Pricing High Dimensional American Options on Maximum

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The following method computes the price of American Options on Maximum in High dimensions using a stochastic grid method. It is based on the paper [2].

1 Introduction

1.1 Problem Formulation

We consider a d-dimensional Brownian motion, whose augmented filtration is denoted $\mathcal{F}_t$, and a d-dimensional $\mathcal{F}_t$-adapted asset $S_t = (S_t^1, \cdots, S_t^d)$. The payoff of the option is represented by $h(t, S_t)$ and the riskless savings account process is denoted $B_t := \exp(\int_0^t r_s ds)$, where r_t denotes the instantaneous risk-free rate of return. The problem is to compute

$$V_0 = \max_\tau \mathbb{E}\left(\frac{h(\tau, S_\tau)}{B_\tau}\right),$$

where τ is a stopping time taking values in a finite set $\{t_0 = 0, t_1, \cdots, t_N = T\}$. The value of the option at terminal time T is $V(T, x) = h(T, x)$. The conditional continuation value $Q(t_i, S_{t_i} = x)$, i.e. the expected future payoff at time t_i and state $S_{t_i} = x$, is given by

$$Q(t_i, S_{t_i} = x) = \frac{B_{t_i}}{B_{t_{i+1}}}\mathbb{E}\left(V(t_{i+1}S_{t_{i+1}})|S_{t_i} = x\right). \tag{1}$$

The Bermudan option value at time t_i and state $S_{t_i} = x$ is given by

$$V(t_iS_{t_i}) = \max\left(h(t_i, S_{t_i}), Q(t_i, S_{t_i})\right).$$

We are interesting in finding the value of the option at the initial state S_0, i.e. $V(0, S_0)$.

1.2 The Stochastic Grid Method

The stochastic grid method (SGM) solves a general optimal stopping problem using a hybrid of dynamic programming and Monte Carlo methods. The method first computes the optimal exercise policy and a direct estimator of the true option price. Then, a lower bound value of the price is obtained by discounting the payoff obtained by following the exercise policy. Unlike the Longstaff-Schwartz algorithm, the conditional expectation appearing in (1) are not computed using a least-square method. In an approach similar to Barraquand and Martineau [1], Jain and Oosterlee reduce the dimensions of the problem by using $g(S_{t_{i+1}})$ (where $g : \mathbb{R}^d \longmapsto \mathbb{R}$) for the regression of the continuation value Q at time t_{i+1}. Then, they use the distribution of the transition $g(S_{t_{i+1}})|S_{t_i}$ to get the approximation of $\mathbb{E}[Q(t_{i+1}, S_{t_{i+1}})|S_{t_i}]$. We refer to Section 2 for more details.

2 Details on the Method

We are interested in computing an approximation $\hat{V}(t_i S_{t_i})$ of $V(t_i S_{t_i})$ backwards in time. By using nested conditional expactations, we have

$$V(t_i S_{t_i}) = \max\left(h(t_i, S_{t_i}), \frac{B_{t_i}}{B_{t_{i+1}}}\mathbb{E}\left[\underbrace{\mathbb{E}\left[V(t_{i+1}S_{t_{i+1}})|(g(S_{t_{i+1}}), S_{t_i})\right]}_{Z(t_{i+1}, g(S_{t_{i+1}}), S_{t_i})}|S_{t_i}\right]\right).$$

Similar to regression-based algorithms, SGM approximates the unknown functional form $Z(t_{i+1}, g(S_{t_{i+1}}), S_{t_i})$ by projecting it on P polynomial basis functions. $Z(t_{i+1}, g(S_{t_{i+1}}), S_{t_i})$ is approximated by

$$\hat{Z}(t_{i+1}, g(S_{t_{i+1}})) = \mathbb{E}[\hat{V}(t_{i+1}S_{t_{i+1}})|g(S_{t_{i+1}})] = \sum_{p=0}^{P-1} a_p \Psi_p(g(S_{t_{i+1}}))$$

at each time step, where $(\Psi_p)_{p=0,\cdots,P-1}$ form a set of basis functions. It remains to compute an approximation of the continuation value $Q(t_i, S_{t_i})$ defined by (1)

$$\begin{aligned}\hat{Q}(t_i, S_{t_i} = x) &= \frac{B_{t_i}}{B_{t_{i+1}}}\mathbb{E}\left[\hat{Z}(t_{i+1}, g(S_{t_{i+1}}))|S_{t_i} = x\right], \\ &= \frac{B_{t_i}}{B_{t_{i+1}}}\int \sum_{p=0}^{P-1} a_p \Psi_p(g(S_{t_{i+1}})) d\mathbb{P}(g(S_{t_{i+1}})|S_{t_i} = x).\end{aligned}$$

2.1 Computation of the distribution of $g(S_{t_{i+1}})$ given state S_{t_i}

The computation of the distribution of $g(S_{t_{i+1}})$ given S_{t_i} can be computed more or less easily

- if the exact transition probability density function $\mathbb{P}(g(S_{t_{i+1}})|S_{t_i} = x)$ is known, for example for a call or a put on a single asset in the Black-Scholes framework, or a call or a put on the geometric mean of d assets,
- if the first four centered moments $(\mu_i)_{i=1\cdots 4}$ of the distribution are known, we can use the Gram-Charlier Series, which approximates the density function $f(x)$ as

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\left(1 + \frac{\kappa_3}{3!\sigma^3}H_3\left(\frac{x-\mu}{\sigma}\right) + \frac{\kappa_4}{4!\sigma^4}H_4\left(\frac{x-\mu}{\sigma}\right)\right),$$

  where H_i is the ith Hermite polynomial and κ_i is the ith cumulant, i.e. $\kappa_1 = \mu_1$, $\kappa_2 = \mu_2 = \sigma^2$, $\kappa_3 = \mu_3$ and $\kappa_4 = \mu_4 - 4\mu_2^2$.
- if the first four moments are unknown, we can compute an approximation of these moments by using Monte Carlo techniques.

2.2 Algorithm

We summarize the SGM algorithm.

1. Generate M sample paths $\{S_{t_0}, \cdots, S_{t_N}\}$ starting from S_0.

2. Compute the option value for the grid points at times $t_N = T$: $V(T, S_T) = h(T, S_T) = (g(S_T) + X)_+$

3. Compute the approximate functional form $\hat{Z}(t_N, S_{t_N}) = \mathbb{E}[V(t_N, S_{t_N})|g(S_{t_N})]$ by regressing the option value at the grid points over polynomial basis functions of $g(S_{t_N})$

4. Perform the following steps for each exercise time t_i moving backward in time, starting from t_{N-1} till t_0, to obtain $V(0, S_0)$

   (a) If necessary, compute the first four centered moments of $g(S_{t_{i+1}})$ to get the density function $\mathbb{P}(g(S_{t_{i+1}})|S_{t_i} = x)$

   (b) Compute the continuation value for grid points at t_i using $\hat{Z}(t_{i+1}, S_{t_{i+1}})$

   $$\hat{Q}(t_i, S_{t_i}) = \frac{B_{t_i}}{B_{t_{i+1}}} \mathbb{E}\left[\hat{Z}(t_{i+1}, g(S_{t_{i+1}}))|S_{t_i}\right]$$

   (c) Compute the option value for grid points at t_i

   $$\hat{V}(t_i, S_{t_i}) = \max(h(t_i, S_{t_i}), \hat{Q}(t_i, S_{t_i}))$$

   (d) Compute the functional approximation for the conditional expectation, i.e.

   $$\hat{Z}(t_i, S_{t_i}) = \mathbb{E}[\hat{V}(t_{i+1}, S_{t_{i+1}})|g(S_{t_i})]$$

   by regressing the option value obtained at each grid point t_i over the polynomial basis functions of $g(S_{t_i})$

5. Using the exercise strategy obtained while computing the direct SGM estimator $\hat{V}$, for each path determine the earliest time to exercise $\tilde{\tau} = \min\{t \in [0, T] : \hat{Q}_t \leq h_t\}$. Obtain the lower bound option value $\mathbb{E}[\frac{h_{\tilde{\tau}}}{B_{\tilde{\tau}}}]$.

3 The Model

For the numerical experiments, we consider the pricing of an American Call on Maximum of d assets. We assume that the asset prices follow correlated geometric Brownian motion processes, i.e.

$$dS_t^i = S_t^i((r - q_i)dt + \sigma_i dW_t), \ S_0 = x$$

where each asset pays a dividend at a continuous rate q_i. r is the riskless short interest rate, and σ is the vector of volatilities. W^i, $i = 1, \cdots, d$ are standard Brownian motions and the instantaneous correlation between W_t^i and W_t^j is ρ. We assume that the option expires at time T and there are $N + 1$ equally spaced exercise dates in the interval $[0, T]$. The strike price of the option is K, and the payoff is

$$h(t, S_t) = \left(\max(S_t^1, \cdots, S_t^d) - K\right)_+ .$$

3.1 Continuation Value for the Single Asset case

In case of a single asset in the Black-Scholes model, we have

$$\mathbb{P}(g(S_{t_{i+1}})|S_{t_i} = x) = x \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)h + \sigma\sqrt{h}G\right),$$

where $h = t_{i+1} - t_i$ and $G \sim \mathcal{N}(0,1)$. Then, if we choose the canonical polynomial basis, we get

$$\begin{aligned}\hat{Q}(t_i, S_{t_i}) &= e^{-rh} \int_{\mathbb{R}} \sum_{p=0}^{P-1} a_p (g(S_{t_{i+1}}))^p d\mathbb{P}(g(S_{t_{i+1}})|S_{t_i}), \\ &= e^{-rh} \sum_{p=0}^{P-1} a_p (S_{t_i})^p e^{ph((r-q-\frac{\sigma^2}{2})+\frac{p}{2}\sigma^2)}.\end{aligned}$$

3.2 Continuation Value for Max options

In this case, we need to know $\mathbb{P}(g(S_{t_{i+1}})|S_{t_i})$, i.e. $\mathbb{P}(\max(S^1_{t_{i+1}}, \cdots, S^d_{t_{i+1}})|S_{t_i})$. In the case of the Black-Scholes model, we can rewrite

$$\mathbb{P}(\max(S^1_{t_{i+1}}, \cdots, S^d_{t_{i+1}}) = X|S_{t_i}) = \mathbb{P}(\max(Y^1_{t_{i+1}}, \cdots, Y^d_{t_{i+1}}) = \log(X)|S_{t_i}),$$

where $(Y^1_{t_{i+1}}, \cdots, Y^d_{t_{i+1}})$ has a multivariate normal distribution. Using Clark's algorithm (see [2, Appendix A]), we can obtain the first four moments of the random variable $Y := \max(Y^1_{t_{i+1}}, \cdots, Y^d_{t_{i+1}})$. The continuation value is given by

$$\begin{aligned}\hat{Q}(t_i, S_{t_i}) &= e^{-rh} \int_{\mathbb{R}} \sum_{p=0}^{P-1} a_p e^{px} d\mathbb{P}(Y = x|S_{t_i}), \\ &= e^{-rh} \sum_{p=0}^{P-1} a_p e^{p\mu + p^2\frac{\sigma^2}{2}} \left(1 + \frac{\kappa^3}{6}p^3 + \frac{\kappa^4}{24}p^4\right)\end{aligned}$$

4 Numerical experiments

In the case of an American Call on Maximum, the numerical data used by default are the following

S_0	K	T	σ	r	q	ρ
90	100	3	0.2	0.05	0.1	0

The number of trajectories of S is $M = 10^4$. The number of discretization time steps used for the discretization of S is $N = 50$. The number of polynomial basis functions is 6.

References

[1] J. Barraquand and D. Martineau. A Primal-Dual Simulation Algorithm for Pricing Multi-Dimensional American Options. *Management Science*, 50:1222–1234, 1995. 1

[2] S. Jain and C.W. Oosterlee. Pricing High-Dimensional American Options Using The Stochastic Grid Method. available at http://ta.twi.tudelft.nl/mf/users/oosterle/oosterlee/shashi1.pdf, 2012. 1, 4