Pricing of Bermudan options under the assumption that the optimal stopping time depends only on the values of the still-alive European component options [Andersen]

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The following is based on [A]. Suppose that, for $i = 0, \ldots, n-1$, we have a martingale $(\widetilde{v}^i_t)_{t\in[0,T_i]}$ representing the discounted value at t of the European option with maturity T_i and discounted payoff $\widetilde{v}^i_{T_i}$. Suppose that we have the following Markov functional forms

$$\widetilde{v}^i_{T_j} \;=\; f_{i,j}(x_{T_j}) \;\;,\;\; j = 0, \ldots, i\,. \tag{1}$$

Here (x_t) is a Markov process with values in $\mathbb{R}^D$. I.e., we have a closed formula (or at least a closed form approximation) for the values of the n European options.

Now consider the Bermudan option given by the payoffs $\widetilde{v}^i_{T_i}$ at the exercise times T_i for $i = 0, \ldots, n-1$. Let $\widetilde{V}_t$ denote its discounted value at t. Then

$$\widetilde{V}_0 \;=\; E(\,\widetilde{V}_{T_0}\,) \tag{2}$$

$$\widetilde{V}_{T_i} \;=\; \sup_{\tau\in\mathcal{T}_{\{i,\ldots,n-1\}}} \widetilde{V}_{T_i}(\tau)\;,\;\text{ where } \widetilde{V}_{T_i}(\tau) \;:=\; E(\,\widetilde{v}^\tau_{T_\tau}\,|\,\mathcal{F}_{T_i}\,) \tag{3}$$

and $\mathcal{T}_{\{i\ldots,n-1\}}$ denotes the set of stopping times with values in $\{i, \ldots, n-1\}$. We introduce the indicator function $I(T_i)$ which is one if exercising at T_i is optimal and zero otherwise; hence

$$\tau^*_i \;=\; \inf\{\, j = i, \ldots, n-1\,;\, I(T_j) = 1\}\;. \tag{4}$$

Here τ_i^* denotes the optimal stopping time in (3), i.e. $\widetilde{V}_{T_i} = \widetilde{V}_{T_i}(\tau_i^*)$. Now suppose that $I(T_i)$ has the following Markov functional form:

$$I(T_i) \;=\; \widetilde{b}_i(\widetilde{v}^{\,i}_{T_i}, \ldots, \widetilde{v}^{\,n-1}_{T_i}) \;=\; b_i(x_{T_i}) \tag{5}$$

for some deterministic boolean functions $\widetilde{b}_i$, b_i. That is, we assume that the exercise decision at T_i depends only on the values at T_i of the still-alive European component options, and this dependance is expressed by $\widetilde{b}_i$. Of course, once that $\widetilde{b}_i$ is chosen, one obtains b_i directly from (1).

Observe that, denoting $\vec{b}_i = (b_i, \ldots, b_{n-1})$, the following definition motivated by (4) and (5) yields an element of $\mathcal{T}_{\{i\ldots,n-1\}}$:

$$\tau_{\vec{b}_i} \;:=\; \inf\{\, j = i, \ldots, n-1 \,;\, b_j(x_{T_j}) = 1 \,\} \;\in\; \mathcal{T}_{\{i\ldots,n-1\}} \;.$$

Obviously, we find functional forms

$$\tau_{\vec{b}_i} \;=\; g_{\vec{b}_i}(x_{T_i}, \ldots, x_{T_{n-1}}) \;.$$

Hence, in view of (1), we obtain functional forms

$$\widetilde{v}^{\,\tau_{\vec{b}_i}}_{T_{\tau_{\vec{b}_i}}} \;=\; f_{\tau_{\vec{b}_i},\tau_{\vec{b}_i}}(x_{T_{\tau_{\vec{b}_i}}}) \;=\; F_{\vec{b}_i}(x_{T_i}, \ldots, x_{T_{n-1}}) \;.$$

This yields the following representation of $E(\,\widetilde{V}_{T_i}(\tau_{\vec{b}_i})\,)$:

$$E(\,\widetilde{V}_{T_i}(\tau_{\vec{b}_i})\,) \;=\; E(\,\widetilde{v}^{\,\tau_{\vec{b}_i}}_{T_{\tau_{\vec{b}_i}}}\,) \;=\; E(\,F_{\vec{b}_i}(x_{T_i}, \ldots, x_{T_{n-1}})\,) \;. \tag{6}$$

Here we use definition (3) in the first step.

0.1 Monte Carlo approximation of $\widetilde{V}_0$, provided the b_i are chosen

Suppose that, for our Markov process (x_t), we are given M Monte Carlo samples $(x^m_{T_0}, \ldots, x^m_{T_{n-1}})$, where $m = 0, \ldots, M-1$. Then, based on (2) , (3) and (6), we have the following Monte Carlo approximation of the discounted present value $\widetilde{V}_0$ of our Bermudan option:

$$\widetilde{V}_0 \;\approx\; \tfrac{1}{M} \sum_{m=0}^{M-1} F_{\vec{b}_0}(x^m_{T_0}, \ldots, x^m_{T_{n-1}}) \;.$$

0.2 Choice of the b_i

Concerning the choice of the $b_0, \ldots, b_{n-1}$, we note first that one reasonably takes

$$\widetilde{b}_{n-1}(v_{n-1}) \;:=\; 1_{v_{n-1}>0}$$

which states that the Bermudan option is exercised at the last exercise date T_{n-1} if and only if the lastEuropean component option (which is the only one being still alive) is in-the-money.
Now the b_i for $i < n-1$ can be chosen (backward) iteratively via Monte Carlo maximization over given parametric classes $\mathcal{B}_i$ of the expected discounted value $E(\widetilde{V}_{T_i})$ of the Bermudan option at time T_i:

$$b_i \;=\; \arg\max_{b\in\mathcal{B}_i} E\Big(\widetilde{V}_{T_i}(\tau_{(b,b_{i+1},\dots,b_{n-1})})\Big) \;=\; \arg\max_{b\in\mathcal{B}_i} \sum_{l=0}^{L-1} F_{(b,b_{i+1},\dots,b_{n-1})}(x^l_{T_i},\dots,x^l_{T_{n-1}}) \;.$$

Here we use (6) in the second step and we consider L Monte Carlo paths independent of those we use for the approximation of $\widetilde{V}_0$; one should take $M >> L$. An example for the choice of the classes $\mathcal{B}_i$ of boolean functions

$$\mathcal{B}_i \;=\; \{\,\widetilde{b}^H_i\,;\,H\geq 0\,\}\;,\;\text{ where } \widetilde{b}^H_i(v_i,\dots,v_{n-1}) := 1_{v_i>H}\;.$$

Hence exercise takes place at T_i if the payoff of the European option maturing at T_i exceeds some barrier H. A second example is

$$\mathcal{B}_i \;=\; \{\,\widetilde{b}^H_i\,;\,H\geq 0\,\}\;,\;\text{ where } \widetilde{b}^H_i(v_i,\dots,v_{n-1}) := 1_{v_i>\max(H,v_{i+1},\dots,v_{n-1})}\;.$$

This second strategy is a refinement that also checks if at least one of the remaining European options has a value exceeding the value of the present European option. If this is the case, the strategy decides that exercise cannot be optimal – a reflection of the fact that the Bermudan option can always be sold at the value of its most expensive European component option.
Examples of well-known one-dimensional optimization algorithms include Golden Section Search and Brent's method.

0.3 Remarks

(1) One might call the general approach chosen here: Monte Carlo pricing of Markov functional Bermudan options under the assumption that the optimal stopping time is also Markov-functional.
(2) We give the precise definition of some of the functional forms used above:

$$b_i(x) \;:=\; \widetilde{b}_i(f_{i,i}(x),\dots,f_{n-1,i}(x))$$

$$g_{\vec{b}_i}(x_i,\dots,x_{n-1}) \;:=\; \inf\{\,j=i,\dots,n-1\,;\,b_j(x_j)=1\,\}$$

$$F_{\vec{b}_i}(x_i,\dots,x_{n-1}) \;:=\; f_{\tau,\tau}(x_\tau)\;,\quad\text{where } \tau := g_{\vec{b}_i}(x_i,\dots,x_{n-1})\;.$$

Here $x, x_i, \dots, x_{n-1} \in \mathbb{R}^D$.

1 Numerical results: Bermudan swaption pricing in the one-factor LIBOR Market Model

We fix a discrete tenor structure

$$0 = \mathcal{T}_0 < \mathcal{T}_1 < \ldots < \mathcal{T}_e \text{ with } \mathcal{T}_{i+1} - \mathcal{T}_i \equiv \delta$$

and define the rightcontinuous function $\eta(t)$ by

$$\mathcal{T}_{\eta(t)-1} \;\le\; t \;<\; \mathcal{T}_{\eta(t)} \quad , \quad \text{in particular} \quad \eta(\mathcal{T}_i) = i+1 \; .$$

Denoting by $P(t,T)$ the time t price of a zero-coupon bond maturing at T, we define for $i = 0, \ldots, e-1$ the forward LIBOR rates for the period $[\mathcal{T}_i, \mathcal{T}_{i+1}]$:

$$L^i_t \;:=\; \delta^{-1}\left(\tfrac{P(t,\mathcal{T}_i)}{P(t,\mathcal{T}_{i+1})} \;-\; 1\right) \quad , \quad t \in [0, \mathcal{T}_i] \; .$$

The method presented in Section 1 will hence be applied for

$$D \;=\; e \quad \text{and} \quad x_t \;=\; (L^i_{t\wedge \mathcal{T}_i})_{i=0,\ldots,e-1} \;\in\; \mathbb{R}^e \; .$$

We assume forward measure dynamics of the following simple type:

$$dL^i_t \;=\; \lambda\, L^i_t dW^i_t \; .$$

This is equivalent to the following spot measure dynamics:

$$dL^i_t \;=\; \lambda\, L^i_t \Big(b_i(t, L_t)dt \;+\; dW_t\Big) \quad , \quad b_i(t,L) \;:=\; \delta\,\lambda \sum_{j=\eta(t)}^{i} \tfrac{L^j}{1+\delta L^j} \; .$$

All simulations will be done under these spot measure dynamics. Recall that the corresponding spot numeraire (N_t) satisfies

$$N_{\mathcal{T}_i} \;=\; \prod_{j=0}^{i-1} (1 + \delta L^j_{\mathcal{T}_j}) \; .$$

Let us consider a (payer) interest rate swap where fixed cashflows $K\delta$ paid at $\mathcal{T}_{s+1}, \ldots, \mathcal{T}_e$ are swapped against floating LIBOR on a unit notional. We will price the corresponding Bermudan swaption with n exercise dates $T_i = \mathcal{T}_{s+i}$, where $i = 0, \ldots, n-1$. Hence, the discounted payoff at T_i is

$$\widetilde{v}^i_{T_i} \;=\; \tfrac{1}{N_{T_i}} \left(1 - P(T_i, \mathcal{T}_e) \;-\; K\,\delta \sum_{j=s+i+1}^{e} P(T_i, \mathcal{T}_j)\right)_+ \quad , \quad T_i \;=\; \mathcal{T}_{s+i} \; .$$

Closed form approximations [corresponding to the $f_{i,j}$ needed in (1)] for European swaption prices can be found e.g. in [AA, § 5]. We consider the following parameter values:

$$\delta \;=\; 0.5 \; , \; L^i_0 = 0.06 \; , \; K = 0.06 \; , \; L = 10000 \; , \; M = 50000 \; .$$

The following table of prices corresponds to Table 1 in [A]. The letters E and B correspond to *European* ($n = 1$) and *Bermudan* ($n = e - s$).
We applied the first strategy presented in Section 1.2 for the choice of the b_i. All Monte Carlo simulations are based on a first-order log-Euler discretization with time step δ. Numbers in parenthesis denote the 95 % confidence interval.
In the European case, we also give the price obtained via the closed form approximation mentioned above.

$T_0 = \mathcal{T}_s$	$\mathcal{T}_e$	λ	E or B	CF	MC
1	4	0.2	E	122.0	120.9 (1.7)
2	4	0.2	E	111.4	109.3 (1.6)
3	4	0.2	E	66.1	65.8 (1.0)
1	4	0.2	B		157.1 (1.7)
2	5	0.2	E	162.4	159.3 (2.3)
3	5	0.2	E	128.4	127.8 (1.9)
4	5	0.2	E	71.8	71.1 (1.1)
2	5	0.2	B		188.4 (2.3)
5	10	0.15	E	253.6	252.0 (3.4)
6	10	0.15	E	215.3	214.8 (2.9)
7	10	0.15	E	169.0	168.3 (2.3)
8	10	0.15	E	116.7	116.7 (1.6)
9	10	0.15	E	60.0	59.8 (0.8)
5	10	0.15	B		283.6 (3.3)

References

[A] L. Andersen; 'A simple approach to the pricing of Bermudan swaptions in the Multi-Factor LIBOR Market Model', Journal of Computational Finance 3, 5-32 (2000). 1, 5

[AA] L. Andersen, J. Andreasen; 'Volatility skews and extensions of the LIBOR Market Model', Mathematical Finance 7, 1-32 (2000). 4