Higher-order interpolation for mesh adaptation

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Summary. This paper addresses the construction of anisotropic metrics from higher-order interpolation error in 2 dimensions [2, 3] for mesh adaptation. Our approach is based on homogeneous polynomials that model a local interpolation error. Optimal orientation and ratios are found by using the Sylvester decomposition [4]. Then we apply a global calculus of variation to get the optimal metric field minimizing the \(L^p\) norm of the interpolation error. We illustrate this approach on a numerical example.

1 Third order interpolation error model

We define the quadratic interpolate \(\Pi^2_h u\) of \(u\) on a mesh \(\mathcal{H}\) by:

\[
\forall \mathbf{x} \in \Omega, \quad \Pi^2_h u(\mathbf{x}) = \sum_{i=1}^{6} u(\mathbf{p}_i) \varphi_i(\mathbf{x})
\]

where \(\mathbf{p}_i\) is the \(i^{th}\) mesh nodes, \(\varphi_i(.)\) is the \(i^{th}\) \(P^2\) Lagrange shape function defined by:

\[
\begin{aligned}
\varphi_i(\mathbf{x}) &= \psi_i(\mathbf{x})(2\psi_i(\mathbf{x}) - 1), \quad i = 1, 2, 3 \\
\varphi_i(\mathbf{x}) &= 4 \psi_i(\mathbf{x}) \psi_{i+1}(\mathbf{x}), \quad i = 4, 5, 6 \quad (edges \; midpoints)
\end{aligned}
\]

and \(\psi_i\) is the \(i^{th}\) \(P^1\) Lagrange shape function defined by:

\[
\psi_i(\mathbf{p}_j) = \delta_{ij}, \quad \mathbf{p}_j \in \mathcal{H}.
\]

We approximate the local interpolation error \(e_h = |u - \Pi^2_h u|\) by an homogeneous polynomial of degree \(k = 3\) in 2 variables:

\[
P_k(\mathbf{x}) = \sum_{i=0}^{k} \binom{k}{i} a_i x^i y^{k-i}.
\]  \(1\)

\(\binom{k}{i}\) are the binomial coefficients and \(a_i\) are the third-order derivatives of \(u\) on each vertex of the mesh. When \(u\) is a numerical solution, they are obtained by using a reconstruction method based on the Clément interpolation operator.
To recover the third-order derivatives $D^{(3)}_R u$, the process starts from the constant hessian $H_{u|K}$ on $K$ given by:

$$H_{u|K}(x) = \sum_{i=1}^{6} u(p_i) H\varphi_i(x).$$

For each $p_i$, we thus have the following hessian reconstruction:

$$H_R u(p_i) = \sum_{K_j \in S_i} \frac{|K_j|}{|S_i|} H_{u|K_j}.$$  \hfill (2)

where $|K_j|$ denote the area of $K_j$ and $S_i$ the area of the stencil of $p_i$. To recover the third-order derivatives $a_i = [D^{(3)}_R u]_i$ from $u$, we apply a gradient reconstruction procedure to each component of the hessian.

$$D^{(3)}_R u(p_i) = \sum_{K_j \in S_i} \frac{|K_j| \nabla(H_R u)|_{K_j}}{|S_i|}.$$  \hfill (3)

2 Local error decomposition and optimal local metric

In this section, we approach locally the variations of $P_e$ by a quadratic definite positive form (a metric tensor) taken at power $\frac{k_2}{2}$, i.e, for $x = (x, y) \in \mathbb{R}^2$,

$$|P_e(x)| \leq \left( ^t x \mathcal{M}_{opt}^e x \right)^{\frac{k_2}{2}}.$$  \hfill (4)

Geometrically, the optimal local metric is the one of maximal area whose unit ball is included in the isoline $1$ of $|P_e|$. To obtain $\mathcal{M}_{opt}$, we propose a local optimization problem based on the Sylvester’s theorem [4]. The Sylvester binary decomposition allows to write the 2-dimensional homogeneous polynomial $P_e(x, y)$ of degree $k$ as a sum of $k$th powers of $r$ distinct linear forms in $\mathbb{C}$:

$$P_e(x) = \sum_{j=1}^{r} \lambda_j (\alpha_j x + \beta_j y)^k,$$  \hfill (5)

$r$ is the decomposition rank, i.e it is the minimal number of linear terms such that (5) holds. We use the algorithm proposed in [4].

Once we obtain the decomposition (5), $\mathcal{M}_{opt}$ is obtained differently according to the nature of the coefficients $(\alpha_j, \beta_j)$:

- Real case: $\mathcal{M}_{opt}^e = \left( ^t Q \left( \begin{array}{cc} \frac{1}{k_2} & 0 \\ 0 & \frac{1}{k_2} \end{array} \right) \right) Q$.

- Complex case: $\mathcal{M}_{opt}^e = 2^{-\frac{k}{2}} \left( ^t \bar{Q} \left( \begin{array}{cc} \frac{1}{k_2} & 0 \\ 0 & \frac{1}{k_2} \end{array} \right) \right) Q$. 

where \( Q = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \) gives the optimal directions of \( M_{opt}^{loc} \). \( \bar{Q} \) is the real transpose and \( \bar{Q}^\dagger \) is the conjugate complex transpose of \( Q \). The sizes along optimal directions are: \( h_i = \frac{1}{|\lambda_i|^\frac{1}{p}} \). We depict in Figure 1 two error models and their corresponding optimal metric \( M_{opt}^{loc} \).

\[ \text{Fig. 1. Examples of error models:} \ P_e = 50x^3 - 120x^2y - 1000xy^2 - y^3 \ (\text{left}), \ P_e = x^3 - 500x^2y - 500xy^2 - y^3 \ (\text{right}). \ Representation of their iso-values and their corresponding optimal ellipse included in the isoline 1(in red). \]

3 Global variational calculus problem

The optimal global metric \( M_{opt} = (M_{opt}(x))_{x \in \mathbb{R}^2} \) is the solution of the following variational calculus problem:

\[
M_{opt} = \min_M E_p(M) = \left( \int_{\Omega} |e_M(x)|^p \, dx \right)^{\frac{1}{p}} = \left( \int_{\Omega} |\bar{x} M(x) x|^{\frac{kp}{2}} \, dx \right)^{\frac{2}{kp}} \tag{6}
\]

under the constraint \( C(M) = \int_{\Omega} (h_1h_2)^{-1} = N \).

To solve this problem, we use the Euler-Lagrange necessary condition which states that there exists \( \alpha \) a constant such that \( \forall \delta M, \delta E(M, \delta M) = \alpha \delta C(M, \delta M) \).

We obtain the optimal global metric \( M_{opt} \), solution of the problem (6):

\[
M_{opt} = N \left( \int_{\Omega} (\lambda_1\lambda_2)^{\frac{k_p}{2(kp+2)}} \right)^{-1} \left( \lambda_1 \lambda_2 \right)^{-\frac{1}{kp+2}} \begin{pmatrix} 1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.
\]

The optimal value of the density is:

\[
d_{opt} = N \left( \int_{\Omega} (\lambda_1\lambda_2)^{\frac{k_p}{2(kp+2)}} \right)^{-1} (\lambda_1\lambda_2)^{\frac{k_p}{2(kp+2)}},
\]
and the optimal value of the error is:

\[ E_p(M_{opt})^p = 2^{\frac{k}{p}} N^{-\frac{k}{2}} \left( \int_{\Omega} (\lambda_1 \lambda_2)^{\frac{k}{p} + \frac{1}{2}} \right)^{\frac{k}{2}} (\lambda_1 \lambda_2)^{\frac{k}{p} + \frac{1}{2}}. \]

The order of convergence for a sequel of continuous meshes \( (M^N_{L^p})_N \) verifies:

\[ \|u - \Pi^2_{M^N_{L^p}} u\|_{L^p(\Omega)} \leq \frac{C_{st}}{N^{\frac{k}{2}}}. \] (7)

Relation (7) points out a global \( k \)-th order of mesh convergence.

4 Numerical example

To validate our approach, we compare an adaptation based on the previous optimal metric (with third order derivatives recovery) with an adaptation only based on the hessian on \( u \) (constant by triangles). For both strategies, the interpolation error level is computed by mean of 5th order gauss interpolation to estimate \( \|u - \Pi^2_{h} u\|_{L^p(\Omega)} \). Consequently, we compare a \( P^1 \)-driven adaptation with a \( P^2 \)-driven adaptation having a \( P^2 \)-Lagrange triangle to represent the function. We consider:

\[ f(x, y) = \begin{cases} 
0.01 \sin(50xy) & \text{if } xy \leq -\frac{\pi}{50} \\
\sin(50xy) & \text{if } -\frac{\pi}{50} < xy \leq \frac{2\pi}{50} \\
0.01 \sin(50xy) & \text{if } xy > \frac{2\pi}{50}
\end{cases} \]

Fig. 2. Convergence curves: \( L^1 \) norm of the error versus the number of dof (left), \( L^2 \) norm of the error versus the number of dof (right) for the function \( f \).
The spatial convergence curves are depicted in Figure 2 and the meshes in Figure 3. The sequence of $P^2$-driven adapted mesh shows a 3rd order of convergence while $P^1$-driven adaptation shows an asymptotic rate of convergence of one (despite the interpolation is still 2nd order) for $L^1$ and $L^2$ norms. This emphasizes the needs to consider higher-order interpolation error in order to achieve an optimal rate of convergence.

5 Conclusion

We have shown that anisotropic mesh adaptation can be extended to higher order interpolations. We are currently extending this approach to the 3 dimensional case by using tensor decomposition methods. Extension to curved isoparametric triangles is also ongoing.

References