

PRICING PARISIAN OPTIONS

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ABSTRACT. In this work, we propose to price Parisian options using Laplace transforms. Not only, do we compute the Laplace transforms of all the different Parisian options, but we also explain how to invert them numerically. We discuss the accuracy of the numerical inversion and present the evolution of the Greeks through a few graphs.

1. INTRODUCTION

With the development of stock exchanges around the world, more and more people have become interested in derivatives and especially in options. Standard options provide its owner with the right to buy or sell a number of stocks for a fixed amount of money at a given time, called the maturity time. There are more complex options, known under the name of exotic or also path-dependent options. These options are valuable only if the stock price has satisfied certain conditions before the maturity time, this is precisely this kind of options we are going to study. More precisely, we will deal with options that give their owners the right to buy (call options) or sell (put options) a number of stocks for a fixed amount of money (the strike) if the stock price has stayed below (or above) a certain level (the barrier) for a certain time (the option window) before the maturity time. This option is called a Parisian down-and-in option (or alternatively a Parisian up-and-in option). This is only one example of all the different Parisian options. Basically, we will only consider European style options, which means that one can only exercise his option at the maturity time. Parisian options are, to some extent, a kind of barrier options. One could influence the value of a barrier option by buying a lot of stocks or on the contrary by selling a lot of them. For instance, let us imagine that we own a lot of up-and-in barrier options which haven't been knocked in yet. If the maturity time is close, then we could be tempted to buy a lot of stocks to have the option knocked in. If we consider a Parisian up-and-in option, this is no longer possible since the asset price has to remain above the level for a much longer period (several days). Therefore, Parisian options can be seen as a guarantee against easy arbitrage.

As one will discover later on, there exist a lot of different Parisian options. There are two different ways of measuring the time spent above or below the barrier. Either, one only counts the time spent in a row and starts counting from 0 each time the stock price crosses the barrier, this type is referred to as the continuous Parisian options, or one adds the time spent below or above the barrier without resuming the counting from 0 each time the stock price crosses the barrier, these options are called cumulative Parisian options. In practice, these two kinds of Parisian options raise different questions about the paths of Brownian motion. Therefore, we will only stick to the continuous style options.

There already exist several studies on the Parisian Options. Basically, two techniques can be used to price Parisian options either Laplace transforms or partial differential equations. The Laplace transform technique was first introduced by [Chesney et al.(1997)Chesney, Jeanblanc-Picqué, and Yor]. [Schröder(2003)] has also tackled these options using Laplace transforms. The PDE method was developed by [Haber et al.(1999)Haber, Schönbauer, and Wilmott(1998)].

In this article, we present a way of computing the prices of Parisian Options. The real issue in pricing options is to be able to hedge them. This can only be done if we are able to compute the prices at any time t smaller than the maturity time. The computation of the prices at time 0 requires to study a little of the excursion theory of Brownian motion. The most complex proofs will only be given in the Appendix. The pricing technique, we expose here, is based on Laplace transforms. In this work, we compute the Laplace transforms of all the different Parisian options and we also discuss in detail the accuracy of the numerical technique used to invert the Laplace transform. The numerical inversion is

based on [Abate et al.(1999)Abate, Choudhury, and Whitt].

The article is divided as follows. First of all in Section 2 we give some definitions concerning Brownian Motions and hitting time. We also explain how to write the price of such options in terms of hitting times. In Section 3, we explain how to compute the Laplace transform of the price of a Parisian Down Call at time 0. Section 4 is devoted to the computation of the Laplace transform of the prices of Parisian Up Calls still at time 0. Some parity relationships are given in Section 5 to deduce the prices of Parisian Puts. At that stage we are able to price any Parisian Options at time 0. In Section 6, we show how to compute the prices at some time t relying on the prices at time 0.

Then, in Section 6 we will expose an algorithm to invert numerically a Laplace transform and we will also discuss its accuracy and efficiency. This method is extremely accurate and fast compared with the PDE method.

To conclude this article we present a few graphs to try to better understand these options. We also give a few hedging simulations.

We have implemented in C the technique presented here. All the prices were computed using this program. The different graphs concerning the hedging of such options were generated using the C code we wrote.

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In this article, we will use the following notations:

| | |
|-----------|---|
| S | the process representing the asset price, |
| K | the strike, |
| T | the maturity of the option, |
| L | the barrier level for process S , |
| D | the option window, |
| x | the initial value of process S , |
| r | the interest rate, |
| δ | the dividend rate, |
| σ | the volatility, |
| k | $1/\sigma \ln(K/x)$, |
| b | $1/\sigma \ln(L/x)$ (<i>i.e.</i> the barrier level for the Brownian motion), |
| λ | the Laplace variable, |
| θ | $\sqrt{2\lambda}$, |
| d | $\frac{b-k}{\sqrt{D}}$, |
| m | $\frac{1}{\sigma} \left(r - \delta - \frac{\sigma^2}{2} \right)$. |

2. DEFINITIONS

First, we will give a few definitions and notations used in the rest of the article. Then, we will present the features of such options. We only focus on the down-and-in and down-and-out calls in this section since the features of the other Parisian options can easily be deduced from these two.

2.1. Some notations. Let us describe an excursion at (or away from) level L for an Itô process S . We define

$$g_{L,t}^S = \sup\{u \leq t \mid S_u = L\}, \quad d_{L,t}^S = \inf\{u \geq t \mid S_u = L\}.$$

The trajectory of S between $g_{L,t}^S$ and $d_{L,t}^S$ is the excursion at level L , straddling time t .

Let $S = \{S_t, t \geq 0\}$ denote the price of the underlying asset. We suppose that under the risk neutral measure \mathcal{Q} , the dynamics of S is given by

$$dS_t = S_t((r - \delta)dt + \sigma dW_t), \quad S_0 = x$$

where $W = \{W_t, t \geq 0\}$ is a \mathcal{Q} Brownian motion and $x > 0$. It follows that

$$S_t = x \exp \left(\left(r - \delta - \frac{\sigma^2}{2} \right) t + \sigma W_t \right).$$

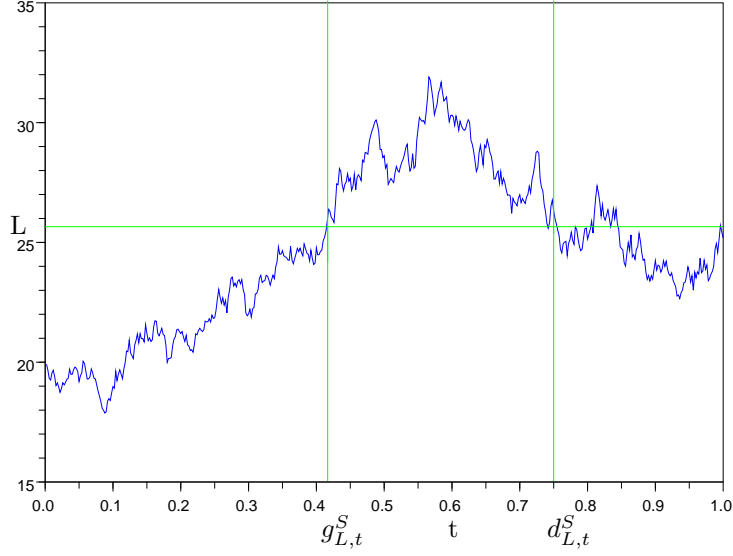


FIGURE 1. Dynamic of an asset

Let us introduce the following notations

$$m = \frac{1}{\sigma} \left(r - \delta - \frac{\sigma^2}{2} \right), \quad b = \frac{1}{\sigma} \ln \left(\frac{L}{x} \right)$$

where L is the excursion level. Under \mathcal{Q} , the dynamics of the asset is given by $S_t = x \exp(\sigma(mt + W_t))$. From now on, we will consider that every option has a maturity time T . Relying on the Cameron-Martin-Girsanov theorem, we can introduce a new probability \mathcal{P} , which makes $Z = \{Z_t = W_t + mt, 0 \leq t \leq T\}$ a \mathcal{P} -Brownian motion and $\frac{d\mathcal{P}}{d\mathcal{Q}}|_{\mathcal{F}_T} = e^{mZ_T - \frac{m^2}{2}T}$. Thus, S rewrites $S_t = x e^{\sigma Z_t}$.

2.2. The Parisian down-and-out call. A down-and-out Parisian option becomes worthless if S reaches L and remains constantly below level L for a time interval longer than D before maturity time T , which is exactly the same as saying that Brownian motion Z makes an excursion below b older than D . Let us introduce

$$\begin{aligned} T_b &= \inf \{t > 0 \mid Z_t = b\}, \\ g_t^b &= \sup \{u \leq t \mid Z_u = b\}, \\ T_b^- &= \inf \{t > 0 \mid (t - g_t^b) \mathbf{1}_{\{Z_t < b\}} > D\}. \end{aligned}$$

One should notice that referring to the previous notations $g_t^b = g_{L,t}^S$.

The price of a down-and-out option at time 0 with payoff $\phi(S_T)$, in an arbitrage free model, is given by

$$e^{-rtT} \mathbb{E}_{\mathcal{Q}} \left(\phi(S_T) \mathbf{1}_{\{T_b^- > T\}} \right) = e^{-(r + \frac{m^2}{2})T} \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_b^- > T\}} \phi(xe^{\sigma Z_T}) e^{mZ_T} \right). \quad (2.1)$$

Let us denote by $PDOC(x, T; K, L; r, \delta)$ the value of a Parisian down-and-out call. From (2.1), we have

$$PDOC(x, T; K, L; r, \delta) = e^{-(r + \frac{1}{2}m^2)T} \mathbb{E}_{\mathcal{P}} (\mathbf{1}_{\{T_b^- > T\}} (xe^{\sigma Z_T} - K)^+ e^{mZ_T}).$$

In many formulae involving a function Π of maturity T , as in (2.1), the discount factor $\exp[-(r + \frac{1}{2}m^2)T]$ appears. In order to give more concise formulae, we introduce the following notation:

$$*\Pi(T) = e^{(r + \frac{1}{2}m^2)T} \Pi(T). \quad (2.2)$$

Hence, we will compute the Laplace transform of $*\Pi$ rather than the one of Π . Any way the following obvious relation between their Laplace transforms hold

$$\widehat{\Pi}(\lambda) = \widehat{*\Pi}(\lambda + (r + \frac{1}{2}m^2)). \quad (2.3)$$

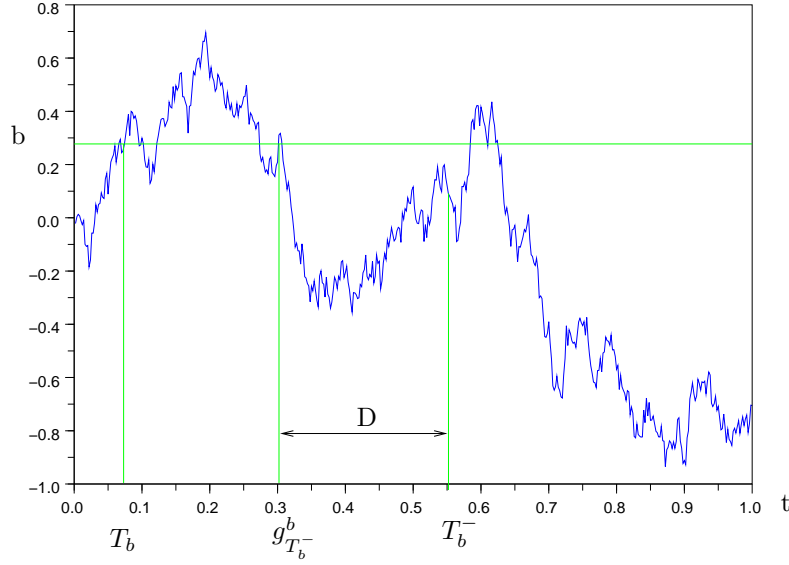


FIGURE 2. Excursion of Brownian Motion

Since the functions Π we will consider will stand for option prices, they are bounded. This remark will enable us to state the accuracy of the numerical inversion in Section 7.

Using notation (2.2), we obtain

$$*PDOC(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- > T\}})(xe^{\sigma Z_T} - K)^+ e^{mZ_T}).$$

2.3. The Parisian down-and-in call. The owner of a down-and-in option receives the pay-off if S makes an excursion below level L older than D before maturity time T , which is exactly the same as saying that Brownian motion Z makes an excursion below b older than D . The price of a down-and-in option at time 0 with payoff $\phi(S_T)$ is given by

$$e^{-rT} \mathbb{E}_{\mathcal{Q}} \left(\phi(S_T) \mathbf{1}_{\{T_b^- < T\}} \right) = e^{-(r + \frac{m^2}{2})T} \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_b^- < T\}} \phi(xe^{\sigma Z_T}) e^{mZ_T} \right). \quad (2.4)$$

Let us denote by $PDIC(x, T; K, L; r, \delta)$ the value of a Parisian down-and-in call. From (2.4), we have

$$PDIC(x, T; K, L; r, \delta) = e^{-(r + \frac{1}{2}m^2)T} \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}})(xe^{\sigma Z_T} - K)^+ e^{mZ_T}).$$

Using notation (2.2), we obtain

$$*PDIC(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}})(xe^{\sigma Z_T} - K)^+ e^{mZ_T}).$$

The following scheme explains how to deduce the prices of the different kinds of Parisian options one from the others.

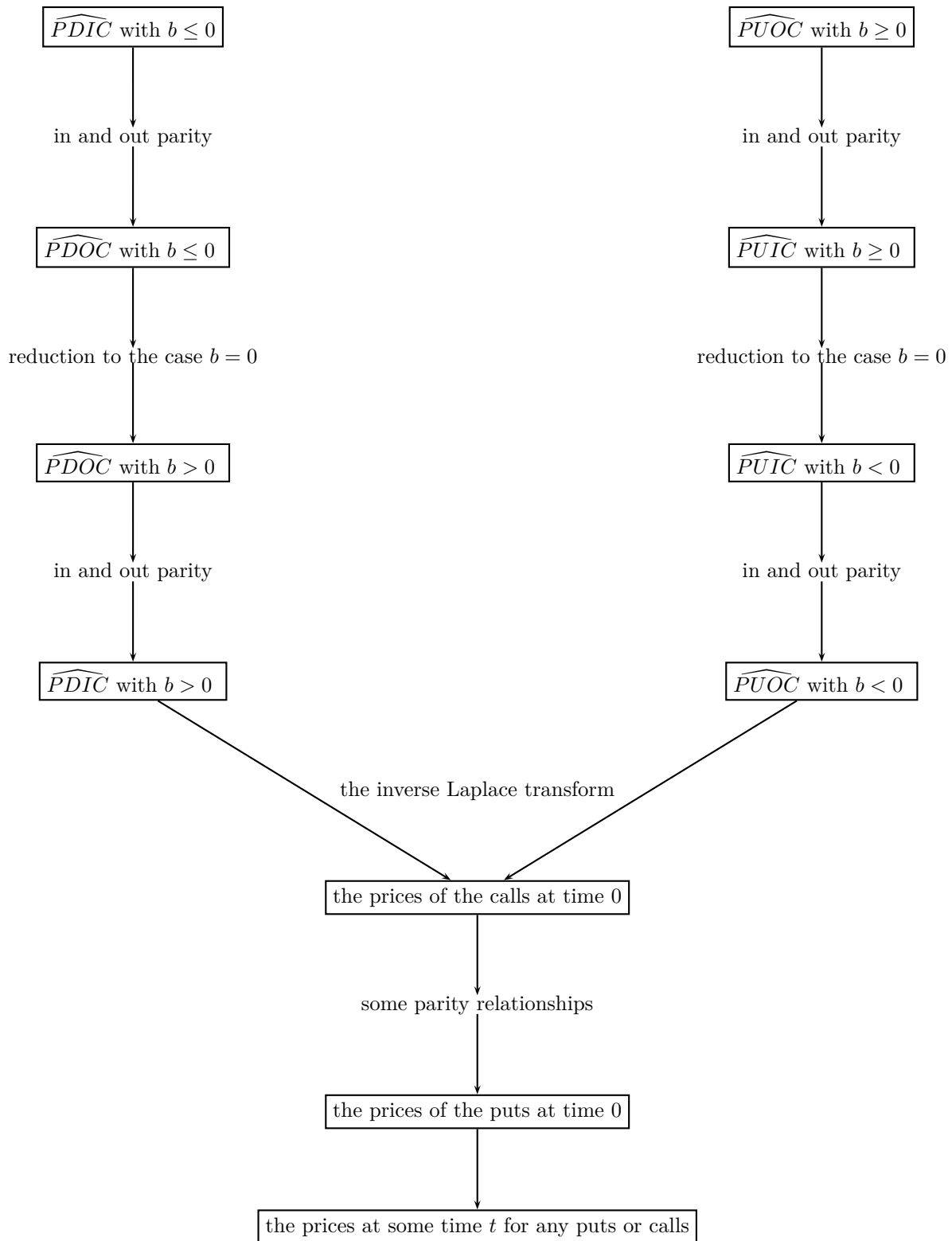


FIGURE 3. Organigram of how to deduce the prices one from the others

3. THE PARISIAN DOWN CALLS

As shown in the previous scheme, all the different prices are deduced from their Laplace transforms. Now, we will explain how to compute these Laplace transforms. In this section, we will only deal with down version of the calls. We will follow exactly the previous scheme to deduce step by step all the needed Laplace transforms.

3.1. The valuation of a Parisian down-and-in call with $b \leq 0$. We want to compute

$*PDIC(x, T; K, L; r, \delta)$. Let us denote by $\mathcal{F}_t = \sigma(Z_s, s \leq t)$ the natural filtration of Brownian motion $Z = \{Z_t; t \geq 0\}$. One notices that T_b^- is an \mathcal{F}_t -stopping time. We have

$$\begin{aligned} *PDIC(x, T; K, L; r, \delta) &= \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}})(xe^{\sigma Z_T} - K)^+ e^{mZ_T}, \\ &= \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}) \mathbb{E}_{\mathcal{P}}[(xe^{\sigma Z_T} - K)^+ e^{mZ_T} | \mathcal{F}_{T_b^-}] \end{aligned}$$

and we can write

$$*PDIC(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}) \mathbb{E}_{\mathcal{P}}[xe^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+ e^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})} | \mathcal{F}_{T_b^-}].$$

$$\begin{aligned} \mathbb{E}_{\mathcal{P}} \left[(xe^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+ e^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})} | \mathcal{F}_{T_b^-} \right] &= \\ \mathbb{E}_{\mathcal{P}} \left[(xe^{\sigma(Z_T - Z_{T_b^-} + z)} - K)^+ e^{m(Z_T - Z_{T_b^-} + z)} | \mathcal{F}_{T_b^-} \right]_{|z=Z_{T_b^-}}. \end{aligned} \quad (3.1)$$

Let W_t denote $Z_{t+T_b^-} - Z_{T_b^-}$. Relying on the strong Markov property, W_t is independent of $\mathcal{F}_{T_b^-}$ and $W_{T-T_b^-} = Z_T - Z_{T_b^-}$. Let Y_t denote $(xe^{\sigma(W_{T-t}+z)} - K)^+ e^{m(W_{T-t}+z)}$, Y_t is independent of $\mathcal{F}_{T_b^-}$. Then, a well-known result on conditional expectations, states that $\mathbb{E}(Y_{T_b^-} | \mathcal{F}_{T_b^-}) = \mathbb{E}(Y_t)_{|t=T_b^-}$. So, we obtain

$$\begin{aligned} \mathbb{E}_{\mathcal{P}} \left[(xe^{\sigma(Z_T - Z_{T_b^-} + Z_{T_b^-})} - K)^+ e^{m(Z_T - Z_{T_b^-} + Z_{T_b^-})} | \mathcal{F}_{T_b^-} \right] &= \\ \mathbb{E}_{\mathcal{P}} \left[(xe^{\sigma(W_{T-\tau}+z)} - K)^+ e^{m(W_{T-\tau}+z)} \right]_{|z=Z_{T_b^-}, \tau=T_b^-}. \end{aligned} \quad (3.2)$$

$$\mathbb{E}_{\mathcal{P}} \left[(xe^{\sigma(W_{T-\tau}+z)} - K)^+ e^{m(W_{T-\tau}+z)} \right] = \frac{1}{\sqrt{2\pi(T-\tau)}} \left(\int_{-\infty}^{\infty} e^{mu} (xe^{\sigma u} - K)^+ \exp \left(-\frac{(u-z)^2}{2(T-\tau)} \right) du \right).$$

So, we get

$$*PDIC(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}) \mathcal{P}_{T-T_b^-}(f_x)(Z_{T_b^-}),$$

with

$$f_x(z) = e^{mz} (e^{\sigma z} - K)^+,$$

and

$$\mathcal{P}_t(f_x)(z) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f_x(u) \exp \left(-\frac{(u-z)^2}{2t} \right) du.$$

As recalled in [Appendix C](#), the random variables $Z_{T_b^-}$ and T_b^- are independent. By denoting the law of $Z_{T_b^-}$ by $\nu(dz)$, we obtain

$$\begin{aligned} *PDIC(x, T; K, L; r, \delta) &= \int_{-\infty}^{\infty} \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- < T\}}) \mathcal{P}_{T-T_b^-}(f_x)(z) \nu(dz), \\ &= \int_{-\infty}^{\infty} f_x(y) h_b(T, y) dy, \end{aligned} \quad (3.3)$$

where

$$h_b(t, y) = \int_{-\infty}^{\infty} \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_b^- < t\}} \frac{\exp \left(-\frac{(z-y)^2}{2(t-T_b^-)} \right)}{\sqrt{2\pi(t-T_b^-)}} \right) \nu(dz).$$

Since we consider the case $b < 0$, we can use the following expression for the law of $Z_{T_b^-}$, as it is proved in [Appendix C](#)

$$\mathbb{P}(Z_{T_b^-} \in dx) = \frac{dx}{D}(b-x) \exp\left(-\frac{(x-b)^2}{2D}\right) \mathbf{1}_{\{x \leq b\}}. \quad (3.4)$$

3.1.1. *The Laplace transform of $*PDIC(x, T; K, L; r, \delta)$.* We can calculate $*PDIC(x, T; K, L; r, \delta)$ by using a Laplace transform. Let $\widehat{*PDIC}(x, \lambda; K, L; r, \delta)$ denote the Laplace transform of $*PDIC(x, T; K, L; r, \delta)$ for any λ with $\operatorname{Re}(\lambda)$ large enough such as all the integrals discussed below are convergent. This condition implies that $m + \sigma - \sqrt{2\lambda} < 0$. We have

$$\begin{aligned} \widehat{*PDIC}(x, \lambda; K, L; r, \delta) &= \int_0^\infty e^{-\lambda t} \int_{-\infty}^\infty f_x(y) h_b(t, y) dy dt, \\ &= \int_{-\infty}^\infty f_x(y) \int_0^\infty e^{-\lambda t} h_b(t, y) dt dy. \end{aligned} \quad (3.5)$$

The Laplace transform of $h_b(T, y)$. We would like to compute

$$\widehat{h}_b(\lambda, y) = \int_0^\infty e^{-\lambda t} h_b(t, y) dt.$$

We know that :

$$h_b(t, y) = \int_{-\infty}^b \frac{b-z}{D} \exp\left(-\frac{(z-b)^2}{2D}\right) \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_b^- < t\}} \frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}} \right) dz.$$

We can write

$$h_b(t, y) = \int_{-\infty}^b \frac{b-z}{D} \exp\left(-\frac{(z-b)^2}{2D}\right) \gamma(t, z-y) dz,$$

where

$$\gamma(t, x) = \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_b^- < t\}} \frac{\exp\left(-\frac{x^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}} \right),$$

and we have

$$\widehat{h}_b(\lambda, y) = \int_{-\infty}^b \frac{b-z}{D} \exp\left(-\frac{(z-b)^2}{2D}\right) \int_0^\infty e^{-\lambda t} \gamma(t, z-y) dt dz. \quad (3.6)$$

So, we need to compute the Laplace transform of $\gamma(t, x)$

$$\int_0^\infty e^{-\lambda t} \gamma(t, x) dt = \mathbb{E}_{\mathcal{P}} \left(\int_{T_b^-}^\infty e^{-\lambda t} \frac{\exp\left(-\frac{x^2}{2(t-T_b^-)}\right)}{\sqrt{2\pi(t-T_b^-)}} dt \right).$$

The change of variables $u = t - T_b^-$ gives

$$\int_0^\infty e^{-\lambda t} \gamma(t, x) dt = \mathbb{E}_{\mathcal{P}}(e^{-\lambda T_b^-}) \int_0^\infty e^{-\lambda u} \frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi u}} du.$$

Using results from [Appendix A](#) and [B](#), we get

$$\int_0^\infty e^{-\lambda t} \gamma(t, x) dt = \frac{\exp[-(|x| - b)\theta]}{\theta \psi(\theta \sqrt{D})}.$$

Thanks to (3.6), we can rewrite

$$\widehat{h}_b(\lambda, y) = \frac{e^{b\theta}}{D\theta\psi(\theta\sqrt{D})} \int_{-\infty}^b (b-z) \exp\left(-\frac{(z-b)^2}{2D} - |z-y|\theta\right) dz.$$

By changing variables $x = b - z$, we have

$$\widehat{h}_b(\lambda, y) = \frac{e^{b\theta}}{D\theta\psi(\theta\sqrt{D})} \int_0^\infty x \exp\left(-\frac{x^2}{2D} - |b-x-y|\theta\right) dx. \quad (3.7)$$

Let $K_{\lambda,D}(b-y)$ denote $\int_0^{+\infty} x \exp\left(-\frac{x^2}{2D} - |b-x-y|\theta\right) dx$.

valuation of $K_{\lambda,D}(b-y)$. Relying on the definition of $f_x(y)$, we know that y is always bigger than $\frac{1}{\sigma} \ln\left(\frac{K}{x}\right)$.

► Let us consider the case $K \geq L$. In this case we have $y-b \geq \frac{1}{\sigma} \ln(\frac{K}{L})$, then $y-b \geq 0$. So we get

$$K_{\lambda,D}(b-y) = \int_0^\infty x \exp\left(-\frac{x^2}{2D} + (b-x-y)\theta\right) dx$$

because $x \geq 0$ and $y-b \geq 0$.

$$\begin{aligned} K_{\lambda,D}(b-y) &= e^{(b-y)\theta} \int_0^\infty x \exp\left(-\frac{x^2}{2D} - x\theta\right) dx, \\ &= D e^{(b-y)\theta} \psi(-\theta\sqrt{D}). \end{aligned}$$

From (3.7) we obtain

$$\widehat{h}_b(\lambda, y) = \frac{\psi(-\theta\sqrt{D}) \exp[(2b-y)\theta]}{\psi(\theta\sqrt{D}) \theta}. \quad (3.8)$$

If we fill in (3.5) with the expression of $\widehat{h}_b(\lambda, y)$, we get

$$*\widehat{PDIC}(x, \lambda; K, L; r, \delta) = \frac{\psi(-\theta\sqrt{D}) e^{2b\theta}}{\theta \psi(\theta\sqrt{D})} \int_{\frac{1}{\sigma} \ln(\frac{K}{x})}^\infty e^{-y\theta} e^{my} (x e^{\sigma y} - K) dy. \quad (3.9)$$

Let k denote $\frac{1}{\sigma} \ln\left(\frac{K}{x}\right)$.

We come up with the following formula for $*\widehat{PDIC}(x, \lambda; K, L; r, \delta)$.

$$*\widehat{PDIC}(x, \lambda; K, L; r, \delta) = \frac{\psi(-\theta\sqrt{D}) e^{2b\theta}}{\theta \psi(\theta\sqrt{D})} K e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right),$$

for $K > L$ and $x \geq L$.

► Let us consider the case $K \leq L$. In this case we have $k < b$. We also have

$$*\widehat{PDIC}(x, \lambda; K, L; r, \delta) = \frac{e^{2b\theta}}{\theta D \psi(\theta\sqrt{D})} \int_k^{+\infty} e^{my} (x e^{\sigma y} - K) K_{\lambda,D}(b-y) dy$$

where

$$K_{\lambda,D}(b-y) = \int_0^{+\infty} z \exp\left(-\frac{z^2}{2D} - |b-z-y|\theta\right) dz.$$

For $y \in [b, +\infty[$ we have $b-y \leq 0$. $K_{\lambda,D}(b-y)$ has already been computed in this case. For $y \in [k, b]$, we have $b-y \geq 0$. We have to compute $K_{\lambda,D}$ in such a case. Let a denote $b-y$, $a > 0$.

$$\begin{aligned} K_{\lambda,D}(a) &= \int_0^\infty z \exp\left(-\frac{z^2}{2D} - |a-z|\theta\right) dz, \\ &= \underbrace{\int_0^a z \exp\left(-\frac{z^2}{2D} - (a-z)\theta\right) dz}_A + \underbrace{\int_a^{+\infty} z \exp\left(-\frac{z^2}{2D} + (a-z)\theta\right) dz}_B. \end{aligned}$$

▷ The valuation of B

$$\begin{aligned}
\int_a^{+\infty} z e^{\left(-\frac{z^2}{2D} + (a-z)\theta\right)} dz &= e^{a\theta} \int_a^{+\infty} z e^{\left(-\frac{z^2}{2D} - z\theta\right)} dz, \\
&= e^{a\theta} \int_a^{+\infty} D \left(\frac{z}{D} + \theta - \theta\right) e^{\left(-\frac{z^2}{2D} - z\theta\right)} dz, \\
&= e^{a\theta} D \left[-e^{\left(-\frac{z^2}{2D} - z\theta\right)} \right]_a^{+\infty} - e^{a\theta} \theta D \int_a^{+\infty} e^{\left(-\frac{z^2}{2D} - z\theta\right)} dz, \\
&= e^{a\theta} D e^{-\frac{a^2}{2D} - a\theta} - e^{a\theta} \theta D \int_a^{+\infty} e^{-\frac{1}{2} \left(\frac{z}{\sqrt{D}} + \theta\sqrt{D}\right)^2 + \lambda D} dz, \\
&= D e^{-\frac{a^2}{2D}} - e^{a\theta} \theta D e^{\lambda D} \int_a^{+\infty} e^{-\frac{1}{2} \left(\frac{z}{\sqrt{D}} + \theta\sqrt{D}\right)^2} dz.
\end{aligned}$$

By changing variables $u = \frac{z}{\sqrt{D}} + \theta\sqrt{D}$, we get

$$\begin{aligned}
B &= D e^{-\frac{a^2}{2D}} - e^{a\theta} \theta D e^{\lambda D} \sqrt{D} \int_{\frac{a}{\sqrt{D}} + \theta\sqrt{D}}^{+\infty} e^{-\frac{1}{2} u^2} du, \\
&= D e^{-\frac{a^2}{2D}} - e^{a\theta} \theta D e^{\lambda D} \sqrt{2\pi D} (1 - \mathcal{N}\left(\frac{a}{\sqrt{D}} + \theta\sqrt{D}\right)).
\end{aligned}$$

We finally obtain :

$$B = D \left[e^{-\frac{a^2}{2D}} - e^{a\theta} \theta \sqrt{2\pi D} e^{\lambda D} \left(1 - \mathcal{N}\left(\frac{a}{\sqrt{D}} + \theta\sqrt{D}\right) \right) \right]. \quad (3.10)$$

▷ The valuation of A

$$\begin{aligned}
\int_0^a z \exp\left(-\frac{z^2}{2D} - (a-z)\theta\right) dz &= e^{-a\theta} \int_0^a z e^{-\frac{z^2}{2D} + z\theta} dz, \\
&= e^{-a\theta} \int_0^a D \left(\frac{z}{D} + \theta - \theta\right) e^{-\frac{z^2}{2D} + z\theta} dz, \\
&= e^{-a\theta} D \left[-e^{-\frac{z^2}{2D} + z\theta} \right]_0^a + D\theta e^{-a\theta} \int_0^a e^{-\frac{z^2}{2D} + z\theta} dz, \\
&= -D e^{-\frac{a^2}{2D}} + D e^{-a\theta} + D\theta e^{\lambda D} e^{-a\theta} \int_0^a e^{-\frac{1}{2} \left(\frac{z}{\sqrt{D}} - \theta\sqrt{D}\right)^2} dz, \\
&= -D e^{-\frac{a^2}{2D}} + D e^{-a\theta} + D\theta e^{\lambda D} e^{-a\theta} \sqrt{D} \int_{-\theta\sqrt{D}}^{\frac{a}{\sqrt{D}} - \theta\sqrt{D}} e^{-\frac{u^2}{2}} du.
\end{aligned}$$

By changing variables $u = \frac{z}{\sqrt{D}}$, we get

$$A = D \left(e^{-a\theta} - e^{-\frac{a^2}{2D}} + \sqrt{2\pi D} \theta e^{\lambda D} e^{-a\theta} \left(\mathcal{N}\left(\frac{a}{\sqrt{D}} - \theta\sqrt{D}\right) - \mathcal{N}(-\theta\sqrt{D}) \right) \right).$$

Finally, in the case $a = b - y \geq 0$ we get

$$\begin{aligned}
K_{\lambda,D}(a) &= D \left[e^{-a\theta} + e^{\lambda D} \theta \sqrt{2\pi D} \left(e^{-a\theta} \left[\mathcal{N}\left(\frac{a}{\sqrt{D}} - \theta\sqrt{D}\right) - \mathcal{N}(-\theta\sqrt{D}) \right] \right. \right. \\
&\quad \left. \left. - e^{a\theta} \left[1 - \mathcal{N}\left(\frac{a}{\sqrt{D}} + \theta\sqrt{D}\right) \right] \right) \right]. \quad (3.11)
\end{aligned}$$

So, we find

$$\begin{aligned}
&*\widehat{PDIC}(x, \lambda; K, L; r, \delta) = \\
&\frac{e^{2b\theta}}{\theta\psi(\theta\sqrt{D})} \left[\int_k^b e^{my} (x e^{\sigma y} - K) \left[e^{-(b-y)\theta} + \theta\sqrt{2\pi D} e^{\lambda D} \left(e^{-(b-y)\theta} \left[\mathcal{N}\left(\frac{b-y}{\sqrt{D}} - \theta\sqrt{D}\right) - \mathcal{N}(-\theta\sqrt{D}) \right] \right. \right. \right. \right. \\
&\quad \left. \left. \left. - e^{(b-y)\theta} \left[1 - \mathcal{N}\left(\frac{b-y}{\sqrt{D}} + \theta\sqrt{D}\right) \right] \right) \right] dy + \int_b^{+\infty} e^{my} (x e^{\sigma y} - K) e^{(b-y)\theta} \psi(\theta\sqrt{D}) dy \right].
\end{aligned}$$

After doing long but not difficult computations we get, for $\mathbf{K} \leq \mathbf{L} \leq \mathbf{x}$,

$$\begin{aligned}
*\widehat{PDIC}(x, \lambda; K, L) &= \frac{e^{(m+\theta)b}}{\psi(\theta\sqrt{D})} \left(\frac{2K}{m^2 - \theta^2} \left[\psi(-\sqrt{D}m) + \sqrt{2\pi D} e^{\frac{Dm^2}{2}} m \mathcal{N}(-d - \sqrt{D}m) \right] \right. \\
&\quad - \frac{2L}{(m + \sigma)^2 - \theta^2} \left[\psi(-\sqrt{D}(m + \sigma)) + \sqrt{2\pi D} e^{\frac{D}{2}(m+\sigma)^2} (m + \sigma) \mathcal{N}(-d - \sqrt{D}(m + \sigma)) \right] \Big) \\
&\quad + \frac{K e^{(m+\theta)k}}{\theta \psi(\theta\sqrt{D})} \left(\frac{1}{m + \theta} - \frac{1}{m + \sigma + \theta} \right) \left[\psi(-\theta\sqrt{D}) + \theta e^{\lambda D} \sqrt{2\pi D} \mathcal{N}(d - \theta\sqrt{D}) \right] \\
&\quad + \frac{e^{\lambda D} \sqrt{2\pi D}}{\psi(\theta\sqrt{D})} K e^{2b\theta} e^{(m-\theta)k} \mathcal{N}(-d - \theta\sqrt{D}) \left(\frac{1}{m + \sigma - \theta} - \frac{1}{m - \theta} \right), \quad (3.12)
\end{aligned}$$

where $d = \frac{b-k}{\sqrt{D}}$.

3.2. The valuation of a Parisian down-and-out call with $b \leq 0$. To find the valuation of a Parisian down-and-out call we can use the relation between

$*PDIC(x, T; K, L; r, \delta)$, $*PDOC(x, T; K, L; r, \delta)$ and the Black-Scholes price of an European call

$$*PDOC(x, T; K, L; r, \delta) = *BSC(x, T; K; r, \delta) - *PDIC(x, T; K, L; r, \delta),$$

where

$$*BSC(x, T; K; r, \delta) = \mathbb{E}_{\mathcal{P}}(e^{mZ_T} (xe^{\sigma Z_T} - K)^+).$$

Therefore, we obtain

$$*\widehat{PDOC}(x, \lambda; K, L; r, \delta) = *\widehat{BSC}(x, \lambda; K; r, \delta) - *\widehat{PDIC}(x, \lambda; K, L; r, \delta).$$

Now, we need to find the valuation of $*\widehat{BSC}(x, \lambda; K, L; r, \delta)$

$$\begin{aligned}
*BSC(x, T; K; r, \delta) &= \mathbb{E}_{\mathcal{P}}(e^{mZ_T} (xe^{\sigma Z_T} - K)^+), \\
&= \int_{-\infty}^{+\infty} e^{mz} (xe^{\sigma z} - K)^+ \frac{1}{\sqrt{2\pi T}} e^{-\frac{z^2}{2T}} dz. \\
*\widehat{BSC}(x, \lambda; K; r, \delta) &= \int_{-\infty}^{+\infty} e^{mz} (xe^{\sigma z} - K)^+ \int_0^{+\infty} \frac{e^{-\lambda t}}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dt dz.
\end{aligned}$$

Thanks to [Appendix B](#) we have

$$\int_0^{+\infty} \frac{e^{-\lambda t}}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dt = \frac{e^{-|z|\theta}}{\theta}.$$

Then, we can write

$$\begin{aligned}
*\widehat{BSC}(x, \lambda; K; r, \delta) &= \int_{-\infty}^{+\infty} e^{mz} (xe^{\sigma z} - K)^+ \frac{e^{-|z|\sqrt{2\lambda}}}{\theta} dz, \\
&= \int_{\frac{1}{\sigma} \ln(\frac{K}{x})}^{+\infty} e^{mz} (xe^{\sigma z} - K) \frac{e^{-|z|\theta}}{\theta} dz.
\end{aligned}$$

3.2.1. case $K \geq x$. In this case, we can easily compute $*\widehat{BSC}(x, \lambda; K)$. Using the previous notations we have

$$*\widehat{BSC}(x, \lambda; K; r, \delta) = \int_K^{+\infty} e^{mz} (xe^{\sigma z} - K) \frac{e^{-|z|\theta}}{\theta} dz$$

and in this case $\frac{1}{\sigma} \ln\left(\frac{K}{x}\right) \geq 0$, so we get

$$\begin{aligned}
*\widehat{BSC}(x, \lambda; K; r, \delta) &= \int_K^{+\infty} e^{mz} (xe^{\sigma z} - K) \frac{e^{-z\theta}}{\theta} dz, \\
&= \frac{1}{\theta} \int_K^{+\infty} e^{(m+\sigma-\theta)z} dz - \frac{K}{\theta} \int_K^{+\infty} x e^{(m-\theta)z} dz, \\
&= -\frac{K}{\theta} \frac{e^{(m-\theta)k}}{m + \sigma - \theta} + \frac{K}{\theta} \frac{e^{(m-\theta)k}}{m - \theta}. \quad (3.13)
\end{aligned}$$

Then, we get the formula for the Laplace transform of the Black-Scholes call in the case $K \geq x$:

$$*\widehat{BSC}(x, \lambda; K; r, \delta) = \frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right), \text{ for } K \geq x.$$

To obtain $*\widehat{PDOC}(x, \lambda; K, L; r, \delta)$ we only need to subtract $*\widehat{PDIIC}(x, \lambda; K, L; r, \delta)$.

$$\begin{aligned} *\widehat{PDOC}(x, \lambda; K, L; r, \delta) &= \frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) - \frac{\psi(-\theta\sqrt{D})e^{2b\theta}}{\theta\psi(\theta\sqrt{D})} \\ &\quad K e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right), \\ &= \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \frac{K}{\theta} e^{(m-\theta)k} \left[1 - \frac{e^{2b\theta}\psi(-\theta\sqrt{D})}{\psi(\theta\sqrt{D})} \right]. \end{aligned} \quad (3.14)$$

Furthermore,

$$\psi(-\theta\sqrt{D}) = \psi(\theta\sqrt{D}) - \theta\sqrt{2\pi D}e^{\lambda D}. \quad (3.15)$$

So, the following formula holds

$$\begin{aligned} *\widehat{PDOC}(x, \lambda; K, L; r, \delta) &= \left[1 - e^{2b\theta} + \frac{\theta e^{2b\theta}\sqrt{2\pi D}e^{\lambda D}}{\psi(\theta\sqrt{D})} \right] \frac{K}{\theta} e^{(m-\theta)k} \\ &\quad \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \text{ for } K \geq x \geq L. \end{aligned} \quad (3.16)$$

3.2.2. case $K \leq x$. In this case the integral has to be split.

$$\begin{aligned} *\widehat{BSC}(x, \lambda; K; r, \delta) &= \int_{\frac{1}{\sigma}\ln(\frac{K}{x})}^{+\infty} e^{mz}(xe^{\sigma z} - K) \frac{e^{-|z|\theta}}{\theta} dz, \\ &= \int_K^0 e^{mz}(xe^{\sigma z} - K) \frac{e^{z\theta}}{\theta} dz + \int_0^{+\infty} e^{mz}(xe^{\sigma z} - K) \frac{e^{-z\theta}}{\theta} dz, \\ &= \frac{1}{\theta} \left(\int_K^0 xe^{(m+\sigma+\theta)z} - Ke^{(m+\theta)z} dz + \int_0^{+\infty} xe^{(m+\sigma-\theta)z} - Ke^{(m-\theta)z} dz \right), \\ &= \frac{1}{\theta} \left(\frac{x}{m+\sigma+\theta} - \frac{K}{m+\theta} - \frac{Ke^{(m+\theta)k}}{m+\sigma+\theta} + \frac{Ke^{(m+\theta)k}}{m+\theta} - \frac{x}{m+\sigma-\theta} + \frac{K}{m+\theta} \right), \\ &= \frac{2K}{m^2 - \theta^2} - \frac{2x}{(m+\sigma)^2 - \theta^2} + \frac{Ke^{(m+\theta)k}}{\theta} \left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right). \end{aligned} \quad (3.17)$$

So, we get

$$\begin{aligned} *\widehat{BSC}(x, \lambda; K; r, \delta) &= \frac{2K}{m^2 - \theta^2} - \frac{2x}{(m+\sigma)^2 - \theta^2} + \frac{Ke^{(m+\theta)k}}{\theta} \\ &\quad \left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right), \text{ for } K \leq x. \end{aligned} \quad (3.18)$$

Finally, we come up with the following formula for the valuation of a Parisian down-and-out call with $b \leq 0$

► Case $K \geq L$.

$$\begin{aligned} *\widehat{PDOC}(x, \lambda; K, L; r, \delta) &= \frac{2K}{m^2 - \theta^2} - \frac{2x}{(m+\sigma)^2 - \theta^2} + \frac{Ke^{(m+\theta)k}}{\theta} \left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right) \\ &\quad - \frac{\psi(-\theta\sqrt{D})e^{2b\theta}}{\theta\psi(\theta\sqrt{D})} K e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right), \text{ for } x \geq K \geq L. \end{aligned} \quad (3.19)$$

► Case $K \leq L$.

$$\begin{aligned}
& * \widehat{PDO C}(x, \lambda; K, L) = \\
& \frac{2K}{m^2 - \theta^2} \left[1 - \frac{e^{(m+\theta)b}}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}m) + \sqrt{2\pi D} e^{\frac{Dm^2}{2}} m \mathcal{N}(-d - \sqrt{D}m) \right) \right] \\
& - \frac{2}{(m + \sigma^2) - \theta^2} \left[x - \frac{Le^{(m+\theta)b}}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}(m + \sigma)) + \sqrt{2\pi D} e^{\frac{D(m+\sigma)^2}{2}} \right. \right. \\
& \quad \left. \left. (m + \sigma) \mathcal{N}(-d - \sqrt{D}(m + \sigma)) \right) \right] \\
& + \frac{Ke^{(m+\theta)b}}{\theta} \left(\frac{1}{m + \theta} - \frac{1}{m + \theta + \sigma} \right) \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\theta\sqrt{D}) + \theta e^{\lambda D} \sqrt{2\pi D} \mathcal{N}(d - \theta\sqrt{D}) \right) \right] \\
& - \frac{e^{\lambda D} \sqrt{2\pi D}}{\psi(\theta\sqrt{D})} Ke^{2b\theta} e^{(m-\theta)b} \mathcal{N}(-d - \theta\sqrt{D}) \left(\frac{1}{m - \theta + \sigma} - \frac{1}{m - \theta} \right),
\end{aligned}$$

for $K \leq L \leq x$.

3.3. The valuation of a Parisian down-and-out call with $b > 0$.

3.3.1. *reduction to the case $b = 0$.* If b is positive and $T_b^- \geq T \geq D$, then $T_b \leq D$.

Therefore, the discounted value of a down-and-out call in the case $L > x$ is

$$* PDO C(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{T_b^- \geq T\}} \mathbf{1}_{\{T_b \leq D\}} [xe^{\sigma Z_T} - K]^+ e^{mZ_T}). \quad (3.20)$$

We can also write :

$$* PDO C(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}} \left(\mathbb{E}_{\mathcal{P}}[\mathbf{1}_{\{T_b^- \geq T\}} \mathbf{1}_{\{T_b \leq D\}}] [xe^{\sigma(Z_T - Z_{T_b} + b)} - K]^+ e^{m(Z_T - Z_{T_b} + b)} \mid \mathcal{F}_{T_b} \right).$$

We have $\mathcal{F}_{T_b} = \{A \in \mathcal{A}, \forall t \geq 0, A \cap \{T_b \leq t\} \in \mathcal{F}_t\}$, then $\{T_b \leq D\} \in \mathcal{F}_{T_b}$, because

$$\{T_b \leq D\} \cap \{T_b \leq t\} = \{T_b \leq D \wedge t\}$$

and

$$\{T_b \leq D \wedge t\} \in \mathcal{F}_{t \wedge D} \subset \mathcal{F}_t.$$

Therefore $\mathbf{1}_{\{T_b \leq D\}}$ is \mathcal{F}_{T_b} -measurable.

So we get

$$* PDO C(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_b \leq D\}} \mathbb{E}_{\mathcal{P}}[\mathbf{1}_{\{T_b^- - T_b \geq T - T_b\}}] [xe^{\sigma Z_T - Z_{T_b} + b} - K]^+ e^{m(Z_T - Z_{T_b} + b)} \mid \mathcal{F}_{T_b} \right).$$

Relying on the strong Markov property we can write that $T_b^- - T_b \stackrel{\text{law}}{=} T_0^-$.

Hence

$$* PDO C(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_b \leq D\}} \mathbb{E}_{\mathcal{P}}[\mathbf{1}_{\{T_0^- \geq T - T_b\}}] [xe^{\sigma(Z_T - Z_{T_b} + b)} - K]^+ e^{m(Z_T - Z_{T_b} + b)} \mid \mathcal{F}_{T_b} \right).$$

Let W_t denote $Z_{T_b+t} - Z_{T_b}$, relying on the strong Markov property W_t is independent of \mathcal{F}_{T_b} .

Let Y_t denote $\mathbf{1}_{\{T_0^- \geq T - t\}} [xe^{\sigma(W_{T-t} + b)} - K]^+ e^{m(W_{T-t} + b)}$.

- Y_t is independent of \mathcal{F}_{T_b} ,
- T_b is \mathcal{F}_{T_b} -measurable so we can write $\mathbb{E}[Y_{T_b} \mid \mathcal{F}_{T_b}] = \mathbb{E}[Y_t]_{t=T_b}$.

Hence we have

$$\begin{aligned}
* PDO C(x, T; K, L; r, \delta) &= \mathbb{E}[\mathbf{1}_{\{T_b \leq D\}} \mathbb{E}[Y_t]_{t=T_b}], \\
&= \int_{-\infty}^{\infty} \mathbf{1}_{\{u \leq D\}} \mathbb{E}_{\mathcal{P}}[Y_u] \mu_b(du)
\end{aligned}$$

where $\mu_b(du)$ is the law of T_b recalled in [Appendix A](#). We get

$$* PDO C(x, T; K, L; r, \delta) = \int_0^D \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_0^- \geq T - u\}} [xe^{\sigma(W_{T-u} + b)} - K]^+ e^{m(W_{T-u} + b)} \right) \mu_b(du).$$

So, we have

$$* PDO C(x, T; K, L; r, \delta) = e^{mb} \int_0^D \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_0^- \geq T - u\}} [xe^{\sigma b} e^{\sigma W_{T-u}} - K]^+ e^{mW_{T-u}} \right) \mu_b(du).$$

As $b = \frac{1}{\sigma} \ln \left(\frac{L}{x} \right)$, we get

$$*PDOC(x, T; K, L; r, \delta) = Le^{mb} \int_0^D \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_0^- \geq T-u\}} [e^{\sigma W_{T-u}} - K/L]^+ e^{mW_{T-u}} \right) \mu_b(du).$$

The price of a Parisian down-and-out call in the case $b > 0$ is given by

$$*PDOC(x, T; K, L; r, \delta) = Le^{mb} \int_0^D *PDOC^0(T-u; K/L; r, \delta) \mu_b(du) \quad (3.21)$$

where

$$*PDOC^0(T; K; r, \delta) = \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{T_0^- \geq T\}} [e^{\sigma Z_T} - K]^+ e^{mZ_T} \right).$$

3.3.2. *The Laplace transform of $*PDOC(x, T; K, L; r, \delta)$.* If we consider the Laplace transform of $*PDOC(x, T; K, L; r, \delta)$ with respect to T , we get

$$\begin{aligned} *\widehat{PDOC}(x, \lambda; K, L; r, \delta) &= \int_0^{+\infty} e^{-\lambda t} Le^{mb} \int_0^D *PDOC^0(t-u; K/L; r, \delta) \mu_b(du) \mathbf{1}_{\{t-u>0\}} dt, \\ &= Le^{mb} \int_0^D \mu_b(du) \int_u^{+\infty} e^{-\lambda t} *PDOC^0(t-u; K/L; r, \delta) dt, \\ &\quad \text{we change variables } (v, u) = (t-u, u) \\ &= Le^{mb} \int_0^D \mu_b(du) e^{-\lambda u} \int_0^{+\infty} e^{-\lambda v} *PDOC^0(v; K/L; r, \delta) dv, \\ &= Le^{mb} \int_0^D \mu_b(du) e^{-\lambda u} *\widehat{PDOC}^0(\lambda; K/L; r, \delta). \end{aligned}$$

If we compute $\int_0^D \mu_b(du) e^{-\lambda u}$, we find $e^{-\theta b} \mathcal{N} \left(\theta \sqrt{D} - \frac{b}{\sqrt{D}} \right) + e^{\theta b} \mathcal{N} \left(-\sqrt{D} \theta - \frac{b}{\sqrt{D}} \right)$ as proved in *Appendix A*, where \mathcal{N} denotes the standard normal cumulative distribution. Finally, we come up with the following formula

$$\begin{aligned} *\widehat{PDOC}(x, \lambda; K, L; r, \delta) &= L \left[e^{(m-\theta)b} \mathcal{N} \left(\sqrt{D} \theta - \frac{b}{\sqrt{D}} \right) + \right. \\ &\quad \left. e^{(m+\theta)b} \mathcal{N} \left(-\sqrt{D} \theta - \frac{b}{\sqrt{D}} \right) \right] *\widehat{PDOC}^0(\lambda; K/L; r, \delta), \text{ for } L \geq x. \quad (3.22) \end{aligned}$$

► Case $K \geq L$. $*\widehat{PDOC}^0(\lambda; K/L)$ has already been computed in (3.16), and we had found

$$*\widehat{PDOC}^0(\lambda; K/L; r, \delta) = \frac{\sqrt{2\pi D} e^{\lambda D}}{\psi(\theta \sqrt{D})} \frac{K}{L} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right), \text{ for } K > L.$$

Then, we now have an explicit formula for the Laplace transform of $*PDOC(x, T; K, L; r, \delta)$ when $K > L$.

$$\begin{aligned} *\widehat{PDOC}(x, \lambda; K, L; r, \delta) &= \left[e^{(m-\theta)b} \mathcal{N} \left(\theta \sqrt{D} - \frac{b}{\sqrt{D}} \right) + e^{(m+\theta)b} \mathcal{N} \left(-\theta \sqrt{D} - \frac{b}{\sqrt{D}} \right) \right] \\ &\quad \frac{\sqrt{2\pi D} e^{\lambda D}}{\psi(\theta \sqrt{D})} K e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right), \text{ for } K \geq L \geq x. \quad (3.23) \end{aligned}$$

► Case $K \leq L$. In this case, we have

$$\begin{aligned}
& * \widehat{PDO C}^0(\lambda; K/L) = \\
& \frac{2K}{L(m^2 - \theta^2)} \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}m) + \sqrt{2\pi D} e^{\frac{Dm^2}{2}} m \mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}m\right) \right) \right] \\
& - \frac{2}{(m + \sigma^2) - \theta^2} \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}(m + \sigma)) \right. \right. \\
& \quad \left. \left. + \sqrt{2\pi D} e^{\frac{D(m+\sigma)^2}{2}} (m + \sigma) \mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}(m + \sigma)\right) \right) \right] \\
& + \frac{K e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})}}{L\theta} \left(\frac{1}{m + \theta} - \frac{1}{m + \theta + \sigma} \right) \\
& \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\theta\sqrt{D}) + \theta e^{\lambda D} \sqrt{2\pi D} \mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \theta\sqrt{D}\right) \right) \right] \\
& - \frac{e^{\lambda D} \sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \frac{K}{L} e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} \mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \theta\sqrt{D}\right) \left(\frac{1}{m - \theta + \sigma} - \frac{1}{m - \theta} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& * \widehat{PDO C}(x, \lambda; K, L; r, \delta) = L \left(e^{(m-\theta)b} \mathcal{N}\left(\sqrt{D}\theta - \frac{b}{\sqrt{D}}\right) + e^{(m+\theta)b} \mathcal{N}\left(-\sqrt{D}\theta - \frac{b}{\sqrt{D}}\right) \right) \\
& \left\{ \frac{2K}{L(m^2 - \theta^2)} \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}m) + \sqrt{2\pi D} e^{\frac{Dm^2}{2}} m \mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}m\right) \right) \right] \right. \\
& - \frac{2}{(m + \sigma^2) - \theta^2} \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}(m + \sigma)) \right. \right. \\
& \quad \left. \left. + \sqrt{2\pi D} e^{\frac{D(m+\sigma)^2}{2}} (m + \sigma) \mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}(m + \sigma)\right) \right) \right] \\
& + \frac{K e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})}}{L\theta} \left(\frac{1}{m + \theta} - \frac{1}{m + \theta + \sigma} \right) \\
& \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\theta\sqrt{D}) + \theta e^{\lambda D} \sqrt{2\pi D} \mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \theta\sqrt{D}\right) \right) \right] \\
& \left. - \frac{e^{\lambda D} \sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \frac{K}{L} e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} \mathcal{N}\left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \theta\sqrt{D}\right) \left(\frac{1}{m - \theta + \sigma} - \frac{1}{m - \theta} \right) \right\},
\end{aligned}$$

for $K \leq L$ and $x \leq L$.

3.4. The valuation of a Parisian down-and-in call with $b > 0$. So far, we have managed to find explicit formulae for the Laplace transforms of the down-and-out call prices with $b > 0$. Now, we will use the relationships existing between down-and-out options and down-and-in options to compute the price of a down-and-in call in the case $b > 0$. In fact, the following formula holds

$$* \widehat{PDIC}(x, \lambda; K, L; r, \delta) = * \widehat{BSC}(x, \lambda, K, r, \delta) - * \widehat{PDO C}(x, \lambda; K, L; r, \delta)$$

where $* \widehat{PDO C}(x, \lambda; K, L; r, \delta)$ has already been computed above in the *Section 3.3.2* for $b > 0$ and $* \widehat{BSC}(x, \lambda, K, r, \delta)$ has been calculated in (3.18) and (3.14). We simply recall the formula

$$* \widehat{BSC}(x, \lambda, K, r, \delta) = \begin{cases} \frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m - \theta} - \frac{1}{m - \theta + \sigma} \right) & \text{if } K \geq x, \\ \frac{2K}{m^2 - \theta^2} - \frac{2x}{(m + \sigma)^2 - \theta^2} + \frac{K}{\theta} e^{(m+\theta)k} \left(\frac{1}{m + \theta} - \frac{1}{m + \theta + \sigma} \right) & \text{if } K \leq x. \end{cases}$$

If we put all the terms together we find the following formula

► Case $K \geq L$.

$$\begin{aligned} * \widehat{PDIC}(x, \lambda; K, L; r, \delta) = & \frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m-\theta+\sigma} \right) \\ & - \left(e^{(m-\theta)b} \mathcal{N} \left(\theta\sqrt{D} - \frac{b}{\sqrt{D}} \right) + e^{(m+\theta)b} \mathcal{N} \left(-\theta\sqrt{D} - \frac{b}{\sqrt{D}} \right) \right) \\ & \frac{\sqrt{2\pi D} e^{\lambda D}}{\psi(\theta\sqrt{D})} K e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right), \text{ for } K \geq L \geq x. \end{aligned}$$

► Case $K \leq L$.

$$\begin{aligned} * \widehat{PDIC}(x, \lambda; K, L; r, \delta) = & \frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m-\theta+\sigma} \right) \\ & - L \left(e^{(m-\theta)b} \mathcal{N} \left(\sqrt{D}\theta - \frac{b}{\sqrt{D}} \right) + e^{(m+\theta)b} \mathcal{N} \left(-\sqrt{D}\theta - \frac{b}{\sqrt{D}} \right) \right) \\ & \left\{ \frac{2K}{L(m^2 - \theta^2)} \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}m) + \sqrt{2\pi D} e^{\frac{Dm^2}{2}} m \mathcal{N} \left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}m \right) \right) \right] \right. \\ & \left. - \frac{(m+\sigma^2) - \theta^2}{(m+\sigma)^2 - \theta^2} \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}(m+\sigma)) + \sqrt{2\pi D} e^{\frac{D(m+\sigma)^2}{2}} (m+\sigma) \mathcal{N} \left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}(m+\sigma) \right) \right) \right] \right. \\ & \left. + \frac{K e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})}}{L\theta} \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right) \right. \\ & \left. \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\theta\sqrt{D}) + \theta e^{\lambda D} \sqrt{2\pi D} \mathcal{N} \left(\frac{\ln(\frac{L}{K})}{\sigma\sqrt{D}} - \theta\sqrt{D} \right) \right) \right] \right. \\ & \left. - \frac{e^{\lambda D} \sqrt{2\pi D} K}{\psi(\theta\sqrt{D}) L} e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} \mathcal{N} \left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \theta\sqrt{D} \right) \left(\frac{1}{m-\theta+\sigma} - \frac{1}{m-\theta} \right) \right\}, \text{ for } x \leq K \leq L. \\ \\ * \widehat{PDIC}(x, \lambda; K, L; r, \delta) = & \frac{2K}{m^2 - \theta^2} - \frac{2x}{(m+\sigma)^2 - \theta^2} + \frac{K}{\theta} e^{(m+\theta)k} \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right) \\ & - L \left(e^{(m-\theta)b} \mathcal{N} \left(\sqrt{D}\theta - \frac{b}{\sqrt{D}} \right) + e^{(m+\theta)b} \mathcal{N} \left(-\sqrt{D}\theta - \frac{b}{\sqrt{D}} \right) \right) \\ & \left\{ \frac{2K}{L(m^2 - \theta^2)} \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}m) + \sqrt{2\pi D} e^{\frac{Dm^2}{2}} m \mathcal{N} \left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}m \right) \right) \right] \right. \\ & - \frac{2}{(m+\sigma^2) - \theta^2} \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\sqrt{D}(m+\sigma)) \right. \right. \\ & \left. \left. + \sqrt{2\pi D} e^{\frac{D(m+\sigma)^2}{2}} (m+\sigma) \mathcal{N} \left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \sqrt{D}(m+\sigma) \right) \right) \right] \right. \\ & \left. + \frac{K e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})}}{L\theta} \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right) \right. \\ & \left. \left[1 - \frac{1}{\psi(\theta\sqrt{D})} \left(\psi(-\theta\sqrt{D}) + \theta e^{\lambda D} \sqrt{2\pi D} \mathcal{N} \left(\frac{\ln(\frac{L}{K})}{\sigma\sqrt{D}} - \theta\sqrt{D} \right) \right) \right] \right. \\ & \left. - \frac{e^{\lambda D} \sqrt{2\pi D} K}{\psi(\theta\sqrt{D}) L} e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} \mathcal{N} \left(\frac{\ln(\frac{K}{L})}{\sigma\sqrt{D}} - \theta\sqrt{D} \right) \left(\frac{1}{m-\theta+\sigma} - \frac{1}{m-\theta} \right) \right\}, \text{ for } K \leq x \leq L. \end{aligned}$$

4. THE PARISIAN UP CALLS

This section will go exactly through the same points as the previous one but considering the Up calls instead of the Down ones this time. Once again the organisation of this section is based on the presentation scheme.

4.1. The valuation of a Parisian Up-and-in call with $b \geq 0$. The owner of an up-and-in option receives the pay-off if S makes an excursion above the level L older than D before the maturity time T , which is exactly the same as saying Brownian motion Z makes an excursion above b older than D . Using the previous notations we can write :

$$*PUIC(x, T; K, L; r, \delta) = \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{\{T_b^+ < T\}\}}(xe^{\sigma Z_T} - K)^+ e^{mZ_T}), \quad (4.1)$$

where

$$T_b^+ = \inf \{t > 0 | \mathbf{1}_{\{\{Z_t > b\}\}}(t - g_t^b) > D\}. \quad (4.2)$$

The computation of $*PUIC(x, T; K, L; r, \delta)$ for $b > 0$ is exactly the same as the computation of $*PDIC(x, T; K, L; r, \delta)$ for $b < 0$. We just have to find the law of T_b^+ . We have

$$*PUIC(x, T; K, L; r, \delta) = \int_{-\infty}^{+\infty} \mathbb{E}_{\mathcal{P}}(\mathbf{1}_{\{\{T_b^+ < T\}\}} \mathcal{P}_{T-T_b^+}(f_x)(z)) \nu(dz), \quad (4.3)$$

where

- $f_x(z) = e^{mz}(xe^{\sigma z} - K)^+$,
- $\mathcal{P}_t(f_x)(z) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} f_x(u) \exp\left(-\frac{(u-z)^2}{2t}\right) du$,
- $\nu(dz)$ is the law of $Z_{T_b^+}$.

We have

$$*PUIC(x, T; K, L; r, \delta) = \int_{-\infty}^{+\infty} f_x(y) h_b(T, y) dy, \quad (4.4)$$

where

$$h_b(t, y) = \int_{-\infty}^{\infty} \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{\{T_b^+ < t\}\}} \frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^+)}\right)}{\sqrt{2\pi(t-T_b^+)}} \right) \nu(dz). \quad (4.5)$$

Since we consider the case $b > 0$, we can use the following expression for the law of $Z_{T_b^+}$, as it is proved in [Appendix C](#)

$$\mathbb{P}(Z_{T_b^+} \in dx) = \frac{dx}{D} (x - b) \exp\left(-\frac{(x-b)^2}{2D}\right) \mathbf{1}_{\{\{x \geq b\}\}}. \quad (4.6)$$

4.1.1. The Laplace transform of $*PUIC(x, T; K, L; r, \delta)$. We still have

$$\widehat{*PUIC}(x, \lambda; K, L; r, \delta) = \int_{-\infty}^{\infty} f_x(y) \int_0^{\infty} e^{-\lambda t} h_b(t, y) dt dy. \quad (4.7)$$

We would like to compute

$$\widehat{h}_b(\lambda, y) = \int_0^{\infty} e^{-\lambda t} h_b(t, y) dt. \quad (4.8)$$

We know that

$$h_b(t, y) = \int_b^{+\infty} \frac{z-b}{D} \exp\left(-\frac{(z-b)^2}{2D}\right) \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{\{T_b^+ < t\}\}} \frac{\exp\left(-\frac{(z-y)^2}{2(t-T_b^+)}\right)}{\sqrt{2\pi(t-T_b^+)}} \right) dz. \quad (4.9)$$

We can write

$$h_b(t, y) = \int_b^{+\infty} \frac{z-b}{D} \exp\left(-\frac{(z-b)^2}{2D}\right) \gamma(t, z-y) dz,$$

where

$$\gamma(t, x) = \mathbb{E}_{\mathcal{P}} \left(\mathbf{1}_{\{\{T_b^+ < t\}\}} \frac{\exp\left(-\frac{x^2}{2(t-T_b^+)}\right)}{\sqrt{2\pi(t-T_b^+)}} \right)$$

and we have

$$\widehat{h}_b(\lambda, y) = \int_b^{+\infty} \frac{z-b}{D} \exp\left(-\frac{(z-b)^2}{2D}\right) \int_0^{\infty} e^{-\lambda t} \gamma(t, z-y) dt dz. \quad (4.10)$$

So, we need to compute the Laplace transform of $\gamma(t, x)$

$$\int_0^\infty e^{-\lambda t} \gamma(t, x) dt = \mathbb{E}_{\mathcal{P}} \left(\int_{T_b^+}^\infty e^{-\lambda t} \frac{\exp\left(-\frac{x^2}{2(t-T_b^+)}\right)}{\sqrt{2\pi(t-T_b^+)}} dt \right).$$

By changing variables $u = t - T_b^+$, we get

$$\int_0^\infty e^{-\lambda t} \gamma(t, x) dt = \mathbb{E}_{\mathcal{P}}(e^{-\lambda T_b^+}) \int_0^\infty e^{-\lambda u} \frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi u}} du.$$

Using results from [Appendix C](#) and [B](#), we come up with

$$\int_0^\infty e^{-\lambda t} \gamma(t, x) dt = \frac{\exp[-(|x| + b)\theta]}{\theta\psi(\theta\sqrt{D})}. \quad (4.11)$$

Thanks to [\(4.10\)](#) we can rewrite

$$\hat{h}_b(\lambda, y) = \frac{e^{-b\theta}}{D\theta\psi(\theta\sqrt{D})} \int_0^\infty x \exp\left(-\frac{x^2}{2D} - |b + x - y|\theta\right) dx. \quad (4.12)$$

Let $K\mathbf{1}_{\{\lambda, D\}}(y - b)$ denote $\int_0^{+\infty} x \exp\left(-\frac{x^2}{2D} - |b + x - y|\theta\right) dx$.

4.1.2. *The valuation of $K\mathbf{1}_{\{\lambda, D\}}(y - b)$.* Let c denote $y - b$.

We have $K\mathbf{1}_{\{\lambda, D\}}(c) = \int_0^{+\infty} x \exp\left(-\frac{x^2}{2D} - |x - c|\theta\right) dx$.

► Case $K \geq L$. In such a case we have, for $y \in [k, +\infty[$, $y - b \geq 0$.

We can use the formula [\(3.11\)](#) to compute $K\mathbf{1}_{\{\lambda, D\}}(c)$. Then for $\hat{h}_b(\lambda, y)$ we get :

$$\begin{aligned} \hat{h}_b(\lambda, y) = \frac{e^{-b\theta}}{\theta\psi(\theta\sqrt{D})} & \left[e^{-(y-b)\theta} + \theta\sqrt{2\pi D} e^{\lambda D} \left(e^{-(y-b)\theta} \left[\mathcal{N}\left(\frac{y-b}{\sqrt{D}} - \theta\sqrt{D}\right) - \mathcal{N}(-\theta\sqrt{D}) \right] \right. \right. \\ & \left. \left. - e^{(y-b)\theta} \left(1 - \mathcal{N}\left(\frac{y-b}{\sqrt{D}} + \theta\sqrt{D}\right) \right) \right) \right]. \quad (4.13) \end{aligned}$$

By plugging this result in [\(4.7\)](#) and by doing long but easy calculations we get:

$$\begin{aligned} * \widehat{PUIC}(x, \lambda; K, L; r, \delta) &= e^{(m-\theta)b} \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{m^2 - \theta^2} e^{\frac{Dm^2}{2}} m \mathcal{N}(d + \sqrt{D}m) \right. \\ &\quad \left. - \frac{2L}{(m + \sigma)^2 - \theta^2} e^{\frac{D(m+\sigma)^2}{2}} (m + \sigma) \mathcal{N}(d + \sqrt{D}(m + \sigma)) \right] \\ &\quad + \frac{e^{-2b\theta}}{\psi(\theta\sqrt{D})} K e^{(m+\theta)k} e^{\lambda D} \sqrt{2\pi D} \mathcal{N}(d - \theta\sqrt{D}) \left(\frac{1}{m + \sigma + \theta} - \frac{1}{m + \theta} \right) \\ &\quad + \frac{e^{(m-\theta)k}}{\theta\psi(\theta\sqrt{D})} K \left(\frac{1}{m - \theta} - \frac{1}{m + \sigma - \theta} \right) \left(\psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D} \mathcal{N}(d - \theta\sqrt{D}) \right), \quad (4.14) \end{aligned}$$

for $x \leq L \leq K$.

► Case $K \leq L$. If $K \leq L$ we have $y - b \geq 0$ for $y \in [b, +\infty[$ and $y - b \leq 0$ for $y \in [k, b]$. So we get

$$\begin{aligned} * \widehat{PUIC}(x, \lambda; K, L; r, \delta) &= \frac{e^{-b\theta}}{D\theta\psi(\theta\sqrt{D})} \left(\int_k^b e^{my} (xe^{\sigma y} - K) \int_0^{+\infty} z \exp\left(-\frac{z^2}{2D} - (z + b - y)\theta\right) dz dy \right. \\ &\quad \left. + \int_b^{+\infty} e^{my} (xe^{\sigma y} - K) \int_0^{+\infty} z \exp\left(-\frac{z^2}{2D} - |z + b - y|\theta\right) dz dy \right). \end{aligned}$$

After doing computations we get

$$\begin{aligned} * \widehat{PUIC}(x, \lambda; K, L; r, \delta) &= \frac{e^{(m-\theta)b}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{m^2 - \theta^2} \psi(\sqrt{D}m) - \frac{2L}{(m+\sigma)^2 - \theta^2} \psi(\sqrt{D}(m+\sigma)) \right] \\ &+ \frac{e^{-2b\theta} \psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})} K e(m+\theta)k \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right), \text{ for } K \leq L \text{ and } x \leq L. \end{aligned} \quad (4.15)$$

4.2. The valuation of a Parisian Up-and-out call with $b \geq 0$. Thanks to the formula of $* \widehat{PUIC}(x, \lambda; K, L; r, \delta)$ we can find $* \widehat{PUOC}(x, \lambda; K, L; r, \delta)$. By using the relations between $* \widehat{PUIC}$ and $* \widehat{PUOC}$ and the Laplace transform of a Call when $x \leq K$ (which has been computed in [3.2.1](#)). So, for $x \leq L \leq K$, we obtain

$$\begin{aligned} * \widehat{PUOC}(x, \lambda; K, L; r, \delta) &= \frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \\ &- e^{(m-\theta)b} \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{m^2 - \theta^2} e^{\frac{Dm^2}{2}} m \mathcal{N}(d + \sqrt{D}m) \right. \\ &\quad \left. - \frac{2L}{(m+\sigma)^2 - \theta^2} e^{\frac{D(m+\sigma)^2}{2}} (m+\sigma) \mathcal{N}(d + \sqrt{D}(m+\sigma)) \right] \\ &- \frac{e^{-2b\theta}}{\psi(\theta\sqrt{D})} K e^{(m+\theta)k} e^{\lambda D} \sqrt{2\pi D} \mathcal{N}(d - \theta\sqrt{D}) \left(\frac{1}{m+\sigma+\theta} - \frac{1}{m+\theta} \right) \\ &- \frac{e^{(m-\theta)k}}{\theta\psi(\theta\sqrt{D})} K \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \left(\psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D} \mathcal{N}(d - \theta\sqrt{D}) \right), \end{aligned}$$

and for $K \leq x \leq L$, we have

$$\begin{aligned} * \widehat{PUOC}(x, \lambda; K, L; r, \delta) &= \\ &\frac{2K}{m^2 - \theta^2} - \frac{2x}{(m+\sigma)^2 - \theta^2} + \frac{K}{\theta} e^{(m+\theta)k} \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right) \\ &- \frac{e^{(m-\theta)b}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{m^2 - \theta^2} \psi(\sqrt{D}m) - \frac{2L}{(m+\sigma)^2 - \theta^2} \psi(\sqrt{D}(m+\sigma)) \right] \\ &- \frac{e^{-2b\theta} \psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})} K e(m+\theta)k \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right). \end{aligned} \quad (4.16)$$

Finally, for the case $x \leq K \leq L$ we get

$$\begin{aligned} * \widehat{PUOC}(x, \lambda; K, L; r, \delta) &= \\ &\frac{K}{\theta} e^{(m-\theta)k} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \\ &- \frac{e^{(m-\theta)b}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{m^2 - \theta^2} \psi(\sqrt{D}m) - \frac{2L}{(m+\sigma)^2 - \theta^2} \psi(\sqrt{D}(m+\sigma)) \right] \end{aligned} \quad (4.17)$$

$$- \frac{e^{-2b\theta} \psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})} K e(m+\theta)k \left(\frac{1}{m+\theta} - \frac{1}{m+\theta+\sigma} \right). \quad (4.18)$$

4.3. The valuation of a Parisian Up-and-out call with $b \leq 0$. We proceed exactly the same way as for the case $b \geq 0$.

We have

$$* PUOC(x, T; K, L; r, \delta) = Le^{mb} \int_0^D * PUOC^0(1, T-u; K/L, 1; r, \delta) \mu_b(du),$$

and for its Laplace transform we get

$$* \widehat{PUOC}(x, T; K, L; r, \delta) = Le^{mb} \int_0^D \mu_b(du) e^{-\lambda u} * \widehat{PUOC}^0(1, \lambda; K/L, 1; r, \delta).$$

To compute $\int_0^D \mu_b(du) e^{-\lambda u}$, we can refer to *Appendix A*, but by plugging $-b$ instead of b . So we find

$$\int_0^D \mu_b(du) e^{-\lambda u} = e^{\theta b} \mathcal{N}(\theta\sqrt{D} + \frac{b}{\sqrt{D}}) + e^{-\theta b} \mathcal{N}(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}).$$

Therefore, for $L \leq x$ we get

$$\begin{aligned} * \widehat{PUOC}(x, T; K, L; r, \delta) &= L \left(e^{(m+\theta)b} \mathcal{N}(\theta\sqrt{D} + \frac{b}{\sqrt{D}}) \right. \\ &\quad \left. + e^{(m-\theta)b} \mathcal{N}(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}) \right) * \widehat{PUOC}^0(1, \lambda; \frac{K}{L}, 1; r, \delta). \end{aligned}$$

Depending on the relative position of K and L , one of the following formula for $* \widehat{PUOC}(x, T; K, L; r, \delta)$ holds.

► Case $K \geq L$.

$$\begin{aligned} * \widehat{PUOC}^0(1, \lambda; \frac{K}{L}, 1; r, \delta) &= \frac{K}{L\theta} e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \\ &\quad - \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{L(m^2-\theta^2)} e^{\frac{Dm^2}{2}} m \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) + \sqrt{D}m\right) \right. \\ &\quad \left. - \frac{2}{(m+\sigma)^2 - \theta^2} e^{\frac{D(m+\sigma)^2}{2}} (m+\sigma) \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) + \sqrt{D}(m+\sigma)\right) \right] \\ &\quad - \frac{1}{\psi(\theta\sqrt{D})} \frac{K}{L} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} e^{\lambda D} \sqrt{2\pi D} \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) - \theta\sqrt{D}\right) \left(\frac{1}{m+\sigma+\theta} - \frac{1}{m+\theta} \right) \\ &\quad - \frac{e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})}}{\theta\psi(\theta\sqrt{D})} \frac{K}{L} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \left(\psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D} \mathcal{N}\left(\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) - \theta\sqrt{D}\right) \right). \end{aligned}$$

$$\begin{aligned} * \widehat{PUOC}(x, T; K, L; r, \delta) &= L \left(e^{(m+\theta)b} \mathcal{N}\left(\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right) + e^{(m-\theta)b} \mathcal{N}\left(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right) \right) \\ &\quad \left\{ \frac{K}{L\theta} e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) - \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{L(m^2-\theta^2)} e^{\frac{Dm^2}{2}} m \right. \right. \\ &\quad \left. \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) + \sqrt{D}m\right) - \frac{2}{(m+\sigma)^2 - \theta^2} e^{\frac{D(m+\sigma)^2}{2}} (m+\sigma) \right. \\ &\quad \left. \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) + \sqrt{D}(m+\sigma)\right) \right] - \frac{1}{\psi(\theta\sqrt{D})} \frac{K}{L} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} e^{\lambda D} \sqrt{2\pi D} \\ &\quad \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) - \theta\sqrt{D}\right) \left(\frac{1}{m+\sigma+\theta} - \frac{1}{m+\theta} \right) - \frac{e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})}}{\theta\psi(\theta\sqrt{D})} \frac{K}{L} \\ &\quad \left. \left(\frac{1}{m-\theta} - \frac{1}{m+\sigma-\theta} \right) \left(\psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D} \mathcal{N}\left(\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) - \theta\sqrt{D}\right) \right) \right\} \end{aligned}$$

for $L \leq K$ and $L \leq x$.

► Case $K \leq L$.

$$\begin{aligned} * \widehat{PUOC}^0(1, \lambda; \frac{K}{L}, 1; r, \delta) &= \\ &\quad \frac{2K}{L(m^2-\theta^2)} - \frac{2}{(m+\sigma)^2 - \theta^2} + \frac{K}{L\theta} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right) \\ &\quad - \left[\frac{1}{\psi(\theta\sqrt{D})} \left(\frac{2K}{L(m^2-\theta^2)} \psi(\sqrt{D}m) - \frac{2}{(m+\sigma)^2 - \theta^2} \psi(\sqrt{D}(m+\sigma)) \right) \right] \\ &\quad - \frac{\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})} \frac{K}{L} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m+\theta} - \frac{1}{m+\sigma+\theta} \right). \end{aligned}$$

$$\begin{aligned}
*\widehat{PUOC}(x, T; K, L; r, \delta) = & \\
& L \left(e^{(m+\theta)b} \mathcal{N}\left(\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right) + e^{(m-\theta)b} \mathcal{N}\left(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right) \right) \\
& \left\{ \frac{2K}{L(m^2 - \theta^2)} - \frac{2}{(m + \sigma)^2 - \theta^2} + \frac{K}{L\theta} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m + \theta} - \frac{1}{m + \theta + \sigma} \right) \right. \\
& - \left[\frac{1}{\psi(\theta\sqrt{D})} \left(\frac{2K}{L(m^2 - \theta^2)} \psi(\sqrt{D}m) - \frac{2}{(m + \sigma)^2 - \theta^2} \psi(\sqrt{D}(m + \sigma)) \right) \right] \\
& \left. - \frac{\psi(-\theta\sqrt{D})}{\theta\psi(\theta\sqrt{D})} \frac{K}{L} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m + \theta} - \frac{1}{m + \sigma + \theta} \right) \right\}, \text{ for } K \leq L \leq x.
\end{aligned}$$

4.4. The valuation of a Parisian Up-and-in call with $b \leq 0$. We will also use the relations between $*\widehat{PUIC}(x, \lambda; K, L; r, \delta)$ and $*\widehat{PUOC}(x, \lambda; K, L; r, \delta)$. We have

$$*\widehat{PUIC}(x, \lambda; K, L; r, \delta) = *\widehat{BSC}(x, \lambda, K, r, \delta) - *\widehat{PUOC}(x, \lambda; K, L; r, \delta)$$

where $*\widehat{PUOC}(x, \lambda; K, L; r, \delta)$ has already been computed above in *Section 4.3* for $b \leq 0$ and $*\widehat{BSC}(x, \lambda, K, r, \delta)$ has been calculated in *Section 3.2.1*.

So we derive the three following formulae

$$\begin{aligned}
*\widehat{PUIC}(x, \lambda; K, L; r, \delta) = & \frac{K}{\theta} e^{(m-\theta)b} \left(\frac{1}{m - \theta} - \frac{1}{m + \sigma - \theta} \right) \\
& - L \left(e^{(m+\theta)b} \mathcal{N}\left(\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right) + e^{(m-\theta)b} \mathcal{N}\left(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right) \right) \\
& \left(\frac{K}{L\theta} e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m - \theta} - \frac{1}{m + \sigma - \theta} \right) \right) \\
& - \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{L(m^2 - \theta^2)} e^{\frac{Dm^2}{2}} m \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln\left(\frac{K}{L}\right) + \sqrt{D}m\right) \right. \\
& \left. - \frac{2}{(m + \sigma)^2 - \theta^2} e^{\frac{D(m+\sigma)^2}{2}} (m + \sigma) \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln\left(\frac{K}{L}\right) + \sqrt{D}(m + \sigma)\right) \right] \\
& - \frac{1}{\psi(\theta\sqrt{D})} \frac{K}{L} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} e^{\lambda D} \sqrt{2\pi D} \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln\left(\frac{K}{L}\right) - \theta\sqrt{D}\right) \left(\frac{1}{m + \sigma + \theta} - \frac{1}{m + \theta} \right) \\
& - \frac{e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})}}{\theta\psi(\theta\sqrt{D})} \frac{K}{L} \left(\frac{1}{m - \theta} - \frac{1}{m + \sigma - \theta} \right) \\
& \left(\psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D} \mathcal{N}\left(\frac{1}{\sigma\sqrt{D}} \ln\left(\frac{K}{L}\right) - \theta\sqrt{D}\right) \right), \text{ for } L \leq x \leq K,
\end{aligned}$$

$$\begin{aligned}
*\widehat{PUIC}(x, \lambda; K, L; r, \delta) = & \frac{2K}{m^2 - \theta^2} - \frac{2x}{(m + \sigma)^2 - \theta^2} + \frac{K}{\theta} e^{(m+\theta)k} \left(\frac{1}{m + \theta} - \frac{1}{m + \theta + \sigma} \right) \\
& - L \left(e^{(m+\theta)b} \mathcal{N}(\theta\sqrt{D} + \frac{b}{\sqrt{D}}) + e^{(m-\theta)b} \mathcal{N}(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}) \right) \\
& \left\{ \frac{K}{L\theta} e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m - \theta} - \frac{1}{m + \sigma - \theta} \right) \right. \\
& - \frac{\sqrt{2\pi D}}{\psi(\theta\sqrt{D})} \left[\frac{2K}{L(m^2 - \theta^2)} e^{\frac{Dm^2}{2}} m \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) + \sqrt{D}m\right) \right. \\
& \left. \left. - \frac{2}{(m + \sigma)^2 - \theta^2} e^{\frac{D(m+\sigma)^2}{2}} (m + \sigma) \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) + \sqrt{D}(m + \sigma)\right) \right] \right. \\
& - \frac{1}{\psi(\theta\sqrt{D})} \frac{K}{L} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} e^{\lambda D} \sqrt{2\pi D} \mathcal{N}\left(-\frac{1}{\sigma\sqrt{D}} \ln(\frac{K}{L}) - \theta\sqrt{D}\right) \left(\frac{1}{m + \sigma + \theta} - \frac{1}{m + \theta} \right) \\
& \left. - \frac{e^{\frac{m-\theta}{\sigma} \ln(\frac{K}{L})} K}{\theta\psi(\theta\sqrt{D}) L} \left(\frac{1}{m - \theta} - \frac{1}{m + \sigma - \theta} \right) \right. \\
& \left. \left(\psi(-\theta\sqrt{D}) + \theta\sqrt{2\pi D} e^{\lambda D} \mathcal{N}\left(\frac{1}{\sigma\sqrt{D}} \ln\left(\frac{K}{L}\right) - \theta\sqrt{D}\right) \right) \right\}, \text{ for } L \leq K \leq x,
\end{aligned}$$

$$\begin{aligned}
*\widehat{PUIC}(x, \lambda; K, L; r, \delta) = & \frac{2K}{m^2 - \theta^2} - \frac{2x}{(m + \sigma)^2 - \theta^2} \\
& + \frac{K}{\theta} e^{(m+\theta)k} \left(\frac{1}{m + \theta} - \frac{1}{m + \theta + \sigma} \right) - L \left(e^{(m+\theta)b} \mathcal{N}\left(\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right) \right. \\
& \left. + e^{(m-\theta)b} \mathcal{N}\left(-\theta\sqrt{D} + \frac{b}{\sqrt{D}}\right) \right) \left\{ \frac{2K}{L(m^2 - \theta^2)} - \frac{2}{(m + \sigma)^2 - \theta^2} + \right. \\
& \left. \frac{K}{L\theta} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m + \theta} - \frac{1}{m + \theta + \sigma} \right) \right. \\
& \left. - \left[\frac{1}{\psi(\theta\sqrt{D})} \left(\frac{2K}{L(m^2 - \theta^2)} \psi(\sqrt{D}m) - \frac{2}{(m + \sigma)^2 - \theta^2} \psi(\sqrt{D}(m + \sigma)) \right) \right] \right. \\
& \left. - \frac{\psi(-\theta\sqrt{D}) K}{\theta\psi(\theta\sqrt{D}) L} e^{\frac{m+\theta}{\sigma} \ln(\frac{K}{L})} \left(\frac{1}{m + \theta} - \frac{1}{m + \sigma + \theta} \right) \right\}, \text{ for } K \leq L \leq x.
\end{aligned}$$

5. SOME PARITY RELATIONSHIPS

Now we will explain how to find all the other prices by simply using the formulae we have established so far and some parity relationships.

Let us consider a Parisian Down and Out Put.

$$PDOP(x, T; K, L, D, r, \delta) = \mathbb{E} \left(e^{mZ_T} (K - xe^{\sigma Z_T})^+ \mathbf{1}_{\{T_b^- > T\}} \right) e^{-(r + \frac{m^2}{2})T}. \quad (5.1)$$

One notices that the first time the Z Brownian motion makes below b an excursion longer than D is the same as the first time Brownian motion $-Z$ makes above $-b$ an excursion longer than D . Therefore, introducing the new Brownian motion $W = -Z$ we can rewrite

$$\begin{aligned}
PDOP(x, T; K, L, D, r, \delta) &= \mathbb{E} \left(e^{-mW_T} (K - xe^{-\sigma W_T})^+ \mathbf{1}_{\{T_{-b}^+ > T\}} \right) e^{-(r + \frac{m^2}{2})T}, \\
&= Kx \mathbb{E} \left(e^{-(m+\sigma)W_T} \left(\frac{1}{x} e^{\sigma W_T} - \frac{1}{K} \right)^+ \mathbf{1}_{\{T_{-b}^+ > T\}} \right) e^{-(r + \frac{m^2}{2})T}.
\end{aligned}$$

Let us introduce $m' = -(m + \sigma)$, $\delta' = r$, $r' = \delta$ and $b' = -b$. With these relations we easily check that $m' = \frac{1}{\sigma} \left(r' - \delta' - \frac{\sigma^2}{2} \right)$ and that $r' + \frac{m'^2}{2} = r + \frac{m^2}{2}$. Moreover, we notice that the barrier L' corresponding to $b' = -b$ is $\frac{1}{L}$. Therefore, $\mathbb{E} \left(e^{-(m+\sigma)W_T} \left(\frac{1}{x} e^{\sigma W_T} - \frac{1}{K} \right)^+ \mathbf{1}_{\{T_b^+ > T\}} \right) e^{-(r+\frac{m^2}{2})T}$ is in fact the price of a Up and Out Call $PUOC \left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r \right)$. Finally, we come up with the following relation

$$PDOP(x, T; K, L, D, r, \delta) = xK PUOC \left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r \right).$$

The same relation holds if we replace a call by a put and vice-versa and if we consider In options instead of Out ones.

$$\begin{aligned} PUOP(x, T; K, L, D, r, \delta) &= xK PDOC \left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r \right), \\ PUIP(x, T; K, L, D, r, \delta) &= xK PDIC \left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r \right), \\ PDIP(x, T; K, L, D, r, \delta) &= xK PUIC \left(\frac{1}{x}, T; \frac{1}{K}, \frac{1}{L}, D, \delta, r \right). \end{aligned}$$

In the previous sections we computed the price of all the Down Calls an Up Calls. From these relationships, we can deduce the prices of all the Parisian Puts. What we still have to find is how to invert the Laplace transform.

6. PRICES AT ANY TIME t

At this stage we can compute all the prices at time 0, but to be able to hedge such an option we besides need the prices at some time $t \leq T$. So we will consider a Down-and-In option to show how the price at some time t can be deduced from the prices at time 0 of the Down-and-In options with different parameters. Relying on this example one can easily prove similar formulae for other options.

6.1. Three different paths for the Brownian motion. The price of a Parisian Down and In Call at time 0 is given by the formula (2.4). From this formula, we can deduce the price of a Down and In call at any time t .

$$PDIC(S_t, t; x, T; K, L, D, r, \delta) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left((x e^{\sigma(W_T + mT)} - K)^+ \mathbf{1}_{\{T_b^- \leq T\}} | \mathcal{F}_t \right). \quad (6.1)$$

Now we can change the probability measure as we did at the beginning to make $Z = \{W_t + mt; t \geq 0\}$ a Brownian motion under the new probability we called \mathcal{P} , (\mathbb{E} will from now on denote the expectation under the probability \mathcal{P}). Then, we can write

$$\begin{aligned} PDIC(S_t, t; x, T; K, L, D, r, \delta) &= e^{-r(T-t)} \frac{\mathbb{E} \left(e^{mZ_T - \frac{1}{2}m^2T} (x e^{\sigma Z_T} - K)^+ \mathbf{1}_{\{T_b^- \leq T\}} | \mathcal{F}_t \right)}{e^{mZ_t - \frac{1}{2}m^2t}}, \\ &= e^{-r(T-t)} \frac{\mathbb{E} \left(e^{mZ_t} e^{m(Z_T - Z_t) - \frac{1}{2}m^2T} (x e^{\sigma Z_T} - K)^+ \mathbf{1}_{\{T_b^- \leq T\}} | \mathcal{F}_t \right)}{e^{mZ_t - \frac{1}{2}m^2t}}, \\ &= e^{-(r+\frac{m^2}{2})(T-t)} \mathbb{E} \left(e^{m(Z_T - Z_t)} (x e^{\sigma Z_T} - K)^+ \mathbf{1}_{\{T_b^- \leq T\}} | \mathcal{F}_t \right). \end{aligned}$$

Let us introduce a few notations

$$T' = T - t \text{ and } b' = \frac{1}{\sigma} \ln \left(\frac{L}{S_t} \right), \quad (6.2)$$

$$T'_{b'} = \inf \{s > 0; Z_{t+s} - Z_t = b'\}. \quad (6.3)$$

In the case $Z_t < b$, we introduce D' the time Z has already spent in the excursion.

$$PDIC(S_t, t; x, T; K, L, D, r, \delta) = e^{-(r+\frac{m^2}{2})T'} \mathbb{E} \left(e^{m(Z_T - Z_t)} (S_t e^{\sigma(Z_T - Z_t)} - K)^+ \mathbf{1}_{\{T_{b'}^- \leq T'\}} | \mathcal{F}_t \right). \quad (6.4)$$

The indicator can be split up in several parts depending on which path you are on. On both paths the excursion has already started. On the red one, the excursion will not last long enough, so the asset still

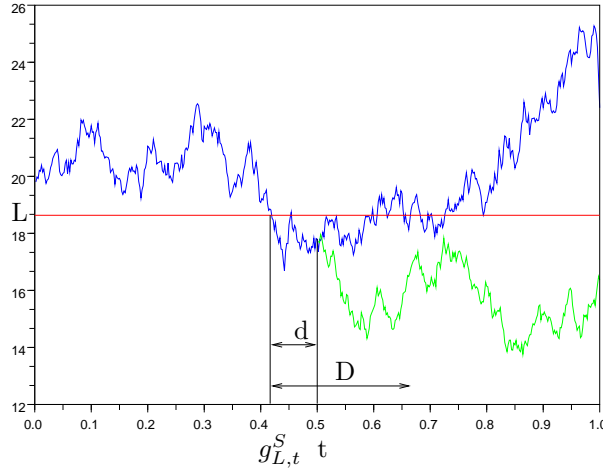


FIGURE 4. Possible evolutions of an asset price

has to do an entirely new excursion below L longer than D , whereas on the green path the process only has to remain below L for a time longer than $D - d$. All these remarks enable us to rewrite the indicator as follows

$$\mathbf{1}_{\{\{T_b^-\leq T\}\}} = \mathbf{1}_{\{\{Z_t > b\}\}} \mathbf{1}_{\{\{T_{b'}'^-\leq T'\}\}} + \mathbf{1}_{\{\{Z_t \leq b\}\}} \left(\mathbf{1}_{\{\{T_{b'}' \geq D-D'\}\}} \mathbf{1}_{\{\{D-D' \leq T'\}\}} + \mathbf{1}_{\{\{T_{b'}' < D-D\}\}} \mathbf{1}_{\{\{T_{b'}'^-\leq T'\}\}} \right). \quad (6.5)$$

$$\begin{aligned} PDIC(S_t, t; x, T; K, L, D, r, \delta) &= e^{-(r+\frac{m^2}{2})T'} \left\{ \mathbb{E} \left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \mathbf{1}_{\{\{Z_t > b\}\}} \mathbf{1}_{\{\{T_{b'}'^-\leq T'\}\}} | \mathcal{F}_t \right), \right. \\ &\quad + \mathbb{E} \left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \mathbf{1}_{\{\{Z_t \leq b\}\}} \mathbf{1}_{\{\{T_{b'}' \geq D-d\}\}} \mathbf{1}_{\{\{D-d \leq T-t\}\}} | \mathcal{F}_t \right), \\ &\quad \left. + \mathbb{E} \left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \mathbf{1}_{\{\{Z_t \leq b\}\}} \mathbf{1}_{\{\{T_{b'}' \leq D-D'\}\}} \mathbf{1}_{\{\{T_{b'}'^-\leq T-t\}\}} | \mathcal{F}_t \right) \right\}, \\ &= e^{-(r+\frac{m^2}{2})T'} \left\{ \mathbf{1}_{\{\{Z_t > b\}\}} \mathbb{E} \left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \mathbf{1}_{\{\{T_{b'}'^-\leq T'\}\}} | \mathcal{F}_t \right), \right. \\ &\quad + \mathbf{1}_{\{\{Z_t \leq b\}\}} \mathbf{1}_{\{\{D-D' \leq T-t\}\}} \mathbb{E} \left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \mathbf{1}_{\{\{T_{b'}' \geq D-d\}\}} | \mathcal{F}_t \right), \\ &\quad \left. + \mathbf{1}_{\{\{Z_t \leq b\}\}} \mathbb{E} \left(e^{m(Z_T-Z_t)} (S_t e^{\sigma(Z_T-Z_t)} - K)^+ \mathbf{1}_{\{\{T_{b'}' \leq D-D'\}\}} \mathbf{1}_{\{\{T_{b'}'^-\leq T-t\}\}} | \mathcal{F}_t \right) \right\}. \end{aligned}$$

$T_{b'}'$ and $T_{b'}'^-$ are both independent of \mathcal{F}_t , so we can write

$$\begin{aligned}
PDIC(S_t, t; x, T; K, L, D, r, \delta) &= e^{-(r+\frac{m^2}{2})T'} \left\{ \mathbf{1}_{\{\{Z_t > b\}\}} \mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'}^- \leq T'\}\}} \right) \right. \\
&\quad + \mathbf{1}_{\{\{Z_t \leq b\}\}} \mathbf{1}_{\{\{D-D' \leq T'\}\}} \mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'}^- \geq D-D'\}\}} \right) \\
&\quad + \mathbf{1}_{\{\{Z_t \leq b\}\}} \mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \mathbf{1}_{\{\{T_{b'}^- \leq T'\}\}} \right) \Big\}, \\
&= e^{-(r+\frac{m^2}{2})T'} \left\{ \mathbf{1}_{\{\{Z_t > b\}\}} PDIC(S_t, T'; K, L; r, \delta) \right. \\
&\quad + \mathbf{1}_{\{\{Z_t \leq b\}\}} \mathbf{1}_{\{\{D-D' \leq T'\}\}} \underbrace{\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \geq D-D'\}\}} \right)}_{(i)} \\
&\quad \left. + \mathbf{1}_{\{\{Z_t \leq b\}\}} \underbrace{\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \mathbf{1}_{\{\{T_{b'}^- \leq T'\}\}} \right)}_{(ii)} \right\}.
\end{aligned}$$

6.2. The computation of the different expectations. Let us calculate (i) in the case $D - D' \leq T'$

$$\begin{aligned}
&\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \geq D-D'\}\}} \right) \\
&= \mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \right) - \mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \right) \\
&= *BSC(S_t, T'; K, r, \delta) - \mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \right)
\end{aligned}$$

The last expectation above can be computed by conditioning with respect to $\mathcal{F}_{T_{b'}}$ since $D - D' \leq T'$.

$$\begin{aligned}
&\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \right) \\
&= \mathbb{E} \left(\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \mid \mathcal{F}_{T_{b'}} \right) \right), \\
&= \mathbb{E} \left(\mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \mathbb{E} \left(e^{m(Z_{T'} - Z_{T_{b'}})} e^{mb'} (S_t e^{\sigma b'} e^{\sigma(Z_{T'} - Z_{T_{b'}})} - K)^+ \mid \mathcal{F}_{T_{b'}} \right) \right).
\end{aligned}$$

If $W_t = Z_{t+T_b'} - Z_{T_b'}$ and Y_t denotes $e^{mW_{T'-t}} (L e^{\sigma W_{T'-t}} - K)^+$, then Y_t is independent of $\mathcal{F}_{T_{b'}}$ and $T_{b'}$ is $\mathcal{F}_{T_{b'}}$ -measurable

$$\begin{aligned}
&\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \right) \\
&= \mathbb{E} \left(\mathbf{1}_{\{\{T_{b'} \leq D-D'\}\}} \mathbb{E} \left(e^{mW_{T'-\tau}} e^{mb'} (S_t e^{\sigma b'} e^{\sigma W_{T'-\tau}} - K)^+ \mid \tau = T_{b'} \right) \right), \\
&= \underbrace{\int_0^{D-D'} e^{mb'} \mathbb{E} \left(e^{mW_{T'-u}} (L e^{\sigma W_{T'-u}} - K)^+ \right) \mu_{b'}(u) du}_{P(L, T')}.
\end{aligned}$$

Now, we will consider the Laplace transform of $P(L, T')$ with respect to T'

$$\begin{aligned}
\widehat{P}(L, \lambda) &= \int_0^{+\infty} e^{-\lambda\tau} \int_0^{D-D'} e^{mb'} \mathbb{E}(e^{mW_{\tau-u}} (Le^{\sigma W_{\tau-u}} - K)^+) \mu_{b'}(u) du d\tau, \\
&= \int_0^{D-D'} \int_0^{+\infty} e^{-\lambda\tau} e^{mb'} \mathbb{E}(e^{mW_{\tau-u}} (Le^{\sigma W_{\tau-u}} - K)^+) d\tau \mu_{b'}(u) du, \\
&\quad \text{a change of variables } (u, \xi) = (u, \tau - u) \text{ gives} \\
&= \int_0^{D-D'} \int_0^{+\infty} e^{-\lambda u} e^{-\lambda\xi} e^{mb'} \mathbb{E}(e^{mW_\xi} (Le^{\sigma W_\xi} - K)^+) \mu_{b'}(u) du d\xi, \\
&\quad \text{relying on Appendix A we can write} \\
&= e^{mb'} \left\{ e^{-\theta|b'|} \mathcal{N} \left(\theta\sqrt{D-D'} - \frac{|b'|}{\sqrt{D-D'}} \right) \right. \\
&\quad \left. + e^{\theta|b'|} \mathcal{N} \left(-\theta\sqrt{D-D'} - \frac{|b'|}{\sqrt{D-D'}} \right) \right\} * \widehat{BSC}(L, \lambda; K, r, \delta).
\end{aligned}$$

Let us now compute (ii). We can condition with respect to $\mathcal{F}_{T_{b'}}$, since $T_{b'}^-$ is bound to be bigger than $D - D'$ so $T_{b'}$ is almost surely smaller than T'

$$\begin{aligned}
&\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{T_{b'} \leq D-D'\}} \mathbf{1}_{\{T_{b'}^- \leq T'\}} \right) \\
&= \mathbb{E} \left(\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{T_{b'} \leq D-D'\}} \mathbf{1}_{\{T_{b'}^- \leq T'\}} \middle| \mathcal{F}_{T_{b'}} \right) \right), \\
&= e^{mb'} \mathbb{E} \left(\mathbf{1}_{\{T_{b'} \leq D-D'\}} \mathbb{E} \left(e^{m(Z_{T'} - Z_{T_{b'}})} (Le^{\sigma(Z_{T'} - Z_{T_{b'}})} - K)^+ \mathbf{1}_{\{T_0^- \leq T' - T_{b'}\}} \middle| \mathcal{F}_{T_{b'}} \right) \right).
\end{aligned}$$

If $W_t = Z_{t+T_b'} - Z_{T_b'}$ and Y_t denotes $e^{mW_{T'-t}} (Le^{\sigma W_{T'-t}} - K)^+ \mathbf{1}_{\{T_0^- \leq T'-t\}}$, Y_t is independent of $\mathcal{F}_{T_{b'}}$ and $T_{b'}$ is $\mathcal{F}_{T_{b'}}$ -measurable. So $\mathbb{E}(Y_t | \mathcal{F}_{T_{b'}}) = \mathbb{E}(Y_t) |_{t=T_{b'}}$ and therefore we can write

$$\begin{aligned}
&\mathbb{E} \left(e^{mZ_{T'}} (S_t e^{\sigma Z_{T'}} - K)^+ \mathbf{1}_{\{T_{b'} \leq D-D'\}} \mathbf{1}_{\{T_{b'}^- \leq T'\}} \right) \\
&= e^{mb'} \mathbb{E} \left(\mathbf{1}_{\{T_{b'} \leq D-D'\}} \mathbb{E} \left(e^{mW_{T'-u}} (Le^{\sigma W_{T'-u}} - K)^+ \mathbf{1}_{\{T_0^- \leq T'-u\}} \right) \middle|_{u=T_{b'}} \right), \\
&= \underbrace{\int_0^{D-D'} e^{mb'} \mathbb{E} \left(e^{mW_{T'-u}} (Le^{\sigma W_{T'-u}} - K)^+ \mathbf{1}_{\{T_0^- \leq T'-u\}} \right) \mu_{b'}(u) du}_{Q(L, T')}.
\end{aligned}$$

Let us consider the Laplace transform of $Q(L, T')$ with respect to T' .

$$\begin{aligned}
\widehat{Q}(L, \lambda) &= \int_0^{+\infty} e^{-\lambda\tau} \int_0^{D-D'} e^{mb'} \mathbb{E} \left(e^{mW_{\tau-u}} (Le^{\sigma W_{\tau-u}} - K)^+ \mathbf{1}_{\{T_0^- \leq \tau-u\}} \right) \mu_{b'}(u) du d\tau, \\
&= Le^{mb'} \int_0^{+\infty} e^{-\lambda\tau} \int_0^{D-D'} *PDIC^0(1, \tau - u, \frac{K}{L}, 1, D, r, \delta) d\tau \mu_{b'}(u) du, \\
&= Le^{mb'} \int_0^{D-D'} \mu_{b'} e^{-\lambda u} du * \widehat{PDIC}^0(1, \lambda, \frac{K}{L}, 1, D, r, \delta).
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
&* \widehat{PDIC}(S_t, t; x, T; K, L, D, r, \delta) \\
&= \mathbf{1}_{\{Z_t > b\}} * \widehat{PDIC}(S_t, T', K, L, D, r, \delta) + \mathbf{1}_{\{Z_t \leq b\}} \mathbf{1}_{\{D-D' \leq T'\}} \\
&\quad \left(Le^{mb'} \int_0^{D-D'} \mu_{b'} e^{-\lambda u} du \left(* \widehat{PDIC}^0(1, \lambda, \frac{K}{L}, 1, D, r, \delta) \right. \right. \\
&\quad \left. \left. - * \widehat{BSC}(1, \lambda, \frac{K}{L}, r, \delta) \right) + * \widehat{BSC}(S_t, T', K, r, \delta) \right).
\end{aligned}$$

If we compute $\widehat{*PUIC}(S_t, t; x, T; K, L, D, r, \delta)$ we get exactly the same result by changing $\widehat{*PDIC^0}(1, \lambda, \frac{K}{L}, 1, D, r, \delta)$ into $\widehat{*PUIC^0}(1, \lambda, \frac{K}{L}, 1, D, r, \delta)$ in the previous formula.

If one wants to value the Put Options, one can rely on the parity relationships given in the previous section and then use again the price of the Calls at time t .

7. THE INVERSE LAPLACE TRANSFORM

This section is devoted to the numerical inversion of the Laplace transforms computed previously. We recall that the Laplace transforms are computed with respect to the maturity time. We explain how to recover a function from its Laplace transform using a contour integral. The real problem is how to numerically evaluate this complex integral. This is done in two separate steps involving two different errors. First, as explained in Section 7.1 we replace the integral by a series. The first step creates a discretisation error, which is handled by Proposition 7.1. Secondly, one has to compute a non-finite series. This can be achieved by simply truncating the series but it leads to a tremendously slow convergence. Here, we prefer to use the Euler acceleration as presented in Section 7.2. Proposition 7.2 states an upper-bound for the error due to the accelerated computation of the non finite series. Theorem 7.2 gives a bound for the global error.

7.1. The Fourier series representation. Thanks to [Widder(1941), Theorem 9.2], we know how to recover a function from its Laplace transform.

Theorem 7.1. *Let f be a continuous function defined on \mathbb{R}^+ and α a positive number. If the function $f(t)e^{-\alpha t}$ is integrable, then given the Laplace transform \hat{f} , f can be recovered from the contour integral*

$$f(t) = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} e^{st} \hat{f}(s) ds, \quad t > 0. \quad (7.1)$$

The variable α has to be chosen greater than the abscissa of convergence of \hat{f} . The abscissa of convergence of the Laplace transforms of the barrier Parisian option prices computed previously is smaller than $(m + \sigma)^2/2$. Hence, α must be chosen strictly greater than $(m + \sigma)^2/2$.

For any real valued function satisfying the hypotheses of Theorem 7.1, we introduce a trapezoidal discretisation of Equation (7.1)

$$f_{\pi/t}(t) = \frac{e^{\alpha t}}{2t} \hat{f}(\alpha) + \frac{e^{\alpha t}}{t} \sum_{k=1}^{\infty} (-1)^k \operatorname{Re} \left(\hat{f} \left(\alpha + i \frac{k\pi}{t} \right) \right). \quad (7.2)$$

Proposition 7.1. *If f is a continuous bounded function satisfying $f(t) = 0$ for $t < 0$, we have*

$$|e_{\pi/t}(t)| = |f(t) - f_{\pi/t}(t)| \leq \|f\|_{\infty} \frac{e^{-2\alpha t}}{1 - e^{-2\alpha t}}. \quad (7.3)$$

To prove Proposition 7.1, we need the following result adapted from [Abate et al.(1999)Abate, Choudhury, and Whitt, Theorem 5]

Lemma 7.1. *For any continuous and bounded function f such that $f(t) = 0$ for $t < 0$, we have*

$$e_{\pi/t}(t) = f_{\pi/t}(t) - f(t) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} f(t(1+2k)) e^{-2k\alpha t}. \quad (7.4)$$

Proof of Proposition 7.1. By performing a change of variables $s = \alpha + iu$ in the integral in (7.1), we can easily obtain an integral of a real variable.

$$f(t) = \frac{e^{\alpha t}}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\alpha + iu) (\cos(ut) + i \sin(ut)) du.$$

Moreover, since f is a real valued function, the imaginary part of the integral vanishes

$$f(t) = \frac{e^{\alpha t}}{2\pi} \int_{-\infty}^{+\infty} \operatorname{Re} \left(\hat{f}(\alpha + iu) \right) \cos(ut) - \operatorname{Im} \left(\hat{f}(\alpha + iu) \right) \sin(ut) du.$$

We notice that

$$\operatorname{Im}(\widehat{f}(\alpha + iu)) = -\operatorname{Im}(\widehat{f}(\alpha - iu)), \quad \operatorname{Re}(\widehat{f}(\alpha + iu)) = \operatorname{Re}(\widehat{f}(\alpha - iu)).$$

So,

$$f(t) = \frac{e^{\alpha t}}{\pi} \int_0^{+\infty} \operatorname{Re}(\widehat{f}(\alpha + iu)) \cos(ut) - \operatorname{Im}(\widehat{f}(\alpha + iu)) \sin(ut) du. \quad (7.5)$$

Using a trapezoidal integral with a step $h = \frac{\pi}{t}$ leads to Equation (7.2). Remembering that $f(t) = 0$ for $t < 0$, we can easily deduce from Lemma 7.1 that

$$e_{\pi/t}(t) = \sum_{k=0}^{\infty} f(t(1+2k)) e^{-2k\alpha t}.$$

Taking the upper bound of f yields (7.3). \square

Remark 7.1. For the upper bound in Proposition 7.1 to be smaller than $10^{-8} \|f\|_{\infty}$, one has to choose $2\alpha t = 18.4$. In fact, this bound holds for any choice of the discretisation step h satisfying $h < 2\pi/t$.

Simply truncating the summation in the definition of $f_{\pi/t}$ to compute the trapezoidal integral is far too rough to provide a fast and accurate numerical inversion. One way to improve the convergence of the series is to use the Euler summation.

7.2. The Euler summation. To improve the convergence of a series S , we use the Euler summation technique as described by [Abate et al.(1999)Abate, Choudhury, and Whitt], which consists in computing the binomial average of q terms from the p -th term of the series S . The binomial average obviously converges to S as p goes to infinity. The following proposition describes the convergence rate of the binomial average to the infinite series $f_{\pi/t}(t)$ when p goes to ∞ .

Proposition 7.2. Let f be a function of class C^{q+4} such that there exists $\epsilon > 0$ s.t. $\forall k \leq q+4$, $f^{(k)}(s) = \mathcal{O}(e^{(\alpha-\epsilon)s})$. We define $s_p(t)$ as the approximation of $f_{\pi/t}(t)$ when truncating the non-finite series in (7.2) to p terms

$$s_p(t) = \frac{e^{\alpha t}}{2t} \widehat{f}(\alpha) + \frac{e^{\alpha t}}{t} \sum_{k=1}^p (-1)^k \operatorname{Re} \left(\widehat{f} \left(\alpha + i \frac{\pi k}{t} \right) \right), \quad (7.6)$$

and $E(q, p, t) = \sum_{k=0}^q C_q^k 2^{-q} s_{p+k}(t)$. Then,

$$|f_{\pi/t}(t) - E(q, p, t)| \leq \frac{te^{\alpha t} |f'(0) - \alpha f(0)|}{\pi^2} \frac{(p+1)! q!}{2^{q-2} (p+q+2)!} + \mathcal{O} \left(\frac{1}{p^{q+3}} \right)$$

when p goes to infinity.

Using Propositions 7.1 and 7.2, we get the following result concerning the global error on the numerical computation of the price of a Parisian call option

Theorem 7.2. Let f be the price of a Parisian call option. Using the notations of Proposition 7.2, we have

$$|f(t) - E(q, p, t)| \leq S_0 \frac{e^{-2\alpha t}}{1 - e^{-2\alpha t}} + \frac{e^{\alpha t} t |f'(0) - \alpha f(0)| (p+1)! q!}{\pi^2 2^{q-2} (p+q+2)!} + \mathcal{O} \left(\frac{1}{p^{q+3}} \right) \quad (7.7)$$

where α is defined in Theorem 7.1.

Proof of Theorem 7.2. f being the price of a Parisian call option, we know that f is bounded by S_0 . Moreover, f is continuous (actually of class C^{∞} , see Appendix E). Hence, Proposition 7.1 yields the first term on the right-hand side of (7.7).

Relying on Proposition E.1, we know that $*f$ is of class C^{∞} and $*f^{(k)}(t) = \mathcal{O}(e^{\frac{(m+\sigma)^2}{2}t})$, $\forall k \geq 0$. Since $f(t) = e^{-(r+m^2/2)t} *f(t)$, it is quite obvious that f is also of class C^{∞} and $f^{(k)}(t) = \mathcal{O}(e^{((m+\sigma)^2/2 - (r+m^2/2))t})$, $\forall k \geq 0$. Since $\alpha > \frac{(m+\sigma)^2}{2}$, we can apply Proposition 7.2 to get the result. \square

Proof of Proposition 7.2. We compute the difference between two successive terms.

$$E(q, p+1, t) - E(q, p, t) = \frac{e^{\alpha t}}{2qt} \sum_{k=0}^q C_q^k (-1)^{p+1+k} a_{p+k+1},$$

where

$$a_p = \int_0^{+\infty} e^{-\alpha s} \cos\left(\frac{p}{t}\pi s\right) f(s) ds. \quad (7.8)$$

Let $g(s) = e^{-\alpha s} f(s)$. Since $g^{(k)}(\infty) = 0$ for $k \leq q+3$ and $g^{(q+4)}$ is integrable, we can perform $(q+3)$ integrations by parts in (7.8) to obtain a Taylor expansion when p goes to infinity

$$a_p = \frac{c_2}{p^2} + \frac{c_4}{p^4} + \cdots + \frac{c_q}{p^{2[(q+3)/2]}} + \mathcal{O}\left(\frac{1}{p^{q+4}}\right) \quad (7.9)$$

with $c_2 = \frac{4t^2(f'(0) - \alpha f(0))}{\pi^2}$.
We can rewrite (7.9)

$$a_p = \frac{c_2}{p(p+1)} + \frac{c'_3}{p(p+1)(p+2)} + \cdots + \frac{c'_q}{p(p+1) \cdots (p+q+2)} + \mathcal{O}\left(\frac{1}{p^{q+4}}\right).$$

Some elementary computations show that for $j \geq 1$

$$\sum_{k=0}^q C_q^k (-1)^{p+1+k} \frac{1}{(p+k)(p+k+1) \cdots (p+k+j)} = (-1)^{p+1} \frac{p! (q+j)!}{j! (p+q+j+1)!}.$$

Computing $\sum_{k=0}^q C_q^k (-1)^{p+1+k} a_{p+k+1}$ leads to

$$E(q, p+1, t) - E(q, p, t) = (-1)^{p+1} c_2 \frac{e^{\alpha t}}{2^q t} \frac{p! (q+1)!}{(p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+4}}\right).$$

Moreover, $\frac{p! (q+1)!}{(p+q+2)!}$ is decreasing w.r.t p , so

$$|E(q, \infty, t) - E(q, p, t)| \leq c_2 \frac{e^{\alpha t}}{2^q t} \frac{p! (q+1)!}{(p+q+2)!} + \mathcal{O}\left(\frac{1}{p^{q+3}}\right).$$

□

Remark 7.2. Whereas Proposition 7.1 in fact holds for any $h < 2\pi/t$, the proof of Proposition 7.2 is essentially based on the choice of $h = \pi/t$ since the key point is to be able to write $E(q, p+1, t) - E(q, p, t)$ as the general term of an alternating series. The impressive convergence rate of $E(q, p, t)$ definitely relies on the choice of this particular discretisation step. For a general step h , it is much more difficult to study the convergence rate and one cannot give an explicit upper-bound.

Remark 7.3. For $2\alpha t = 18.4$ and $q = p = 15$, the global error is bounded by $S_0 10^{-8} + t |f'(0) - \alpha f(0)| 10^{-11}$. As one can see, the method we use to invert Laplace transforms provides a very good accuracy with few computations.

Remark 7.4. Considering the case of call options in Theorem 7.2 is sufficient since put prices are computed using parity relations and their accuracy is hung up to the one of call prices. Theorem 7.2 also holds for single barrier Parisian options.

APPENDIX A. THE VALUATION OF $\int_0^D \mu_b(du)e^{-\lambda u}$ IN THE CASE $b > 0$

We already know that $\mu_b(du) = \frac{|b|}{\sqrt{2\pi}u^3} e^{\left(\frac{-b^2}{2u}\right)} du$.

$$\begin{aligned}
\int_0^D e^{-\lambda u} \mu_b(du) &= \int_0^D e^{-\lambda u} \frac{b}{\sqrt{2\pi}u^3} e^{\frac{-b^2}{2u}} du \\
&\quad \text{with a change of variable } t = \frac{1}{\sqrt{u}} \text{ we get,} \\
&= \int_{1/\sqrt{D}}^{+\infty} b \sqrt{\frac{2}{\pi}} e^{\frac{-\lambda}{t^2}} e^{\frac{-b^2 t^2}{2}} dt, \\
&= \int_{1/\sqrt{D}}^{+\infty} b \sqrt{\frac{2}{\pi}} e^{\frac{-\lambda}{t^2}} e^{\frac{-b^2 t^2}{2}} dt, \\
&\quad \text{let } \theta \text{ denote } \sqrt{2\lambda} \\
&= \int_{1/\sqrt{D}}^{+\infty} b \sqrt{\frac{2}{\pi}} \exp\left(\frac{-\theta b}{2} \left(\frac{1}{(\sqrt{\frac{b}{\theta}}t)^2} + (\sqrt{\frac{b}{\theta}}t)^2\right)\right) dt, \\
&\quad \text{let's change variable again } u = \sqrt{\frac{b}{\theta}}t \\
&= \int_{\frac{\sqrt{b}}{\sqrt{\theta D}}}^{+\infty} \sqrt{\frac{2b\theta}{\pi}} \exp\left(\frac{-\theta b}{2} \left(\frac{1}{u^2} + u^2\right)\right) du, \\
&= \int_{\frac{\sqrt{b}}{\sqrt{\theta D}}}^{+\infty} \sqrt{\frac{2b\theta}{\pi}} \exp\left(\frac{-\theta b}{2} \left(\frac{1}{u} - u\right)^2\right) e^{-\theta b} du, \\
&\quad \text{a new change of variable } v = \frac{1}{u} - u \text{ gives} \\
&= \sqrt{\frac{b\theta}{2\pi}} e^{-\theta b} \int_{-\infty}^{\sqrt{\frac{\theta D}{b}} - \frac{\sqrt{b}}{\sqrt{\theta D}}} e^{\frac{-\theta b}{2} v^2} \left(1 - \frac{v}{\sqrt{v^2 + 4}}\right) dv, \\
&\quad \text{one more change of variable } u = \sqrt{\theta b}v \text{ provides the following expression} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\theta b} \int_{-\infty}^{\theta\sqrt{D} - \frac{b}{\sqrt{D}}} e^{-u^2/2} \left(1 - \frac{u}{\sqrt{u^2 + 4\theta b}}\right) du, \\
&\quad \text{a last change of variable } v = \sqrt{u^2 + 4\theta b} \text{ ends the calculation} \\
&= e^{-\theta b} \mathcal{N}\left(\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\theta b} \int_{\theta\sqrt{D} + \frac{b}{\sqrt{D}}}^{+\infty} e^{-\frac{v^2 - 4\theta b}{2}} dv, \\
&= e^{-\theta b} \mathcal{N}\left(\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right) + e^{\theta b} \mathcal{N}\left(-\theta\sqrt{D} - \frac{b}{\sqrt{D}}\right).
\end{aligned}$$

If we let D go to infinity, we can deduce the Laplace transform of T_b , for any real b

$$\mathbb{E}[e^{-\lambda T_b}] = e^{-\theta|b|}.$$

APPENDIX B. THE VALUATION OF $\int_0^{+\infty} e^{-\lambda u} \frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi}u} du$

Once again we introduce $\theta = \sqrt{2\lambda}$.

A change of variable $u = \frac{|x|t^2}{\theta}$ straightforward gives the new expression

$$\begin{aligned}
\int_0^{+\infty} e^{-\lambda u} \frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi}u} du &= \int_0^{+\infty} \sqrt{\frac{2|x|}{\pi\theta}} \exp\left(-\frac{\theta|x|}{2} \left(\frac{1}{t^2} + t^2\right)\right) dt, \\
&= \sqrt{\frac{2|x|}{\pi\theta}} e^{-\theta|x|} \int_0^{+\infty} \exp\left(-\frac{\theta|x|}{2} \left(\frac{1}{t} - t\right)^2\right) dt.
\end{aligned}$$

Once again, we can use the change of variable $s = u - \frac{1}{u}$, which maps $[0, +\infty[$ into $]-\infty, +\infty[$ and we have $du = \frac{ds}{2} \left(1 + \frac{s}{\sqrt{s^2 + 4}}\right)$. The second of the last term is odd, so its integral over \mathbb{R} cancels and we get

$$\sqrt{\frac{|x|}{2\pi\theta}} e^{-\theta|x|} \int_{-\infty}^{+\infty} \exp\left(-\frac{\theta|x|}{2} s^2\right) ds.$$

So finally we obtain

$$\int_0^{+\infty} e^{-\lambda u} \frac{\exp\left(-\frac{x^2}{2u}\right)}{\sqrt{2\pi u}} du = \frac{1}{\theta} e^{-\theta|x|}. \quad (\text{B.1})$$

APPENDIX C. THE LAW OF $(T_b^-, Z_{T_b^-})$ AND $(T_b^+, Z_{T_b^+})$

In this Section, we recall some useful results on the law of the couples $(T_b^-, Z_{T_b^-})$ and $(T_b^+, Z_{T_b^+})$ from [Revuz and Yor(1999)] and [Chesney et al.(1997)Chesney, Jeanblanc-Picqué, and Yor].

C.1. Case $b = 0$. In this case, we denote $T^- = T_0^-$.

The first important result is that T^- and Z_{T^-} are independent.

$$\mathbb{P}(Z_{T^-} \in dx) = -\frac{x}{D} \exp\left(-\frac{x^2}{2D}\right) 1_{\{x < 0\}} dx. \quad (\text{C.1})$$

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T^-)) = \frac{1}{\psi(\lambda\sqrt{D})}. \quad (\text{C.2})$$

Similarly,

$$\mathbb{P}(Z_{T^+} \in dx) = \frac{x}{D} \exp\left(-\frac{x^2}{2D}\right) 1_{\{x > 0\}} dx. \quad (\text{C.3})$$

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T^+)) = \frac{1}{\psi(\lambda\sqrt{D})}. \quad (\text{C.4})$$

C.2. Case $b < 0$. This case study can be reduced to the previous one, with the help of the stopping time T_b .

We can write $T_b^- = T_b + T^-(W)$, with

$$\begin{aligned} T_0^-(W) &= \inf\{t \geq 0; 1_{\{W_t \leq 0\}}(t - g_t^W) \geq D\} \stackrel{law}{=} T_0^-, \\ W &= \{W_t = Z_{T_b+t} - b; t \geq 0\}, \\ g_t^W &= \sup\{u \leq t; W_u = 0\}. \end{aligned}$$

Moreover using the strong Markov property it is clear that T_b and $T_0^-(W)$ are independent.

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b^-)) = \mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b)) \mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_0^-(W))).$$

As $\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b)) = \exp(-|b|\lambda)$, we get

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b^-)) = \frac{\exp(b\lambda)}{\psi(\lambda\sqrt{D})}. \quad (\text{C.5})$$

Now, we are trying to find the law of $Z_{T_b^-}$

$$\begin{aligned}
\mathbb{P}(Z_{T_b^-} \in dx) &= \mathbb{P}(Z_{T_b^- - T_b} \circ \theta_{T_b} \in dx), \\
&= \mathbb{E}[\mathbf{1}_{\{Z_{T_b^- - T_b} \circ \theta_{T_b} \in dx\}}], \\
&= \mathbb{E}\left[\mathbb{E}[\mathbf{1}_{\{Z_{T_b^- - T_b} \circ \theta_{T_b} \in dx\}} | \mathcal{F}_{T_b}]\right], \\
&= \mathbb{E}\left[\mathbb{E}^b[\mathbf{1}_{\{Z_{T_b^- - T_b} \in dx\}} | \mathcal{F}_{T_b}]\right], \\
&= \mathbb{E}\left[\mathbb{E}^b[\mathbf{1}_{\{Z_{T_b^- - T_b} \in dx\}}]\right], \\
&= \mathbb{E}\left[\mathbb{E}^b[\mathbf{1}_{\{Z_{T^-} \in dx\}}]\right], \\
&= \mathbb{E}[\mathbb{P}^b[Z_{T^-} \in dx]], \\
&= \mathbb{E}[\mathbb{P}[Z_{T^-} + b \in dx]], \\
&= \mathbb{E}[\mathbb{P}[Z_{T^-} \in (dx - b)]], \\
&= \mathbb{P}[Z_{T^-} \in (dx - b)].
\end{aligned}$$

Finally we obtain

$$\mathbb{P}(Z_{T_b^-} \in dx) = \frac{b - x}{D} \exp\left(-\frac{(x - b)^2}{2D}\right) \mathbf{1}_{\{x < b\}} dx. \quad (\text{C.6})$$

C.3. Case $b > 0$. If $b > 0$, we can write $T_b^+ = T_b + T_0^+(W)$ with

$$\begin{aligned}
T_0^+(W) &= \inf\{t \geq 0; \mathbf{1}_{\{W_t \geq 0\}}(t - g_t^W) \geq D\} \stackrel{\text{law}}{=} T_0^+, \\
W &= \{W_t = Z_{T_b+t} - b; t \geq 0\}, \\
g_t^W &= \sup\{u \leq t; W_u = 0\}.
\end{aligned}$$

It follows, from the independence of T_b and $T_0^+(W)$ by using the strong Markov property, that

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b^+)) = \mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b)) \mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_0^+(W))).$$

As $\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b)) = \exp(-|b|\lambda)$, we get

$$\mathbb{E}(\exp(-\frac{1}{2}\lambda^2 T_b^-)) = \frac{\exp(-b\lambda)}{\psi(\lambda\sqrt{D})}. \quad (\text{C.7})$$

The law of $Z_{T_b^+}$ can be computed in the same way as the law of $Z_{T_b^-}$

$$\mathbb{P}(Z_{T_b^+} \in dx) = \mathbb{P}[Z_{T^+} \in (dx - b)].$$

Finally, we have

$$\mathbb{P}(Z_{T_b^+} \in dx) = \frac{x - b}{D} \exp\left(-\frac{(x - b)^2}{2D}\right) \mathbf{1}_{\{x > b\}} dx. \quad (\text{C.8})$$

APPENDIX D. AROUND BROWNIAN MOTION

Let us consider a standard Brownian motion $W = \{W_t; t \geq 0\}$. First of all, we recall two results on the joint law of the Brownian motion and its extrema. A proof can be found in [Revuz and Yor(1994)].

D.1. Law of $(W_t, \sup_{0 \leq u \leq t} W_u)$.

$$\mathbb{P}(W_t \in dx, \sup_{0 \leq u \leq t} W_u \in dy) = \mathbf{1}_{\{0 \leq y\}} \mathbf{1}_{\{x \leq y\}} \frac{2(2y - x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y - x)^2}{2t}\right) dx dy.$$

D.2. Law of $(W_t, \inf_{0 \leq u \leq t} W_u)$.

$$\mathbb{P}(W_t \in dx, \inf_{0 \leq u \leq t} W_u \in dy) = \mathbf{1}_{\{y \leq 0\}} \mathbf{1}_{\{y \leq x\}} \frac{2(2y - x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2y - x)^2}{2t}\right) dx dy$$

D.3. Hitting time. The law of the hitting time T_b defined by

$$T_b = \inf\{t \geq 0 \mid W_t = b\}.$$

is given by

$$\mathbb{P}(T_b \in dx) = \frac{|b|}{\sqrt{2\pi x^3}} e^{-\frac{b^2}{2x}} dx. \quad (\text{D.1})$$

D.4. Excursion. Let g_t denote the last time before t that W hit the level 0.

$$g_t = \sup\{u \leq t \mid W_u = 0\}. \quad (\text{D.2})$$

The purpose is to find the law of (g_t, W_t) . Let \mathbb{P}^x denote the probability starting at level x . The probability starting at the level 0 is simply denoted by \mathbb{P} .

First we would like to calculate $\mathbb{P}^x(W_t \in dy, T_0 > t)$, with $x > 0$ and $y > 0$.

$$\mathbb{P}^x(W_t \in dy, T_0 > t) = \mathbb{P}^x(W_t \in dy) - \mathbb{P}^x(W_t \in dy, T_0 < t). \quad (\text{D.3})$$

Using the reflexion principle, we can stop the Brownian motion at time T_0 and reflect the rest of the trajectory. So it is the same for the Brownian motion issued from x to cross 0 before time t and to end up in the neighbourhood of y as to end up in the neighbourhood of $-y$. Thanks to the almost sure continuity of the Brownian motion paths we can drop the condition that the Brownian motion has hit 0 before time t . So we come up with the following equality

$$\mathbb{P}(W_{t-T_0} \in -dy, T_0 < t) = \mathbb{P}^x(W_t \in -dy). \quad (\text{D.4})$$

So, putting all the different terms together and using the law of the Brownian motion at time t , we come up with the following formula :

$$\mathbb{P}^x(W_t \in dy, T_0 > t) = \frac{1}{\sqrt{2\pi t}} \left(e^{-(x-y)^2/2t} - e^{-(x+y)^2/2t} \right) \mathbf{1}_{\{xy>0\}} dy. \quad (\text{D.5})$$

Now, we can try to compute the law of (g_t, W_t) . Let's calculate $\mathbb{P}(W_t \in dy, g_t \leq s)$. If $t < s$, then g_t is always smaller than s because g_t is bounded by t , so the probability does not depend on s anymore. Thus, its partial differential with respect to s is identically null. Now we assume that $s \leq t$, $y > 0$.

$$\begin{aligned} \mathbb{P}(W_t \in dy, g_t \leq s) &= \mathbb{E}(\mathbf{1}_{\{W_t \in dy, g_t \leq s\}}), \\ &= \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{W_t \in dy, W_u \neq 0 \forall u \in [s, t]\}} \mid \mathcal{F}_s)), \\ &= \mathbb{E}(\mathbb{E}(\mathbf{1}_{\{W_{t-s} \circ \theta_s \in dy, W_u \circ \theta_s \neq 0 \forall u \in [0, t-s]\}} \mid \mathcal{F}_s)), \\ &\quad \text{Relying on the Markov property, we may write} \\ &= \mathbb{E}(\mathbb{E}^{W_s}(\mathbf{1}_{\{W_{t-s} \in dy, W_u \neq 0 \forall u \in [0, t-s]\}})), \\ &\quad \text{we calculated the second expectation above, so we get} \\ &= \mathbb{E} \left(\frac{1}{\sqrt{2\pi(t-s)}} \left(e^{-(W_s-y)^2/2(t-s)} - e^{-(W_s+y)^2/2(t-s)} \right) dy \right), \\ &= \int_0^\infty dx \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s} \frac{1}{\sqrt{2\pi(t-s)}} \left(e^{-(x-y)^2/2(t-s)} - e^{-(x+y)^2/2(t-s)} \right) dy, \\ &= \sqrt{\frac{s(t-s)}{t}} \int_{y \frac{s}{t(t-s)}}^\infty e^{-z^2/2} dz e^{-y^2/2t}. \end{aligned} \quad (\text{D.6})$$

Finally, we only have to differentiate with respect to s to come up with the formula of the density of (g_t, W_t) .

$$\mathbb{P}(W_t \in dy, g_t \in ds) = \frac{y}{2\pi \sqrt{s(t-s)^3}} \exp \left(-\frac{y^2}{2(t-s)} \right) \mathbf{1}_{\{s \leq t\}} ds dy. \quad (\text{D.7})$$

If we assume that $y < 0$ then, since W and $-W$ follow the same law, we can write

$$\mathbb{P}(W_t \in dy, g_t \leq s) = \mathbb{P}(W_t \in -dy, g_t \leq s), \quad (\text{D.8})$$

which enables us to refer to the previous case and the final formula for the law of the couple (g_t, W_t) is given by

$$\mathbb{P}(W_t \in dy, g_t \in ds) = \frac{|y|}{2\pi\sqrt{s(t-s)^3}} \exp\left(-\frac{y^2}{2(t-s)}\right) \mathbf{1}_{\{s \leq t\}} ds dy. \quad (\text{D.9})$$

APPENDIX E. REGULARITY OF OPTION PRICES

Proposition E.1. *Let $f(t)$ be the “star” price of a Parisian option of maturity t . If $b_1 < 0$ and $b_2 > 0$, f is of class C^∞ and for all $k \geq 0$, $f^{(k)}(t) = \mathcal{O}\left(e^{\frac{(m+\sigma)^2}{2}t}\right)$ when t goes to infinity.*

For the sake of clearness, we will only prove Proposition E.1 for single barrier Parisian options as the scheme of the proof is still valid for Parisian options. Once again, we can restrict to calls. Let $f(t) = PDIC(x, t; K, L; r, \delta)$.

$$f(t) = \mathbb{E} \left[e^{mZ_t} (S_t - K)_+ \mathbf{1}_{\{T_b^- < t\}} \right].$$

Let W_t denote $Z_{t+T_b^-} - Z_{T_b^-}$. Relying on the strong Markov property,

$$f(t) = \mathbb{E} \left(\mathbf{1}_{\{T_b^- < t\}} \mathbb{E} \left[(xe^{\sigma(W_{t-\tau}+z)} - K)_+ e^{m(W_{t-\tau}+z)} \right]_{|z=Z_{T_b^-}, \tau=T_b^-} \right). \quad (\text{E.1})$$

Let ν denote the density of $Z_{T_b^-}$ (see [Chesney et al.(1997)Chesney, Jeanblanc-Picqué, and Yor] for its expression) and μ the density of T_b^- (see Proposition F.1 for a proof of existence). Since $Z_{T_b^-}$ and T_b^- are independent, Equation (E.1) can be written

$$f(t) = \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw (xe^{\sigma(w\sqrt{t-\tau}+z)} - K)_+ e^{m(w\sqrt{t-\tau}+z)} p(w) \nu(z) \mu(\tau)$$

where $p(w) = \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}}$. A change of variable on τ gives

$$f(t) = \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw (xe^{\sigma(w\sqrt{\tau}+z)} - K)_+ e^{m(w\sqrt{\tau}+z)} p(w) \nu(z) \mu(t-\tau).$$

Since μ is of class C^∞ and all its derivatives are null at 0 and bounded on any interval $[0, T]$ (see Appendix F), one can easily prove that f is of class C^∞ and that for all $k \geq 0$

$$f^{(k)}(t) = \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw (xe^{\sigma(w\sqrt{\tau}+z)} - K)_+ e^{m(w\sqrt{\tau}+z)} p(w) \nu(z) \mu^{(k)}(t-\tau).$$

This proves the first part of Proposition E.1. From Proposition F.1, we know that μ and all its derivatives are bounded. Then, we can bound $f^{(k)}$

$$\begin{aligned} |f^{(k)}(t)| &\leq \int_0^t d\tau \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dw x e^{(m+\sigma)(w\sqrt{\tau}+z)} p(w) \nu(z) \|\mu^{(k)}\|_\infty, \\ &\leq \int_{-\infty}^{\infty} x e^{(m+\sigma)z} \nu(z) dz \|\mu^{(k)}\|_\infty \int_0^t e^{\frac{(m+\sigma)^2}{2}\tau} d\tau, \\ &\leq e^{\frac{(m+\sigma)^2}{2}t} \frac{2x}{(m+\sigma)^2} \|\mu^{(k)}\|_\infty \int_{-\infty}^{\infty} e^{(m+\sigma)z} \nu(z) dz. \end{aligned}$$

APPENDIX F. REGULARITY OF THE DENSITY OF T_b^-

In this section, we assume $b < 0$.

Proposition F.1. *The r.v. T_b^- has a density μ w.r.t to Lebesgue’s measure. μ is of class C^∞ and for all $k \geq 0$, $\mu^{(k)}(0) = \mu^{(k)}(\infty) = 0$.*

To prove this proposition, we need the two following lemmas.

Lemma F.1. *Let \mathcal{N} be the analytic prolongation of the cumulative normal distribution function on the complex plane. The following equivalent holds*

$$\mathcal{N}(r(1+i)) \sim 1 \text{ when } r \rightarrow \infty.$$

Lemma F.2. *For $b < 0$, we have for $u \in \mathbb{R}$*

$$\mathbb{E} \left(e^{-iuT_b^-} \right) = \mathcal{O} \left(e^{-|b|\sqrt{|u|}} \right) \text{ when } |u| \rightarrow \infty.$$

Proof of Proposition F.1. We recall that

$$\mathbb{E} \left(e^{-\frac{\lambda^2}{2} T_b^-} \right) = \frac{e^{\lambda b}}{\psi(\lambda\sqrt{D})}. \quad (\text{F.1})$$

We define $\mathcal{O} = \{z \in \mathbb{C}; -\frac{\pi}{4} < \arg(z) < \frac{\pi}{4}\}$. One can easily prove that the function $z \mapsto \mathbb{E} \left(e^{-\frac{z^2}{2} T_b^-} \right)$ is holomorphic on the open set \mathcal{O} and hence analytic. Moreover, $z \mapsto \frac{e^{zb}}{\psi(z\sqrt{D})}$ is also analytic on \mathcal{O} except perhaps in a countable number of isolated points. These two functions coincide on \mathbb{R}^+ , so they are equal on \mathcal{O} .

Consequently, we can derive the following equality. For all $z \in \mathbb{C}$ with positive real part, we have

$$\mathbb{E} \left(e^{-zT_b^-} \right) = \frac{e^{\sqrt{2z}b}}{\psi(\sqrt{2z}D)}. \quad (\text{F.2})$$

We use the following convention: for any $z \in \mathbb{C}$ with positive real part, \sqrt{z} is the only complex number $z' \in \mathcal{O}$ such that $z = z'^2$.

Thanks to the continuity of both terms in (F.2), the equality also holds for pure imaginary numbers. Hence, by setting $z = iu$ for $u \in \mathbb{R}$ in Equation (F.2), we obtain the Fourier transform of T_b^-

$$\mathbb{E} \left(e^{-iuT_b^-} \right) = \frac{e^{\sqrt{2ui}b}}{\psi(\sqrt{2iu}D)}.$$

From Lemma F.2, we know that the Fourier transform of T_b^- is integrable on \mathbb{R} , thus the r.v. T_b^- has a density μ w.r.t. the Lebesgue measure given by

$$\mu(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\sqrt{2ui}b}}{\psi(\sqrt{2iu}D)} e^{-iut} du.$$

Moreover, thanks to Lemma F.2, $u \mapsto u^k \frac{e^{\sqrt{2ui}b}}{\psi(\sqrt{2iu}D)}$ is integrable and continuous. Hence, μ is of class C^∞ . Since $\mu(t) = 0$ for $t < D$, for all $k \geq 0$, $\mu^{(k)}(0) = 0$. Lemma F.3 yields that for all $k \geq 0$, $\lim_{t \rightarrow \infty} \mu^{(k)}(t) = 0$. \square

Proof of Lemma F.1.

$$\mathcal{N}(x + iy) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(v+iy)^2}{2}} dv.$$

It is easy to check that $\partial_x \mathcal{N}(x + iy) - \partial_y \mathcal{N}(x + iy) = 0$ and this definition coincides with the cumulative normal distribution function on the real axis, so it is the unique analytic prolongation. We write $\mathcal{N}(x + iy) = \mathcal{N}(x) + \int_0^y \partial_y \mathcal{N}(x + iy) dy$, to get

$$\begin{aligned} \mathcal{N}(x + iy) &= \mathcal{N}(x) - i \frac{1}{\sqrt{2\pi}} \int_0^y \int_{-\infty}^x (v + iu) e^{-\frac{(v+iu)^2}{2}} dv du, \\ &= \mathcal{N}(x) + i \frac{1}{\sqrt{2\pi}} \int_0^y e^{-\frac{(x+iu)^2}{2}} du. \end{aligned}$$

Taking $x + iy = r(1 + i)$ gives

$$\begin{aligned} \mathcal{N}(r(1 + i)) &= \mathcal{N}(r) + i \frac{1}{\sqrt{2\pi}} \int_0^r e^{-\frac{(r+iu)^2}{2}} du, \\ &= \mathcal{N}(r) + i \frac{1}{\sqrt{2\pi}} \int_0^1 e^{\frac{r^2}{2}(t^2-1)} e^{-itr^2} r dt. \end{aligned} \quad (\text{F.3})$$

For $t \in [0, 1]$, $e^{\frac{r^2}{2}(t^2-1)} r$ tends to 0 when r goes to infinity. The function $r \mapsto e^{\frac{r^2}{2}(t^2-1)} r$ is maximum for $r = \frac{1}{1-t^2}$, hence the following upper bound holds

$$e^{\frac{r^2}{2}(t^2-1)} r \leq \frac{1}{1-t^2} e^{\frac{1}{2(t^2-1)}} \quad \text{for all } t \in [0, 1].$$

The upper bound is integrable on $[0, 1]$, so by using the bounded convergence theorem, we can assert that the integral on the right hand side of (F.3) tends to 0 when r goes to infinity. \square

Proof of Lemma F.2. We only do the proof for $u > 0$. For $r > 0$,

$$\psi(r(1+i)) = 1 + r(1+i)\sqrt{2\pi}e^{r^2i}\mathcal{N}(r(1+i)).$$

Using the equivalent of $\mathcal{N}(r(1+i))$ when r goes to infinity (see Lemma F.1) enables to establish that $|\psi(r(1+i))| \sim 2r\sqrt{\pi}$ when r goes to infinity. Noticing that $\sqrt{iu} = \frac{\sqrt{2u}}{2}(1+i)$ ends the proof. \square

Here is a quite obvious lemma we used in the proof of Proposition F.1.

Lemma F.3. *Let g be an integrable function on \mathbb{R} , then*

$$\lim_{t \rightarrow \infty} \int_{-\infty}^{\infty} g(u) e^{iut} du = 0.$$

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