

# Asian options pricing with finite elements

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February 18, 2016

## Premia 18

In [1] Ben Hameur, Breton and Lécuyer are interested in the pricing of fixed strike Amerasian options based on arithmetic average. They approximate the Amerasian option by a Bermudean Asian option with  $n$  possible exercise dates. The pricing of Bermudean style options is naturally formulated as a stochastic dynamic programming (DP) problem. At each exercise date the price of the option is obtained as the maximum of the exercise value of the option and the holding value. Ben Hameur, Breton and Lécuyer propose a numerical approach based on piecewise bilinear interpolation over rectangular finite elements to solve the (DP) equations under the Black and Scholes model. Their method can be adapted to the pricing of European Asian option by keeping the computation of the holding value but removing the maximisation with the exercise value at each exercise date. So from now on, we focus on the American case.

### Notations

We consider a single primitive asset whose price  $(S_t)_{0 \leq t \leq T}$  satisfies the Black and Scholes equation under the risk-neutral probability measure :

$$\begin{aligned} S_0 &> 0 \\ dS_t &= S_t((r - d)dt + \sigma dW_t) \quad 0 \leq t \leq T \end{aligned} \tag{1}$$

where

- $T$  is the maturity date
- $r$  is the risk free rate

- $d$  is the dividend rate
- $\sigma$  is the volatility parameter
- $(W_t)_{0 \leq t \leq T}$  is a standard brownian motion
- $\mu = r - d - \sigma^2/2$

Let  $h = T/n$  and  $t_m = mh$  be the observation dates for  $0 \leq m \leq n$ ; for  $1 \leq m \leq n$ ,  $t_m$  is a possible exercise date and the corresponding payoff is

- for the American call :  $(\bar{S}_m - K)^+$
- for the American put :  $(K - \bar{S}_m)^+$

where  $(x)^+ = \max(x, 0)$  and  $\bar{S}_m = \frac{S_{t_1} + \dots + S_{t_m}}{m}$ . We note at time  $t_m : S_{t_m} = s$  et  $\bar{S}_m = \bar{s}$ , and we focus only on the Amerasian call.

## The DP equation

### The value function

For  $m = 0, \dots, n$  the price  $v_m(s, \bar{s})$  of the Bermudean Asian option at time  $t_m$  when spot is  $s$  and average  $\bar{s}$  is obtained as the maximum of *the exercise value* :

$$v_m^e(\bar{s}) = (\bar{s} - K)^+ \quad (2)$$

and the *the holding value* (which is the value of the option if it is not exercised at time  $t_m$  but is exercised optimally in the future) :

$$v_m^h(s, \bar{s}) = \begin{cases} \rho E[v_1(S_{t_1}, S_{t_1}) | S_0 = s] & \text{if } m = 0 \\ \rho E\left[v_{m+1}\left(S_{t_{m+1}}, \frac{m\bar{s} + S_{t_{m+1}}}{m+1}\right) | S_{t_m} = s\right] & \text{if } 1 \leq m \leq n-1 \end{cases} \quad (3)$$

where  $\rho = e^{-rh}$ .

Hence :

$$v_m(s, \bar{s}) = \begin{cases} \max\{v_m^h(s, \bar{s}), v_m^e(\bar{s})\} & \text{if } 0 \leq m \leq n-1 \\ v_m^e(\bar{s}) & \text{if } m = n \end{cases} \quad (4)$$

The change of variable  $\bar{s}' = \frac{m\bar{s} - s}{m-1}$  if  $m > 1$  and 0 if not simplifies the dynamic programming equations. If we set  $w_m^h(s, \bar{s}') = v_m^h\left(s, \frac{(m-1)\bar{s}' + s}{m}\right)$  and  $w_m(s, \bar{s}') = v_m\left(s, \frac{(m-1)\bar{s}' + s}{m}\right)$  we have for  $1 \leq m$ ,

$$w_m^h(s, \bar{s}') = \rho E\left[w_{m+1}\left(s e^{\sigma(W_{t_{m+1}} - W_{t_m}) + \mu(t_{m+1} - t_m)}, \frac{(m-1)\bar{s}' + s}{m}\right)\right] \quad (5)$$

and

$$w_m(s, \bar{s}') = \max\left\{\left(\frac{(m-1)\bar{s}' + s}{m} - K\right)^+, w_m^h(s, \bar{s}')\right\} \quad (6)$$

For  $m = 0$ ,

$$w_0(S_0) = \rho E\left[w_1\left(S_0 e^{\sigma W_{t_1} + \mu t_1}, 0\right)\right] \quad (7)$$

The advantage of the change of variable is that the second variable of the function  $w_{m+1}$  in the expectation giving  $w_m^h$  becomes deterministic.

### The closed form for $w_{n-1}$

Setting  $\bar{K} = nK - ((n-2)\bar{s}' + s)$  one obtains :

$$\begin{aligned} w_{n-1}^h(s, \bar{s}') &= \rho E \left[ \left( \frac{(n-2)\bar{s}' + s(1 + e^{\sigma(W_{t_n} - W_{t_{n-1}}) + \mu(t_n - t_{n-1})})}{n} - K \right)^+ \right] \\ &= \frac{\rho}{n} E \left[ \left( s e^{\sigma(W_{t_n} - W_{t_{n-1}}) + \mu(t_n - t_{n-1})} - \bar{K} \right)^+ \right] \end{aligned}$$

Hence :

$$w_{n-1}^h(s, \bar{s}') = \begin{cases} \frac{s}{n} e^{-dh} - \rho \frac{\bar{K}}{n} & \text{if } \bar{K} \leq 0 \\ \frac{1}{n} (N(d_1) s e^{-dh} - \rho \bar{K} N(d_1 - \sigma\sqrt{h})) & \text{if } \bar{K} > 0 \end{cases} \quad (8)$$

where  $d_1 = \frac{\ln(s/\bar{K}) + (r-d+\sigma^2/2)h}{\sigma\sqrt{h}}$  and  $N$  denotes the standart normal cumulative distribution function. One can finally express  $w_{n-1}$  :

$$w_{n-1}(s, \bar{s}') = \max \left\{ w_{n-1}^h(s, \bar{s}'), \left( \frac{(n-2)\bar{s}' + s}{n-1} - K \right)^+ \right\} \quad (9)$$

## Numerical Solution

### Notations

For  $0 \leq m \leq n-2$ , the computation of  $w_m$  from  $w_{m+1}$  is based on backward recursion using finite elements over  $(s, \bar{s}') \in [0, \infty[^2$ . Ben Hameur, Breton and Lecuyer propose to take the grid defined by the points  $(a_i, b_j)$  with  $i = 0, \dots, p+1$  and  $j = 0, \dots, q+1$  where :

- $a_0 = 0$
- $a_1 = S_0 e^{\mu t_{n-1} - 3\sigma\sqrt{t_{n-1}}}$
- $a_i$  = the quantile of order  $\frac{i-1}{p-2}$  of the lognormal distribution with parameters  $\mu t_{n-1}$  and  $\sigma\sqrt{t_{n-1}}$  for  $i = 2, \dots, p-2$
- $a_{p-1} = S_0 e^{\mu t_{n-1} + 3\sigma\sqrt{t_{n-1}}}$
- $a_p = S_0 e^{\mu t_{n-1} + 4\sigma\sqrt{t_{n-1}}}$
- $a_{p+1} = +\infty$

and

- $b_0 = 0$
- $b_1 = S_0 e^{\mu t_{n-1} - 2\sigma\sqrt{t_{n-1}}}$
- $b_{q/4} = K \frac{(n-1)\rho-1}{n-2}$
- $b_{3q/4} = \frac{nK}{n-2}$
- $b_q = S_0 e^{\mu t_{n-1} + 3.9\sigma\sqrt{t_{n-1}}}$
- $b_{q+1} = +\infty$
- the other  $b_j$  are equally spaced.

The function  $w_m$  is approximated by a bilinear function over each rectangle:

$$\tilde{w}_m(s, \bar{s}') = \sum_{i=0}^p \sum_{j=0}^q I_{s, \bar{s}'}^{i,j} (\alpha_{ij}^m + \beta_{ij}^m s + \gamma_{ij}^m \bar{s}' + \delta_{ij}^m s \bar{s}') \quad (10)$$

where  $I_{s, \bar{s}'}^{i,j} = 1$  if  $a_i \leq s < a_{i+1}$  and  $b_j \leq \bar{s}' < b_{j+1}$  and 0 if not.

### How to proceed

The backward recursion is initialized by choosing the coefficient  $\alpha_{ij}^{n-1}$ ,  $\beta_{ij}^{n-1}$ ,  $\gamma_{ij}^{n-1}$  and  $\delta_{ij}^{n-1}$  so that the values of  $\tilde{w}_{n-1}$  coincides with the values given by the closed form expression of  $w_{n-1}$  at the points of the grid.

The first step to obtain  $\tilde{w}_m$  from  $\tilde{w}_{m+1}$  consists in computing  $\tilde{w}_m$  at the points of the grid.

$$\begin{aligned} \tilde{w}_m^h(a_k, b_l) &= \rho \sum_{i=0}^p \left( [\alpha_{i\xi}^{m+1} + \gamma_{i\xi}^{m+1} c_{kl}] P_{ik} + [\beta_{i\xi}^{m+1} + \delta_{i\xi}^{m+1} c_{kl}] a_k Q_{ik} \right) \\ \tilde{w}_m(a_k, b_l) &= \max\{\tilde{w}_m^h(a_k, b_l), (c_{kl} - K)^+\} \end{aligned} \quad (11)$$

where<sup>1</sup>

- $\tau$  is lognormal with parameters  $\mu h$  and  $\sigma\sqrt{h}$
- $I$  is the indicator function
- $P_{ik} = E[I_{\{a_i \leq a_k \tau < a_{i+1}\}}] = N\left(\frac{\ln(a_{i+1}/a_k) - h\mu}{\sigma\sqrt{h}}\right) - N\left(\frac{\ln(a_i/a_k) - h\mu}{\sigma\sqrt{h}}\right)$
- $Q_{ik} = E[\tau I_{\{a_i \leq a_k \tau < a_{i+1}\}}] = e^{h\sigma^2/2 + \mu h} \left[ N\left(\frac{\ln(a_{i+1}/a_k) - h\mu}{\sigma\sqrt{h}}\right) - N\left(\frac{\ln(a_i/a_k) - h\mu}{\sigma\sqrt{h}}\right) \right]$
- $c_{kl} = \frac{(m-1)b_l + a_k}{m} \quad k = 0, \dots, p \quad l = 0, \dots, q$

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<sup>1</sup>with the conventions  $N(-\infty) = 0$  and  $N(+\infty) = 1$

- $\xi$  is such that  $c_{kl} \in [b_\xi, b_{\xi+1}[$

The second step is devoted to the computation of  $\alpha_{ij}^m$ ,  $\beta_{ij}^m$ ,  $\gamma_{ij}^m$  and  $\delta_{ij}^m$ . Setting for  $i = 0, \dots, p-1$  and  $j = 0, \dots, q-1$  :  $w_1 = \tilde{w}_m(a_i, b_j)$ ,  $w_2 = \tilde{w}_m(a_{i+1}, b_j)$ ,  $w_3 = \tilde{w}_m(a_i, b_{j+1})$  and  $w_4 = \tilde{w}_m(a_{i+1}, b_{j+1})$ . We obtain by an easy computation:

- $\delta_{ij}^m = \frac{w_4 - w_3 - w_2 + w_1}{(b_{j+1} - b_j)(a_{i+1} - a_i)}$
- $\beta_{ij}^m = \frac{w_2 - w_1}{a_{i+1} - a_i} - \delta_{ij}^m b_j$
- $\gamma_{ij}^m = \frac{w_3 - w_1}{b_{j+1} - b_j} - \delta_{ij}^m a_i$
- $\alpha_{ij}^m = w_1 - \beta_{ij}^m a_i - \gamma_{ij}^m b_j - \delta_{ij}^m a_i b_j$

By extrapolation  $\alpha_{p+1,j}^m$ ,  $\beta_{p+1,j}^m$ ,  $\gamma_{p+1,j}^m$ ,  $\delta_{p+1,j}^m$  (resp.  $\alpha_{i,q+1}^m$ ,  $\beta_{i,q+1}^m$ ,  $\gamma_{i,q+1}^m$ ,  $\delta_{i,q+1}^m$ ) are chosen equal to  $\alpha_{pj}^m$ ,  $\beta_{pj}^m$ ,  $\gamma_{pj}^m$ ,  $\delta_{pj}^m$  (resp.  $\alpha_{iq}^m$ ,  $\beta_{iq}^m$ ,  $\gamma_{iq}^m$ ,  $\delta_{iq}^m$ ).

At time  $t_1$  the variable  $\bar{s}'$  is identically equal to 0. The approximation  $\tilde{w}_1(s, 0)$  of  $w_1(s, 0)$  is chosen affine on each interval  $[a_i, a_{i+1}]$  and obtained after computation of  $\tilde{w}_1(a_i, 0)$  and  $\tilde{w}_1(a_{i+1}, 0)$  as above. Let  $\eta$  be such that the initial spot  $S_0$  belongs to  $[a_\eta, a_{\eta+1}[$ . Then  $\tilde{w}_0(a_\eta)$  and  $\tilde{w}_0(a_{\eta+1})$  are computed by :

$$\begin{aligned} \tilde{w}_0(a_\eta) &= \rho \sum_{i=0}^p (\alpha_{i0}^1 P_{i\eta} + \beta_{i0}^1 a_\eta Q_{i\eta}) \\ \tilde{w}_0(a_{\eta+1}) &= \rho \sum_{i=0}^p (\alpha_{i0}^1 P_{i\eta+1} + \beta_{i0}^1 a_{\eta+1} Q_{i\eta+1}) \end{aligned} \quad (12)$$

The approximated price and delta of the option returned by the method are respectively  $\tilde{w}_0(a_\eta) + \frac{\tilde{w}_0(a_{\eta+1}) - \tilde{w}_0(a_\eta)}{a_{\eta+1} - a_\eta} (S_0 - a_\eta)$  and  $\frac{\tilde{w}_0(a_{\eta+1}) - \tilde{w}_0(a_\eta)}{a_{\eta+1} - a_\eta}$ .

## References

- [1] H.Ben Hameur M.Breton P. L'Ecuyer. A numerical procedure for pricing american-style asian option. *preprint*, 1999. [1](#)