Surface circulation from satellite images: reduced model of the Black Sea

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Oceanography: Black Sea circulation

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Objective
Estimation of motion $\mathbf{W}(\mathbf{x}, t)$ on an image sequence $I(\mathbf{x}, t)$ by Data Assimilation.

Input Data
Sequence of satellite images $I$:

Result of estimation
Estimation of apparent velocity $\mathbf{W}$:
Data assimilation (4D-Var)

\( \mathbf{X}(t) \) state vector, \( \mathbf{Y}(t) \) observation vector.

Find \( \mathbf{X} \) such as:

\[
\begin{align*}
\frac{\partial \mathbf{X}}{\partial t}(t) + \mathbf{M}((\mathbf{X}))(t) &= 0 \\
\mathbf{X}(0) &= \mathbf{X}_b + \mathbf{E}_b \\
\mathbf{H}(\mathbf{X}, \mathbf{Y})(t) &= \mathbf{E}_O(t)
\end{align*}
\]

- \( \mathbf{X}(t) \) depends only from \( \mathbf{X}(0) \) and the integration from (1)
- The problem is then to minimize the two error terms \( \mathbf{E}_b \) and \( \mathbf{E}_O \). This is solved using a steepest descent method (L-BFGS).
Model

- Lagrangian constancy of the velocity $\mathbf{w}(t)$

$$\frac{d\mathbf{w}}{dt} = \frac{\partial \mathbf{w}}{\partial t} + (\mathbf{w} \cdot \nabla)\mathbf{w} = 0$$

- Optical flow on the pseudo image $I_s(t)$

$$\frac{\partial I_s}{\partial t} + \mathbf{w} \cdot \nabla I_s = 0$$
Reduction

- Reduced state: less memory
- Regularity: applied on basis elements
- Boundary conditions: imposed to the basis elements
- Numerical schemes: ODE vs PDE
Full and reduced models

**Full model**

\[
\begin{align*}
\frac{\partial w}{\partial t}(x, t) + (w \cdot \nabla) w(x, t) &= 0 \\
\frac{\partial l_s}{\partial t}(x, t) + w \cdot \nabla l_s(x, t) &= 0
\end{align*}
\]

\[
\begin{align*}
w(x, t) &\approx \sum_{k=1}^{K} a_k(t) \phi_k(x) \\
l_s(x, t) &\approx \sum_{l=1}^{L} b_l(t) \psi_l(x)
\end{align*}
\]

**Reduced model**

\[
\begin{align*}
\frac{da_k}{dt}(t) + a^T B(k) a &= 0, \quad k = [1, K] \\
\frac{db_l}{dt}(t) + a^T G(l) b &= 0, \quad l = [1, L]
\end{align*}
\]

\[
\begin{align*}
B(k)_{i,j} &= \frac{(\phi_i \nabla) \phi_j, \phi_k}{\langle \phi_k, \phi_k \rangle} \\
G(l)_{i,j} &= \frac{\phi_i \cdot \nabla \psi_j, \psi_l}{\langle \psi_l, \psi_l \rangle}
\end{align*}
\]
Motion basis

\( \phi_i \) are obtained by sequentially solving systems \( S_i \):

\[
S_i = \begin{cases} 
\phi_i = \min_{f \in L^2(\Omega)^2} \langle \nabla f, \nabla f \rangle \\
\text{div} (\phi_i(x)) = 0 \quad \forall x \in \Omega \\
\phi_i(x) \cdot n(x) = 0 \quad \forall x \in \partial\Omega \\
\langle \phi_i, \phi_k \rangle = \delta_{i,k}, \quad k \in [1, i]
\end{cases}
\]
Motion basis
Motion basis
\[ S_i = \begin{cases} 
\psi_i = \min_{f \in L^2(\Omega)} \langle \nabla f, \nabla f \rangle \, dx \\
\nabla \psi_i(x) \cdot n(x) = 0 \quad \forall x \in \partial \Omega \\
\langle \psi_i, \psi_k \rangle = \delta_{i,k}, \quad k \in [1, i] 
\end{cases} \]
Image basis
Data assimilation system

State vector: \( \mathbf{X}(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \)

with:
\[
\begin{align*}
a(t) &= (a_1(t), \ldots, a_K(t))^T \\
b(t) &= (b_1(t), \ldots, b_L(t))^T
\end{align*}
\]

- Background equation: \( b(0) - b^{obs}(t_1) = \mathcal{E}_B \)
- Observation equation: \( b(t_k) - b^{obs}(t_k) = \mathcal{E}_R(t_k) \).

Energy function:
\[
J(\mathbf{X}(0)) = \int \mathcal{E}_R(t)^T R^{-1}(t) \mathcal{E}_R(t) dt + \mathcal{E}_B^T B^{-1} \mathcal{E}_B
\]
Whole Black Sea bassin

Simulation of full model from chosen initial conditions:

Observation dates chosen:

\[ t_0 \rightarrow t_1 \rightarrow t_2 \rightarrow t_3 \rightarrow \ldots \rightarrow t_6 \]
Assimilation of Black Sea simulations

Ground-truth:

Results of Assimilation in the reduced model:
Satellite images

SST images of the Black Sea acquired with NOAA/AVHRR between May 14th, 2005 and May 15th, 2005
Satellite images

SST images of the Black Sea acquired with NOAA/AVHRR between May 14th, 2005 and May 15th, 2005
Characteristic points

Ground-truth

Reduced model

Full model

Sun
Assimilation of satellite images on the whole basin

Clime
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Conclusions and perspective

Advantages and drawbacks

- Reduction
- Boundary conditions applied to the basis elements
- Choice of coefficients number linked to studied spatial scale

Perspectives

- Adaptative waveforms
- POD or seasonal POD
- Krylov approximation