APPROXIMATE FORMULAS FOR EUROPEAN OPTIONS WITH STOCHASTIC VOLATILITY MODELS

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Premia 18

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The approximate formulas are based on the paper of Elisa Alos [1].

1. Extension of the Hull and White formula

1.1. Model and notations. We assume that the stock price follows the following model:

$$dX_t = rX_t dt + \sigma_t X_t(\rho dW_t + \sqrt{1-\rho^2} dZ_t) \qquad (1)$$

We have to define some notations for this model:

Notations 1.1. *r: instantaneous interest rate*
W et Z: independent standard Brownian motions
$\rho \in [-1, 1]$: correlation factor
σ_t: volatility process independent of Z
K: Strike
X: stock price
$BSCall(t, X, \sigma)$: price of a European Call with maturity T, spot X and strike K given by the Black-Scholes formula
$BSPut(t, X, \sigma)$: price of a European Call with maturity T, spot X and strike K given by the Black-Scholes formula
$BS(t, X, \sigma)$: we use this notation when the result is true for the Call and the Put

V_t: instantaneous value of the option
$D_s^W A$: Malliavin derivative of the random variable A with respect to the Brownian motion W

We also have to define the following functions:

Notations 1.2.

$$d_1(x,\sigma) = \frac{\ln(\frac{x}{K}) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \tag{2}$$

$$H(s,X_s,v_s) = (\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2})BS(s,X_s,\nu_s) = \frac{X_s}{\sigma\sqrt{2\pi(T-s)}}\exp(-\frac{d_1^2(X_s,v_s)}{2})(1-\frac{d_1(X_s,v_s)}{\sigma\sqrt{T-s}}) \tag{3}$$

$$\Lambda_s = \sigma_s(\int_s^T D_s^{W^*}\sigma_r^2 dr) \tag{4}$$

$$v_t^* = \sqrt{\frac{1}{T-t}\int_t^T \mathbb{E}[\sigma_s^2|\mathcal{F}_t]ds} \tag{5}$$

$$v_t = \sqrt{\frac{1}{T-t}\int_t^T \sigma_s ds} \tag{6}$$

1.2. **Extended formula.** The following proposition is the main result in [1].

Proposition 1.1. *Considering the previous model, we have this formula:*

$$\boxed{V_t = \mathbb{E}[BS(t,X_t,v_t)|\mathcal{F}_t] + \frac{\rho}{2}\mathbb{E}[\int_t^T \exp[-r(s-t)]H(s,X_s,v_s)\Lambda_s ds|\mathcal{F}_t]} \tag{7}$$

To prove it, we have to use Itô's formula for anticipated processes: We consider an Itô process $X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds$ and the anticipated process $Y_t = \int_t^T \theta_s ds$. Let be F in $\mathcal{C}^2(\mathbb{R}^3)$

$$\begin{aligned} F(t,X_t,Y_t) &= F(0,X_0,Y_0) + \int_0^t \frac{\partial F}{\partial s}(s,X_s,Y_s)ds + \\ &+ \int_0^t \frac{\partial F}{\partial x}(s,X_s,Y_s)dX_s + \int_0^t \frac{\partial F}{\partial y}(s,X_s,Y_s)dY_s \\ &+ \int_0^t \frac{\partial^2 F}{\partial x \partial y}(s,X_s,Y_s)[\int_s^T D_s^W \theta_r dr]u_s ds + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial^2 x}(s,X_s,Y_s){u_s}^2 ds \end{aligned}$$

The proof is based on the application of the previous formula to the process $\exp(-rt)BS(t,X_t,v_t)$ and it is given in [1].

2. Approximations for pricing

Proposition 2.1. *The value of the option can be approximated by the following expression:*

$$\boxed{V_t^{approx} = BS(t,X_t,v_t^*) + \frac{\rho}{2}H(t,X_t,v_t^*)\mathbb{E}(\int_t^T \Lambda_s|\mathcal{F}_t)} \tag{8}$$

3. Study of several continuous models

3.1. **Stein's and Scott's models.**

3.1.1. *Presentation.* In these models, the volatility process σ can be written $\sigma_t = f(Y_t)$ where Y is an Ornstein-Ulhenbeck process which is solution of the following stochastic equation:

$$dY_t = \alpha(m - Y_t)dt + \lambda\sqrt{\alpha}dW_t \tag{9}$$

We have:

$$\begin{aligned} D_s^W \sigma_r^2 &= 2f(Y_r)f'(Y_r)D_s^W(Y_r) \\ &= 2\lambda\sqrt{\alpha}f(Y_r)f'(Y_r)\exp[-\alpha(r-s)]\mathbf{1}_{s<r} \end{aligned}$$

So we have, in these models, the following approximated formula:

$$V_t^{approx} = BS(t, X_t, v_t^*) + \rho\lambda\sqrt{\alpha}H(t, X_t, v_t^*)\mathbb{E}(\int_t^T \int_s^t f(Y_r)f'(Y_r)f(Y_s)\exp[-\alpha(r-s)]drds|F_t) \tag{10}$$

3.1.2. *Stein's model.* In the model of stein, we consider the function f defined by:$f: x \to x$. We want to price at t=0,so we have:

$$V_{approx} = BS(0, X_0, v_0^*) + \rho\lambda\sqrt{\alpha}H(0, X, v_0^*)\mathbb{E}[\int_0^T \int_s^T Y_r \exp(-\alpha(r-s))Y_s drds]$$

It just remains to evaluate v_0^* and I:
We have:

$$v_0^{*2} = \frac{1}{T}\int_0^T [m + (\sigma_0 - m)\exp(-\alpha s)]^2 ds \tag{11}$$

So, we have:

$$\begin{aligned} v_0^{*2} \quad = \quad & \frac{1}{4T\alpha}[-\sigma_0^2 - \lambda^2\exp(4T\alpha) + 2\sigma_0 m - m^2 - 4\sigma_0 m\exp(T\alpha) + \\ & 4m^2\exp(T\alpha) + 2m^2T\alpha\exp(2T\alpha) + 2\lambda^2T\alpha\exp(2T\alpha) + \exp(2T\alpha)\sigma_0^2 + \\ & 2\exp(2T\alpha)\sigma_0 m - 3\exp(2T\alpha)m^2 + \lambda^2\exp(2T\alpha)]\exp(-2T\alpha) \end{aligned}$$

Now, we consider the quantity I:

$$I = \mathbb{E}[\int_0^T \int_s^T Y_r \exp(-\alpha(r-s))Y_s drds] \tag{12}$$

By linearity, we have:

$$I = \int_0^T \int_s^T \mathbb{E}[Y_r Y_s]\exp(-\alpha(r-s))drds$$

So, we can write:

$$I = I_1 + I_2$$

and:

$$I_1 = \int_0^T \int_s^T cov[Y_r, Y_s]\exp(-\alpha(r-s))drds \tag{13}$$

$$I_2 = \int_0^T \int_s^T \mathbb{E}[Y_r]\mathbb{E}[Y_s]\exp(-\alpha(r-s))drds \tag{14}$$

We obtain the following results:

$$I_1 = -\frac{\lambda^2}{8}(2T\exp(-2T\alpha)\alpha + \frac{\exp(-2T\alpha) - 1)}{\alpha^2})$$

$$
\begin{aligned}
I_2 &= \frac{1}{4\alpha^2}[-9\exp(2T\alpha)m^2 - 2m^2T\alpha - 4m\sigma_0T\alpha\exp(T\alpha) + 4m^2T\alpha\exp(2T\alpha) \\
&+4m^2T\alpha\exp(T\alpha) - 8m\sigma_0\exp(T\alpha) + 12m^2\exp(T\alpha) - 2\sigma_0^2T\alpha - \sigma_0^2 \\
&+4m\sigma_0T\alpha + 4m\sigma_0 - 3m^2 + \sigma_0^2\exp(2T\alpha) + 4\exp(2T\alpha)m\sigma_0]\exp(-2T\alpha)
\end{aligned}
$$

3.1.3. *Scott's model.* In this model, the function f is defined by: $f : x \to \exp(x)$.
At t=0, we have:

$$V_{approx} = BS(0, X_0, v_0^*) + \rho\lambda\sqrt{\alpha}H(0, X, v_0^*)\mathbb{E}[\int_0^T \int_s^T \exp[2Y_r + Y_s]\exp[-\alpha(r-s)]drds]$$

It just remains to evaluate this term:

$$I = \mathbb{E}[\int_0^T \int_s^T \exp[2Y_r + Y_s]\exp[-\alpha(r-s)]drds] \tag{15}$$

we denote g the the function define by:

$$g(r,s) = 2\mathbb{E}(Y_r) + \mathbb{E}(Y_s) + 2Var(Y_r) + 2Cov(Y_r, Y_s) + \frac{Var(Y_s)}{2} - \alpha(r-s) \tag{16}$$

$$
\begin{aligned}
g(r,s) = \quad & 3m + 2(\ln(\sigma_0) - m)\exp(-\alpha r) + (\ln(\sigma_0) - m)\exp(-\alpha s) \\
& +\lambda^2(1 - \exp(2\alpha r)) \\
& +\lambda^2\exp(-\alpha(r+s))(\exp(2\alpha s) - 1) \\
& +1/4\lambda^2(1 - \exp(2\alpha s)) - \alpha(r-s)
\end{aligned}
$$

So we can write:

$$\boxed{I = \int_0^T \int_s^T \exp g(r,s)drds} \tag{17}$$

This expression can't be formally calculated. So we have to use the Riemman approximation to evaluate this quantity. It is the same thing to calculate v_0^*. Indeed, we easily show that if we use the following notation,

Notations 3.1.

$$h(s) = 2(\mathbb{E}[Y_s] + Var(Y_s)) \tag{18}$$

So, we have:

$$\boxed{{v_0^*}^2 = \frac{1}{T}\int_0^T \exp[h(s)]ds} \tag{19}$$

3.2. **Heston's model.** In this model, the volatility can be written $\sigma_t = \sqrt{Y_t}$ where Y is a Cox-Ingersoll-Ross process which is solution of the following stochastic equation:

$$dY_t = k(\theta - Y_t)dt + \nu\sqrt{Y_t}dW_t \tag{20}$$

We want to know the price of the option at t=0. We note $I = \mathbb{E}(\int_0^T \Lambda_s)$. It's easy to show this:

$$I = \int_0^T \int_s^T \mathbb{E}[\mathbb{E}[D_s^W\sigma_r^2|\mathcal{F}_s]\sigma_s]drds \tag{21}$$

It's very interesting because of the Clark-Ocone formula. Indeed, we can write:

$$Y_t = \theta + (Y_0 - \theta)\exp(-kt) + \nu\int_0^t \exp[k(s-t)]\sqrt{Y_s}dW_s \tag{22}$$

Using the uniqueness of the decomposition of Clark-Ocone, we have:

$$\mathbb{E}[D_s^W\sigma_r^2|\mathcal{F}_s] = \nu\sigma_s\exp[k(s-r)] \tag{23}$$

So, we have:

$$I = \nu \mathbb{E}[\int_0^T [\int_s^T \exp(-k(r-s))dr]\sigma_s^2 ds]$$

We obtain:

$$\boxed{I = \frac{\nu}{k^2}(\theta(kT-2) + \sigma_0^2 + \exp(-kT)(kT(\theta - \sigma_0^2) + 2\theta - \sigma_0^2))} \quad (24)$$

Now, we just have to calculate v_0^*. In this case, it's very easy because we have:

$$\mathbb{E}[\sigma_s^2] = \mathbb{E}[v_s] = \theta + (\sigma_0^2 - \theta)\exp(-ks) \quad (25)$$

To finish, we have:

$$\boxed{{v_0^*}^2 = \theta + \frac{1}{kT}(\sigma_0^2 - \theta)(1 - \exp(-kT))} \quad (26)$$

4. Evaluation of δ

Differentiating the extended formula (7), we can write:

$$\delta = \mathbb{E}[\frac{\partial BS}{\partial x}(0, X_0, v_0)|\mathcal{F}_0] + \frac{\rho}{2}\mathbb{E}^*[\int_0^T \exp[-r(s-t)]\frac{\partial H}{\partial x}(s, X_s, v_s)\Lambda_s ds] \quad (27)$$

So, using the same reasoning than that we have done for the price, we can write:

$$\boxed{\delta_{approx} = \frac{dBS}{dx}(0, X_0, v_0*) + \frac{\rho}{2}\frac{\partial H}{\partial x}(t, X_0, v_0^*)\mathbb{E}(\int_0^T \Lambda_s|\mathcal{F}_t)} \quad (28)$$

Using the previous calculus, we can easily evaluate this quantity which gives us a good approximation for the evaluation of δ.

5. Validity domain of the approximation

First, we can show that the approximation is valid for all commonly used initial values of the volatility σ_0. So, we have to consider the different maturity T. We will make different observations for each model:

5.1. **Stein's and Heston's models.** We obtain the same evolution of the error for these models. We consider for instance the evolution of the error in Heston's model when the maturity T increases.
Fixing the parameters to the values: $X_0 = 100$ $r = 0.0953$ $\sigma_0 = 0.2$ $k = 8$ $\theta = 0.04$ $\nu = 0.1$ $\rho = -0.5$
We have, for a Call:

	K	Monte-Carlo price	Approximated price	Error (percent)
T=0.25	90	12.5915	12.5885	0.0238
	95	8.5333	8.53245	0.0099
	100	5.2411	5.2419	0.0152
	105	2.8769	2.8785	0.0556
	110	1.3986	1.3995	0.0643

	K	Monte-Carlo price	Approximated price	Error (percent)
T=0.5	90	15.1768	15.1669	0.0652
	95	11.3839	11.3861	0.0193
	100	8.1617	8.1648	0.0380
	105	5.5711	5.5762	0.0915
	110	3.6173	3.6213	0.1106

T=1

K	Monte-Carlo price	Approximated price	Error (percent)
90	19.741	19.7276	0.0678
95	16.2029	16.1876	0.0944
100	13.0392	13.0269	0.0943
105	10.2855	10.2733	0.1186
110	7.9389	7.9543	0.1939

T=5

K	Monte-Carlo price	Approximated price	Error (percent)
90	46.3924	45.6478	1.6179
95	43.5861	42.0262	1.2846
100	40.8455	40.4956	0.8566
105	38.1858	38.0600	0.3294
110	35.6192	35.725	0.2970

So we can see that the approximation gives good results when T is small. When T increases, to have admissible results, we have to increase the strike K. We can note that we can have some very good results for $T < 1$.

5.2. **Scott's model.** In this model, the validity domain of the approximation is smaller. Indeed, if we fix the different parameters to the values:
$X_0 = 100$ $r = 0.0953$ $\sigma_0 = 0.2$ $\alpha = 4$ $m = 0.02$ $\lambda = 0.05$ $\rho = -0.5$
We have, for a Call:

T=0.1

K	Monte-Carlo price	Approximated price	Error (percent)
90	11.2586	11.2635	0.0435
95	7.1924	7.1940	0.0222
100	4.0409	4.0441	0.0792
105	1.9586	1.9673	0.4442
110	0.8246	0.8209	0.4487

T=0.35

K	Monte-Carlo price	Approximated price	Error (percent)
90	19.0901	19.0601	0.1571
95	16.2326	16.2237	0.0548
100	13.7233	13.7124	0.0794
105	11.5109	11.5135	0.0225
110	9.6189	9.6082	0.0110

T=0.5

K	Monte-Carlo price	Approximated price	Error (percent)
90	24.9514	24.5138	1.7538
95	22.4683	22.0016	2.0771
100	20.2452	19.7117	2.6352
105	17.2243	17.6322	2.3682
110	16.2040	15.7501	2.7999

So, to have some admissible results, we have to work with very small maturities: $T < 0.3$ gives very good results.

6. Bates' model

Now we will interest ourselves to the process with jump-diffusion and, more particularly, to the Bates Model.

6.1. **The model.** Using the same notations as before, Bates model is driven by the following stochastic equations:

$$X_t = X_0 + (r - \lambda k)t - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s(\rho dW_s + \sqrt{1-\rho^2}dB_s) + Z_t$$

Z is a composed Poisson process with intensity λ and ν is the Levy measure indépendant of W and B and such as $k = \frac{1}{\lambda}\int_{\mathbb{R}}(\exp(y)-1)\nu(dy) < \infty$. The volatility σ is such as $\sigma_t = \sqrt{Y_t}$ where Y is a Cox-Ingersoll-Ross process:

$$dY_t = k(\theta - Y_t) + \mu\sqrt{Y_t}dW_t$$

In this model, we choose that $\frac{1}{\lambda}\nu$ is following a normal law $\mathcal{N}(m, w^2)$.

6.2. **Options valuation.**

6.2.1. *Exact formula.*

Notations 6.1.

$$G(s, X_s, v_s) = (\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x})BS(s, X_s, v_s)$$

Using the Itô formula for anticipated process, we can show that the instant value of the option V_t can be written as:

$$V_t = \mathbb{E}[BS(t, X_t, v_t)|\mathcal{F}_t] + \frac{\rho}{2}\mathbb{E}[\int_t^T \exp[-r(s-t)]\frac{\partial G}{\partial x}(s, X_s, v_s)\Lambda_s ds|\mathcal{F}_t]$$
$$+\mathbb{E}[\int_t^T \int_{\mathbb{R}} \exp[-r(s-t)][BS(s, X_s + y, v_s) - BS(s, X_s, v_s)]\nu(dy)ds|\mathcal{F}_t]$$
$$-\lambda k\mathbb{E}[\int_t^T \exp[-r(s-t)]\frac{\partial BS}{\partial x}(s, X_s, v_s)ds|\mathcal{F}_t]$$

The proof of its result is given in [2].

6.2.2. *Approximated formula.* Using the same methods as the ones used in the continuous case, we can show that we can approximate the instant value of the options by the following expression:

$$V_t^{approx} = \hat{V}_t^{approx} + (T-t)\lambda BS(t, X_t + m + \frac{\omega^2(T-t)}{2}, \sqrt{\frac{w^2}{T-t} + v_t^{*2}})$$
$$-\lambda(T-t)BS(t, X_t, v_t^*) - \lambda k(T-t)\frac{\partial BS}{\partial x}(t, X_t, v_t^*)$$

where $\hat{V}_t^{approx}$ is the approximate value of the option in the continuous case.

6.3. **Delta calculation.**

6.3.1. *Exact formula.* As we did in the continuous case, we can differentiate the instant value of the option to obtain the value of δ_t. So we can write:

$$\delta_t = \mathbb{E}[\frac{\partial BS}{\partial x}(t, X_t, v_t)|\mathcal{F}_t] + \frac{\rho}{2}\mathbb{E}[\int_t^T \exp[-r(s-t)]\frac{\partial^2 G}{\partial x^2}(s, X_s, v_s)\Lambda_s ds|\mathcal{F}_t]$$
$$+\mathbb{E}[\int_t^T \int_{\mathbb{R}} \exp[-r(s-t)][\frac{\partial BS}{\partial x}(s, X_s + y, v_s) - \frac{\partial BS}{\partial x}(s, X_s, v_s)]\nu(dy)ds|\mathcal{F}_t]$$
$$-\lambda k\mathbb{E}[\int_t^T \exp[-r(s-t)]\frac{\partial^2 BS}{\partial x^2}(s, X_s, v_s)ds|\mathcal{F}_t]$$

6.3.2. *Approximated formula.* Now, using the same methods as the one used for approximate the value of the option, we can write:

$$\delta_t^{approx} = \hat{\delta}_t^{approx} + (T-t)\lambda \frac{\partial BS}{\partial x}(t, X_t + m + \frac{\omega^2(T-t)}{2}, \sqrt{\frac{w^2}{T-t} + v_t^{*2}}) \\ -\lambda(T-t)\frac{\partial BS}{\partial x}(t, X_t, v_t^*) - \lambda k(T-t)\frac{\partial^2 BS}{\partial x^2}(t, X_t, v_t^*)$$

where $\hat{\delta}_t^{approx}$ is the approximated value of δ_t in the continuous case.

6.4. **Approximation validity.** For continuous models, we have seen that the maturity value T is very important for the precision of the approximation. (It's visible in the proofs when the error between the real and the approximated values is bounded by T^n). To apply the Îto's formula for anticipated and continuous process and to considerate that the lemma in [1] is also true in our case, we have to considerate process with low jumps intensity. So, we have to considerate the influence of λ in the precision of the approximation. We consider a call with the following parametrization:
$\sigma_0^2 = 0.01$ $\kappa = 2$ $\theta = 0.01$ $\mu = 0.2$ $m = 0.7$ $w^2 = 0.16$ $\rho = -0.5$, $X_0 = 100$ $K = 90$, $T = 0.5$

λ	Approximated price	Fourier price	Error (percent)
0.1	14.7041	14.5521	1.045
0.05	14.4957	14.4146	0.562
0.01	14.3289	14.3420	0.09

So, it appears that we can have very good results as $\lambda <= 0.05$. To conclude, we can say that to have very good results in the approximation we have to work with $T <= 1$ and $\lambda <= 0.05$.

References

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