Pricing and hedging American-style options: a simple simulation-based approach, by Y. Wang and R. Caflisch

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1 Discrete optimal stopping problem

Let us consider a discrete non-negative random process \((H_i)_{0 \leq i \leq k}\) adapted to a filtration \((\mathcal{F}_i)_{0 \leq i \leq k}\). We suppose that this process is a function of another underlying process \(X_i\): \(H_i = h_i(X_i)\) for some function \(h_i\).

We are interested in the problem of finding a stopping time that maximizes the \(\mathbb{F}_i\)-conditional expectation \(\mathbb{E}_i[H_\tau]\) over stopping time \(\tau\) taking value in \(\{i, ..., k\}\)

\[
\sup_{\tau \in \{i, ..., k\}} \mathbb{E}_i[H_\tau] \tag{1}
\]

We call an optimal stopping time for this problem an \(\mathcal{F}_i\)-stopping time \(\tau^*_i\) such that

\[
\mathbb{E}_i[H_{\tau^*_i}] = \sup_{\tau \in \{i, ..., k\}} \mathbb{E}_i[H_\tau]
\]
We define the process $Q^*_i := E_i[H_{\tau^*_i}]$. The process $(Q^*_i)_{0 \leq i \leq k}$ is called the Snell envelope process of $(H_i)_{0 \leq i \leq k}$. It can be constructed using the well known backward dynamic programming algorithm. In fact, by definition we have $Q^*_k = H_k$

$$
\begin{cases}
Q^*_k &= H_k \\
Q^*_i &= \max (H_i, E_i[Q^*_{i+1}]) 
\end{cases}
\quad (2)
$$

Then $\tau^*_i$ can be represented by

$$
\tau^*_i = \inf \{j, i \leq j \leq k : H_j \geq E_j[Q^*_{j+1}]\} = \inf \{j, i \leq j \leq k : H_j \geq E_j[H_{\tau^*_j+1}]\}
$$

or

$$
\begin{cases}
\tau^*_k &= k \\
\tau^*_i &= i I_A + \tau^*_{i+1} I_{A^c}
\end{cases}
\quad (3)
$$

where $A = \{H_i \geq E_i[H_{\tau^*_i+1}]\}$

## 2 Regression method to price American option: Longstaff-Schwartz algorithm

The optimal stopping time family $\{\tau^*_i\}_{i}$ (or equivalently the price process $(Q^*_i)$) can be estimated via plain Monte Carlo simulation of the backward dynamic program (3). But this raises the problem of how to estimate the conditional expectation $E_i[H_{\tau^*_i+1}]$. Using Monte Carlo method to estimate this latter implies nested simulations, which makes the algorithm potentially very slow.

In fact, to estimate $Q^*_i = \max (H_i, E_i[Q^*_{i+1}])$, we need the conditional expectation $E_i[Q^*_{i+1}]$: We estimate this latter by sampling conditional on the state at time $i$, hence the nested simulations.

One possible way to circumvent this problem is the well known Longstaff-Schwartz algorithm. We note the price given by this algorithm by $Q^{LS}_i$.

The idea is to estimate the conditional expectations $E_i[Q^{LS}_{i+1}]$ using some regression methods, without any nested simulation. In fact, in the case of a Markovian setting, the conditional expectation is a function of the underlying process $X_i$

$$E_i[Q^{LS}_{i+1}] = \phi_i(X_i)$$

The function $\phi_i$ is then approximated by an orthogonal projection $\langle \alpha_i, g_i \rangle$. We then choose $\alpha_i$ as the one that minimize the second order moment
\[
\begin{align*}
\min_{\alpha_i \in \mathbb{R}^d} \mathbb{E} \left[ (Q_{i+1}^{LS} - \langle \alpha_i, g_i \rangle(X_i))^2 \right]
\end{align*}
\]

Using Monte Carlo method, this minimization problem can be rewritten, for a set of \(M\) simulated paths \(\{w_1, w_2, ..., w_M\}\):

\[
\begin{align*}
\min_{\alpha_i \in \mathbb{R}^d} \sum_m \left[ Q_{i+1}^{LS}(w_m) - \langle \alpha_i, g_i \rangle(X_i(w_m)) \right]^2
\end{align*}
\]

Then \(Q_i^{LS}\) is estimated by \(Q_i^{LS}(w_m) = \max(h_i(X_i(w_m)), \langle \alpha_i, g_i \rangle(X_i(w_m)))\).

The stopping time given by this method is

\[
\tau_i^{LS} = \inf \{j, i \leq j \leq k : h_j(X_j) \geq \langle \alpha_j, g_j \rangle(X_j) \}
\]

for \(1 \leq i \leq k\). At initial time \(i = 0\), the continuation value is just the plain expectation \(\mathbb{E}_0 \left[ Q_1^{LS} \right]\), so we don’t need regression to evaluate it. The option price at initial time is then given by \(Q_0^{LS} = \max \left( H_0, \mathbb{E}_0 \left[ Q_1^{LS} \right] \right)\).

However, this method gives only an approximation of the price, and no information about the sensitivities of American option, also called the greeks. The method proposed by [Wang, Caflisch 2009], based on the Least Squares approximation of conditional expectations by [Longstaff, Schwartz 2001], gives an estimation of these sensitivities.

3 Modified Longstaff-Schwartz algorithm to estimate the greeks

The approach proposed in [Wang, Caflisch 2009] is based on the least squares regression method in Longstaff-Schwartz algorithm (LS). We will note this algorithm (MLSM) as in [Wang, Caflisch 2009]. The key idea is to use a random set of initial stock price and approximate the option price function at initial date by an additional regression. This means that instead of having \(X_0\) as deterministic, we consider that it has some pre-defined distribution.

Hence, by using the Longstaff-Schwartz algorithm, we get the option price at initial time 0 on this random set of initial stock price, then by an additional regression we can have an option price functional approximation of the form \(Q_0^{MLSM}(X) = \langle \alpha_0, g_0 \rangle(X)\). Therefore, to obtain option price and its the sensitivities to \(X\), we just evaluate the function \(\langle \alpha_0, g_0 \rangle\) at the actual initial stock price \(X_0\) and use the explicit derivative of
this basis function \( \frac{\partial \langle \alpha_0, g_0 \rangle}{\partial X} \).

We can get for instance the option price, the delta and the gamma by

\[
Q_0^{MLSM}(X_0) = \langle \alpha_0, g_0 \rangle(X_0)
\]

\[
\Delta_0^{MLSM}(X_0) = \frac{\partial Q_0^{MLSM}}{\partial X}(X_0) = \frac{d\langle \alpha_0, g_0 \rangle}{dX}(X_0)
\]

\[
\Gamma_0^{MLSM}(X_0) = \frac{\partial^2 Q_0^{MLSM}}{\partial X^2}(X_0) = \frac{d^2\langle \alpha_0, g_0 \rangle}{dX^2}(X_0)
\]

As noted in [Wang, Caflisch 2009], this algorithm can be interpreted as starting the stock price process from a date before initial time 0.

**Numerical implementation**

- Taking into account the previous remark, we generate the random set of initial stock price from an initial distribution of the form

\[
X(w, t = 0) = X_0 e^{-\alpha \sigma^2 + \sigma \sqrt{\alpha} w}, \quad w \sim \mathcal{N}(0, 1)
\]

where \( \sigma, T \) are the corresponding stock volatility and option maturity. \( \alpha \) should be used to adjust the variance of the distribution. We note that the mean of \( X(w, t = 0) \) is \( X_0 \).

- For the regression step, we used the least squares algorithm implemented in the Premia Numerical Library (PNL) via the function \texttt{pnl\_basis\_fit\_ls}. It permits to use canonical, Hermite and Tchebychev polynomials as regression basis and provides also the derivatives value.
References
