

# The Heston Stochastic-Local Volatility Model: Efficient Particle Method

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## 1 Introduction

The following method proposed by [1] deals with an efficient particle method for computing the price of a stochastic volatility model.

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## 2 Theoretical framework

We consider the following local stochastic volatility model

$$dS_t = rS_t dt + S_t \sigma(t, S_t) a_t dW_t$$

where  $a$  is a stochastic process. From Dupire, this model is exactly calibrated to market smiles if we have

$$\sigma_{LV}^2(t, s) = \sigma^2(t, s) \mathbb{E}[a_t^2 | S_t = s].$$

where  $\sigma_{LV}$  represents the Dupire local volatility. The volatility function depends on the joint PDF  $p(t, s, a)$  of  $(s_t, a_t)$

$$\sigma(t, s) = \sigma_{LV}(t, s) \sqrt{\frac{\int p(t, s, a') da'}{\int (a')^2 p(t, s, a') da'}} \quad (1)$$

## 3 Particle method

Let us introduce this method by considering the following McKean SDE

$$dX_t = b(t, X_t, \mathbb{P}_t) dt + \sigma(t, X_t, \mathbb{P}_t) dW_t, \quad \mathbb{P}_t = Law(X_t)$$

The simulation of such an equation consists of replacing the law  $\mathbb{P}_t$  by the empirical distribution

$$\mathbb{P}_t^M := \frac{1}{M} \sum_{i=1}^M \delta_{X_t^{i,M}}$$

where  $(X_t^{i,M})_{1 \leq i \leq M}$  are the solutions to the  $\mathbb{R}^M$  dimensional linear SDE

$$\begin{aligned} dX_t^{i,M} &= b(t, X_t^{i,M}, \mathbb{P}_t^M)dt + \sigma(t, X_t^{i,M}, \mathbb{P}_t^M)dW_t^i, \\ Law(X_0^{i,M}) &= \mathbb{P}_0 \end{aligned}$$

where  $(W^i)_{1 \leq i \leq M}$  are  $M$  independent Brownian motions and  $\mathbb{P}_t^M$  is a random measure on  $\mathbb{R}$ . We get

$$dX_t^{i,M} = \frac{1}{M} \sum_{j=1}^M b(t, X_t^{i,M}, X_t^{j,M})dt + \frac{1}{M} \sum_{j=1}^M \sigma(t, X_t^{i,M}, X_t^{j,M})dW_t^i.$$

From (Snitzman, 1991), we know that if at  $t = 0$   $X_0^{i,M}$  are independent r.v., then as  $M$  tends to infinity, for any fixed  $t > 0$ ,  $(X_t^{i,M})_{1 \leq i \leq M}$  are asymptotically independent and their empirical measure  $\mathbb{P}_t^M$  converges in distribution to the true measure  $\mathbb{P}_t$ .

## 4 Numerical algorithm

### 4.1 Particle method to compute $\mathbb{E}[a_t^2 | S_t = s]$

We approximate  $\mathbb{E}[a_t^2 | S_t = s]$  by  $\mathbb{E}^{\mathbb{P}_t^M}[a_t^2 | S_t = s]$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_t^M}[a_t^2 | S_t = s] &= \frac{\int (a')^2 p_M(t, s, a') da'}{\int p_M(t, s, a') da'} \\ &= \frac{\sum_{i=1}^M (a_t^{i,M})^2 \delta(S_t^{i,M} - s)}{\sum_{i=1}^M \delta(S_t^{i,M} - s)} \end{aligned}$$

We use a regularizing kernel  $\delta_{t,M}(\cdot)$  that converges to the Dirac function as  $M$  tends to infinity. It is natural to choose

$$\delta_{t,M}(x) = \frac{1}{h_{t,M}} K\left(\frac{x}{h_{t,M}}\right)$$

where  $K$  is a fixed, symmetric kernel with a bandwidth  $h_{t,M}$  tending to 0 as  $M$  tends to infinity. For example we can choose  $K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ ,  $K(x) = \frac{15}{16}(1-x^2)^2 \mathbb{1}_{\{|x| \leq 1\}}$  or an indicator function  $K(x) = \mathbb{1}_{\{|x| \leq 1\}}$ . Using this method, we approximate  $\sigma(t, s)$  defined in (1) by

$$\sigma_M(t, s) = \sigma_{LV}(t, s) \sqrt{\frac{\sum_{i=1}^M \delta(S_t^{i,M} - s)}{\sum_{i=1}^M (a_t^{i,M})^2 \delta(S_t^{i,M} - s)}}. \quad (2)$$

### 4.2 Simulation scheme

We consider the following dynamics of the Heston SLV model expressed in terms of independent Brownian motions:

$$\begin{aligned} dS_t &= rS_t dt + \sigma(t, S_t)S_t \sqrt{a_t} \left( \rho_{x,v} d\tilde{W}_t^v + \sqrt{1 - \rho_{x,v}^2} d\tilde{W}_t^x \right), \\ da_t &= \kappa(\bar{v} - a_t)dt + \gamma \sqrt{a_t} d\tilde{W}_t^v \end{aligned}$$

where  $\tilde{W}^x$  and  $\tilde{W}^v$  are independent Brownian motions. We discretize  $[0, T]$  on a regular grid of size  $N$ , with step size  $\Delta = \frac{T}{N}$ . An Euler scheme gives the following approximation

$$s_{i+1,j} = s_{i,j} + rs_{i,j}\Delta + \sigma_M(t_i, s_{i,j})s_{i,j}\sqrt{(a_t)_+}\sqrt{\Delta}Z_x, \quad s_{0,j} = S_0,$$

$$a_{i+1,j} = a_{i,j} + a_v(t_i, a_{i,j})\Delta + a_v(t_i, a_{i,j})\sqrt{\Delta}Z_v, \quad a_{0,j} = a_0,$$

for  $j = 1, \dots, M$  and  $i = 0, \dots, N$  and  $\sigma_M(t, s)$  is given by (2).  $Z_x = Z_1$  and  $Z_v = \rho_{x,v}Z_1 + \sqrt{(1 - \rho_{x,v}^2)}Z_2$  where  $Z_1$  and  $Z_2$  are two independent standard normal variables.

## 5 Numerical experiments

We test the algorithm on a Call option with payoff  $e^{-rT}(S_T - K)_+$  with the following parameters, for different maturities and strikes :

$r$	$s_0$	$a_0$	$\kappa$	$\gamma$	$\rho_{x,v}$
0	1	0.0945	1.05	0.95	-0.315

We use  $N = 100$  times steps and  $M = 5000$  particles and the local volatility  $\sigma_{LV}$  is given by

$$\sigma_{LV}(t, x) = 0.01 + 0.1e^{-x/s_0} + 0.01t.$$

K T	2	8
0.7	0.300711	0.310436
0.9	0.107903	0.152974
1.1	0.005151	0.053920
1.3	0.000345	0.021565
1.5	0.000227	0.012228

## References

- [1] J. Guyon, P.H. Labordère. Being particular about calibration. Risk, January 2012.