B_{ls}})$ and we have:
  \[
  \langle \varepsilon_B, B^{-1}\varepsilon_B \rangle = \langle \varepsilon_{B_{ls}}, B_{ls}^{-1}\varepsilon_{B_{ls}} \rangle
  \]
- Matrix $B_{ls}$: chosen diagonal, each element is set to 1 (1 Celsius degree means 25% of image dynamics)
- Covariance matrix $R$: chosen diagonal, each element is set to 1
- $\gamma$: empirically fixed
Results
A first satellite experiment

- A sequence of four SST images was acquired over Black Sea on October 10\textsuperscript{th}, 2007

Figure: October 10\textsuperscript{th} 2007, over Black Sea

(a) 30\textit{min}  
(b) 6\textit{h}  
(c) 15\textit{h}  
(d) 30\textit{h}
Results
Velocity retrieved

Figure: Motion computed between the first and second observation
No ground-truth on satellite data, how to evaluate?

We propose to compute the trajectory of some characteristic points in order to evaluate algorithms in term of transport.

A comparison with state-of-the-art is performed.

We proceed as follow:

- Manual selection of a characteristic point in the first observation.
- A map of signed distance is computed.
- Distance map is transported by the velocity field that we want to evaluate [Lepoittevin et al., 2013].
- Computation of local maximum in transported map gives the characteristic point position along the sequence.
Result

Characteristic points

(a) First observation

(b) Last observation. Blue = our method, red = Sun et al.

Figure: Evolution of some characteristic points
Satellite Experiment #2
Observations

- Sequence acquired on October 8th, 2005

Figure: October 8th, 2005, over Black Sea
Satellite Experiment #2

Motion results

(a) [Sun et al., 2010]  (b) Proposed method

Figure: Motion computed between the first and second observation
Satellite Experiment #2

Characteristic points

(a) First observation

Figure: Evolution of some characteristic points
Satellite Experiment #2

Characteristic points

(a) Second observation

Figure: Evolution of some characteristic points
Satellite Experiment #2

Characteristic points

(a) Third observation

**Figure**: Evolution of some characteristic points
Satellite Experiment #2

Characteristic points

(a) Last observation

Figure: Evolution of some characteristic points
Satellite Experiment #3

Observations

- Sequence acquired on July 27th, 2007

(a) 30min  
(b) 8h15min  
(c) 13h  
(d) 22h30min  
(e) 24h30min
Satellite Experiment #3
Motion results

(a) [Sun et al., 2010]        (b) Proposed method

Figure: Motion computed between the first and second observation
Satellite Experiment #3

Characteristic points

Figure: Evolution of some characteristic points

(a) First observation
Satellite Experiment #3

Characteristic points

(a) Second observation

Figure: Evolution of some characteristic points
Satellite Experiment #3

Characteristic points

(a) Third observation

Figure: Evolution of some characteristic points
Satellite Experiment #3

Characteristic points

(a) Fourth observation

Figure: Evolution of some characteristic points
Satellite Experiment #3

Characteristic points

(a) Last observation

Figure: Evolution of some characteristic points
Satellite Experiment #4

Observations

- Sequence acquired on May 14th, 2005

(a) 30min

(b) 2h55min

(c) 5h15min

(d) 7h15min

(e) 16h15min
Satellite Experiment #4

Motion results

(a) [Suter, 1994]

(b) Proposed method

Figure: Motion computed between the first and second observation
Satellite Experiment #4

Characteristic points

(a) First observation

Figure: Evolution of some characteristic points
Satellite Experiment #4

Characteristic points

Figure: Evolution of some characteristic points
Satellite Experiment #4

Characteristic points

(a) Third observation

Figure: Evolution of some characteristic points
Satellite Experiment #4

Characteristic points

(a) Fourth observation

**Figure**: Evolution of some characteristic points
Satellite Experiment #4

Characteristic points

(a) Last observation

Figure: Evolution of some characteristic points
Concluding remarks

- Determination of surface circulation using a rough model $\mathbb{M}$
- Dynamics not modelled by $\mathbb{M}$ is captured in $a$
- Analysis of $a$ retrieved and comparison with a shallow water model
- Experiments on synthetic models (Temperature and upper layer thickness) with ground truth


