Hull and White Two-factor model

Ismail Laachir

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Contents

1 Hull and White two-factor model 1

2 Trinomial Tree method 2
   2.1 Tree for a one-factor process: 2
   2.2 Tree for a two-factor process: 3
   2.3 Calibration of the tree to the market yield curve: 5

3 Pricing of a security using the tree: 5

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1 Hull and White two-factor model

Hull and White model is a short-rate model. One of its main characteristics
is its ability to match the initial yield curve by using time-varying param-
eter. A one factor version of this model was first proposed in [1] (already
implemented in Premia). In this project we consider the two-factor version
proposed in [2].

Hull&White two-factor model is defined by an EDS which describes the
evolution of the spot rate \( r(t) \):

\[
\begin{align*}
   dr(t) &= [\theta(t) + u(t) - a r(t)] dt + \sigma_1 dW_1(t) \\
   du(t) &= -b u(t) dt + \sigma_2 dW_2(t), \quad u(0) = 0
\end{align*}
\]
The two processes $W_1$ and $W_2$ are brownian motions with instantaneous correlation $\rho$, and $\theta$ is a deterministic function totally given by the market value of the zero coupon bonds.

Let us denote by $P_M(0,T)$ the market zero coupon bond value maturing at time $T$ and $f_M(t) = -\frac{\partial \log(P_M(0,t))}{\partial t}$ the market present instantaneous forward rate, then with an appropriate choice for the function $\theta$ (see Hull&White 1994 for details), the model exactly fits the market bonds curve and we have several analytical formulas:

Zero coupon bond at time $t$ knowing that $r(t) = r_t$ and $u(t) = u_t$:

$$P(t, T) = A(t, T)e^{-B(t,T)r_t-C(t,T)u_t}.$$  

Explicit formulations for $A$, $B$ and $C$ can be found in [2].

The price at time $t$ for a European Call on a ZC bond:

$$C_t = \mathbb{E}_t \left[ e^{-\int_t^T r(s) ds} (P(T,S) - K)_+ \right] = P(t, S)N(h) - KP(t, T)N(h - \sigma_p).$$

Where $N$ is the cumulative function of the normal law,

$$h = \frac{1}{\sigma_p} \log \left( \frac{P(t, S)}{P(t, T)K} \right) + \frac{\sigma_p^2}{2}$$

and $\sigma_p$ is given in [2].

This closed formula for european option on bond also leads to closed formula for cap and floor.

## 2 Trinomial Tree method

### 2.1 Tree for a one-factor process:

We first recall the procedure for the construction of a trinomial tree that approximates a process $x$ of the form:

$$dx(t) = -a x(t) dt + \sigma dW(t), \quad x(0) = 0$$

Let $0 = t_0 < t_1 < .. < t_n = T$ be a time scale for our tree in $[0, T]$, $\Delta t_i = t_{i+1} - t_i$, and $x_{i,j}$ the $x$ node value at time $t_i$ for the $j^{th}$ space step of the tree (starting from the down). We need then:
\[
\begin{align*}
\{ & \mathbb{E} [x(t_{i+1})|x(t_i) = x_{i,j}] = M_{i,j} \\
& \text{Var} [x(t_{i+1})|x(t_i) = x_{i,j}] = V^2_{i,j} = V^2_i
\end{align*}
\]

Knowing that \(x\) has is a gaussian process, \(M_{i,j}\) and \(V^2_i\) can be computed:

\[
\begin{align*}
M_{i,j} &= x_{i,j} e^{-a \Delta t_i} \\
V^2_i &= \frac{\sigma^2}{2a} \left[ 1 - e^{-a \Delta t_i} \right]
\end{align*}
\]

At time \(t_i\), the nodes are equally spaced, so: \(x_{i,j} = j \Delta x_i\), with \(\Delta x_i = V_{i-1} \sqrt{3}\).

Starting at time \(t_i\) from node \(x_{i,j}\), the process can move to three node at time \(t_{i+1}\):

\[
\begin{align*}
\{ & \text{Up with probability } p_u(i, j) \text{ to the node } x_{i+1,k+1} \\
& \text{Middle with probability } p_m(i, j) \text{ to the node } x_{i+1,k} \\
& \text{Drown with probability } p_d(i, j) \text{ to the node } x_{i+1,k-1}
\end{align*}
\]

The index \(k\) is chosen so that \(x_{i+1,k}\) is as close as possible to the mean \(M_{i,j}\), ie:

\[
k = \text{round} \left( \frac{M_{i,j}}{\Delta x_{i+1}} \right) = \text{round} \left( j \beta_i \right)
\]

, with \(\beta_i = \frac{\Delta x_i}{\Delta x_{i+1}} e^{-a \Delta t_i}\).

The probabilities \(p_u(i, j), p_m(i, j)\) and \(p_d(i, j)\) are chosen so that the conditional mean and variance of the discrete process in the tree match those of the continuous process \((M_{i,j} \text{ and } V^2_i)\). See [3] for the obtained formulas.

### 2.2 Tree for a two-factor process:

Now that we know how to construct a trinomial tree for a process of the kind 
\[dx(t) = -a x(t) dt + \sigma(t) dW(t),\]
we will use this technique for the two-factor process \(r\).

First, we consider the process \(x\) verifying the same equation as \(r\), with \(\theta = 0\):

\[
\begin{align*}
dx(t) &= [u(t) - a x(t)] dt + \sigma_1 dW_1(t), \quad x(0) = 0 \\
du(t) &= -b u(t) dt + \sigma_2 dW_2(t), \quad u(0) = 0
\end{align*}
\]

\[
\begin{align*}
\{ & \mathbb{E} [x(t_{i+1})|x(t_i) = x_{i,j}] = M_{i,j} \\
& \text{Var} [x(t_{i+1})|x(t_i) = x_{i,j}] = V^2_{i,j} = V^2_i
\end{align*}
\]
If we suppose that $a \neq b$, then the dependance of $x$ on $u$ can be eliminated by defining

$$y = x + \frac{u}{b - a}$$

so that

$$\begin{cases} 
dy(t) = -a \, y(t) \, dt + \sigma_3 \, dW_3(t), & y(0) = 0 \\
du(t) = -b \, u(t) \, dt + \sigma_2 \, dW_2(t), & u(0) = 0 
\end{cases}$$

where

$$\sigma_3^2 = \sigma_3^2 + \frac{\sigma_2^2}{(b-a)^2} + \frac{2 \rho \sigma_1 \sigma_2}{b-a}$$

and $W_3$ is a brownian motion. The correlation between $W_2$ and $W_3$ is

$$\rho_{uy} = \frac{\rho \sigma_1 + \sigma_2/(b-a)}{\sigma_3}$$

The first step to construct a tree for $x$ is to construct two trinomial trees, with the technique explained above, for the two processes $u$ and $y$, then use the formula

$$x = y - \frac{u}{b - a}$$

The tree obtained for $x$ will be a two-dimensional trinomial tree, where every node will have nine branches, result of the combination of the tree branches of $u$ and $y$.

At time $t_i$, we define the nodes $y(i, h)$ and $u(i, l)$ so the node for $x$ is $x(i, h, l)$. We define $j$ the index of middle branche (in the tree of $y$) emanating from $y(i, h)$ and the corresponding probabilities $pu$, $pm$, $pd$, and define $k$ the index of middle branche (in the tree of $u$) emanating from $u(i, l)$ and the corresponding probabilities $qu$, $qm$, $qd$. Then, starting from $x(i, h, l)$, the process move to nine branches $x(i, j + \epsilon_1, k + \epsilon_2)$, where $\epsilon_1$ and $\epsilon_2$ take the values 0, 1 or -1.

Finally we have to decide the probabilities associated with every node of the nine:

In case of zero correlation between $u$ and $y$ (ie $\rho_{uy} = 0$), the matrix of probabilities for the nine branches of $x$ is simply:
In the case of correlated processes, the elements of the matrix above are shifted in such a way that the sum of shifts in each row and column is zero (to preserve the law of \( u \) and \( y \)) and to have the good correlation \( \rho_{uy} \) between \( u \) and \( y \). See [2] for further details.

2.3 Calibration of the tree to the market yield curve :

After the construction of the tree for the process \( x \), the process \( r \) can be defined as \( r(t) = x(t) + \alpha(t) \) where \( \alpha \) is a deterministic function. It is calculated using the Arrow-Debreu node prices and the market price of Zero Coupon Bonds.

We denote by \( Q_{i+1,j,k} \) the present value of a security that pays 1 if the node \((i + 1, j, k)\) is attained and zero otherwise. These quantities are calculated recursively, knowing \( \alpha_i \) and \( Q_{i,h,l} \) for all \((h, l)\), by :

\[
Q_{i+1,j,k} = \sum_{h,l} Q_{i,h,l} q_i(h,l,j,k) \exp\{-(\alpha_i + x_{i,h,l}) \Delta t_i}\]

where \( q_i(h,l,j,k) \) is the probability of moving from \((i, h, l)\) to \((i + 1, j, k)\).

Then, \( \alpha_{i+1} \) is calculated by solving :

\[
P_M(0,t_{i+2}) = \sum_{i,j} Q_{i+1,j,k} \exp\{-\alpha_{i+1} + x_{i+1,j,k}\} \Delta t_{i+1}\]

ie :

\[
\alpha_{i+1} = \frac{1}{\Delta t_{i+1}} \ln \frac{\sum_{i,j} Q_{i+1,j,k} \exp\{-x_{i+1,j,k}\} \Delta t_{i+1}}{P_M(0,t_{i+2})}
\]

The initial value for \( \alpha \) and \( Q \) are : \( Q_{0,0,0} = 1 \) and \( \alpha_0 = -\ln(P_M(0,t_1))/t_1 \).

3 Pricing of a security using the tree :

Now that we have a trinomial tree of the spot rate \( r_{i,j,k} \) with their transition probabilities we can compute the price \( h(t, r(t), u(t)) \) of any european option.
with payoff $H(T, r(T), u(T))$ thanks to a backward induction, starting with $h(T, r(T), u(T)) = H(T, r(T), u(T))$:

$$h_{i,j,k} = e^{-r_{i,j,k} \Delta t_i} \sum_{\epsilon_1, \epsilon_2 \in \{-1,0,1\}} h_{i+1,j^*,k^*+\epsilon_2} q_i(j,k,j^*+\epsilon_1,k^*+\epsilon_2)$$

Where $(i+1, j^*, k^*)$ is the index of middle branch emanating from $(i,j,k)$ and $q_i(j,k,j^*+\epsilon_1,k^*+\epsilon_2)$ is the probability of moving from $(i,j,k)$ to $(i+1,j^*+\epsilon_1,k^*+\epsilon_2)$.

In the case of an american payoff, we compare the result of the backward induction with the payoff $H(t_i, r_{i,j,k}, u_{i,k})$:

$$h_{i,j,k} = \max \left( H(t_i, r_{i,j,k}, u_{i,k}), e^{-r_{i,j,k} \Delta t_i} \sum_{\epsilon_1, \epsilon_2 \in \{-1,0,1\}} h_{i+1,j^*,k^*+\epsilon_2} q_i(j,k,j^*+\epsilon_1,k^*+\epsilon_2) \right)$$

References

