Backward Convolution Algorithm for Discretely Sampled Asian Options

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Abstract

This note gives a summary of the backward price convolution algorithm used in [1] to price discretely sampled Asian options. For more details see [1].

1 Introduction

Let \((\Omega, \mathcal{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a probability space. We consider an underlying price of a risky asset given by

\[
S_j = S_0 \exp \left( \sum_{k=1}^{j} Z_k \right) \text{ for } j = \{1, \ldots, n\},
\]  

and a process of partial sums defined by

\[
I_j = \sum_{k=0}^{j} \lambda_k S_k \text{ for } j = \{1, \ldots, n\},
\]

where \((Z_k)_{1 \leq k \leq n}\) is a collection of independent random variables and the deterministic process \(\lambda\) depends on the type of the Asian option (see Table 1).

As proved by [1], the price of an Asian option amounts to calculating the quantity

\[
\mathbb{E} \left( I_n^+ \right).
\]

Introduce a new filtration \(\mathcal{G} := (G_i)_{1 \leq i \leq n}\)

\[
G_i = \sigma (Z_n, Z_{n-1}, \ldots, Z_{n-i+1}),
\]

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Table 1: Choice of $\lambda$ corresponding to different types of Asian options. $\alpha > 0$ is the coefficient of partiality for floating strike options. Coefficient $\gamma$ takes value 1 (0) when $S_0$ is (is not) included in the average.

<table>
<thead>
<tr>
<th>Option type</th>
<th>$\lambda_0$</th>
<th>$\lambda_1, \ldots, \lambda_{n-1}$</th>
<th>$\lambda_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Call, fixed strike</td>
<td>$\frac{1}{n+\gamma} - \frac{\alpha}{S_0}$</td>
<td>$\frac{1}{n+\gamma}$</td>
<td>$1 - \frac{\alpha}{n+\gamma}$</td>
</tr>
<tr>
<td>Call, floating strike</td>
<td>$\frac{1}{n+\gamma}$</td>
<td>$\frac{1}{n+\gamma}$</td>
<td>$\frac{\alpha}{n+\gamma} - 1$</td>
</tr>
<tr>
<td>Put, fixed strike</td>
<td>$\frac{\alpha}{S_0} - \frac{1}{n+\gamma}$</td>
<td>$\frac{\alpha}{n+\gamma} - 1$</td>
<td></td>
</tr>
<tr>
<td>Put, floating strike</td>
<td>$\frac{\alpha}{n+\gamma}$</td>
<td>$\frac{\alpha}{n+\gamma} - 1$</td>
<td></td>
</tr>
</tbody>
</table>

and a process $X$ defined by

$$X_k := \lambda_{n-k} + X_{k-1} \exp (Z_{n+1-k})$$

$$X_0 := \lambda_n :$$

Proposition 2.1 of [1] shows that $X$ is a $\mathcal{G}$-Markov process under $\mathbb{P}$.

## 2 Backward Price Convolution Algorithm

To price Asian options [1] have used a backward algorithm described in the following theorem (Theorem 3.1 in [1]). Note that we should have $\lambda_k > 0$ for any $k \in \{1, \ldots, n\}$. This is the case for the fixed strike Asian call.

**Theorem 2.1.** Assume that for all $k$ the CDF of $Z_{n+1-k}$ has a probability density function $f_k$ with respect to the Lebesgue measure on $\mathbb{R}$, satisfying

$$\mu_k := \int_{\mathbb{R}} e^z f_k(z) dz < \infty.$$ 

Consider constants $\lambda_k > 0$, $0 < k \leq n$ and $\lambda_0 \in \mathbb{R}$. Define functions $p_k : \mathbb{R} \to \mathbb{R}$ for $0 < k \leq n$ and $q_k, h_k : \mathbb{R} \to \mathbb{R}$ for $0 \leq k < n$ as follows

$$p_n(y) := (e^y + \lambda_0)^+;$$

$$h_k(y) := \log (e^y + \lambda_{n-k}), \quad 0 < k < n,$$

$$q_{k-1}(x) := \int_{\mathbb{R}} p_k(x + z) f_k(z) dz, \quad 0 < k \leq n,$$

$$p_{k-1}(y) := q_{k-1}(h_{k-1}(y)), \quad 1 < k \leq n.$$ 

The following statements hold:

1. The forward price of an Asian call contract with parameters $(\lambda_j)_{0 \leq j \leq n}$ is given by

$$\mathbb{E} \left( I_n^+ \right) = S_0 \mathbb{E} \left( X_n^+ \right) = S_0 q_0 \left( \log (\lambda_n) \right).$$
2. There are positive constants $a_k, b_k$ such that for all $x, y \in \mathbb{R}$

\[
0 \leq p_k(y) \leq a_k e^y + b_k,
\]

\[
0 \leq q_k(x) \leq a_k e^x + b_k + 1.
\]

These constants are given recursively by

\[
a_n = 1, \quad b_n = \lambda^+_0
\]

\[
a_{k-1} = a_k \mu_k
\]

\[
b_{k-1} = b_k + a_{k-1} \lambda_{n-k+1}.
\]

The range of integration in the above theorem must be curtailed. So functions $p_k$ and $q_k$ are approximated by functions $\bar{p}_k$ and $\bar{q}_k$ defined by

\[
\bar{q}_{k-1}(x) := \int_{\mathbb{R}} \bar{p}_k(x + z) f_k(z) d\mathbb{I}_{[\bar{L}_{k-1}, \bar{U}_{k-1}]}(z), \quad 0 < k \leq n
\]

\[
\bar{p}_{k-1}(y) := \bar{q}_{k-1}(h_{k-1}(y)) \mathbb{I}_{[L_{k-1}, U_{k-1}]}(y), \quad 1 < k \leq n
\]

\[
\bar{p}_n(y) := p_n(y) \mathbb{I}_{[L_n, U_n]}(y).
\]

The curtailed ranges $[\bar{L}_k, \bar{U}_k]$ for $k \in \{0, \ldots, n - 1\}$ and $[L_k, U_k]$ for $k \in \{1, \ldots, n\}$ are defined in Theorem 3.2 in [1]. The pricing error caused by the curtailment is also estimated in the latter. The idea in [1] is to evaluate $\bar{p}_k$ and $\bar{q}_k$ in the Fourier space. Then, for $0 \leq k < n$ and $x \in (\bar{L}_k, \bar{U}_k)$

\[
\bar{q}_{k-1}(x) = \mathcal{F}^{-1} \left( \mathcal{F}(\bar{p}_k) \bar{\phi}_k \right)(x),
\]

where $\mathcal{F}$ denotes the Fourier transform, $\phi_k$ is the characteristic function of $Z_{n-k}$ and $\bar{\phi}_k$ denotes its complex conjugate. The Fourier transform is approximated by the general discrete Fourier transform (see [1], Definition 4.3). The backward algorithm for numerical implementations is summarized by Proposition 4.3 of [1].

References


References