The double exponential jump model, initiated by Steven Kou (see [1]), is an exponential Levy model, which is a compromise between reality and tractability. It gives an explanation of the two empirical phenomena which received much attention in financial markets: the asymmetric leptokurtic feature and the volatility smile. It permits to obtain analytical solutions to the prices of many derivatives: European call and put options; interest rate derivatives, such as swaptions, caps, floors, and bond options; as well as path-dependent options, such as perpetual American options, barrier, and lookback options.

1 The model

The behaviour of the asset price, $S_t$, under the risk neutral probability is modeled as followed:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t + d \left( \sum_{i=1}^{N_t} e^{Y_i} - 1 \right)$$

Where $W$ is a standard brownian motion, $N$ is a poisson process with rate $\lambda$, the constants $\mu$ and $\sigma > 0$ are drift and volatility of the diffusion part and the jump sizes $\{Y_1, Y_2, \ldots\}$ are i.i.d random variables with a common asymmetric double exponential distribution, of density:

$$f_Y(y) = p\eta_1 e^{-\eta_1 y} 1_{\{y \geq 0\}} + q\eta_2 e^{\eta_2 y} 1_{\{y < 0\}}$$

where $p, q \geq 0$ are constants, $p + q = 1$, $\eta_1 > 1$ and $\eta_2 > 0$. The random processes $(W_t)_{t \geq 0}$, $(N_t)_{t \geq 0}$, and random variables $\{Y_1, Y_2, \ldots\}$ are independant. Furthermore we have $\mu = r - \lambda \xi$ with:

$$\xi = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1$$
The condition on $\mu$ hold in order to obtain $(e^{-rS_t})_{t \geq 0}$ is a martingale. The characteristic exponent $G$ of log $(S_t)$ (i.e. $\mathbb{E}[e^{\theta \log(S_t)}] = e^{G(\theta)t}$) is defined as:

$$G(x) = x \left( r - \frac{1}{2} - \lambda \xi \right) + \frac{1}{2} x^2 \sigma^2 + \lambda \left( \frac{\mu \eta_1}{\eta_1 - x} + \frac{\eta_2}{\eta_2 + x} - 1 \right)$$

The equation $G(x) = \alpha$ has exactly four roots (see [2]): $\beta_{1,\alpha}$, $\beta_{2,\alpha}$, $-\beta_{3,\alpha}$, $-\beta_{4,\alpha}$, where

$$0 < \beta_{1,\alpha} < \beta_{2,\alpha} < \infty, \quad 0 < \beta_{3,\alpha} < \beta_{4,\alpha} < \infty. \quad (1.4)$$

## 2 European call and put

Let us define some special functions (see pp. 1094 and 1099 in [1]):

$$Hh_{-1}(x) = e^{-\frac{x^2}{2}}$$

$$Hh_0(x) = \sqrt{2\pi} \Phi(-x)$$

$$Hh_n(x) = \int_x^{+\infty} Hh_{n-1}(y) dy = \frac{1}{n!} \int_x^{+\infty} (t-x)^n e^{-\frac{t^2}{2}} dt \quad \forall n \geq 0$$

$$I_n(c; \alpha, \beta, \gamma) = \int_c^{+\infty} e^{\alpha x} Hh_n(\beta c - \gamma) dx \quad \forall n \geq -1$$

where $\Phi$ is the standard normal cumulative distribution. Then we have:

$$n Hh_n(x) = Hh_{n-2}(x) - x Hh_{n-1}(x) \quad \forall n \geq 1$$

And $\forall n \geq -1$:

$$I_n(c; \alpha, \beta, \gamma) = -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^{n} \left( \frac{\beta}{\alpha} \right)^{n-i} Hh_i(\beta c - \beta) + \left( \frac{\beta}{\alpha} \right)^{n+1} \frac{\sqrt{2\pi} e^{\frac{a^2}{2} + \frac{c^2}{2\pi}}}{\beta} \Phi \left( -\beta c + \frac{a}{\beta} \right) \quad \beta > 0, \alpha \neq 0$$

$$I_n(c; \alpha, \beta, \gamma) = -\frac{e^{\alpha c}}{\alpha} \sum_{i=0}^{n} \left( \frac{\beta}{\alpha} \right)^{n-i} Hh_i(\beta c - \beta) - \left( \frac{\beta}{\alpha} \right)^{n+1} \frac{\sqrt{2\pi} e^{\frac{a^2}{2} + \frac{c^2}{2\pi}}}{\beta} \Phi \left( \beta c - \frac{a}{\beta} \right) \quad \beta < 0, \alpha < 0$$

Introduce the following notation : For any given probability $P$, define :

$$\psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) = \mathbb{P}[Z_T \geq a] \quad (2.5)$$

where $Z_T = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i$ and $Y$ has a double exponential distribution with density as in (1.2), and $N$ is a poisson process with rate $\lambda$. Theorem B.1. in [1] gives us :

$$\psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, T) = \begin{cases} \frac{e^{(\sigma \eta_1)^2 T}}{\sigma \sqrt{2\pi T}} \sum_{n=1}^{+\infty} \pi_n \sum_{k=1}^{n} P_{n,k} \left( \sigma \sqrt{T \eta_1} \right)^k I_{k-1} \left( a - \mu T; -\eta_1, -\frac{1}{\sigma \sqrt{T}} \right) \\ + \frac{e^{(\sigma \eta_2)^2 T}}{\sigma \sqrt{2\pi T}} \sum_{n=1}^{+\infty} \pi_n \sum_{k=1}^{n} Q_{n,k} \left( \sigma \sqrt{T \eta_2} \right)^k I_{k-1} \left( a - \mu T; \eta_2, -\frac{1}{\sigma \sqrt{T}} \right) \\ + \pi_0 Phi \left( a - \mu T \frac{1}{\sigma \sqrt{T}} \right) \end{cases}$$
where
\[
P_{n,i} := \sum_{j=i}^{n-1} p^j q^{n-j} \left( \frac{n-i-1}{j-i} \right) \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{j-i} \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{n-j}, \quad 1 \leq i \leq n-1
\]
\[
Q_{n,i} := \sum_{j=i}^{n-1} q^j p^{n-j} \left( \frac{n-i-1}{j-i} \right) \left( \frac{\eta_2}{\eta_1 + \eta_2} \right)^{j-i} \left( \frac{\eta_1}{\eta_1 + \eta_2} \right)^{n-j}, \quad 1 \leq i \leq n-1
\]
\[
P_{n,n} := p^n; \quad Q_{n,n} := q^n, \quad \pi_n = \frac{e^{-\lambda T} \lambda^n}{n!}
\]

Using theorem 2, in [1], we know that the price of european call at inception and with maturity $T$ is:
\[
S_0 \psi \left[ r + \frac{1}{2} \sigma^2 - \lambda \xi, \sigma, \lambda, \bar{p}, \eta_1, \bar{\eta}_2; \log \left( \frac{K}{S_0} \right), T \right] - K e^{-rT} \psi \left[ r + \frac{1}{2} \sigma^2 - \lambda \xi, \sigma, \lambda, p, \eta_1, \eta_2; \log \left( \frac{K}{S_0} \right), T \right]
\]
where
\[
\bar{p} = \frac{p}{1 + \xi \eta_1 - 1}, \quad \bar{\lambda} = \lambda(1 + \xi), \quad \bar{\eta}_1 = \eta_1 - 1, \quad \bar{\eta}_2 = \eta_2 + 1
\]
The put price can be obtain by using the call-put parity.

3 Finite time horizon american put option

Let $EuP(v, t)$ be the price of a european put option with initial stock price $v$ and maturity $t$, $\mathbb{P}^v [S_t \leq K]$ the probability that the stock price at $t$ is below $K$ with initial stock price $v$, $z = 1 - e^{-rT}$, $\beta_3 \equiv \beta_{3, r/2}$, $\beta_4 \equiv \beta_{5, r/2}$, $C_\beta = \beta_3 \beta_4 (1 + \eta_2)$ (see (1.4)), $D_\beta = \eta_2 (1 + \beta_3)(1 + \beta_4)$, $v_0 \equiv v_0(t) \in (0, K)$ the unique solution to the equation
\[
C_\beta K - D_\beta (v_0 + EuP(v_0, t)) = (C_\beta - D_\beta) K e^{-rT} \mathbb{P}^{v_0} [S_t \leq K] \tag{3.6}
\]
and
\[
A = \frac{v_0^{\beta_4}}{\beta_4 - \beta_3} \left\{ \beta_4 K - (1 + \beta_4) [v_0 + EuP(v_0, t)] + K e^{-rT} \mathbb{P}^{v_0} [S_t \leq K] \right\} > 0,
\]
\[
B = \frac{v_0^{\beta_4}}{\beta_3 - \beta_4} \left\{ \beta_4 K - (1 + \beta_3) [v_0 + EuP(v_0, t)] + K e^{-rT} \mathbb{P}^{v_0} [S_t \leq K] \right\} > 0,
\]

Then the price of a finite-horizon american put option with maturity $t$ and strike $K$ can be approximated by $\psi(S_0, t)$ which is given by (see $\bar{g}3$ in [3])
\[
\psi(v, t) = \begin{cases} 
EuP(v, t) + Av^{-\beta_3} + Bv^{-\beta_4}, & \text{if } v \geq v_0 \\
K - v, & \text{if } v \leq v_0
\end{cases}
\]
4 Lookback option

The price of a lookback floating strike put option is given by:

\[ LP(T) = \mathbb{E} \left[ e^{-rT} \left( \max_{0 \leq t \leq T} \{ M, \max_{0 \leq t \leq T} S_t \} - S_T \right) \right] \]

\[ = \mathbb{E} \left[ e^{-rT} \left( \max_{0 \leq t \leq T} \{ M, \max_{0 \leq t \leq T} S_t \} \right) \right] - S_0 \]

where \( M \geq S_0 \) is a fixed constant representing the prefixed maximum at time 0. The Laplace transform of the lookback put, using notations in 1.4, is given by (see theorem 1 in [3])

\[
\int_0^{+\infty} e^{-\alpha T} LP(T) dT = \frac{S_0 A_\alpha}{C_\alpha} \left( \frac{S_0}{M} \right)^{\beta_{1,a+r}-1} + \frac{S_0 B_\alpha}{C_\alpha} \left( \frac{S_0}{M} \right)^{\beta_{2,a+r}-1} + \frac{M}{\alpha + r} \frac{S_0}{\alpha} \quad \forall \alpha > 0
\]

where

\[ A_\alpha = \frac{(\eta_1 - \beta_{1,a+r})^{\beta_{2,a+r}}}{\beta_{1,a+r}-1} \]

\[ B_\alpha = \frac{(\beta_{2,a+r} - \eta_1)^{\beta_{1,a+r}}}{\beta_{2,a+r}-1} \]

\[ C_\alpha = (\alpha + r) \eta_1 (\beta_{2,a+r} - \beta_{1,a+r}) \]

The put price is obtained by using an inversion of the Laplace transform. The call option price follows just by symmetry. For the lookback fixed strike, when we have \( M \geq \max(S_0, K) \) for the put or \( m \leq \min(S_0, K) \) for the call, we get similar results to those for floatings.

5 Barrier option

Since all eight types of barrier can be solved in similar way, we focus only on the price Up and In Call option defined as followed

\[ UIC = \mathbb{E} \left[ e^{-rT} (S_T - K) \mathbb{1}_{\{\max_{0 \leq t \leq T} S_t \geq H\}} \right] \quad (5.7) \]

where \( H > S_0 \) is the barrier level. For any given probability \( P \), define:

\[ \Psi(\mu, \sigma, \lambda, p, \eta_1, \eta_2; a, b, T) = P \left[ Z_T \geq a, \max_{0 \leq t \leq T} Z_t \geq b \right] \quad (5.8) \]

where \( Z_T = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i \) and \( Y \) has a double exponential distribution with density as in (1.2), and \( N \) is a poisson process with rate \( \lambda \). Using formula (3.1) and the result before remark 3.1 in [2], we get

\[
\int_0^{+\infty} e^{-\alpha T} P \left[ \max_{0 \leq t \leq T} Z_t \geq b \right] = \frac{1}{\alpha} \left( \frac{\eta_1 - \beta_{1,a}}{\eta_1 (\beta_{2,a} - \beta_{1,a})} e^{-b \beta_{1,a}} + \frac{\beta_{2,a} - \eta_1}{\eta_1 (\beta_{2,a} - \beta_{1,a})} e^{-b \beta_{2,a}} \right)
\]
By Inverting the Laplace transform we get \( \mathbb{P}[\max_{0 \leq t \leq T} Z_t \geq b] \), which is useful for some types of barrier options. Let us now define some functions

\[
H_i(a, b, c; n) := \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{\left(\frac{4c^2 - b}{2}\right)t + \frac{i}{2} H_i(t + \frac{a}{\sqrt{b}})} dt \quad i \geq 1, n \geq 0
\]

\[
A_\alpha := \mathbb{E}\left[e^{-\alpha \tau_b} \mathbb{1}_{X_{\tau_b} = b}\right] = \frac{\eta_1 - \beta_1, \alpha e^{-b \beta_1, \alpha} + \beta_2, \alpha - \eta_1}{\beta_2, \alpha - \beta_1, \alpha} e^{-b \beta_2, \alpha}
\]

\[
B_\alpha := \mathbb{E}\left[e^{-\alpha \tau_b} \mathbb{1}_{X_{\tau_b} > b}\right] = \frac{(\eta_1 - \beta_1, \alpha)(\beta_2, \alpha - \eta_1)}{\eta_1(\beta_2, \alpha - \beta_1, \alpha)} [e^{-b \beta_1, \alpha} - e^{-b \beta_2, \alpha}]
\]

where \( \tau_b = \inf \{t \geq 0; X_t \geq b\} \). Hh functions are defined in \( \S \ 2 \), and \( \beta \) variables in (1.4). For \( i \geq 1 \), under assumption that \( b > 0 \) and \( c > -\sqrt{2b} \), we have

\[
H_i(a, b, c; n) = \frac{1}{i} H_{i-2}(a, b, c; n + 1) - \frac{c}{i} H_{i-1}(a, b, c; n + 1) - \frac{a}{i} H_{i-1}(a, b, c; n)
\]

By knowing \( H_{-1}(a, b, c; n) \) and \( H_0(a, b, c; n) \), this recursive formula allows us to determine all values of \( H_i \). Lemmas A.1 and A.2 in [2] give us

\[
H_{-1}(a, b, c; n) = e^{-\alpha c - \sqrt{2a^2}b} \sqrt{\frac{1}{2b} \left(\frac{\alpha^2}{4b}\right)} \sum_{j=0}^{n} \frac{(-n)_j (n + 1)_j}{j! \left(\sqrt{\frac{2a^2}{b^2}}\right)} j!, \quad a \neq 0, n \geq 0
\]

\[
H_{-1}(a, b, c; n) = e^{-\alpha c - \sqrt{2a^2}b} \sqrt{\frac{1}{2b} \left(\frac{\alpha^2}{4b}\right)} \sum_{j=0}^{n-1} \frac{(-n)_j (n + 1)_j}{j! \left(\sqrt{\frac{2a^2}{b^2}}\right)} j!, \quad a \neq 0, n \leq -1
\]

\[
H_{-1}(0, b, c; n) = \frac{(2n)!}{n!} \frac{1}{4b^2}, \quad n \geq 0
\]

\[
H_0(a, b, c; n) = \frac{c}{2(n + 1)^2} \frac{H_1(a, b, c; n + 1)}{H_{-1}(a, b, c; n + 1)} - \frac{a}{2(n + 1)^2} \frac{H_{-1}(a, b, c; n)}{H_{-1}(a, b, c; n + 1)}, \quad b = \frac{1}{2} c^2, n \geq 0
\]

And \( \forall n \geq 0 \) et \( b \neq \frac{1}{2} c^2 \)

\[
H_0(a, b, c; n) = \frac{n!}{(b - \frac{1}{2} c^2)^{n+1}} \sum_{i=0}^{n} \frac{(b - \frac{1}{2} c^2)^i}{i!} \left(\frac{a}{2} H_{-1}(a, b, c; i - 1) - \frac{c}{2} H_{-1}(a, b, c; i)\right), \quad a > 0
\]

\[
H_0(a, b, c; n) = \frac{n!}{(b - \frac{1}{2} c^2)^{n+1}} \left(1 + \sum_{i=0}^{n} \frac{(b - \frac{1}{2} c^2)^i}{i!} \left(\frac{a}{2} H_{-1}(a, b, c; i - 1) - \frac{c}{2} H_{-1}(a, b, c; i)\right)\right), \quad a < 0
\]

\[
H_0(a, b, c; n) = \frac{n!}{(b - \frac{1}{2} c^2)^{n+1}} \left(\frac{1}{2} + \sum_{i=0}^{n} \frac{(b - \frac{1}{2} c^2)^i}{i!} \frac{a}{2} H_{-1}(a, b, c; i)\right), \quad a = 0
\]
where \((n)_j = n(n+1) \ldots (n+j-1)\), with convention \((n)_0 = 1\).
We can now determine the exact expression of the Laplace transform of \(\Psi\) when \(b > 0\) and \(a \leq b\) (see theorem 4.1 in [2]):

\[
\int_0^{+\infty} e^{-\alpha T} P \left[ Z_T \geq a, \max_{0 \leq t \leq T} Z_t \geq b \right] dT = A_\alpha \int_0^{+\infty} e^{-\alpha T} P \left[ Z_T + \xi^+ \geq a - b \right] dT
\]

\[
+ B_\alpha \int_0^{+\infty} e^{-\alpha T} P \left[ Z_T + \xi^\gamma \geq a - b \right] dT
\]

\[
= (A_\alpha + B_\alpha) \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} H_0 \left(-h, \gamma_\alpha, -\frac{\mu}{\sigma}; n\right)
\]

\[
+ e^{h \eta_1} \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{\lambda^n}{n!} (A_\alpha P_{n,j} + B_\alpha P_{n,j}) \sum_{i=0}^{j-1} (\sigma \eta_1)^i H_i (h, \gamma_\alpha, c_+; n)
\]

\[
- e^{-h \eta_1} \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{\lambda^n}{n!} (A_\alpha Q_{n,j} + B_\alpha Q_{n,j}) \sum_{i=0}^{j-1} (\sigma \eta_1)^i H_i (-h, \gamma_\alpha, c_-; n)
\]

\[
+ e^{h \eta_1} \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{\lambda^n}{n!} (\eta_1)^i H_i (h, \gamma_\alpha, c_+; n)
\]

\[
+ e^{h \eta_1} B_\alpha H_0 (h, \gamma_\alpha, c_+; 0)
\]

where \(\xi^+\) has an exponential law with rate \(\eta_1\), matrix \(P\) and \(Q\) are as defined in \(\gamma\) 2, and

\[
\overline{P}_{n,i} := \sum_{j=i}^{n-1} Q_{n,j} \left(\frac{\eta_2}{\eta_1 + \eta_2}\right)^i, \ \overline{Q}_{n,i} := P_{n,i-1}, \ 2 \leq i \leq n + 1
\]

\[
c_+ := \sigma \eta_1 + \frac{\mu}{\sigma}, \ c_- := \sigma \eta_2 - \frac{\mu}{\sigma}, \ \gamma_\alpha := \alpha + \lambda + \frac{\mu^2}{2 \sigma^2}, \ h := \frac{b - a}{\sigma}
\]

For to get numerically \(P \left[ Z_T \geq a, \max_{0 \leq t \leq T} Z_t \geq b \right]\) for a given \(T, i\) find that is better to inverse the right term in the first equality above, using some properties of the Laplace inversion. Note that \(P \left[ Z_T \geq a - b \right]\) is given in \(\gamma\) 2 and \(P \left[ Z_T + \xi^+ \geq a - b \right]\) is given in [2] (pp. 528, formula B.5):

\[
P \left[ Z_T + \xi^+ \geq a \right] = \frac{e^{(\sigma \eta_1)^2 T}}{\sqrt{2\pi T}} \sum_{n=1}^{+\infty} \pi_n \sum_{k=1}^{n+1} \overline{P}_{n,k} \left(\sigma \sqrt{T} \eta_1\right)^k I_{k-1} \left(a - \mu T; -\eta_1, -\frac{1}{\sigma \sqrt{T}}; -\sigma \sqrt{T} \eta_1\right)
\]

\[
+ \frac{e^{(\sigma \eta_2)^2 T}}{\sqrt{2\pi T}} \sum_{n=1}^{+\infty} \pi_n \sum_{k=1}^{n} \overline{Q}_{n,k} \left(\sigma \sqrt{T} \eta_2\right)^k I_{k-1} \left(a - \mu T; -\eta_2, \frac{1}{\sigma \sqrt{T}}; -\sigma \sqrt{T} \eta_2\right)
\]

\[
+ \pi_0 \eta_1 \frac{e^{(\sigma \eta_1)^2 T}}{\sqrt{2\pi}} I_0 \left(a - \mu T; -\eta_1, -\frac{1}{\sigma \sqrt{T}}; -\eta_1 \sigma \sqrt{T}\right)
\]
The price of the UIC option is obtained by, thanks to Kou and Wang (see theorem 2 in [3])

\[
UIC = S_0 \Psi \left( r + \frac{1}{2} \sigma^2 - \lambda \xi, \sigma, \lambda, p, \eta_1, \eta_2; \log \left( \frac{K}{S_0} \right), \log \left( \frac{H}{S_0} \right), T \right) \\
-Ke^{-rT} \Psi \left( r - \frac{1}{2} \sigma^2 - \lambda \xi, \sigma, \lambda, p, \eta_1, \eta_2; \log \left( \frac{K}{S_0} \right), \log \left( \frac{H}{S_0} \right), T \right)
\]

where

\[
\tilde{p} = \frac{p}{1 + \xi \eta_1 - 1}, \quad \tilde{\lambda} = \lambda (1 + \xi), \quad \tilde{\eta}_1 = \eta_1 - 1, \quad \tilde{\eta}_2 = \eta_2 + 1
\]

References

