Premia 18

Libor market model with volatility skews

Let $P(t,T)$ denote the price at time $t \leq T$ of the zero-coupon bond maturing at time T. With the maturity structure $0 = T_0 < T_1 < \ldots < T_{K+1}$, one associates the forward rates

$$F_k(t) = \frac{1}{T_{k+1} - T_k}\left(\frac{P(t,T_k)}{P(t,T_{k+1})} - 1\right),\ 0 \leq k \leq K.$$

Under the forward measure $\mathbb{Q}^{k+1}$ induced by using $P(t,T_{k+1})$ as a numeraire, the process $(F_k(t))_{t \leq T_k}$ is a martingale. This martingale is assumed to be governed by a $\mathbb{Q}^{k+1}$-Brownian motion W_t^{k+1} and in order to generate volatility skews, Andersen and Andreasen [1] propose to generalize the usual lognormal dynamics $dF_k(t) = F_k(t)\lambda_k dW_t^{k+1}$ by using CEV processes

$$dF_k(t) = F_k^{\alpha}(t)\lambda_k dW_t^{k+1} \tag{1}$$

where $\alpha > 0$. In this model, the initial price $C_k(t)$ of the caplet paying $(T_{k+1} - T_k)(F_k(T_k) - H)^+$ at time T_{k+1} is given by the following formula generalizing Black's formula which holds for $\alpha = 1$:

$$\frac{C_k(t)}{(T_{k+1} - T_k)P(t,T_{k+1})} = \begin{cases} [F_k(t)(1 - \chi^2(a, b+2, c)) - H\chi^2(c,b,a)] \text{ if } \alpha \in (0,1), \\ [F_k(t)N(x^+) - HN(x^-)] \text{ if } \alpha = 1, \\ [F_k(t)(1 - \chi^2(c, -b, a)) - H\chi^2(a, 2-b, c)] \text{ if } \alpha > 1, \end{cases}$$

where

$$\begin{cases} a = \frac{H^{2(1-\alpha)}}{(1-\alpha)^2\lambda_k^2(T_k - t)} \\ b = \frac{1}{1-\alpha} \\ c = \frac{F_k(t)^{2(1-\alpha)}}{(1-\alpha)^2\lambda_k^2(T_k - t)} \\ x_{\pm} = \frac{\ln(F_k(t)/H) \pm \frac{1}{2}\lambda_k^2(T_k - t)}{\sqrt{\lambda_k^2(T_k - t)}} \end{cases}.$$

and $N(.)$ and $\chi^2(., \theta, \gamma)$ respectively denote the cumulative distribution functions of the normal law and of the non-central χ^2 distribution with non-centrality parameter γ and θ degrees of freedom.
Of course, in case the Brownian motion W_t^{k+1} is assumed to be m-dimensional and the constant real volatility coefficient λ_k is replaced by a $\mathbb{R}^m$-valued function $\lambda_k(.)$ of the time variable, the above formulas are easily extended by replacing $\lambda_k^2(T_k - t)$ by $\int_t^{T_k} \|\lambda_k\|^2(u)du$.
Since

$$C_k(t) = (T_{k+1} - T_k)P(t, T_{k+1})\mathbb{E}^{\mathbb{Q}_{k+1}}\left((F_k(T_k) - H)^+ | \mathcal{F}_t\right)$$

Monte-Carlo valuation of caplets is easily implemented by means of a suitable time-discretization of the stochastic differential equation (1).
In order to price a more complex derivative security with payout V_T at time $T \leq T_{K+1}$ depending on the path of all forwards $F_k(u)$, $0 \leq k \leq K$ on the time interval $[t,T]$, one uses the spot measure $\mathbb{Q}$ under which for $n(t) =$

$\sum_{k=1}^{K+1} k1_{\{T_{k-1}\leq t<T_k\}}$, $W_t^{n(t)}$ is a Brownian motion . The numeraire of the spot measure is

$$B_t = P(t, T_{n(t)}) \prod_{j=0}^{n(t)-1} \frac{1}{P(T_j, T_{j+1})} = P(t, T_{n(t)}) \prod_{j=0}^{n(t)-1} (1 + (T_{j+1} - T_j)F_j(T_j)).$$

The brownian motions W_t^{k+1} and $W_t^{n(t)}$ respectively associated with the forward measure $\mathbb{Q}_{k+1}$ and the spot measure $\mathbb{Q}$ are related by

$$dW_t^{k+1} = dW_t^{n(t)} + \sum_{j=n(t)}^{k} \frac{(T_{j+1} - T_j)F_j^{\alpha}(t)\lambda_j}{1 + (T_{j+1} - T_j)F_j(t)}.$$

Since the price V_t at time $t \leq T$ of the derivative security is given by

$$V_t = B_t \mathbb{E}^{\mathbb{Q}} \left(\frac{V_T}{B_T} \middle| \mathcal{F}_t \right)$$

Monte-Carlo computation of the expectation is possible after discretization of the stochastic differential equations (1) under the spot measure $\mathbb{Q}$.

References

[1] L. Andersen and J. Andreasen, *Volatility skews and extensions of the Libor market model.* Applied Mathematical Finance 7, 1-32, 2000 1