Hedging with options in models with jumps

Rama Cont, Peter Tankov, Ekaterina Voltchkova

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Abstract

We consider the problem of hedging a contingent claim, in a market where prices of traded assets can undergo jumps, by trading in the underlying asset and a set of traded options. We give a general expression for the hedging strategy which minimizes the variance of the hedging error, in terms of integral representations of the options involved. This formula is then applied to compute hedge ratios for common options in various models with jumps, leading to easily computable expressions. The performance of these hedging strategies is assessed through numerical experiments.

Keywords: quadratic hedging, option pricing, barrier option, integro-differential equations, Markov processes with jumps, Lévy process.

1 Introduction

The Black-Scholes model and generalizations of it where the dynamics of prices $X_t = (X_t^1, \dots, X_t^m)$ of several assets is described by a diffusion process driven by Brownian motion

$$dX_t = X_t \sigma(t, X_t) dW_t + X_t \mu_t dt \tag{1}$$

have strongly influenced risk management practices in derivatives markets since the 1970s. In such models, the question of hedging a given contingent claim with payoff Y paid at a future date T can be theoretically tackled via a representation theorem for Brownian martingales: by switching to a (unique) equivalent martingale measure Q, we obtain a unique self-financing strategy ϕ_t such that

$$Y = E^Q[Y|\mathcal{F}_0] + \int_0^T \phi_t dX_t \quad Q - a.s. \tag{2}$$

This representation then holds almost surely under any measure equivalent to Q, thus yielding a strategy ϕ_t with initial capital $c = E^Q[Y|\mathcal{F}_0]$ which "replicates" the terminal payoff Y almost-surely. On the computational side, ϕ_t can be computed by differentiating the option price $C(t, S_t) = E^Q[Y|\mathcal{F}_t]$ with respect to the underlying asset(s) X_t. These ideas are central to the use of diffusion models in option pricing and hedging.

Stochastic processes with discontinuous trajectories are being increasingly considered, both in the research literature and in practice, as realistic alternatives to the Black-Scholes model and its diffusion-based generalizations. A natural question is therefore to examine what becomes of the above assertions in presence of discontinuities in asset prices. It is known that, except in very special cases [25], martingales with respect to the filtration of a discontinuous process X cannot be represented in the form (2), leading to *market incompleteness.* Far from being a shortcoming of models with jumps, this property corresponds to a genuine feature of real markets: the impossibility of "replicating" an option by trading in the underlying asset.

A natural extension, due to Föllmer and Sondermann [18], has been to *approximate* the target payoff Y by optimally choosing the initial capital c and a self-financing trading strategy $(\phi_t^1, \ldots, \phi_t^m)$ in the assets $X^1, \ldots, X^m$ in order to minimize the quadratic hedging error [7, 18]:

$$\text{minimize} \quad E\left(c + \sum_{i=1}^{m} \int_0^T \phi_t^i dX_t^i - Y\right)^2, \tag{3}$$

Unlike approaches based on other (non-quadratic) loss functions, quadratic hedging has the (great) advantage of yielding linear hedging rules, which correspond to observed market practices.

The expectation in (3) can be understood either as being computed under an "objective" measure meant as a statistical model of price fluctuations [2, 17, 19, 27] or as being computed under a martingale ("risk-adjusted") measure [6, 7, 16, 18, 24]. Whereas the first choice may seem more natural, there are practical *and* theoretical motivations for using a risk-adjusted (martingale) measure fitted to market prices of options for computing the hedging performance.

- When X is a martingale, problem (3) is related to the *Kunita-Watanabe decomposition* of Y, which has well-known properties guaranteeing the existence of a solution under mild conditions [22]. By contrast, quadratic hedging with discontinuous processes under an arbitrary measure may lead to negative "prices" or not have a solution in general [2].

- Ideally, the probability measure used to compute expectations in (3) should reflect future uncertainty over the lifetime of the option. When using the "statistical" measure as estimated from historical data, this only holds if increments are stationary. On the other hand, the risk-adjusted measure retrieved from quoted option prices using a "calibration" procedure [4, 9, 10] is naturally interpreted as encapsulating the market anticipation of future scenarios.

- More generally, the use of "statistical" measures of risk such as variances or quantiles computed with "statistical" models has been questioned by Aït-Sahalia and Lo [1], who advocate instead the use of corresponding quantities computed using a risk-adjusted measure, estimated non-parametrically from prices of options observed in the market. These quantities, they argue, not only reflect probabilities of occurrence but also the risk premia attached to them by the market so are more natural as criteria for measuring risk.

The purpose of this work is to study the quadratic hedging problem (3) when underlying asset prices are modeled by a process with jumps. In accordance with the above remarks, we will assume that the expectation in (3) is computed using a martingale measure estimated from observed prices of options. With respect to the existing literature, our contribution can be seen as follows:

- Though quadratic hedging with the underlying asset in presence of jumps has been previously studied by several authors, the corresponding expressions for hedging strategies are not always explicit and involve for instance the carré-du-champ operator [7], the Malliavin derivative [6] or various Laplace transforms and path-dependent quantities [20].
- While previous work has focused on hedging with the underlying asset(s), we will see in section 4, switching from naive delta-hedging to the optimal quadratic hedging strategy reduces the risk only marginally and leads to an important residual risk. By contrast, we study hedging strategies combining underlying assets with a set of available options that allow to further reduce the residual risk.
- While previous work has exclusively focused on European options without path-dependence (calls and puts), hedging exotic options is often more important in practice than hedging call and puts. We provide easy to compute expressions for hedge ratios for Asian and barrier options.
- We implement numerically the proposed hedging strategies and compare their performance based on Monte Carlo simulations.

The paper is structured as follows. In Section 2 we derive a general expression for the strategy which minimizes the variance of the hedging error, as computed under a risk-adjusted measure, in the general framework of Itô processes with jumps. Section 3 explains how the problem of hedging with options fits into the framework described in section 2: we provide sufficient conditions under which the prices of various options possess the representation needed to apply the hedging formula. Finally, in Section 4 we apply the general hedging formula of Section 2 to construct hedging strategies for some common options and give numerical examples of their performance.

2 Minimal variance hedging in the jump-diffusion framework

2.1 Model setup

Consider a d-dimensional Brownian motion W and a Poisson random measure J on $[0, \infty) \times \mathbb{R}$ with intensity measure $dt \times \nu(dx)$ defined on a probability space

$(\Omega, \mathcal{F}, P)$, where ν is a positive measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$. $\tilde{J}$ denotes the compensated version of J:

$$\tilde{J}(dt \times dz) = J(dt \times dz) - dt\nu(dz).$$

Let $(\mathcal{F}_t)_{t\geq 0}$ stand for the natural filtration of W and J completed with null sets.

We consider a market consisting of m traded assets $X^i,\ i = 1, \ldots, m$ that can be used for hedging a contingent claim $Y \in \mathcal{F}_T$ with $E[Y^2] < \infty$. We suppose that the prices of traded assets are expressed using the money market account, continuously compounded at the risk-free rate, as numeraire. We assume that, using market prices of options, we have identified a pricing measure under which the prices of traded assets $X^1, \ldots, X^m$ are local martingales. This can be done using for instance methods described in [4, 9]. The evolution of prices under this probability measure will be described by the following stochastic integrals:

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_{[0,t]\times\mathbb{R}} \gamma_s(z)\tilde{J}(ds \times dz). \tag{4}$$

We denote $Y_t = E[Y|\mathcal{F}_t]$ the value of the option and assume that Y_t can be represented by a stochastic integral:

$$Y_t = Y_0 + \int_0^{t\wedge\tau} \sigma^0_s dW_s + \int_{[0,t\wedge\tau]\times\mathbb{R}} \gamma^0_s(z)\tilde{J}(ds \times dz). \tag{5}$$

The initial values X_0 and Y_0 are deterministic, τ is a stopping time which denotes the (possibly random) termination time of the contract (to account for path-dependent features such as barriers). Such a representation can be formally obtained by expressing the option price $Y_t = f(t, X_t)$ applying an Ito formula to the function f. In section 3, we will give various condition under which such a representation can indeed be derived, the main obstacle being the smoothness of f.

We assume the coefficients satisfy the following assumptions:

(i) $\sigma : [0,\infty) \to \mathbb{R}^m \otimes \mathbb{R}^d$ and $\sigma^0 : [0,\infty) \to \mathbb{R}^d$ are càglàd $\mathcal{F}_t$-adapted processes.

(ii) $\gamma : [0,\infty) \times \mathbb{R} \to \mathbb{R}^m$ and $\gamma^0 : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ are càglàd $\mathcal{F}_t$-adapted processes such that

$$\forall t \in [0,T], \forall z \in \mathbb{R}^d, \|\gamma_s(z)\|^2 \leq \rho(z)A_s \qquad \text{and} \qquad |\gamma^i_s(z)\gamma^0_s(z)| \leq \rho(z)A_s$$

hold almost surely for some finite-valued adapted process A and some deterministic function ρ satisfying $\int_{\mathbb{R}} \rho(z)\nu(dz) < \infty$.

(iii) We fix a time horizon T and assume

$$E\int_0^T (\|\sigma_s\|^2 + A_s)ds < \infty$$

These assumptions imply in particular that the stochastic integrals (4)–(5) exist and define square-integrable martingales. Below we give several examples of stock price models satisfying (4) and the assumptions (i)–(iii) and in section 3 we will show that European and many exotic options can indeed be represented in the form (5).

Example 1 (Exponential Lévy models)**.** Let L be a Lévy process with characteristic triplet (σ, ν, γ). For e^L to be a martingale, the characteristic triplet must satisfy

$$\int_{|y|>1} e^y \nu(dy) < \infty, \quad \text{and} \quad \gamma + \frac{\sigma^2}{2} + \int (e^y - 1 - y 1_{|y|\leq 1})\nu(dy) = 0.$$

In this case, $X_t = X_0 e^{L_t}$ satisfies the following stochastic differential equation:

$$X_t = X_0 + \int_0^t \sigma X_s dW_s + \int_{[0,t]\times\mathbb{R}} X_{s-}(e^z - 1)\tilde{J}_L(ds \times dz),$$

where $\tilde{J}_L$ is the compensated jump measure of L. From the Lévy-Khinchin formula,

$$E[X_t^2] = X_0^2 \exp\{t\sigma^2 + t\int_{\mathbb{R}} (e^x - 1)^2 \nu(dx)\},$$

hence, X_t is square integrable if $\int_{|x|\geq 1} e^{2x}\nu(dx) < \infty$.

Example 2 (Markov jump diffusions)**.** Let $\sigma : [0, \infty) \times \mathbb{R}^m \to \mathbb{R}^m \otimes \mathbb{R}^d$ and $\gamma : [0, \infty) \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ be deterministic functions satisfying the conditions of Lipschitz continuity and sublinear growth (see [23, Theorem III.2.32]). Then the following stochastic differential equation

$$X_t = X_0 + \int_0^t \sigma(s, X_{s-})dW_s + \int_{[0,t]\times\mathbb{R}} \gamma(s, z, X_{s-})\tilde{J}(ds \times dz), \tag{6}$$

admits a unique strong solution X satisfying the assumptions (i)–(iii) above. Processes of this type are referred to as (martingale) Markov jump diffusions.

Some authors define jump-diffusions by allowing the intensity measure ν to depend on the state [14]. However, whenever the intensity measure ν in Equation (6) is infinite and has no atom, a model with state-dependent intensity measure can be transformed to the form (6) by choosing appropriate coefficients [21, Theorem 14.80].

Example 3 (Barndorff-Nielsen and Shephard stochastic volatility model)**.** Under the martingale probability the stochastic volatility model proposed by Barndorff-Nielsen and Shephard [3] has the following form:

$$dY_t = (-l(\rho) - \frac{1}{2}\sigma_t^2)dt + \sigma_t dW_t + \rho dZ_t \tag{7}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ_t, \quad \sigma_0^2 > 0 \tag{8}$$

where $l(\theta) = \log E(e^{\theta Z_1})$, $\rho \le 0$, $\lambda > 0$ are constant parameters, W is a standard Brownian motion and Z is a subordinator without drift, independent from W. The stock price process $X_t = X_0 e^{Y_t}$ satisfies the following :

$$X_t = X_0 + \int_0^t \sigma_s X_s dW_s + \int_0^t \int_{\mathbb{R}} X_{s-}(e^{\rho z} - 1)\tilde{J}(ds \times dz),$$

where $\tilde{J}$ is the compensated jump measure of Z. To check the integrability of X, we use the formula for the Laplace transform of Y_t [8, p. 489]:

$$E[X_t^2] = X_0^2 E[e^{2Y_t}] = X_0^2 \exp\left(-2l(\rho)t + \sigma_0^2 \varepsilon(\lambda, t) + \int_0^t l(2\rho + \varepsilon(\lambda, t-s))ds\right)$$

where $\varepsilon(\lambda, t) = \frac{1-e^{-\lambda t}}{\lambda}$. A sufficient condition for this to be finite is

$$l(2\rho + 1/\lambda) < \infty \quad \Longleftrightarrow \quad \int_1^\infty e^{(2\rho+1/\lambda)x}\nu(dx) < \infty,$$

where ν is the Lévy measure of Z. Under this condition Barndorff-Nielsen and Shephard's stochastic volatility model satisfies the hypotheses (i)–(iii) above.

2.2 Minimal variance hedging

Consider an agent who has sold at $t = 0$ the contingent claim with terminal payoff Y for the price c and wants to hedge the associated risk by trading in assets $(X^1, \dots, X^m) = X$. We call an *admissible hedging strategy* a predictable process $\phi : \Omega \times [0, T] \to \mathbb{R}^m$ such that $\int_0^{\cdot} \phi_t dX_t$ is a square integrable martingale. Denote by A the set of such strategies. The residual hedging error of $\phi \in A$ at time T is then given by

$$\epsilon_T(c, \phi) = c - Y + \int_0^T \phi_t dX_t. \tag{9}$$

Proposition 1. *Let X and Y be as in* (4) *and* (5) *satisfying the hypotheses (i)–(iii) on page 4 and suppose in addition that the matrix*

$$M_t = \sigma_t \sigma_t^* + \int_{\mathbb{R}} \nu(dz)\gamma_t(z)\gamma_t(z)^*$$

is almost surely nonsingular for all $t \in [0, T]$, where the star denotes the matrix transposition. Then the minimal variance hedge $(\hat{c}, \hat{\phi})$, solution of

$$E[\epsilon_T(\hat{c}, \hat{\phi})^2] = \inf_{(c,\phi) \in \mathbb{R} \times A} E[(\epsilon_T(c, \phi))^2]$$

is given by

$$\hat{c} = E[Y] = Y_0 \tag{10}$$

$$\hat{\phi}_t = M_t^{-1}\left(\sigma_t^0 \sigma_t^* + \int_{\mathbb{R}} \nu(dz)\gamma_t^0(z)\gamma_t(z)^*\right) 1_{[0,\tau]}(t). \tag{11}$$

Proof. First, for every admissible strategy ϕ,

$$E[(\epsilon_T(\phi))^2] = (c - E[Y])^2 + E\left(E[Y] - Y + \int_0^T \phi_t dX_t\right)^2.$$

This shows that the initial capital is given by $\hat{c} = E[Y]$. Substituting $c = \hat{c}$ yields

$$\begin{aligned}
E[\epsilon(\phi)_T^2] = & \int_0^{T\wedge\tau} E\|\phi_t\sigma_t - \sigma_t^0\|^2 dt + \int_{T\wedge\tau}^T E\|\phi_t\sigma_t\|^2 dt \\
& + \int_0^{T\wedge\tau} dt \int_{\mathbb{R}} \nu(dz) E(\phi_t\gamma_t(z) - \gamma_t^0(z))^2 \\
& + \int_{T\wedge\tau}^T dt \int_{\mathbb{R}} \nu(dz) E(\phi_t\gamma_t(z))^2.
\end{aligned}$$

This expression is clearly minimized by the strategy $\hat{\phi}$. Moreover, under the assumptions of this proposition, almost surely, $(\hat{\phi}_t)_{0\le t\le T}$ is *càglàd* and therefore admissible. □

Remark 1. The left-continuity of hedging strategies in other settings, in particular when explicit representations are not available, is discussed in [24].

Remark 2 (Tikhonov regularization)**.** Although in the above result we suppose that the matrix M_t is nonsingular, in some cases it may be badly conditionned leading to numerically unstable results. To avoid this problem, one can regularize M by adding to it some fraction of the unit matrix: this corresponds to minimizing

$$J(\phi) = E[(\epsilon_T(\phi))^2] + \alpha E\int_0^T \|\phi_t\|^2 dt$$

for some $\alpha > 0$. It is easy to check that the solution to the minimization problem is then given by

$$\hat{\phi}_t^{reg}(\omega) = \{M_t + \alpha I\}^{-1} \times \left(\sigma_t^0\sigma_t^* + \int_{\mathbb{R}} \nu(dz)\gamma_t^0(z)\gamma_t(z)^*\right).$$

This procedure is also equivalent to adding α to each eigenvalue of M. Following the literature on regularization of inverse problems [15], we choose the regularization parameter α in such way that the hedging error with the regularized strategy $E[(\epsilon_T(\hat{\phi}^{reg}))^2]$ is at its highest acceptable level.

3 Martingale representations for option prices

In this section we obtain martingale representations of type (4) for the prices of various options. This will allow us to apply the general formula for hedge ratios (11) in the case when the asset to be hedged and / or the traded assets used for hedging are options on other assets.

To obtain explicit formulas for martingale representations, we assume that the price process X is a Markov process of the form (6). When we need to mention explicitly the starting value of a Markov process, we denote by $(X_t^x)_{t\geq 0}$ the process started from the initial value $X_0 = x$ and by $(X_t^{(\tau,x)})_{t\geq\tau}$ the same process started from the value $X_\tau = x$ at time $t = \tau$.

In some cases (Asian options, stochastic volatility, ...), one has to introduce additional non-traded factors $\tilde{X} \in \mathbb{R}^{\tilde{m}}$ such that the extended state process $(X, \tilde{X})$ is Markovian:

$$
\begin{aligned}
X_t &= X_0 + \int_0^t \sigma(s, X_{s-}, \tilde{X}_{s-}) dW_s + \int_{[0,t]\times\mathbb{R}} \gamma(s, z, X_{s-}, \tilde{X}_{s-}) \tilde{J}(ds \times dz), \\
\tilde{X}_t &= \tilde{X}_0 + \int_0^t \tilde{\mu}(s, X_s, \tilde{X}_s) ds + \int_0^t \tilde{\sigma}(s, X_{s-}, \tilde{X}_{s-}) dW_s \\
&\quad + \int_{[0,t]\times\{z:|z|\leq 1\}} \tilde{\gamma}(s, z, X_{s-}, \tilde{X}_{s-}) \tilde{J}(ds \times dz) \\
&\quad + \int_{[0,t]\times\{z:|z|>1\}} \tilde{\gamma}(s, z, X_{s-}, \tilde{X}_{s-}) J(ds \times dz).
\end{aligned}
$$

Note that the components of $\tilde{X}$ are not necessarily martingales because they do not represent prices of tradables. For simplicity, unless otherwise mentioned, in the rest of this section we assume there are no non-traded factors and that the price process is one-dimensional ($m = 1$). We treat separately the case of general Markov jump diffusions and the case of Lévy processes.

3.1 European options

Let X be defined by (6) and H be a measurable function with $E[H(X_T)^2] < \infty$. The price of a European-type contingent claim is then a deterministic function of time t and state X_t:

$$
C_t = E[H(X_T)|\mathcal{F}_t] = C(t, X_t), \tag{12}
$$

where $C(t,x) = E[H(X_T^{(t,x)})]$.

Suppose that the option price $C(t,x)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x. The Itô formula can then be applied to show that the price of a European option satisfies a stochastic differential equation of type (6):

$$
\begin{aligned}
dC_t &= \frac{\partial C(t,X_t)}{\partial x}\sigma(t,X_t)dW_t + \int_{\mathbb{R}} (C(t, X_{t-} + \gamma(t,z,X_{t-})) - C(t,X_{t-}))\tilde{J}(dt \times dz) \\
&+ \left\{ \frac{\partial C(t,X_t)}{\partial t} + \frac{1}{2}\sigma(t,X_t)^2 \frac{\partial^2 C(t,X_t)}{\partial x^2} \right. \\
&+ \left. \int_{\mathbb{R}} \left(C(t, X_t + \gamma(t,z,X_t)) - C(t,X_t) - \gamma(t,z,X_t)\frac{\partial C(t,X_t)}{\partial x} \right) \nu(dz) \right\} dt
\end{aligned} \tag{13}
$$

Note that the first line of the above expression is a local martingale. On the other hand, from (12), C_t itself is a martingale. Therefore, the sum of the second and the third line is a finite variation continuous local martingale. This means that it is zero and we obtain the following martingale representation for C_t:

$$C(T,X_T) \equiv H(X_T) = C(0,X_0) + \int_0^T \frac{\partial C(t,X_t)}{\partial x}\sigma(t,X_t)dW_t \\ + \int_0^T \int_{\mathbb{R}} (C(t,X_{t-} + \gamma(t,z,X_{t-})) - C(t,X_{t-}))\tilde{J}(dt \times dz). \quad (14)$$

Despite the simplicity of this heuristic argument, a rigorous proof of this formula requires some work. We start with the case of Lévy processes and discontinuous payoffs.

Proposition 2. *Let H be a measurable function with at most polynomial growth: $\exists p \geq 0$, $|H(x)| \leq K(1+|x|^p)$, and X be a Lévy process with characteristic triplet (σ,ν,γ) satisfying the following conditions:*

$$(i) \qquad \sigma > 0 \quad or \quad \exists \beta \in (0,2), \quad \liminf_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^{2-\beta}} \int_{-\varepsilon}^{\varepsilon} |y|^2 \nu(dy) > 0; \quad (15)$$

$$(ii) \qquad \int_{|y|>1} |y|^{p+1}\nu(dy) < \infty. \quad (16)$$

Denote $X_t^x \equiv x + X_t$. Then

1. *The European option price $C(t,x) = E[H(X_{T-t}^x)]$ belongs to the class $C^\infty([0,T) \times \mathbb{R})$ with $\left|\frac{\partial^{n+m} C}{\partial x^n \partial t^m}(x)\right| \leq K(1+|x|^p)$, for all $n,m \geq 0$.*[1]

2. *Suppose in addition that the set of discontinuities of H has Lebesgue measure zero and that*

$$\int_{|y|>1} |y|^{2p}\nu(dy) < \infty. \quad (17)$$

Then the process $(C(t,X_t^x))_{0 \leq t \leq T}$ is a square integrable martingale with the following representation

$$C(t,X_t^x) = C(0,x) + \int_0^t \frac{\partial C(s,X_s^x)}{\partial x}\sigma dW_s \\ + \int_0^t \int_{\mathbb{R}} (C(s,X_{s-}^x + z) - C(s,X_{s-}^x))\tilde{J}(ds \times dz). \quad (18)$$

Proof. Part 1. Let $\phi_t(u) = E[e^{iuX_t}]$. Condition (15) implies

$$|\phi_t(u)| \leq K_1 \exp(-K_2|u|^\alpha) \quad (19)$$

[1] Here and in the proof, K denotes a constant which may depend on n, m et τ and vary from line to line.

for some positive constants K_1, K_2, α and all $t > 0$, and therefore that X_t has a C^∞ density $p_t(x)$ for all $t > 0$. For $\sigma > 0$ this is straightforward and for $\sigma = 0$ see [26, proposition 28.3].

The derivatives of C can now be estimated as follows (we denote $\tau = T - t$)

$$\begin{aligned}
\left|\frac{\partial^{n+m} C(t,x)}{\partial x^n \partial t^m}\right| &\leq \int |H(x+z)| \left|\frac{\partial^{n+m} p_\tau(z)}{\partial z^n \partial \tau^m}\right| dz \\
&\leq K \int (1 + |x+z|^p) \left|\frac{\partial^{n+m} p_\tau(z)}{\partial z^n \partial \tau^m}\right| dz \\
&\leq K(1+|x|^p) \left\|(1+|z|^p)\frac{\partial^{n+m} p_\tau(z)}{\partial z^n \partial \tau^m}\right\|_{L^1} \\
&\leq K(1+|x|^p) \left\|\frac{1}{1+|z|}\right\|_{L^2} \left\|(1+|z|^{p+1})\frac{\partial^{n+m} p_\tau(z)}{\partial z^n \partial \tau^m}\right\|_{L^2} \\
&\leq K(1+|x|^p) \left(\left\|u^n \frac{\partial^m \phi_\tau(u)}{\partial \tau^m}\right\|_{L^2} + \left\|u^n \frac{\partial^{p+1+m} \phi_\tau(u)}{\partial u^{p+1} \partial \tau^m}\right\|_{L^2}\right)
\end{aligned} \tag{20}$$

From the Lévy-Khinchin formula, $\phi_t(u) = e^{t\psi(u)}$ with $|\psi(u)| \leq K(1+|u|^2)$. Moreover,

$$\begin{aligned}
\psi'(u) &= -\sigma^2 u + i\gamma + \int iy(e^{iyu} - 1_{|y|\leq 1})\nu(dy), \\
\psi''(u) &= -\sigma^2 + \int (iy)^2 e^{iyu} \nu(dy), \\
\psi^{(k)}(u) &= \int (iy)^k e^{iyu} \nu(dy), \qquad 3 \leq k \leq p+1.
\end{aligned}$$

Due to the condition (16), the integrals in the above expressions are finite and we have $|\psi'(u)| \leq K_1(1+|u|)$ and $|\psi^{(q)}(u)| \leq K_q$, $2 \leq q \leq p+1$. Therefore,

$$\left|\frac{\partial^{p+1+m} \phi_\tau(u)}{\partial u^{p+1} \partial \tau^m}\right| \leq K(1 + |u|^{p+1+2m})|\phi_\tau(u)|$$

and by (19), both terms in (20) are finite.

Part 2. By Part 1, representation (18) is valid for every $t < T$. By Corollary 25.8 in [26], (17) implies that $E[H^2(X_T^x)] < \infty$. Denote

$$M_t = \int_0^t \frac{\partial C(s, X_s^x)}{\partial x} \sigma dW_s + \int_0^t \int_{\mathbb{R}} (C(s, X_{s-}^x + z) - C(s, X_{s-}^x)) \tilde{J}(ds \times dz).$$

Then, by Jensen's inequality,

$$E\langle M \rangle_t = E\left(C(t, X_t^x) - C(0,x)\right)^2 \leq 2E[H^2(X_T^x)].$$

This implies that the stochastic integrals

$$\int_0^T \frac{\partial C(s, X_s^x)}{\partial x} \sigma dW_s \quad \text{and} \quad \int_0^T \int_{\mathbb{R}} (C(s, X_{s-}^x + z) - C(s, X_{s-}^x)) \tilde{J}(ds \times dz)$$

exist and since X has no jumps at fixed times, $M_t \to M_T$ a.s. when $t \to T$ and one can pass to the limit $t \to T$ in the right-hand side of (18).

It remains to prove that $\lim_{t\to T} C(t, X_t^x) = C(T, X_T^x) \equiv H(X_T^x)$ a.s. Let Z be a Lévy process independent from $\mathcal{F}_T$ and with the same law as X. We need to show $\lim_{t\to T} E[H(X_t + Z_{T-t})|\mathcal{F}_T] = H(X_T)$. Since X has no jumps at fixed times, $X_t + Z_{T-t} \to X_T$ a.s. Recalling from part 1 that X_T has an absolutely continuous density and since the set of discontinuities of H has Lebesgue measure zero, we see that $H(X_t + Z_{T-t}) \to H(X_T)$ a.s. Now the polynomial bound on H enables us to use the dominated convergence theorem and conclude that $\lim_{t\to T} E[H(X_t + Z_{T-t})|\mathcal{F}_T] = H(X_T)$ a.s. $\square$

The above result covers, for example, digital and put options in exponential Lévy models with either a non-zero diffusion component or stable-like behavior of small jumps (e.g. the tempered stable process [8]) and can be trivially extended to call options using the put-call parity. To treat other exponential Lévy models more regularity is needed for the payoff. The following result applies to Lévy processes with no diffusion component and finite second moment, but also to more general Markov processes with jumps:

Proposition 3. *Let X be as in (6) with $m = 1$, $\sigma(s,x) \equiv 0$ and $\gamma(s,z,x)$ satisfying*

$$|\gamma(s,z,x) - \gamma(s,z,x')| \le \rho(z)|x - x'| \quad \text{with} \quad \int_{\mathbb{R}} \rho^2(z)\nu(dz) < \infty$$
$$|\gamma(s,z,x)| \le \rho(z)(1 + |x|).$$

Suppose that the payoff function H is Lipschitz continuous: $|H(x) - H(y)| \le K|x - y|$. Then the process $(C(t, X_t^x))_{0\le t\le T}$ with $C(t,x) = E[H(X_T^{(t,x)})]$, is a square integrable martingale with the representation

$$C(t, X_t^x) = C(0,x) + \int_0^T \int_{\mathbb{R}} (C(t, X_{t-}^x + \gamma(t,z,X_{t-}^x)) - C(t, X_{t-}^x))\tilde{J}(dt \times dz). \tag{21}$$

Proof. By Theorem III.2.32 in [23], the stochastic differential equation (6) admits a non-explosive solution. Since

$$E[(X_t^x)^2] = x^2 + \int_0^t E[\gamma^2(s,z,X_{s-})]\nu(dz)ds \le x^2 + K\int_0^t E[1 + |X_s|^2]ds,$$

it follows from Gronwall's inequality that

$$E[(X_t^x)^2] \le (x^2 + Kt)e^{Kt}. \tag{22}$$

We also note for future use that the same method can be used to obtain

$$E[(X_T^{(t,x)} - X_T^{(t,y)})^2] \le (x-y)^2 e^{K(T-t)}. \tag{23}$$

The estimate (22) implies that $(C(t, X_t^x))_{0\le t\le T}$ is a square integrable martingale and by Theorems III.4.29, III.2.33 in [23], it admits a martingale representation: there exists a measurable function $Z : \Omega \times \mathbb{R} \times [0, T] \to \mathbb{R}$ such that

$$C(t, X_t^x) = C(0, x) + \int_0^t \int_{\mathbb{R}} Z_s(z)\tilde{J}(ds \times dz). \tag{24}$$

If we are able to show the existence of

$$\tilde{C}_t = C(0, x) + \int_0^t \int_{\mathbb{R}} (C(s, X_{s-}^x + \gamma(s, z, X_{s-}^x)) - C(s, X_{s-}^x))\tilde{J}(ds \times dz), \tag{25}$$

then (21) will follow since the jumps of (24) and (25) are indistinguishable and hence $C = \tilde{C}$ (note that this part of the argument does not carry over to the case $\sigma > 0$). By Jensen's inequality and (23),

$$\begin{aligned}(C(t, x) - C(t, y))^2 &\le E[(H(X_T^{(t,x)}) - H(X_T^{(t,y)}))^2] \\ &\le KE[(X_T^{(t,x)} - X_T^{(t,y)})^2] \le K(x - y)^2.\end{aligned}$$

The existence and square integrability of (25) now follows from

$$\begin{aligned}&\int_0^t \int_{\mathbb{R}} E(C(s, X_s^x + \gamma(s, z, X_s^x)) - C(s, X_s^x))^2\nu(dz)ds \\ &\qquad \le \int_0^t \int_{\mathbb{R}} E(\gamma^2(s, z, X_s^x))\nu(dz)ds \le K \int_0^t (1 + |X_s|)^2 ds < \infty.\end{aligned}$$

□

3.2 Asian options

Exotic options can be introduced via the process of non-traded factors $\tilde{X}$ (see the beginning of this section): for Asian options, one can take

$$\tilde{X}_t = \int_0^t X_s ds$$

and the option's price is then given by

$$C_t = E[H(\tilde{X}_T)|\mathcal{F}_t] = C(t, X_t, \tilde{X}_t),$$

that is, the option's price is now a function of time and (extended) state and one can use the theory developed for European options.

3.3 Barrier options

The value of a knock-out barrier option can be represented as

$$C_t^B = E[H(X_T)1_{\tau>T}|\mathcal{F}_t],$$

where τ is the first exit time of X from an interval B. For example, in the case of an up-and-out option with barrier b we have $\tau = \inf\{t \geq 0 : X_t > b\}$. If X is a Markov jump-diffusion then, conditionally on the event that the barrier has not been crossed, the price of a knock-out barrier option only depends on time and state. Therefore, in all cases

$$C_t^B = 1_{\tau > t} C^B(t, X_t),$$
$$\text{where} \quad C^B(t,x) = \begin{cases} 0, & x \notin B \\ E[H(X_T^{(t,x)}) 1_{\tau_t > T}], & x \in B \end{cases} \tag{26}$$

and τ_t is defined by $\tau_t = \inf\{s \geq t : X_s^{(t,x)} \notin B\}$. Furthermore, the above is equivalent to $C_t^B = C^B(t \wedge \tau, X_{t\wedge\tau})$. Supposing that $C(t,x)$ possesses the required differentiability properties, we can apply the Itô formula *up to time* τ obtaining an SDE of type (5):

$$\begin{aligned} C_t^B = C_0^B &+ \int_0^{t\wedge\tau} \frac{\partial C^B(t, X_t)}{\partial x} \sigma(t, X_t) dW_t \\ &+ \int_{[0,t\wedge\tau]\times\mathbb{R}} (C^B(t, X_{t-} + \gamma(t,z,X_{t-})) - C^B(t, X_{t-})) \tilde{J}(dt \times dz). \end{aligned} \tag{27}$$

However, in the case of barrier options the proof of regularity is much more involved [12] than for European ones. The following result, based on Bensoussan and Lions [5], allows to obtain a martingale representation for barrier options under further assumptions:

Proposition 4. *Let X be as in* (6) *with $m = 1$ and σ and γ satisfying the following hypotheses:*

(i) σ is bounded from below by a positive constant and σ and $\frac{\partial \sigma}{\partial x}$ are uniformly bounded from above.

(ii) There exists a Radon measure m on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} |z| m(dz) < \infty$$

and $\forall A \in \mathcal{B}(\mathbb{R})$, $\forall (t,x)$

$$m(A) \geq \nu(\{z : \gamma(t,z,x) \in A\})$$

(iii) B is a bounded open interval on $\mathbb{R}$.

(iv) The payoff function H satisfies $H \in W_0^{1,p}(B)$, $4 < p < \infty$. The space $W_0^{1,p}(B)$ denotes the $W^{1,p}(B)$-closure of $C_0^\infty(B)$, the space of smooth functions with compact support in B. This implies that the payoff must tend to zero as one approaches the barrier.

Then the barrier option price $C^B(t,x)$ *defined by* (26) *belongs to the space*

$$W^{1,2,p} = \{z \in L^p([0,T]\times B) : \frac{\partial z}{\partial t}, \frac{\partial z}{\partial x}, \frac{\partial^2 z}{\partial x^2} \in L^p([0,T]\times B)\}$$

and the process $(C_t^B)_{0\le t\le T}$ *is a square integrable martingale satisfying the representation* (27).

Proof. The regularity result is a corollary of theorems 3.4 and 8.2 in [5]. To obtain the martingale representation, we can approximate $C^B(t,x)$ in $W^{1,2,p}$ by a sequence of smooth functions (C_n^B), apply the Itô formula to each C_n and then pass to the limit using the inequality (III.7.32) in [5]:

$$\left| E\left[\int_t^T f(s,X_s)ds\right]\right| \le C_{T,p}|f|_{L^p} \quad \forall f \in L^p(\mathbb{R}^d), \quad p>2, \tag{28}$$

where X satisfies the hypotheses (i) and (ii) of the above proposition. □

Corollary 1. *Let* X *be a Lévy process with* $\sigma > 0$ *and* $\int |x|\nu(dx) < \infty$, *let* B *be a bounded open interval and suppose that the payoff function* H *satisfies* $H \in W_0^{1,p}(B)$, $4 < p < \infty$. *Then the barrier option price* $C^B(t,x)$ *defined by* (26) *belongs to the space* $W^{1,2,p}$ *and the process* $(C_t^B)_{0\le t\le T}$ *is a square integrable martingale satisfying the representation* (27).

4 Hedging with options: examples and applications

In this section we apply the general hedging formula (11) to options and analyze numerically the performance of the minimal variance hedging strategy in different settings.

Hedging with the underlying in an exponential Lévy model Suppose that the price of the underlying asset follows an exponential Lévy model

$$dX_t = X_t\sigma dW_t + \int_{\mathbb{R}} X_{t-}(e^z-1)\tilde{J}(dt\times dz).$$

The European option price (12) can then be written as the expectation of a Lévy process $Z_t = \log(X_t/X_0)$: $C(t,x) = E[H(xe^{Z_{T-t}})]$. Supposing that the model parameters and the new payoff function $h(z) = H(e^z)$ of the option satisfy either the hypotheses of Proposition 2 or those of Proposition 3, we can compute a martingale representation for $C(t,X_t)$. Applying the general formula (11) we then obtain the following hedge ratio:

$$\phi_t = \frac{\sigma^2 \frac{\partial C}{\partial X}(t,X_{t-}) + \frac{1}{X_{t-}}\int \nu(dz)(e^z-1)[C(t,X_{t-}e^z) - C(t,X_{t-})]}{\sigma^2 + \int (e^z-1)^2\nu(dz)}. \tag{29}$$

Note that the above equation makes sense in a much more general setting than for instance the delta-hedging strategy which requires that the option price be differentiable, a property which can fail in pure-jump models [12].

Case of a single jump size Assume that the stock price process X^1 follows an exponential Lévy model with a non-zero diffusion component and a single possible jump size:

$$dX_t^1 = X_t^1 \sigma dW_t + X_{t-}^1 (e^{z_0} - 1) d\tilde{N}_t,$$

where $\tilde{N}$ is a compensated Poisson process with intensity λ. We want to hedge a European option $Y_t = C(t, X_t^1)$ with the stock and another European option $X_t^2 = C^*(t, X_t^1)$. In this case, Proposition 1 applies due to the presence of a non-degenerate diffusion component. Denoting $\Delta X = (e^{z_0} - 1)X$ and $\Delta C(t, X) = C(t, Xe^{z_0}) - C(t, X)$ we obtain the following hedge ratios:

$$\phi_t^1 = \frac{\Delta C^*(t, X_{t-}^1) \frac{\partial C(t, X_{t-}^1)}{\partial X} - \Delta C(t, X_{t-}^1) \frac{\partial C^*(t, X_{t-}^1)}{\partial X}}{\Delta C^*(t, X_{t-}^1) - \Delta X_{t-}^1 \frac{\partial C^*(t, X_{t-}^1)}{\partial X}},$$

$$\phi_t^2 = \frac{\Delta C(t, X_{t-}^1) - \Delta X_{t-}^1 \frac{\partial C(t, X_{t-}^1)}{\partial X}}{\Delta C^*(t, X_{t-}^1) - \Delta X_{t-}^1 \frac{\partial C^*(t, X_{t-}^1)}{\partial X}}.$$

It is easy to see that with these hedge ratios *the residual hedging error* $\epsilon(\phi)_T$ *is equal to zero.*

When the jump size ΔX^1 is small, the optimal hedge is approximated by delta-gamma hedge ratios

$$\phi_t^1 = \delta_t = \frac{\partial C(t, X_{t-}^1)}{\partial X} - \frac{\partial C^*(t, X_{t-}^1)}{\partial X} \left(\frac{\partial^2 C(t, X_{t-}^1)}{\partial X^2} / \frac{\partial^2 C^*(t, X_{t-}^1)}{\partial X^2} \right)$$

$$\phi_t^2 = \gamma_t = \frac{\partial^2 C(t, X_{t-}^1)}{\partial X^2} / \frac{\partial^2 C^*(t, X_{t-}^1)}{\partial X^2},$$

obtained by setting to zero the first and the second derivative of the hedged portfolio with respect to the stock price. Note however that in general (jump size not small) the delta-gamma hedging strategy does not eliminate the risk completely, although the optimal quadratic hedging strategy does.

Barndorff-Nielsen and Shephard model Let us reconsider the BNS model introduced in example 3. This model is not covered by results of section 3 but we can check the differentiability of the option price directly along the lines of the proof of proposition 2, using the explicit form of the Fourier transform of the log-price Y [8, p. 489]:

$$\phi_t(u) = E\{e^{iuY_t}\} = \exp\Big\{ -iut\, l(\rho) - \frac{\sigma_0^2}{2}(iu + u^2)\varepsilon(\lambda, t) + \int_0^t l\left(i\rho u - \frac{1}{2}(iu + u^2)\varepsilon(\lambda, t - s) \right) ds \Big\}.$$

Because the diffusion coefficient is bounded from below on any finite time interval,

$$|\phi_t(u)| \leq \exp\Big\{ -\frac{\sigma_0^2 u^2}{2}\varepsilon(\lambda, t) + \int_0^t l\left(-\frac{1}{2}u^2 \varepsilon(\lambda, t - s) \right) ds \Big\} \leq \exp\Big\{ -\frac{\sigma_0^2 u^2}{2}\varepsilon(\lambda, t) \Big\}.$$

In addition if $\int_0^\infty z^{m+n}\nu(dz) < \infty$, we can show, as in the proof of proposition 2, that

$$\left|\frac{\partial^{m+n}\phi_t(u)}{\partial u^m \partial t^n}\right| \leq C(t)(1+|u|^{m+2n})\exp\Big\{-\frac{\sigma_0^2 u^2}{2}\varepsilon(\lambda,t)\Big\}$$

and this in turn implies that the price of an option whose payoff does not grow at infinity faster than $|x|^{m-1}$ will be n times differentiable in t and infinitely differentiable in x (at every point with $\sigma > 0$). Applying Proposition 1 yields the following optimal ratio for hedging with the underlying:

$$\phi_t = \frac{\sigma_{t-}^2 \frac{\partial C}{\partial X} + \frac{1}{X_{t-}}\int \nu(dz)(e^{\rho z}-1)[C(t, X_{t-}e^{\rho z}, \sigma_{t-}^2 + z) - C(t, X_{t-}, \sigma_{t-}^2)]}{\sigma_{t-}^2 + \int (e^{\rho z}-1)^2 \nu(dz)}.$$

When there are no jumps in the stock price ($\rho = 0$) the optimal hedging strategy is just delta-hedging: $\phi_t = \frac{\partial C}{\partial X}$; even though there are jumps in the option price, they cannot be hedged. On the other hand, when $\rho \neq 0$, the above formula has the same structure as equation (29) for exponential Lévy models, with the difference that we also have to take into account the effect of jumps in $\sigma(t)$ on the option price. The impact of the stochastic volatility on the optimal hedging strategy with the underlying asset is thus rather limited: for example, the mean reversion parameter λ does not appear in the hedging formula.

Numerical example: hedging a European put in Merton's model In this example we suppose that the asset X^1 follows the Merton (1976) model, which is an exponential Lévy model with $\sigma > 0$ and $\nu(x) = \frac{\lambda}{\delta\sqrt{2\pi}}e^{-\frac{(x-\theta)^2}{2\delta^2}}$. We simulate 10000 trajectories of stock in this model with two different parameter sets given below:

		μ	σ	λ	jump mean	jump stddev
Model 1:	Risk-neutral	—	0.1	5	-0.05	0.1
Bullish market	Historical	0.2	0.1	5	-0.05	0.1
Model 2:	Risk-neutral	—	0.1	10	-0.2	0.2
Fear of crash	Historical	0.2	0.1	5	-0.05	0.1

For each price trajectory we compute the residual error for hedging an out-of-the-money European put with strike $K = 1.2$ and time to maturity $T = 1$ with the underlying and one at-the-money European put with strike $K = 1$ and time to maturity $T = 1$, using three different hedging strategies: delta hedging, optimal quadratic hedging with stock only, optimal quadratic hedging with stock and another option).

It is important to note that, in this and the following example, the hedge ratios were precomputed on a grid of time and stock price values with formula (11) before simulating the price trajectories; details of computations can be found in [11]. The option prices were evaluated using the following result [12]:

Proposition 5. *Let the payoff function H verify the Lipschitz condition and let $h(x) = H(S_0 e^x)$ have polynomial growth at infinity. Then forward value*

$$f_e(t,x) = E[h(x + X_t)]$$

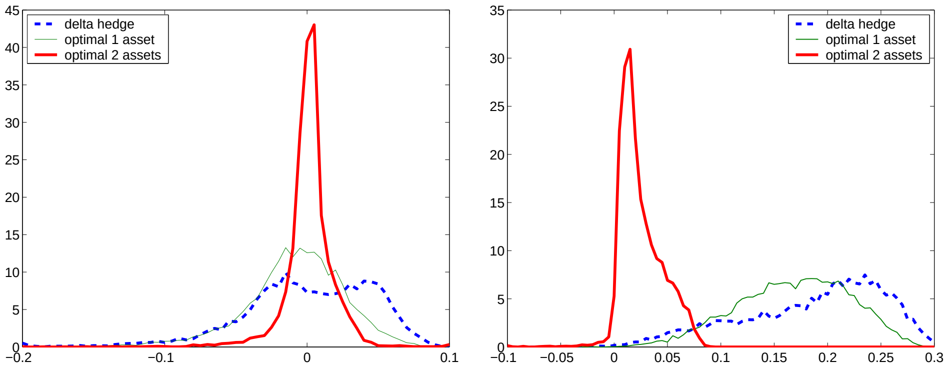

Figure 1: Histograms of the residual hedging error for a European put with strike $K = 1.2$.

of a European option is a viscosity solution of the Cauchy problem

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2}\left[\frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial x}\right] + \int_{\mathbb{R}} \nu(dy)\left[f(x+y) - f(x) - (e^y - 1)\frac{\partial f}{\partial x}\right] \tag{30}$$

with the initial condition $f(0, x) = h(x)$.

We have used a similar representation for the price of a barrier option [12] and the numerical scheme proposed in [13] for solving the associated PIDE (30).

The histograms of the hedging error are shown in Figure 1, left graph, for model 1 and in the right graph for model 2. The table below gives the variance of the residual hedging error for the two models and the three hedging strategies used.

	Bullish market (left)	Fear of crash (right)
Delta hedging:	0.0464	0.1974
Optimal 1 asset:	0.0373	0.1762
Optimal 2 assets:	0.0182	0.0319

First, one can observe that the performance of the optimal quadratic hedging strategy is very similar to that of delta hedging in both models. The performance of both strategies using the underlying only is very sensitive to the difference of Lévy measures under the historical and the risk-neutral probability: when this difference is important as in model 2, both strategies have a very poor performance. On the other hand, this numerical example shows that using options for hedging allows to reduce this sensitivity and achieve an acceptable performance even in presence of an important jump risk premium, that is, when the Lévy measure is very different under the "objective" and the risk-neutral probability.

Hedging a barrier option in Merton's model In this example we continue to work in Merton's model and we want to hedge a barrier put with strike

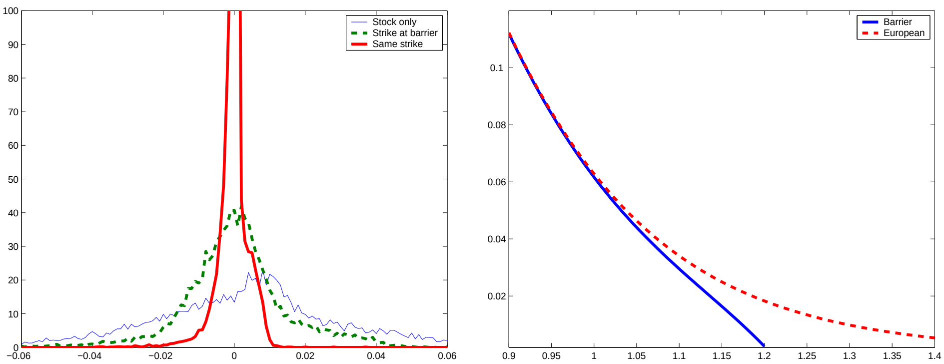

Figure 2: Left: histograms of the residual hedging error for an up and out barrier put with strike $K = 1$ and barrier $B = 1.2$. Right: option price profiles at $T = 0.5$.

$K = 1$ and barrier at $B = 1.2$ with the underlying and a European put option. Figure 2, left graph, depicts the histograms of the residual hedging error for three strategies: hedging with the underlying asset only; hedging with the underlying asset and a European put with strike at the barrier; hedging with stock and a European put with strike $K = 1$. The model parameters correspond to model 1 of previous example. This example shows that a much better hedging performance is achieved by using a European option with the same strike as that of the barrier option. The right graph shows the option price profiles of these two options at time $T = 0.5$. Using a European option for hedging allows to better reproduce the convexity of the barrier option price but it does not take into account the discontinuity of derivative at the barrier.

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