Pricing of Bermudan options under the assumption that the optimal stopping time depends only on the values of the still-alive European component options [Andersen]

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The following is based on [A]. Suppose that, for $i = 0, \ldots, n - 1$, we have a martingale $(\tilde{v}_i^j)_{t \in [0, T_i]}$ representing the discounted value at $t$ of the European option with maturity $T_i$ and discounted payoff $\tilde{v}_i^{T_i}$. Suppose that we have the following Markov functional forms

$$\tilde{v}_i^{T_j} = f_{i,j}(x_{T_j}), \quad j = 0, \ldots, i.$$  

Here $(x_t)$ is a Markov process with values in $\mathbb{R}^D$. I.e., we have a closed formula (or at least a closed form approximation) for the values of the $n$ European options.

Now consider the Bermudan option given by the payoffs $\tilde{v}_i^{T_i}$ at the exercise times $T_i$ for $i = 0, \ldots, n - 1$. Let $\tilde{V}_t$ denote its discounted value at $t$. Then

$$\tilde{V}_0 = E(\tilde{V}_{T_0})$$

$$\tilde{V}_{T_i} = \sup_{\tau \in \mathcal{T}_{\{i, \ldots, n-1\}}} \tilde{V}_{T_i}(\tau), \quad \text{where} \quad \tilde{V}_{T_i}(\tau) := E(\tilde{v}_{T_i}^{\tau} | \mathcal{F}_{T_i})$$

and $\mathcal{T}_{\{i, \ldots, n-1\}}$ denotes the set of stopping times with values in $\{i, \ldots, n-1\}$. We introduce the indicator function $I(T_i)$ which is one if exercising at $T_i$ is optimal and zero otherwise; hence

$$\tau_i^* = \inf\{ j = i, \ldots, n - 1; I(T_j) = 1 \}.$$
Here $\tau^*_i$ denotes the optimal stopping time in (3), i.e. $\tilde{V}_{T_i} = \tilde{V}_{T_i}(\tau^*_i)$. Now suppose that $I(T_i)$ has the following Markov functional form:

\begin{equation}
I(T_i) = \tilde{b}_i(\tilde{v}^1_{T_i}, \ldots, \tilde{v}^{n-1}_{T_i}) = b_i(x_{T_i})
\end{equation}

for some deterministic boolean functions $\tilde{b}_i$, $b_i$. That is, we assume that the exercise decision at $T_i$ depends only on the values at $T_i$ of the still-alive European component options, and this dependence is expressed by $\tilde{b}_i$. Of course, once that $\tilde{b}_i$ is chosen, one obtains $b_i$ directly from (1).

Observe that, denoting $\vec{b}_i = (b_i, \ldots, b_{n-1})$, the following definition motivated by (4) and (5) yields an element of $\mathcal{T}_{\{i,\ldots,n-1\}}$:

$$
\tau_{\vec{b}_i} := \inf\{ j = i, \ldots, n-1 ; b_j(x_{T_j}) = 1 \} \in \mathcal{T}_{\{i,\ldots,n-1\}}.
$$

Obviously, we find functional forms

$$
\tau_{\vec{b}_i} = g_{\vec{b}_i}(x_{T_i}, \ldots, x_{T_{n-1}}).
$$

Hence, in view of (1), we obtain functional forms

$$
\tilde{V}_{T_i,\tau_{\vec{b}_i}} = f_{\tau_{\vec{b}_i},\tau_{\vec{b}_i}}(x_{T_{\tau_{\vec{b}_i}}}) = F_{\vec{b}_i}(x_{T_i}, \ldots, x_{T_{n-1}}).
$$

This yields the following representation of $E(\tilde{V}_{T_i}(\tau_{\vec{b}_i}))$:

\begin{equation}
E(\tilde{V}_{T_i}(\tau_{\vec{b}_i})) = E(\tilde{V}_{T_{\tau_{\vec{b}_i}}}) = E(F_{\vec{b}_i}(x_{T_i}, \ldots, x_{T_{n-1}})).
\end{equation}

Here we use definition (3) in the first step.

0.1 Monte Carlo approximation of $\tilde{V}_0$, provided the $b_i$ are chosen

Suppose that, for our Markov process $(x_t)$, we are given $M$ Monte Carlo samples $(x^m_{T_0}, \ldots, x^m_{T_{n-1}})$, where $m = 0, \ldots, M-1$. Then, based on (2), (3) and (6), we have the following Monte Carlo approximation of the discounted present value $\tilde{V}_0$ of our Bermudan option:

$$
\tilde{V}_0 \approx \frac{1}{M} \sum_{m=0}^{M-1} F_{\vec{b}_0}(x^m_{T_0}, \ldots, x^m_{T_{n-1}}).
$$

0.2 Choice of the $b_i$

Concerning the choice of the $b_0, \ldots, b_{n-1}$, we note first that one reasonably takes

$$
\tilde{b}_{n-1}(v_{n-1}) := 1_{v_{n-1} > 0}
$$
which states that the Bermudan option is exercised at the last exercise date $T_{n-1}$ if and only if the last European component option (which is the only one being still alive) is in-the-money.

Now the $b_i$ for $i < n - 1$ can be chosen (backward) iteratively via Monte Carlo maximization over given parametric classes $B_i$ of the expected discounted value $E(\hat{V}_{T_i})$ of the Bermudan option at time $T_i$:

$$b_i = \arg \max_{b \in B_i} E\left( \hat{V}_{T_i}(\tau(b, b_{i+1}, \ldots, b_{n-1})) \right) = \arg \max_{b \in B_i} \sum_{l=0}^{L-1} F_{T_i}^{T_{i+l-1}}(x^l_{T_{i+l-1}}) \cdot \tau(b) .$$

Here we use (6) in the second step and we consider $L$ Monte Carlo paths independent of those we use for the approximation of $\hat{V}_0$; one should take $M >> L$. An example for the choice of the classes $B_i$ of boolean functions

$$B_i = \{ \tilde{b}_i^H; H \geq 0 \} , \text{ where } \tilde{b}_i^H(v_i, \ldots, v_{n-1}) := 1_{v_i > H} .$$

Hence exercise takes place at $T_i$ if the payoff of the European option maturing at $T_i$ exceeds some barrier $H$. A second example is

$$B_i = \{ \tilde{b}_i^H; H \geq 0 \} , \text{ where } \tilde{b}_i^H(v_i, \ldots, v_{n-1}) := 1_{v_i > \max(H, v_{i+1}, \ldots, v_{n-1})} .$$

This second strategy is a refinement that also checks if at least one of the remaining European options has a value exceeding the value of the present European option. If this is the case, the strategy decides that exercise cannot be optimal - a reflection of the fact that the Bermudan option can always be sold at the value of its most expensive European component option.

Examples of well-known one-dimensional optimization algorithms include Golden Section Search and Brent’s method.

### 0.3 Remarks

(1) One might call the general approach chosen here: Monte Carlo pricing of Markov functional Bermudan options under the assumption that the optimal stopping time is also Markov-functional.

(2) We give the precise definition of some of the functional forms used above:

$$b_i(x) := \tilde{b}_i(f_{x, i}(x), \ldots, f_{x, n-1, i}(x))$$

$$g_{b_i}^{\tau}(x_i, \ldots, x_{n-1}) := \inf\{ j = i, \ldots, n - 1 ; b_j(x_j) = 1 \}$$

$$F_{b_i}^{\tau}(x_i, \ldots, x_{n-1}) := f_{\tau, \tau}(x_\tau) , \text{ where } \tau := g_{b_i}^{\tau}(x_i, \ldots, x_{n-1}) .$$

Here $x, x_i, \ldots, x_{n-1} \in \mathbb{R}^D$. 
1 Numerical results: Bermudan swaption pricing in the one-factor LIBOR Market Model

We fix a discrete tenor structure

\[ 0 = T_0 < T_1 < \ldots < T_e \text{ with } T_{i+1} - T_i \equiv \delta \]

and define the rightcontinuous function \( \eta(t) \) by

\[ \mathcal{T}_{\eta(t)} - 1 \leq t < \mathcal{T}_{\eta(t)} \text{ , in particular } \eta(T_i) = i + 1 . \]

Denoting by \( P(t, T) \) the time \( t \) price of a zero-coupon bond maturing at \( T \), we define for \( i = 0, \ldots, e - 1 \) the forward LIBOR rates for the period \( [T_i, T_{i+1}] \):

\[ L^i_t := \delta^{-1} \left( \frac{P(t, T_i)}{P(t, T_{i+1})} - 1 \right) \text{ , } t \in [0, T_i] . \]

The method presented in Section 1 will hence be applied for \( D = e \) and \( x_t = (L^i_{t \wedge T_i})_{i=0,\ldots,e-1} \in \mathbb{R}^e \). We assume forward measure dynamics of the following simple type:

\[ dL^i_t = \lambda L^i_t dW^i_t . \]

This is equivalent to the following spot measure dynamics:

\[ dL^i_t = \lambda L^i_t \left( b^i(t, L^i_t) dt + dW^i_t \right) , \quad b^i(t, L) := \delta \lambda \sum_{j=\eta(t)}^{i} \frac{L^j_t}{1 + \delta L^j_t} . \]

All simulations will be done under these spot measure dynamics. Recall that the corresponding spot numeraire \( (N_t) \) satisfies

\[ N_{T_i} = \prod_{j=0}^{i-1} (1 + \delta L^j_{T_j}) . \]

Let us consider a (payer) interest rate swap where fixed cashflows \( K \delta \) paid at \( T_{s+1}, \ldots, T_e \) are swapped against floating LIBOR on a unit notional. We will price the corresponding Bermudan swaption with \( n \) exercise dates \( T_i = T_{s+i} \), where \( i = 0, \ldots, n - 1 \). Hence, the discounted payoff at \( T_i \) is

\[ \tilde{v}^i_{T_i} = \frac{1}{N^i_{T_i}} \left( 1 - P(T_i, T_e) - K \delta \sum_{j=s+i+1}^{e} P(T_i, T_j) \right) , \quad T_i = T_{s+i} . \]

Closed form approximations [corresponding to the \( f_{i,j} \) needed in (1)] for European swaption prices can be found e.g. in [AA, §5]. We consider the following parameter values:

\[ \delta = 0.5 , \quad L_0^i = 0.06 , \quad K = 0.06 , \quad L = 10000 , \quad M = 50000 . \]
The following table of prices corresponds to Table 1 in [A]. The letters E and B correspond to European \((n = 1)\) and Bermudan \((n = e - s)\).

We applied the first strategy presented in Section 1.2 for the choice of the \(b_i\). All Monte Carlo simulations are based on a first-order log-Euler discretization with time step \(\delta\). Numbers in parenthesis denote the 95% confidence interval.

In the European case, we also give the price obtained via the closed form approximation mentioned above.

<table>
<thead>
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<th>(T_0 = T_s)</th>
<th>(T_e)</th>
<th>(\lambda)</th>
<th>E or B</th>
<th>CF</th>
<th>MC</th>
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<td>E</td>
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<td>4</td>
<td>0.2</td>
<td>E</td>
<td>111.4</td>
<td>109.3 (1.6)</td>
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<td>4</td>
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<td>E</td>
<td>66.1</td>
<td>65.8 (1.0)</td>
</tr>
<tr>
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<td>0.2</td>
<td>B</td>
<td></td>
<td>157.1 (1.7)</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>0.2</td>
<td>E</td>
<td>162.4</td>
<td>159.3 (2.3)</td>
</tr>
<tr>
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<td>E</td>
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<td>71.1 (1.1)</td>
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<tr>
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<td></td>
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References
