A Finite Volume Method for Pricing American options on two stocks.

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Introduction

The valuation of American options on two stocks, also called two-colours Rainbow options by practitioners, is an important problem in financial economics since a wide variety of contracts that are traded in the O.T.C. market involve such options (Exchange options, Best-of options). Unlike European options, American options cannot be valued by closed-form formulae, even in the Black-Scholes model, and require the use of numerical methods.

1 American Options on Two Stocks

The price at time 0 of an American option on two stocks in the Black-Scholes setting is given by

\[ P_A(0, s_1, s_2) = \sup_{\tau \in [0, T]} E \left[ e^{-r\tau} \psi(S^1_\tau, S^2_\tau) \right]. \]

This price can be formulated, after a logarithm change of variable, in terms of the solution \( u \) to the following variational inequality (see e.g. [10]),

\[
\begin{cases}
\max (\psi - u, \frac{\partial u}{\partial t} + \frac{\sigma_1^2}{2} \frac{\partial^2 u}{\partial x_1^2} + \frac{\sigma_2^2}{2} \frac{\partial^2 u}{\partial x_2^2} + \alpha_1 \frac{\partial u}{\partial x_1} + \alpha_2 \frac{\partial u}{\partial x_2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} - ru) = 0, & (t, x_1, x_2) \text{ in } [0, T] \times \mathbb{R}^2 \\
u(T, x_1, x_2) = \psi(e^{x_1}, e^{x_2})
\end{cases}
\]

by \( P_A(t, s_1, s_2) = u(t, \ln s_1, \ln s_2) \).

With the time change of variable \( t' = T - t \) and the following geometrical transformation :

\[(x, y) \rightarrow (X, Y) = \left( x \cos(\theta) + y \sin(\theta), \left( \frac{\alpha}{\beta} \right)^{\frac{1}{2}} (y \cos(\theta) - x \sin(\theta)) \right)\]
with, if $\sigma_1^2 - \sigma_2^2 \neq 0$

\[
\begin{align*}
\tan(2\theta) &= \frac{2\rho_0 \sigma_1 - \sigma_2}{\sigma_1^2 - \sigma_2^2}, \\
\alpha &= \frac{(\sigma_1^2 + \sigma_2^2) \cos(2\theta) + \sigma_1^2 - \sigma_2^2}{4 \cos(2\theta)}, \\
\beta &= \frac{(\sigma_1^2 + \sigma_2^2) \cos(2\theta) + \sigma_2^2 - \sigma_1^2}{4 \cos(2\theta)}.
\end{align*}
\]

if, $\sigma_1^2 - \sigma_2^2 = 0$,

\[
\begin{align*}
\theta &= \frac{\pi \rho}{|\rho|}, \\
\alpha &= \frac{\sigma_2^2}{2} (1 + |\rho|), \\
\beta &= \frac{\sigma_1^2}{2} (1 - |\rho|).
\end{align*}
\]

one obtains,

\[
\begin{cases}
\min \left( \psi - u, \frac{\partial u}{\partial t} - \alpha \Delta u - \text{grad}(\bar{v}u) + ru \right) = 0, & (t,x_1,x_2) \text{ in } [0,T] \times \mathbb{R}^2 \\
u(0,x_1,x_2) = \psi(e^{x_1},e^{x_2})
\end{cases}
\]

with,

\[
\bar{v} = ((r-\lambda_1-\frac{\sigma_1^2}{2}) \cos(\theta) + (r-\lambda_2-\frac{\sigma_2^2}{2}) \sin(\theta), \left( (r-\lambda_2-\frac{\sigma_2^2}{2}) \cos(\theta) - (r-\lambda_1-\frac{\sigma_1^2}{2}) \sin(\theta) \right) \left( \frac{\alpha}{\beta} \right)^{\frac{1}{2}}).
\]

2 The finite volume schemes

Let us consider the problem :

\[
\begin{align*}
\frac{\partial u}{\partial t}(x,t) + \text{div}(u(x,t)\bar{v}) - \alpha \Delta u + ru(x,t) &\geq 0, & (x,t) \in \Omega \times ]0,T[ \\
u(x,t) &\geq \psi(x), & (x,t) \in \Omega \times ]0,T[ \\
\end{align*}
\]

under the following assumptions

**Assumption 1.**

1. $d \in \mathbb{N}^*$,
2. $\Omega \subset \mathbb{R}^d$ is a bounded open polygonal,
3. $\psi \in H^1_0(\Omega) \cap C^2(\overline{\Omega})$,
4. $\psi \geq 0$ a.e on $\Omega$,
5. $T > 0$.

A weak form of the problem (3)-(6) yields the following variational inequality :

\[
\begin{cases}
u \in L^2(0,T;H^1_0(\Omega)), \frac{\partial u}{\partial t} \in L^2(\Omega \times ]0,T[), u(x,0) = \psi(\Omega), & \text{p.p. } x \in \Omega, \text{satisfying} : \\
\int_{\Omega} \left( \frac{\partial u}{\partial t}(x,t) + ru(x,t) + \text{div}(u(x,t)\bar{v}) \right) (v(x,t) - u(x,t)) + \alpha \nabla u(x,t) \nabla (v(x,t) - u(x,t)) dx \geq 0 \\
\text{p.p } t \in ]0,T[, \forall v \in H^1(\Omega), v \geq \psi.
\end{cases}
\]
By \cite{12}, there exits a unique solution of (7).
In order to obtain a numerical approximation of the solution of (7), let us now describe the space and time discretization of $\Omega \times [0,T]$.

**Definition 1** (Admissible meshes). An admissible mesh of $\Omega$ is given by a set $\tau$ of open bounded polygonal convex subsets of $\Omega$ called control volumes, a family $\varepsilon$ of subsets of $\Omega$ contained in hyperplanes of $\mathbb{R}^d$ with strictly positive measure, and a family of point $(x_K)_{K \in \tau}$ (the ‘centers’ of control volumes) satisfying the following properties:

(i) The closure of the union of all control volumes is $\bar{\Omega}$.

(ii) For any $K \in \tau$, there exists a subset $\varepsilon_K$ of $\varepsilon$ such that $\partial K = \cup_{\sigma \in \varepsilon_K} \bar{\sigma}$. Furthermore, $\varepsilon = \cup_{K \in \tau} \varepsilon_K$.

(iii) For any $(K,L) \in \tau^2$ with $K \neq L$, either the ‘length’ (i.e. the (d-1) Lebesgue measure) of $K \cap L$ is 0 or $K \cap L = \bar{\sigma}$ for some $\sigma \in \varepsilon$. In the latter case, we shall write $\sigma = K \setminus L$ and $\varepsilon_{int} = \varepsilon \cap \exists(K \setminus L) \in \tau^2, \sigma = K \setminus L$. For any $K \in \tau$, we shall denote by $\mathcal{N}_K$ the set of boundary control volumes of $K$, i.e. $\mathcal{N}_K = \{L \in \tau, K \setminus L \in \varepsilon_K\}$.

(iv) The family of points $(x_K)_{K \in \tau}$ is such that $x_K \in K$ for all $K \in \tau$ and, if $\sigma = K \setminus L$, it is assumed that the straight line $(x_K, x_L)$ is orthogonal to $\sigma$.

For a control volume $K \in \tau$, we will denote by $m(K)$ its measure and $\varepsilon_{ext,K}$ the subset of the edges of $K$ included in the boundary $\partial \Omega$. If $L \in \mathcal{N}_K$, $m(K \setminus L)$ will denote the measure of the edge between $K$ and $L$, $\tau_{K \setminus L}$ the ‘transmissibility’ through $K \setminus L$, defined by $\tau_{K \setminus L} = \frac{m(K \setminus L)}{|d(x_K, x_L)|}$. Similarly, if $\sigma \in \varepsilon_{ext,K}$, we will denote by $m(\sigma)$ its measure and $\tau_{\sigma}$ the ‘transmissibility’ through $\sigma$, defined by $\tau_{\sigma} = \frac{m(\sigma)}{|d(x_K, \sigma)|}$. One denotes $\varepsilon_{ext} = \cup_{K \in \tau} \varepsilon_{ext,K}$ and for $\sigma \in \varepsilon_{ext}$, one denotes by $K_{\sigma}$ the control volume $K$ such that $\sigma \in \varepsilon_{ext,K}$. The size of the mesh $\tau$ is defined by $\text{size}(\tau) = \max_{K \in \tau} \text{diam}(K)$, (8)

and a geometrical factor, linked with the regularity of the mesh, is defined by

\[
\text{reg}(\tau) = \max_{K \in \tau} \left( \text{card} \varepsilon_K, \max_{\sigma \in \varepsilon_K} \frac{\text{diam}(K)}{|d(x_K, \sigma)|} \right). \tag{9}
\]

**Definition 2** (Time discretization of $(0,T)$). A time discretization of $(0,T)$ is given by an integer value $N$ and by an increasing sequence of real values $(t^n)_{n \in [0,N+1]}$ with $t^0 = 0$ and $t^{N+1} = T$. The time step is uniform and defined by $\delta t = t^{n+1} - t^n$, for $n \in [0,N]$.

**Definition 3** (Space-time discretization of $\Omega \times (0,T)$). A finite volume discretization $\mathcal{D}$ of $\Omega \times (0,T)$ is a family $\mathcal{D} = (\tau, \varepsilon, (x_K)_{K \in \tau}, N, (t^n)_{n \in [0,N]})$, where $\tau, \varepsilon, (x_K)_{K \in \tau}$ is an admissible mesh of $\Omega$ in the sense of Definition 1 and $N, (t^n)_{n \in [0,N+1]}$ is a time discretization of $(0,T)$ in the sense of Definition 2. For a given mesh $\mathcal{D}$, one defines : $\text{size}(\mathcal{D}) = \max(\text{size}(\tau), \delta t)$, $\text{reg}(\mathcal{D}) = \text{reg}(\tau)$.

Let us now introduce the space of piecewise constant functions associated with an admissible mesh and some "discrete $H^1_0$" norm for this space. This discrete norm will be used to obtain some estimates on the approximate solution given by a finite volume scheme.

**Definition 4.** Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, and $\tau$ an admissible mesh. Define $X(\tau)$ as the set of functions from $\Omega$ to $\mathbb{R}$ which are constant over each control volume of the mesh.
Definition 5 (Discrete norms). Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^d$, and $\tau$ an admissible finite volume mesh in the sense of Definition 1. For $u, v \in X(\tau)$ we define a scalar product by
\[
[u, v]_1, \tau = \sum_{\sigma \in \mathcal{E}} T_{\sigma_{\text{int}}} D_\sigma u D_\sigma v = \sum_{\sigma \in \mathcal{E}_{\text{ext}}} T_{\sigma} (u_{K} - u_{K})(v_{L} - v_{K}) + \sum_{\sigma \in \mathcal{E}_{\text{ext}}} T_{\sigma} u_{K} v_{\sigma} v_{K}\tag{10}
\]
where, for any $\sigma \in \mathcal{E}$, $T_\sigma = \frac{\text{m(\sigma)}}{d_\sigma}$ and
\[
D_\sigma u = [u_K u_L] \text{ if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, \quad D_\sigma u = [u_K] \text{ if } \sigma \in \mathcal{E}_{\text{ext}},
\]
where $u_K$ denotes the value taken by $u$ on the control volume $K$ and the sets $\mathcal{E}, \mathcal{E}_{\text{int}}, \mathcal{E}_{\text{ext}}, \mathcal{E}_{\text{ext}}$ are defined in definition 1. We note $\| \|_{1, \tau}$ the discrete $H^1_{0}$ norm associated.

The schemes:

Let $\mathcal{D}$ be a finite volume discretization of $\Omega \times (0, T)$. Let us now define an implicit upwind finite volume scheme, the discrete unknowns are $u = (u_K^{n+1})_{K \in \tau}, n \in [0, T]$ and $\bar{u} = (\bar{u}_K^{n+1})_{K \in \tau}, n \in [0, T]$ and verify:
\[
u_0^0 = \psi(x_K) = \psi_K, \tag{11}\]
\[
u_K^{n+1} = \max \left( \bar{u}_K^{n+1}, u_K^0 \right), \tag{12}\]
\[m_K \left( \bar{u}_K^{n+1} - u_K^0 \right) + \Delta t \sum_{\sigma \in \mathcal{E}_K} v_{K, \sigma} \nu_{\sigma, +}^{n+1} + \alpha \Delta t [u_K^{n+1}, 1_K]|_1, \tau + r \Delta t m_K u_K^{n+1} = 0, \tag{13}\]
with,
\[
u_K^{n+1} = \begin{cases} u_K^n & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, u_K^{n+1} \geq 0 \\ u_L^n & \text{if } \sigma \in \mathcal{E}_{\text{int}}, \sigma = K|L, u_K^{n+1} < 0 \\ u_K^n & \text{if } \sigma \in \mathcal{E}_{\text{ext}}, v_{K, \sigma} \geq 0 \\ 0 & \text{if } \sigma \in \mathcal{E}_{\text{ext}}, v_{K, \sigma} < 0 \end{cases}
\]

Remark 1. We can also consider a implicit central finite volume scheme:
\[
u_0^0 = \psi(x_K) = \psi_K, \tag{14}\]
\[
u_K^{n+1} = \max \left( \bar{u}_K^{n+1}, u_K^0 \right), \tag{15}\]
\[m_K \left( \bar{u}_K^{n+1} - u_K^0 \right) + \Delta t \left[ \sum_{\sigma \in \mathcal{E}_K, \text{int}} v_{K, \sigma} \frac{u_K^{n+1} + u_K^L}{2} + \sum_{\sigma \in \mathcal{E}_K, \text{ext}} v_{K, \sigma} \frac{u_K^{n+1} - u_K^L}{2} \right] + \alpha \Delta t [u_K^{n+1}, 1_K]|_1, \tau + r \Delta t m_K u_K^{n+1} = 0. \tag{16}\]

Definition 6 (Approximate solution). Let $\mathcal{D}$ be an admissible discretization of $\Omega \times (0, T)$ in the sense of definition 3. The approximate solution ($C^\infty$ in time on $\Omega \times (0, T)$) of (3) - (6) associated to the discretization $\mathcal{D}$ is defined almost everywhere in $\Omega \times (0, T)$ by:
\[u_{\mathcal{D}}(x, t) = \frac{t - n \Delta t}{\Delta t} u_K^{n+1} + \frac{(n+1) \Delta t - t}{\Delta t} u_K^n, \forall (x, t) \in K \times [n \Delta t, (n+1) \Delta t], \forall n = 0 \ldots N, \forall K \in \tau.\]
Thanks to this Definition, one gets almost everywhere in $\Omega \times (0, T)$:
\[
\frac{\partial u^n(x,t)}{\partial t} = u^{n+1}_K - u^n_K, \forall t \in [n\Delta t, (n+1)\Delta t], \forall x \in K, \forall n = 0 \ldots N, \forall K \in \tau.
\]

3 Existence of the solution and stability results for the implicit schemes

Lemma 1. Under Assumptions 1, let $\mathcal{D}$ be a discretization of $\Omega \times (0, T)$ in the sense of Definition 3. If $(u^n_K)_{K \in \tau \ n \in \mathbb{N}}$ is a solution of the implicit upwind finite volume scheme (11) - (13), then there exists a sequence $(\theta^n_K)_{n=0 \ldots N} \in [0, 1]$ such that:
\[
u^n_K + u^n_K = \theta^n_K \left( u^{n+1}_K - u^n_K \right), \forall K \in \tau, \forall n = 0 \ldots N.
\]

Lemma 2 (Existence and uniqueness). Under Assumptions 1, let $\mathcal{D}$ be a discretization of $\Omega \times (0, T)$ in the sense of Definition 3. Then there exists a unique solution $(u^n_K)_{K \in \tau \ n \in \mathbb{N}}$ to the system of equations (11) - (13).

Proposition 1 ($L^\infty$ and $L^2$ estimate). Under Assumptions 1, let $\mathcal{D}$ be a discretization of $\Omega \times (0, T)$ in the sense of Definition 3 and let $(u^n_K)_{K \in \tau \ n \in [0,N+1]}$ be the unique solution of the scheme (11) - (13). Then,
\[
|u^n_K| \leq \|\psi\|_{L^\infty(\Omega)}, \forall K \in \tau, \forall n \in [0, N + 1],
\]
and
\[
\frac{1}{2} \sum_{K \in \tau} m_K (u^{l+1}_K)^2 + \Delta t \sum_{n=0}^{l} (u^{n+1}_K - u^n_K)^2 + \alpha \sum_{n=0}^{l} \Delta t |u^{n+1}_K - u^n_K|_1, \forall l \leq N
\]
\[
\leq \|\psi\|^2_{L^\infty(\Omega)} m(\Omega) + \alpha T [u^0, u^0]_1, \forall l \leq N
\]

4 Estimate

Corollary 1. Under Assumptions 1, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be sequence of discretization of $\Omega \times (0, T)$ in the sense of Definition 3 such that $\Delta t_m \xrightarrow{m \rightarrow +\infty} 0$, size($\tau_m$) $\xrightarrow{m \rightarrow +\infty} 0$, $\zeta \in \mathbb{R}$ such that $\zeta \geq \operatorname{reg}(D_m)$ $\forall m \in \mathbb{N}$, and let $(u^n_K)_{K \in \tau \ n \in [0,N+1]}$ be the unique solution of the scheme (11) - (13). Then, there exists $U \in L^2 (\Omega \times [0, T])$, and a subsequence noted $(u_{D_m})_{m \in \mathbb{N}}$ such that $u_{D_m} \xrightarrow{m \rightarrow +\infty} U$
for the weak topology of $L^2 (\Omega \times [0, T])$.

Proposition 2. Under Assumptions 1, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be sequence of discretization of $\Omega \times (0, T)$ in the sense of Definition 3, $\zeta \in \mathbb{R}$ such that $\zeta \geq \operatorname{reg}(D)$, and let $(u^n_K)_{K \in \tau \ n \in [0,N+1]}$ be the unique solution of the scheme (11) - (13). Then, there exists $C > 0$ only depending on $\psi$, $\alpha$, $\Omega$, $T$, $\bar{v}$, $r$, $\zeta$, such that:
\[
\alpha [u^{N+1}, u^{N+1}]_1, \tau + \sum_{n=0}^{N} \Delta t \sum_{K \in \tau} m_K \left( \frac{u^{n+1}_K - u^n_K}{\Delta t} \right)^2 \leq C(u^0, \psi, \alpha, \Omega, T, \bar{v}, r).
\]
Corollary 2. Under Assumptions 1, let \((D_m)_{m \in \mathbb{N}}\) be sequence of discretization of \(\Omega \times (0, T)\) in the sense of Definition 3 such that size\((\tau_m) \xrightarrow{m \to +\infty} 0\), \(\zeta \in \mathbb{R}\) such that \(\zeta \geq \text{reg}(D)\), and let \((u^n_K)_{K \in \tau, n \in [0,N+1]}\) be the unique solution of the scheme (11) - (13). Then, the set \(\{\frac{\partial u^n_K}{\partial t}\}_{m \in \mathbb{N}}\) is borned in \(L^2(\Omega \times ]0,T[)\) and so, there exists \(Z \in L^2(\Omega \times ]0,T[)\) such that, up to a subsequence, \(\{\frac{\partial u^n_K}{\partial t}\}_{m \in \mathbb{N}}\) tends to \(Z\) in the weak topology of \(L^2(\Omega \times ]0,T[)\) as \(m \to +\infty\).

Corollary 3 (Space-translate and Time-Translate estimate). Under Assumptions 1, let \(\mathcal{D}\) be a discretization of \(\Omega \times (0, T)\) in the sense of Definition 3, \(\zeta \in \mathbb{R}\) such that \(\zeta \geq \text{reg}(D)\) and let \(u_\mathcal{D}\) the approximate solution in the sense of Definition 6 be prolonged by zero on \(\mathbb{R}^{d+1} \setminus \Omega \times ]0,T[\). Then there exists \(C_2\) only depending on \(T, \psi, \alpha, \tau, d\) and \(C_3\) only depending on \(T, \psi, \alpha, d, r, \bar{v}\) such that:

\[
\|u_\mathcal{D}(\cdot, \eta, \cdot) - u_\mathcal{D}(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq C_2|\eta| (|\eta| + 4\text{size}(\tau)), \quad \forall \eta \in \mathbb{R}^d.
\]

and

\[
\|u_\mathcal{D}(\cdot, \cdot + \lambda) - u_\mathcal{D}(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq \lambda C_3, \quad \forall \lambda \in ]0,T[.
\]

With this preceding estimates, one can apply the Riesz-Frechet-Kolmogorov compactness criterion.

5 Compactness

Corollary 4. Under Assumptions 1, let \((D_m)_{m \in \mathbb{N}}\) be sequence of discretization of \(\Omega \times (0, T)\) in the sense of Definition 3 such that \(\Delta t_m \xrightarrow{m \to +\infty} 0\), size\((\tau_m) \xrightarrow{m \to +\infty} 0\), \(\zeta \in \mathbb{R}\) such that \(\zeta \geq \text{reg}(D_m)\) \(\forall m \in \mathbb{N}\), and let \((u^n_K)_{K \in \tau, n \in [0,N+1]}\) be the unique solution of the scheme (11) - (13). Then, up to a subsequence,

\[
u_\mathcal{D}_m \xrightarrow{m \to +\infty} U \text{ dans } L^2(\Omega \times ]0,T[).
\]

where \(U\) is defined by the Corollary 1 and checks \(U \in L^2\left(0,T; H^1(\Omega)\right), U(t, .) = 0\) a.e. on \(\partial \Omega\), a.e.\(t \in ]0,T[,\) and \(\frac{\partial U}{\partial t} = Z\) a.e. on \(]0,T[\times\Omega\).

6 Convergence

Proposition 3. Under Assumptions 1, let \((D_m)_{m \in \mathbb{N}}\) be sequence of discretization of \(\Omega \times (0, T)\) in the sense of Definition 3 such that \(\Delta t_m \xrightarrow{m \to +\infty} 0\), size\((\tau_m) \xrightarrow{m \to +\infty} 0\), \(\zeta \in \mathbb{R}\) such that \(\zeta \geq \text{reg}(D_m)\) \(\forall m \in \mathbb{N}\), and let \((u^n_K)_{K \in \tau, n \in [0,N+1]}\) be the unique solution of the scheme (11) - (13). Let \(\nu_\mathcal{D}_m\) the sequence of approximate solution in the sense of the Definition ??, and let \(U\) the limit of a subsequence \((\nu_\mathcal{D}_m)_{m \in \mathbb{N}}\) thanks to Corollary 4. Then, for all function \(w \in L^2\left(0,T; H^1_0(\Omega)\right)\), such that \(w(t, .) \geq \psi\) a.e. on \(\Omega\), the following inequality holds:

\[
\int_{\Omega} \frac{\partial U}{\partial t}(x,t)(w(x,t) - U(x,t)) + \alpha \nabla U(x,t) \nabla (w(x,t) - U(x,t)) + rU(x,t)(w(x,t) - U(x,t)) + (w(x,t) - U(x,t)) \text{div}(U(x,t)\bar{v}) dx \geq 0 \text{ p.p } t \in ]0,T[.
\]
7 Numerical Results

7.1 Localization and Stability

We define three localizations.

Lemma 3 (Localization). We consider,

\[ P_{A_{loc}}(0, s_1, s_2) = \sup_{\tau \in T_\tau} E \left[ e^{-r\tau + T_0^{s_1}} \psi(S_1^{s_1, s_2}, S_2^{s_1, s_2}) \right], \]

with \( T_0^{s_1} = \inf \{ t > 0, |S_t^{s_1} - s_1| < loc, |S_t^{s_2} - s_2| < loc \}. \)

Then, if \( \psi(s_1, s_2) = (K - \min(s_1, s_2))^+, \)

\[ |P_A(0, s_1, s_2) - P_{A_{loc}}(0, s_1, s_2)| \leq 4K \left[ 2 - N \left( \frac{loc - a_1}{\sqrt{T}\sigma_1} \right) - N \left( \frac{loc - a_2}{\sqrt{T}\sigma_2} \right) \right] \]

if \( \psi(s_1, s_2) = (\max(s_1, s_2) - K)^+, \)

\[ |P_A(0, s_1, s_2) - P_{A_{loc}}(0, s_1, s_2)| \leq 8 \left[ \exp \left( s_1 + T(|r - \lambda_1| + \frac{\sigma_1^2}{2}) \right) + \exp \left( s_2 + T(|r - \lambda_2| + \frac{\sigma_2^2}{2}) \right) \right] \sqrt{2 - N \left( \frac{loc - a_1}{\sqrt{T}\sigma_1} \right) - N \left( \frac{loc - a_2}{\sqrt{T}\sigma_2} \right)} \]

if \( \psi(s_1, s_2) = (s_1 - \mu s_2)^+, \)

\[ |P_A(0, s_1, s_2) - P_{A_{loc}}(0, s_1, s_2)| \leq 8 \exp \left( s_1 + T(|r - \lambda_1| + \frac{\sigma_1^2}{2}) \right) \sqrt{2 - N \left( \frac{loc - a_1}{\sqrt{T}\sigma_1} \right) - N \left( \frac{loc - a_2}{\sqrt{T}\sigma_2} \right)} \]

where, \( ai = T|ri - \lambda_i - \frac{\sigma_i^2}{2}| \) and \( N \) is the repartition function of a standard normal distribution.

One uses the following polynomial approximation for the repartition function of a standard normal distribution:

\[ N(x) \approx 1 - \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \left( b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5 \right), \text{ si } x > 0, \]

with

\[ p = 0.2316419 \]
\[ b_1 = 0.319381530 \]
\[ b_2 = -0.356563782 \]
\[ b_3 = 1.781477937 \]
\[ b_4 = -1.821255978 \]
\[ b_5 = 1.330274429 \]
\[ t = \frac{1}{1+px}. \]

Explicit central finite volume scheme:

\[ u_K^0 = \psi_K, \]
\[ u_K^{n+1} = \max \left( u_K^{n+1}, u_K^n \right), \]
\[ m_K \left( \tilde{u}_K^{n+1} - u_K^n \right) + \Delta t \sum_{L \in N_K} \left( \frac{u_K^n + u_L^n}{2} \right) \int_{K_L} -\tilde{v}.\tilde{n}_K d\gamma(x) + \Delta t \sum_{\sigma \in e_{K,ext}} \frac{u_K^n}{2} \int_{\sigma} -\tilde{v}.\tilde{n}_{K\sigma} d\gamma(x) + \right. \]
\[ \alpha \Delta t [u^n, 1_K]_1, \tau + r \Delta t m_K u_K^n = 0. \]

(17)

(18)

(19)
Explicit upwind finite volume scheme

\[ u^0_K = \psi_K, \]  
\[ u^{n+1}_K = \max \left( \tilde{u}^{n+1}_K, u^0_K \right), \]  
\[ m_K \left( \tilde{u}^{n+1}_K - u^0_K \right) + \Delta t \sum_{\sigma \in \varepsilon_K} v_{K\sigma} u_{n+1}^{\sigma} + \alpha \Delta t |u^n_{1K}|_1, \tau + r \Delta t m_K u^n_K = 0. \]  

with,

\[ \tilde{v} = \left( (r-\lambda_1-\frac{\sigma_1^2}{2}) \cos(\theta) + (r-\lambda_2-\frac{\sigma_2^2}{2}) \sin(\theta) \right) \left( (r-\lambda_1-\frac{\sigma_1^2}{2}) \cos(\theta) - (r-\lambda_1-\frac{\sigma_1^2}{2}) \sin(\theta) \right). \]

Lemma 4 \((L^\infty\) stability\). Let \(D\) be an admissible discretization of \(\Omega \times (0,T)\) in the sense of definition 3.

Let \((u^n_K)_{K,n}\) be the unique solution of the explicit central scheme (17) - (19).

If, \(\Delta t \leq \frac{m_K}{m_K^2 + m_L^2 \sum_{\sigma \in \varepsilon_K} T_{K\sigma}\sum_{L \in N_K} (\alpha T_{KL} + \frac{|v_{KL}|}{4})}, \forall K \in \tau\) and if \(T_{K\sigma} \geq \frac{1}{2\alpha} \left| \int_{\sigma} -\tilde{v} \cdot \vec{n}_{K\sigma} d\gamma(x) \right|, \forall K \in \tau, \forall \sigma \in \varepsilon_K\), then :

\[ \|u^{n+1}_\tau\|_{L^\infty} \leq \|u^n_\tau\|_{L^\infty}, \forall n = 0 \ldots N. \]

Let \((u^n_K)_{K,n}\) be the unique solution of the explicit upwind scheme (20) - (22).

If \(\Delta t \leq \frac{m_K}{r m_K + \left( \sum_{\sigma \in \varepsilon_K} (v_{K\sigma})^2 + \alpha T_{K\sigma} \right)}, \forall K \in \tau\), then :

\[ \|u^{n+1}_\tau\|_{L^\infty} \leq \|u^n_\tau\|_{L^\infty}, \forall n = 0 \ldots N. \]

7.2 Practical implementation and results

We choose to evaluate the American Put option on the minimum of two underlying assets with payoff \(\psi(S^1, S^2) = (K - \min(S^1, S^2))^+\). We assume that the initial values of the stock prices are \(s^1 = 100, s^2 = 100\), the volatility \(\sigma_1 = 0.2, \sigma_2 = 0.2\), the interest rate \(r = \log(1.05)\), the continuous dividend rates \(\delta_1 = 0, \delta_2 = 0\), the exercise price \(K = 100\), the maturity \(T = 1\) and the correlation \(\rho = 0\). We take as the "true" reference price, the one issued of the multinomial BEG tree-method with 3000 step and compare it with the following algorithm:

1. the explicit DP algorithm
2. the DPADI algorithm
3. the BEG algorithm
4. the explicit finite volume algorithm
5. the explicit finite volume algorithm with a smaller time step

For the last algorithm, we multiply by 0.6 the time step obtained by the stability condition. All computation was performed on a PC Pentium IV 2.4 GH computer with 512 Mb of RAM. The "centers" of control volumes are defined as follow:

\[ x_K = x[i \ast N + j] = (v1 - loc + i \ast h, v2 - loc + j \ast g), \forall (i, j) \in [0, N]^2 \]
Table 1: American Put option on the minimum of two underlying assets

<table>
<thead>
<tr>
<th>$N \times M$</th>
<th>DP-EXP</th>
<th>DP-ADI</th>
<th>BEG</th>
<th>FV-EXP</th>
<th>FV-EXPdt*0.6</th>
<th>TRUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>100 $\times$ 100</td>
<td>10.3065;1s</td>
<td>10.2947;1s</td>
<td>10.2974;1s</td>
<td>10.3196;1s</td>
<td>10.3081;1s</td>
<td>10.3080</td>
</tr>
<tr>
<td>200 $\times$ 200</td>
<td>10.3054;6s</td>
<td>10.3031;2s</td>
<td>10.3030;1s</td>
<td>10.3108;3s</td>
<td>10.3095;4s</td>
<td>10.3080</td>
</tr>
<tr>
<td>300 $\times$ 300</td>
<td>10.3073;32s</td>
<td>10.3050;7s</td>
<td>10.3048;1s</td>
<td>10.3098;11s</td>
<td>10.3084;19s</td>
<td>10.3080</td>
</tr>
<tr>
<td>400 $\times$ 400</td>
<td>10.3082;100s</td>
<td>10.3058;18s</td>
<td>10.3057;2s</td>
<td>10.3085;35s</td>
<td>10.3082;59s</td>
<td>10.3080</td>
</tr>
</tbody>
</table>

where the space steps are defined by $h = \frac{2 \text{loc}}{N}$ and $g = \frac{2 \text{loc} \sqrt{\alpha}}{N}$. The control volume $K$ is the rectangle centered in $x_K$ with the measure $m(K) = hg$.

It appears that the numerical FV-EXP method are finally faster than DP-EXP and slower than DP-ADI or BEG.

References