1 The quadratic interest rate model

1.1. Description of the model

In the quadratic interest rate model, the evolution of the spot interest rate $r(t)$ is described by the following SDE:

\[
\begin{align*}
\frac{dx(t)}{dt} &= (\alpha(t) - \beta x(t)) dt + \sigma dW(t), \\
r(t) &= \frac{1}{2} x(t)^2, \\
x(0) &= \sqrt{2r(0)},
\end{align*}
\]

where $\beta$ and $\sigma$ are constants, $\alpha$ is a time-dependent function determined by the values of $\beta$, $\sigma$ and the curve of the $s$-maturity zero-coupon prices at time $t = 0$. Notice that $(x(t), t \geq 0)$ is a gaussian process.

If $\mathbb{E}_t$ denote the conditional expectation at time $t$ under the risk-neutral measure, for $s$-maturity zero-coupon bond at time $t$, we have:

\[
P_s(t) = \mathbb{E}_t [\exp (- \int_0^s r(u)du)]
= \exp \left( - \left( \frac{1}{2} B_s(t)x(t)^2 + b_s(t)x(t) + c_s(t) \right) \right),
\]

where $B_s(t)$, $b_s(t)$ and $c_s(t)$ are described in Section 2 and computed using equations given in Appendix (see 8.2.1).
1.2. The $T$-forward risk adjusted measure

For options on bonds, caplet and call on futures, we will have to use the $T$-forward risk adjusted measure $E^T_t$ defined by:

$$E^T_t[Z(T)] = E_t[e^{-\int_t^T r(u) du} Z(T)] / P_T(t)$$

for all non-negative Itô process $Z$.

1.3. Notations

We write $Y \sim \Omega(B, b, c, \mu, V)$ if $Y = \frac{1}{2}BX^2 + bX + c$, where $X = \mu + VG$ and $G \sim N(0,1)$ is a centered reduced gaussian. We also have $Y = \alpha + \beta(G + \sqrt{\lambda})^2$ where:

$$\alpha = c - \frac{1}{2}B^2, \quad \beta = \frac{1}{2}BV, \quad \lambda = \frac{(\mu + b/B)^2}{V}.$$  \hfill (3)

Notice that $(G + \sqrt{\lambda})^2$ is distributed as a non-central chi-square with 1 degree of freedom and non-centrality parameter $\lambda$. Let $\chi^2(y; \lambda, \beta)$ denote the cumulative distribution of $\beta(G + \sqrt{\lambda})^2$ and $\omega(y; B, b, \mu, V)$ the cumulative distribution of $\frac{1}{2}BX^2 + bX$. Hence, we have

$$\mathbb{P}(Y \leq y) = \mathbb{P} (\beta(G + \sqrt{\lambda})^2 \leq y - \alpha) = \chi^2(y - \alpha; \lambda, \beta),$$

as well as

$$\mathbb{P}(Y \leq y) = \mathbb{P} \left( \frac{1}{2}BX^2 + bX \leq y - c \right) = \omega(y - c; B, b, \mu, V).$$

In particular:

$$\omega(y; B, b, \mu, V) = \chi^2(y + \frac{1}{2}B^2; \lambda, \beta).$$

For a function of two variables written as $f_s(t)$, we write $\dot{f}_s(t) \equiv \partial f_s(t) / \partial s$ and $\ddot{f}_s(t) \equiv \partial^2 f_s(t) / \partial s^2$.

We also use the following conventions:

- $T$ denotes the maturity of an option,
- $t$ denotes the maturity of a futures or a forward contract,
- $s$, or $s'$ denote maturities of zero-coupon bonds,
- $K$ denotes the strike of an option.

All prices are given at initial time $t = 0$. Hence, we have $0 \leq T \leq t \leq s$, $s'$. 
2 Calibration and computation of bond coefficients

2.1. Initial values of bond coefficients

To compute the time-dependent function $\alpha$, we must first compute the forward interest rate at time $t = 0$ from the initial zero-coupon curve $P_s(0)$ as described in the Appendix. Now, for any $s$ and $t$, using equations given in 8.1.1, we can compute $B_s(0), \dot{b}_s(0), \dot{c}_s(0)$ and $\alpha(t)$ to fit the initial yield curve. We get $b_s(0)$ and $c_s(0)$ for any $s$ integrating $\dot{b}_s(0)$ and $\dot{c}_s(0)$ with means of trapezoidal rule.

2.2. Transport equations

Transport equations yield formulas for $B_s(t), b_s(t)$ and $c_s(t)$ for any $t$ and $s$ using their initial values. These equations are given in 6.1.2.

3 Closed formulae for european options on bonds

3.1. European call

Price : $E_0 \left[ e^{-\int_0^T r(u) du} (P_s(T) - K)_+ \right] = P_T(0) \mathbb{E}^T_0 [(P_s(T) - K)_+]$.

Under the $T$-forward risk adjusted measure, we have $-\log(P_s(T)) \sim \Omega(B, b, c, \mu, V)$ where $B = B_s(T)$, $b = b_s(T)$, $c = c_s(T)$ and :

$$\mu = \sqrt{B_T(0)x(0)} + \frac{\dot{b}_T(0)}{\sqrt{B_T(0)}}, \quad V = \sigma^2 B_T(0). \quad (4)$$

Using (3), we compute the coefficients $\alpha, \beta$ and $\lambda$ corresponding to $B, b, c, \mu$ and $V$. Then, the price of a $T$-maturity call option on the $s$-bond is given by:

$$E_0 \left[ e^{-\int_0^T r(u) du} (P_s(T) - K)_+ \right]$$

$$= P_s(0) \chi^2(-\alpha - \log(K); \frac{\lambda}{1+2\beta}, \frac{\beta}{1+2\beta})$$

$$-K P_T(0) \chi^2(-\alpha - \log(K); \lambda, \beta).$$
3.2. Caplet

Price : \( E_0 \left[ e^{-\int_0^T r(u)du} (r(T) - K)_+ \right] = P_T(0) E_0^T \left[ (r(T) - K)_+ \right]. \)

Under the \( T \)-forward risk adjusted measure, we have \( r(T) \sim \Omega(B, b, c, \mu, V) \) with \( B = 1, b = 0, c = 0 \) and \( \mu \) and \( V \) given by (4). Thanks to (3), we compute the coefficients \( \alpha, \beta \) and \( \lambda \) corresponding to \( B, b, c, \mu \) and \( V \). Then, the price of a \( T \)-maturity caplet is given by:

\[
E_0 \left[ e^{-\int_0^T r(u)du} (r(T) - K)_+ \right] = P_T(0) \left[ \frac{1}{2} (r_T(0) - K) + C(K - \alpha; \lambda, \beta) \right],
\]

where:

\[
C(K - \alpha; \lambda, \beta) = \frac{1}{\pi} \int_0^{+\infty} \left[ 1 - \Psi(\lambda, 2\xi^2\beta^2) \cos(\xi(K - \alpha) - \Phi(\lambda, \xi\beta)) \right] \frac{d\xi}{\xi^2},
\]

with

\[
\Psi(\lambda, z) = (1 + 2z)^{-1/4} \exp\left( \frac{\lambda z}{1 + 2z} \right) \quad \text{and} \quad \Phi(\lambda, z) = \frac{1}{2} \arctan(2z) + \frac{\lambda z}{1 + 4z^2}.
\]

3.3. Exchange option

Price : \( E_0 \left[ e^{-\int_0^T r(u)du} (kP_s(T) - k'P_{s'}(T))_+ \right] = P_T(0) E_0^T \left[ (kP_s(T) - k'P_{s'}(T))_+ \right]. \)

Under the \( T \)-forward risk adjusted measure : \( -\log(P_s(T)) = \frac{1}{2} BX^2 + bX + c \) and \( -\log(P_{s'}(T)) = \frac{1}{2} B'X^2 + b'X + c' \) where \( X \sim N(\mu, V) \), \( B = B_s(T), b = b_s(T), c = c_s(T), B' = B_{s'}(T), b' = b_{s'}(T), c' = c_{s'}(T) \) and \( \mu, V \) given by (4). In particular, we have \( -\log(P_s(T)) \sim \Omega(B, b, c, \mu, V) \) and \( -\log(P_{s'}(T)) \sim \Omega(B', b', c', \mu, V) \). Then, the price of the exchange option to put \( k' \) \( s' \)-bonds and call \( k \) \( s \)-bond is :

\[
E_0 \left[ e^{-\int_0^T r(u)du} (kP_s(T) - k'P_{s'}(T))_+ \right] = kP_s(0) \omega(c' - c - \log(\frac{k'}{k}); B - B', b - b', \frac{\mu - bV}{1 + BV}, \frac{V}{1 + BV}) - k'P_{s'}(0) \omega(c' - c - \log(\frac{k'}{k}); B - B', b - b', \frac{\mu - b'V}{1 + B'V}, \frac{V}{1 + B'V}).
\]
3.4. European call on forward contract

\[ \text{Price}^1 = \mathbb{E}_0 \left[ e^{-\int_0^T r(u) du} (P_s(T) - KP_t(T))_+ \right] = P_t(0) \mathbb{E}^t_0 \left[ \left( \frac{P_s(T)}{P_t(T)} - K \right)_+ \right]. \]

Under the \( t \)-forward risk adjusted measure, we have \(-\log(P_s(T)/P_t(T)) \sim \Omega(B, b, c, \mu, V)\) with \( B = B_s(T) - B_t(T), \quad b = b_s(T) - b_t(T), \quad c = c_s(T) - c_t(T)\) and:

\[ \mu = \sqrt{\frac{\dot{B}_t(0)}{B_t(T)}} x(0) + \frac{\dot{b}_t(0)}{\sqrt{\dot{B}_t(0)\dot{B}_t(T)}} - \frac{\dot{b}_t(T)}{B_t(T)}, \quad \text{and} \quad V = \frac{V_t(0) - V_t(T)}{B_t(T)}. \]

Thanks to (3), we compute the coefficients \( \alpha, \beta \) and \( \lambda \) corresponding to \( B, b, c, \mu \) and \( V \). Then, the price of a call option on the \( t \)-delivery forward contract on the \( s \)-bond is given by:

\[ \mathbb{E}_0 \left[ e^{-\int_0^T r(u) du} (P_s(T) - KP_t(T))_+ \right] = P_s(0) \chi^2(-\alpha - \log(K); \frac{\lambda}{1 + 2\beta}, \frac{\beta}{1 + 2\beta}) - KP_t(0) \chi^2(-\alpha - \log(K); \lambda, \beta). \]

4 Closed formulae for futures and european options on futures

For any \( 0 \leq T \leq t \leq s \), we set \( F_{t,s}(T) \equiv \mathbb{E}_T[P_s(t)] \).

4.1. Futures

\[ \text{Price} : \quad F_{t,s}(0) = \mathbb{E}_0[P_s(t)] \]

Under the risk-neutral measure, \(-\log(P_s(t)) \sim \Omega(B, b, c, \mu, V)\) with \( B = B_t(s), \quad b = b_t(s), \quad c = c_t(s), \) and:

\[ \mu = p_t + q_t x(0), \quad V = v_t, \]

Recall that \( x(0) = \sqrt{2r_0} \). Subsection 8.2.1 of the Appendix gives expressions to compute \( B_{t,s}(0), b_{t,s}(0), c_{t,s}(0), p_t \) and \( q_t \). Then, the price of the \( t \)-delivery futures contract on the \( s \)-bond is given by:

\[ F_{t,s}(0) = \mathbb{E}_0[P_s(t)] = \exp \left( -B_{s,t}(0)x(0)^2 + b_{s,t}(0)x(0) + c_{s,t}(0) \right). \]

---

1Evidently, the option to exchange two bonds as in section 3.4. is equivalent to an option on a bond forward contract, as in section 3.4. We thus have two different formulae for this option which agree under the assumption of the model.
4.2. European call option on futures

Price : $\mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (F_{t,s}(T) - K)_+ \right] = P_0(0) \mathbb{E}^T_0 \left[ (F_{t,s}(T) - K)_+ \right]$

Under the $T$-forward risk adjusted measure, $-\log(F_{t,s}(T)) \sim \Omega(B, b, c, \mu, V)$ with $B = B_{t,s}(T), b = b_{t,s}(T), c = c_{t,s}(T)$ and $\mu, V$ given by (4). The coefficients $B_{t,s}(T), b_{t,s}(T),$ and $c_{t,s}(T)$ are computed from $B_{t,s}(0), b_{t,s}(0),$ and $c_{t,s}(0),$ as described in subsection 6.2.2 of the appendix. Thanks to (3), we compute the coefficient $\alpha, \beta$ and $\lambda$ corresponding to $B, b, c, \mu$ and $V$. Then, the price of a $T$-maturity european call option on the $t$-delivery futures on s-bond is:

$$\mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (F_{t,s}(T) - K)_+ \right] = P_0(0) \mathbb{E}^T_0 [F_{t,s}(T)] \chi^2(-\alpha - \log(K); \frac{\lambda}{1 + 2\beta} , \frac{\beta}{1 + 2\beta})$$

$$-KP_0(0)\chi^2(-\alpha - \log(K); \lambda, \beta),$$

with $\mathbb{E}_0^T [F_{t,s}(0)] = e^{-F_{s,t}(0) - G_{s,t}(0) - H}$. Formula for $F, G$ and $H$ are given in section 8.2.3.

4.3. Delivery option

Price : $\mathbb{E}_0 [\min(kP_s(T), k'P_{s'}(T))]$

Under the risk-neutral measure, we have $-\log(P_s(t)) \sim \Omega(B, b, c, \mu, V)$ and $-\log(P_{s'}(t)) \sim \Omega(B', b', c', \mu, V)$ with $B = B_s(t), b = b_s(t), c = c_s(t), B' = B_{s'}(t), b' = b_{s'}(t), c' = c_{s'}(t),$ and $\mu, V$ given by (4). Then, the price of the $T$-maturity futures contract to deliver the cheapest between $k$ s-bond or $k'$ s'-bond is:

$$\mathbb{E}_0 [\min(kP_s(T), k'P_{s'}(T))] = kF_{t,s}(0) \omega(c - c' + \log(\frac{k'}{k}); B' - B, b' - b, \frac{\mu - b'V}{1 + B'V}, \frac{V}{1 + B'V})$$

$$+ k'F_{t,s'}(0) \omega(c' - c - \log(\frac{k'}{k}); B - B', b - b', \frac{\mu - bV}{1 + BV}, \frac{V}{1 + BV}).$$

Notice that $F_{s,t}(0)$ and $F_{s',t}(0)$ are given by (6).
5 Monte Carlo methods for european options on futures and bonds

For each option, we know that \((x(t), t \geq 0)\) is a gaussian process both under the risk-neutral measure and under the \(T\)-forward risk adjusted measure and we have expression for its mean and variance. Therefore, to compute Monte Carlo methods, we simulate the variable \(x(t)\) and we simply use the relationships between \(x(t)\) and the price of zero-coupon bond, the spot interest rate or the price of a futures.

For options on bonds, caplet, and call on futures, we use the distribution of \(x\) under the \(T\)-forward risk adjusted measure. Indeed, we have, \(x \sim N(\mu, V)\) with \(\mu\) and \(V\) given by (4) and for any pay-off \(X(T)\) at maturity \(T\)
\[
E_0 \left[ e^{-\int_0^T r(s)ds} X(T) \right] = P_T(0)E_0^T[X(T)].
\]
Hence, with a Monte Carlo method, we get the desired price.

For futures and for delivery options (section 4.1 and 4.3), we directly get the desired price by a Monte Carlo method using the distribution of \(x\) in the risk-neutral measure: \(x \sim N(\mu, V)\) is again a gaussian process with \(\mu\), and \(V\) given by (5) for futures and by (4) for delivery options.

6 Algorithms

The functions below are common to all programs:

- **void bond_coeffs(ZCMarketData* ZCMarket, Data *data, double T, double beta, double sigma, double x0);**
  This function computes the coefficients \(B_T(0), b_T(0), c_T(0), \dot{B}_T(0), \dot{b}_T(0)\) and stores them in structure data.
  Integrations are done with means of trapezoidal rule.

- **void transport(Omega *om, Data data1, Data data2, double alpha, double beta, double sigma, double x0)**
  This function computes the coefficients of \(P_s(T)\) knowing those of \(P_T(0)\) (contained in data1) and \(P_s(0)\) (contained in data2).
  Results are stored in om.B, om.b and om.c.

- **void om2chn(Omega om, Chn *chn)**
  Transforms an Omega structure into a Chn structure using equations (3).
7 Results and conclusions

To check the accuracy of the computed prices, we did the following tests.

• First, we checked the put prices computed with closed forms agree with the prices given by Pelsser in [2]. Computed prices are exactly the same.

• Then, to check the efficiency of the quadratic interpolation, we have taken a function for $P_s(0)$ ($P_s(0) = \exp (- s (0.08 - 0.05e^{-0.18s}))$) and we have discretized it successively with a time-step of 0.05 and a time-step of 0.25. Prices for each kind of options are nearly the same: the error is always lower than 1 basis point.

• We also checked the prices using Monte Calo method. We launched the program 1000 times and around 95% of computed 95-per-cent-confidence intervals contain the closed form price.

• We also passed the following test: for $\alpha$ given and constant, we have computed prices of zero-coupon bond for several maturity. These prices were stored in the file initialyield.dat. Then, we launched the program for $\alpha$ time-dependent and we checked if computed prices with these values for bond were the same than for $\alpha$ constant. The prices are always the same with an error lower than one basis point.

8 Appendix

In all the following equations, we set $\gamma = \sqrt{\beta^2 + \sigma^2}$.

6.1. Bond coefficients

• 6.1.1. Equations to compute initial values of bond coefficients

For all $s$ and $t$:

$$B_s(0) = \frac{e^{2\gamma s} - 1}{(\gamma + \beta)e^{2\gamma s} + \gamma - \beta}$$
If $\alpha$ is constant, we have closed forms for $b_s(0)$ and $\dot{c}_s(0)$:\n\n\[ h(s) = (\gamma + \beta)e^{2\gamma s} + \gamma - \beta)^{-1}, \]
\[ b_s(0) = \frac{\alpha}{\gamma} h(s)(e^{\gamma s} - 1)^2, \]
\[ \dot{c}_s(0) = \alpha b_s(0) + \frac{1}{2}\sigma^2 B_s(0) - \frac{1}{2}\sigma^2 b_s^2(0). \]

Else, for all $s$, the forward interest rate at time $t = 0$ is:\n\[ r_s(0) = -\frac{\partial \log(P_s(0))}{\partial s}. \]

Then, we have:\n\[ \dot{b}_s(0) = \dot{B}_s(0)x(0) + \sqrt{B_s(0)(2r_s(0) - \frac{1}{2}\sigma^2 B_s(0))} \]
\[ \dot{c}_s(0) = \frac{1}{2}\left( \frac{(\dot{b}_s(0))^2}{B_s(0)} + \sigma^2 B_s(0) \right) \]
\[ \alpha(t) = (\dot{B}_t(0))^{-3/2}(\dot{B}_t(0)\dot{b}_t(0) - \dot{B}_t(0)\dot{b}_t(0)). \]

- **6.1.2. Transport equations for bond coefficients**

For all $T$, and $s$, we have:\n\[ B_s(T) = \frac{B_s(0) - B_T(0)}{B_T(0) - \sigma^2 B_T(0)(B_s(0) - B_T(0))}, \]
\[ \dot{B}_s(T) = \frac{\dot{B}_s(0)\dot{B}_T(0)B_s^2(T)}{(B_s(0) - B_T(0))^2}, \]
\[ b_s(T) = B_s(T)\sqrt{B_s(T)}\left( \frac{b_s(0) - b_T(0)}{B_s(0) - B_T(0)} - \frac{b_T(0)}{B_T(0)} \right), \]
\[ \dot{b}_s(T) = \frac{\dot{b}_s}{\sqrt{B_T(0)}}\left( 1 + \sigma^2 B_T(0)b_s(T) + \dot{B}_s(T)\left( \sigma^2 B_T(0)(b_s(0) - b_T(0)) - \dot{b}_T(0) \right) \right), \]
\[ c_s(T) = c_s(0) - c_T(0) - \dot{c}(B_s(T), b_s(T), \dot{b}_T(0))/\sqrt{B_T(0)}, \sigma^2 B_T(0)). \]

with $\dot{c}(B, b, a, V) = \frac{1}{2}\left( \log(1 + BV) + \frac{Ba^2 + 2ab - Vb^2}{1 + BV} \right)$.

---

**6.2. Futures coefficients**

\(^2\)There is a misprint in the formula given by Jamshidian in [1] for $\dot{c}_s(0)$ which is corrected here.
• 6.2.1. Equations to compute initial values of futures coefficients

For all $s$ and $t$, we set:

$$p_t = \int_0^t \alpha(u)e^{-\beta(t-u)}du, \quad q_t = e^{-\beta t}, \quad v_t = \frac{\sigma^2(1-e^{-2\beta t})}{2\beta}.$$  

Then, for all $s$ and $t$, we have:

$$B_{t,s}(0) = \frac{q_t^2 B_s(t)}{1 + B_s(t)v_t},$$

$$b_{t,s}(0) = \frac{q_t(b_s(t) + B_s(t)p_t)}{1 + B_s(t)v_t},$$

$$c_{t,s}(0) = c_s(t) + \frac{1}{2} \log(1 + v_t B_s(t)) + \frac{B_s(t)p_t^2 + 2b_s(t)p_t - v_t q_t^2}{2(1 + v_t B_s(t))}.$$  

• 6.2.2. Transport equations for futures coefficients

For all $s$, $t$ and $T$, we set:

$$B_{t,s}(T) = \frac{B_{t,s}(0)}{q_T^2 - v_T B_{t,s}(0)},$$

$$b_{t,s}(T) = B_{t,s}(T) \left( \frac{b_{t,s}(0)}{B_{t,s}(0)} q_T - p_T \right),$$

$$c_{t,s}(T) = c_{t,s}(0) - \frac{1}{2} \log(1 + v_T B_{t,s}(T)) + \frac{B_{t,s}(T)p_T^2 + 2b_{t,s}(T)p_T - v_T q_T^2}{2(1 + v_T B_{t,s}(T))}.$$  

For all $t$, $s$ and $T$, we set:

• 6.2.3. Other formulas for futures

For all $s$, $t$ and $T$, we set:

$$p_t = \sqrt{B_T(0)} \quad \text{and} \quad q_t = \frac{b_T(0)}{\sqrt{B_T(0)}},$$

Then, we have$^3$: $E_T^0 [F_{t,s}(0)] = e^{-Fx^2(0)-Gx(0)-H}$ with:

$$F = \frac{q_T^2 B_{t,s}(T)}{1 + B_{t,s}(T)v_T},$$

$$G = \frac{q_T(b_{t,s}(T) + B_{t,s}(T)p_T)}{1 + B_{t,s}(T)v_T},$$

$$H = c_{t,s}(T) + \frac{1}{2} \log(1 + v_T B_{t,s}(T)) + \frac{B_{t,s}(T)p_T^2 + 2b_{t,s}(T)p_T - v_T q_T^2}{2(1 + v_T B_{t,s}(T))}.$$  

$^3$We recall that $x(0) = \sqrt{2r(0)}$. 


References

