The Libor Market Model with Jumps

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Premia 14

Abstract

The aim of this note is to use a Lévy-driven model to describe the joint arbitrage-free dynamics of a set of forward Libor rates. Such model is called a Libor market model. This note is based on the paper of Tankov and Kohatsu-Higa (so for more details see [4]).

1 Preliminaries

We consider a $d$-dimensional Lévy process $Z$ without diffusion component. Thus $(\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^+$, and $\nu$ is a Radon measure on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty.$$  

By the Lévy-Itô decomposition, $X$ can be written in the form

$$Z_t = \gamma t + \int_{|x| > 1, s \in [0,t]} xJ(dx \times ds) + \lim_{\delta \downarrow 0} \int_{\delta \leq |x| \leq 1, s \in [0,t]} x\tilde{J}(dx \times ds)$$  

(1.1)

Here $\gamma \in \mathbb{R}^d$, $J$ is a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity $\nu(dx)dt$, $\tilde{J}(dx \times ds) = J(dx \times ds) - \nu(dx)ds$ and $\nu$ is a Radon measure on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}} (1 \wedge x^2) \nu(dx) < \infty$. Given $\epsilon > 0$, we define the process $R^\epsilon$ by

$$R^\epsilon_t = \int_{0 \leq |x| \leq \epsilon, s \in [0,t]} x\tilde{J}(dx \times ds), \ t \geq 0.$$  

(1.2)

Note that we have

$$\mathbb{E}R^\epsilon_t = 0.$$  

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On the other hand we denote by $\Sigma^\varepsilon$ the covariance matrix of $R^\varepsilon_1$, and thus for any $i, j \in \{1, \ldots, d\}$

$$\Sigma^\varepsilon_{i,j} = \int_{|x|\leq \varepsilon} x_ix_j \nu(dx).$$

Define the process $Z^\varepsilon$ by

$$Z^\varepsilon_t = \int_{|x|> \varepsilon, s \in [0,t]} xJ(dx \times ds), \ t \geq 0.$$ 

Then we have

$$Z_t = \gamma_\varepsilon t + Z^\varepsilon_t + R^\varepsilon_t, \ t \geq 0, \quad (1.3)$$

where

$$\gamma_\varepsilon = \gamma - \int_{|x|\leq 1} x\nu(dx). \quad (1.4)$$

We will call $(T^\varepsilon_i)_{i \geq 1}$ the jump times of the process $Z^\varepsilon$.

## 2 Approximation of multidimensional SDE

Let $X$ be a $n$-dimensional stochastic process, and the unique solution of the stochastic differential equation

$$dX_t = h(X_t^-)dZ_t, \ t \in [0,1], \quad (2.5)$$

where $h$ is a $n \times d$ matrix. A suitable approximation of $X$ is $\bar{X}$ defined by

$$d\bar{X}_t = h\left(\bar{X}_t^-\right)(\gamma_\varepsilon dt + dW^\varepsilon_t + dZ^\varepsilon_t), \quad (2.6)$$

where $W^\varepsilon$ is a $d$-dimensional Brownian motion with covariance matrix $\Sigma^\varepsilon$. The choice of this approximation is explain in [4]. The process $\bar{X}$ can be also written in this form

$$\bar{X}_t = \bar{X}_t^\eta + \int_{\eta}^t h\left(\bar{X}_s\right) dW^\varepsilon_s + \int_{\eta}^t h\left(\bar{X}_s\right) \gamma_\varepsilon ds,$$

$$\bar{X}_{T^\varepsilon_i} = \bar{X}_{T^\varepsilon_i^-} + h\left(\bar{X}_{T^\varepsilon_i^-}\right) \Delta Z_{T^\varepsilon_i}.$$ 

where $\eta_t = \sup T^\varepsilon_i, \ T^\varepsilon_i \leq t$. The idea of [4] is to approximate $\bar{X}$ by

$$Y^0 + \left. \frac{\partial}{\partial \alpha} Y^\alpha \right|_{\alpha = 0},$$

where the family of processes $(Y^\alpha)_{0\leq \alpha \leq 1}$ is defined by

$$Y^\alpha_t = \bar{X}_t^\eta + \int_{\eta}^t h\left(Y^\alpha_s\right) dW^\varepsilon_s + \int_{\eta}^t h\left(Y^\alpha_s\right) \gamma_\varepsilon ds.$$
Hence a new approximation of $X$, called $\bar{X}$, is defined by
\[
\bar{X}_t = Y_{0,t} + Y_{t,1}, \quad t > \eta \\
\bar{X}_{T^i_t} = \bar{X}_{T^i_t} - h(\bar{X}_{T^i_t}) \Delta Z^i_t \\
Y_{0,t} = \bar{X}_m + \int_{\eta}^{t} h(Y_{0,s}) \gamma_t ds \\
Y_{1,t} = \int_{\eta}^{t} h(Y_{0,s}) dW^c_s + \sum_{i=1}^{n} \int_{\eta}^{t} \frac{\partial h}{\partial x_i}(Y_{0,s}) Y^i_s \gamma_t ds.
\]
The random vector $Y_{1,t}$ is Gaussian with mean zero and covariance matrix $\Omega_t$ satisfying
\[
\Omega_t = \int_{\eta}^{t} \left( \Omega_s M_s + M_s^\perp \Omega_s^\perp + N_s \right) ds,
\]
where $M^\perp$ is the transpose of the matrix $M$ and
\[
M^{ij}_{t} = \frac{\partial h^{ij}(Y_{0,t})}{\partial x_j} \gamma^j_t, \quad N_t = h(Y_{0,t}) \Sigma^t h^\perp(Y_{0,t}).
\]

### 3 Libor market model

Let $T_i = T_1 + (i-1)\delta$, $i = 1, \ldots, n + 1$ be a set dates, called tenor dates. The Libor rate $L_i^t$ is the forward interest rate, defined at date $t$ for the period $[T_i, T_{i+1}]$. The Libor rate can be expressed with respect to prices of zero-coupon bonds.

\[
L_i^t = \frac{1}{\delta} \left( \frac{B_t(T_i)}{B_t(T_{i+1})} - 1 \right),
\]

where $B_t(T)$ is the price at time $t$ of a zero-coupon bond with maturity $T$. A arbitrage-free dynamics of $L_1^t, \ldots, L_n^t$ (see [3]) is

\[
\frac{dL_i^t}{dL_i^\tau} = \sigma_{i,t} dZ_t - \int_{\mathbb{R}^d} \sigma_{i,t} z \left[ \prod_{j=i+1}^{n+1} \left( 1 + \frac{\delta L_j^t \sigma_j^t z}{1 + \delta L_j^t} \right) - 1 \right] \nu(dz) dt, \tag{3.7}
\]

where $Z$ is a $d$-dimensional martingale pure jump Lévy process, with Lévy measure $\nu$, and $\sigma_{i,t}$ are $d$-dimensional deterministic volatility functions. The dynamics are given under the so-called terminal measure. This means the last zero-coupon bond, $B_t(T_{n+1})$, is used as the numéraire. So the price at time $t$ of an option with payoff $H = f(L_{T_1}^1, \ldots, L_{T_1}^n)$ at time $T_1$ is given by

\[
\pi_t(H) = \frac{B_t(T_1)}{\prod_{i=1}^{n} (1 + \delta L_i^t)} \mathbb{E} \left[ f(L_{T_1}^1, \ldots, L_{T_1}^n) \prod_{i=1}^{n} \left( 1 + \delta L_i^t \right) / \mathcal{F}_t \right].
\]
We introduce the process \((n + 1)\)-dimensional \(X\) with \(X^0_t = t\) and \(X^i_t = L^i_t\) (for \(i = 1, \ldots, n\)), a \((d + 1)\)-dimensional process \(\tilde{Z} = (t, Z_t)^\perp\), and a \((n + 1) \times (d + 1)\)-dimensional function \(h\) with \(h^{11} = 1\), \(h^{1j} = 0\) for \(2 \leq j \leq d + 1\), \(h^{i1} = f^i(x)\) and \(h^{ij} = \sigma^{j-1}_{i,x_0}\) (for \(2 \leq j \leq d + 1\)) with
\[
\begin{align*}
f^i(x) &= -\int_{\mathbb{R}^d} \sigma_{i,x_0}^j z \left[ \prod_{j=i+1}^{n+1} \left( 1 + \frac{\delta x_j \sigma^{j-1}_{i,x_0} z}{1 + \delta x_j} \right) - 1 \right] \nu(dz) dt,
\end{align*}
\]
so that the equation (3.7) takes the form
\[
dX_t = h(X_{t^-}) d\tilde{Z}_t.
\]
For details about this model, see [4].

References


1, 2, 4

References