

# Quadratic interest rate model

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### 1 The quadratic interest rate model

#### 1.1. Description of the model

In the quadratic interest rate model, the evolution of the spot interest rate  $r(t)$  is described by the following SDE :

$$\begin{cases} dx(t) = (\alpha(t) - \beta x(t)) dt + \sigma dW(t), \\ r(t) = \frac{1}{2}x(t)^2, \\ x(0) = \sqrt{2r(0)}, \end{cases}$$

where  $\beta$  and  $\sigma$  are constants.  $\alpha$  is a time-dependent function determined by the values of  $\beta$ ,  $\sigma$  and the curve of the  $s$ -maturity zero-coupon prices at time  $t = 0$ . Notice that  $(x(t), t \geq 0)$  is a gaussian process.

If  $\mathbb{E}_t$  denote the conditional expectation at time  $t$  under the risk-neutral measure, for  $s$ -maturity zero-coupon bond at time  $t$ , we have :

$$\begin{aligned} P_s(t) &= \mathbb{E}_t [\exp(-\int_t^s r(u)du)] \\ &= \exp\left(-\left(\frac{1}{2}B_s(t)x(t)^2 + b_s(t)x(t) + c_s(t)\right)\right), \end{aligned} \tag{1}$$

where  $B_s(t)$ ,  $b_s(t)$  and  $c_s(t)$  are described in Section 2 and computed using equations given in Appendix (see **8.2.1**).

### 1.2. The $T$ -forward risk adjusted measure

For options on bonds, caplet and call on futures, we will have to use the  $T$ -forward risk adjusted measure  $\mathbb{E}_t^T$  defined by :

$$\mathbb{E}_t^T[Z(T)] = \mathbb{E}_t[e^{-\int_t^T r(u)du} Z(T)]/P_T(t) \quad (2)$$

for all non-negative Itô process  $Z$ .

### 1.3. Notations

We write  $Y \sim \Omega(B, b, c, \mu, V)$  if  $Y = \frac{1}{2}BX^2 + bX + c$ , where  $X = \mu + VG$  and  $G \sim \mathcal{N}(0, 1)$  is a centered reduced gaussian. We also have  $Y = \alpha + \beta(G + \sqrt{\lambda})^2$  where :

$$\alpha = c - \frac{1}{2} \frac{b^2}{B}, \quad \beta = \frac{1}{2}BV, \quad \lambda = \frac{(\mu + b/B)^2}{V}. \quad (3)$$

Notice that  $(G + \sqrt{\lambda})^2$  is distributed as a non-central chi-square with 1 degree of freedom and non-centrality parameter  $\lambda$ . Let  $\chi^2(y; \lambda, \beta)$  denote the cumulative distribution of  $\beta(G + \sqrt{\lambda})^2$  and  $\omega(y; B, b, \mu, V)$  the cumulative distribution of  $\frac{1}{2}BX^2 + bX$ . Hence, we have

$$\mathbb{P}(Y \leq y) = \mathbb{P}(\beta(G + \sqrt{\lambda})^2 \leq y - \alpha) = \chi^2(y - \alpha; \lambda, \beta),$$

as well as

$$\mathbb{P}(Y \leq y) = \mathbb{P}\left(\frac{1}{2}BX^2 + bX \leq y - c\right) = \omega(y - c; B, b, \mu, V).$$

In particular :

$$\omega(y; B, b, \mu, V) = \chi^2(y + \frac{1}{2} \frac{b^2}{B}; \lambda, \beta).$$

For a function of two variables written as  $f_s(t)$ , we write  $\dot{f}_s(t) \equiv \partial f_s(t)/\partial s$  and  $\ddot{f}_s(t) \equiv \partial^2 f_s(t)/\partial s^2$ .

We also use the following conventions :

- $T$  denotes the maturity of an option,
- $t$  denotes the maturity of a futures or a forward contract,
- $s$ , or  $s'$  denote maturities of zero-coupon bonds,
- $K$  denotes the strike of an option.

All prices are given at initial time  $t = 0$ . Hence, we have  $0 \leq T \leq t \leq s, s'$ .

## 2 Calibration and computation of bond coefficients

### 2.1. Initial values of bond coefficients

To compute the time-dependent function  $\alpha$ , we must first compute the forward interest rate at time  $t = 0$  from the initial zero-coupon curve  $P_s(0)$  as described in the Appendix. Now, for any  $s$  and  $t$ , using equations given in 8.1.1, we can compute  $B_s(0)$ ,  $\dot{b}_s(0)$ ,  $\dot{c}_s(0)$  and  $\alpha(t)$  to fit the initial yield curve. We get  $b_s(0)$  and  $c_s(0)$  for any  $s$  integrating  $\dot{b}_s(0)$  and  $\dot{c}_s(0)$  with means of trapezoidal rule.

### 2.2. Transport equations

Transport equations yield formulas for  $B_s(t)$ ,  $b_s(t)$  and  $c_s(t)$  for any  $t$  and  $s$  using their initial values. These equations are given in 6.1.2.

## 3 Closed formulae for european options on bonds

### 3.1. European call

$$Price : \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (P_s(T) - K)_+ \right] = P_T(0) \mathbb{E}_0^T [(P_s(T) - K)_+].$$

Under the  $T$ -forward risk adjusted measure, we have  $-\log(P_s(T)) \sim \Omega(B, b, c, \mu, V)$  where  $B = B_s(T)$ ,  $b = b_s(T)$ ,  $c = c_s(T)$  and :

$$\mu = \sqrt{\dot{B}_T(0)}x(0) + \frac{\dot{b}_T(0)}{\sqrt{\dot{B}_T(0)}}, \quad V = \sigma^2 B_T(0). \quad (4)$$

Using (3), we compute the coefficients  $\alpha$ ,  $\beta$  and  $\lambda$  corresponding to  $B$ ,  $b$ ,  $c$ ,  $\mu$  and  $V$ . Then, the price of a  $T$ -maturity call option on the  $s$ -bond is given by:

$$\begin{aligned} \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (P_s(T) - K)_+ \right] \\ = P_s(0) \chi^2(-\alpha - \log(K); \frac{\lambda}{1+2\beta}, \frac{\beta}{1+2\beta}) \\ - K P_T(0) \chi^2(-\alpha - \log(K); \lambda, \beta). \end{aligned}$$

### 3.2. Caplet

$$Price : \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (r(T) - K)_+ \right] = P_T(0) \mathbb{E}_0^T [(r(T) - K)_+].$$

Under the  $T$ -forward risk adjusted measure, we have  $r(T) \sim \Omega(B, b, c, \mu, V)$  with  $B = 1$ ,  $b = 0$ ,  $c = 0$  and  $\mu$  and  $V$  given by (4). Thanks to (3), we compute the coefficients  $\alpha$ ,  $\beta$  and  $\lambda$  corresponding to  $B$ ,  $b$ ,  $c$ ,  $\mu$  and  $V$ . Then, the price of a  $T$ -maturity caplet is given by:

$$\mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (r(T) - K)_+ \right] = P_T(0) \left[ \frac{1}{2}(r_T(0) - K) + C(K - \alpha; \lambda, \beta) \right],$$

where:

$$C(K - \alpha; \lambda, \beta) = \frac{1}{\pi} \int_0^{+\infty} \left[ 1 - \Psi(\lambda, 2\xi^2\beta^2) \cos(\xi(K - \alpha) - \Phi(\lambda, \xi\beta)) \right] \frac{d\xi}{\xi^2},$$

with

$$\Psi(\lambda, z) = (1 + 2z)^{-1/4} \exp\left(\frac{\lambda z}{1 + 2z}\right) \text{ and } \Phi(\lambda, z) = \frac{1}{2} \arctan(2z) + \frac{\lambda z}{1 + 4z^2}.$$

### 3.3. Exchange option

$$Price : \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (kP_s(T) - k'P_{s'}(T))_+ \right] = P_T(0) \mathbb{E}_0^T [(kP_s(T) - k'P_{s'}(T))_+].$$

Under the  $T$ -forward risk adjusted measure :  $-\log(P_s(T)) = \frac{1}{2}BX^2 + bX + c$  and  $-\log(P_{s'}(T)) = \frac{1}{2}B'X^2 + b'X + c'$  where  $X \sim \mathcal{N}(\mu, V)$ ,  $B = B_s(T)$ ,  $b = b_s(T)$ ,  $c = c_s(T)$ ,  $B' = B_{s'}(T)$ ,  $b' = b_{s'}(T)$ ,  $c' = c_{s'}(T)$  and  $\mu$ ,  $V$  given by (4). In particular, we have  $-\log(P_s(T)) \sim \Omega(B, b, c, \mu, V)$  and  $-\log(P_{s'}(T)) \sim \Omega(B', b', c', \mu, V)$ . Then, the price of the exchange option to put  $k'$   $s'$ -bonds and call  $k$   $s$ -bond is :

$$\begin{aligned} & \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (kP_s(T) - k'P_{s'}(T))_+ \right] \\ &= kP_s(0) \omega\left(c' - c - \log\left(\frac{k'}{k}\right); B - B', b - b', \frac{\mu - bV}{1 + BV}, \frac{V}{1 + BV}\right) \\ & \quad - k'P_{s'}(0) \omega\left(c' - c - \log\left(\frac{k'}{k}\right); B - B', b - b', \frac{\mu - b'V}{1 + B'V}, \frac{V}{1 + B'V}\right). \end{aligned}$$

### 3.4. European call on forward contract

$$Price^1 : \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (P_s(T) - K P_t(T))_+ \right] = P_t(0) \mathbb{E}_0^t \left[ \left( \frac{P_s(T)}{P_t(T)} - K \right)_+ \right].$$

Under the  $t$ -forward risk adjusted measure, we have  $-\log(P_s(T)/P_t(T)) \sim \Omega(B, b, c, \mu, V)$  with  $B = B_s(T) - B_t(T)$ ,  $b = b_s(T) - b_t(T)$ ,  $c = c_s(T) - c_t(T)$  and :

$$\mu = \sqrt{\frac{\dot{B}_t(0)}{\dot{B}_t(T)}} x(0) + \frac{\dot{b}_t(0)}{\sqrt{\dot{B}_t(0)\dot{B}_t(T)}} - \frac{\dot{b}_t(T)}{\dot{B}_t(T)}, \text{ and } V = \frac{V_t(0) - V_t(T)}{B_t(T)}.$$

Thanks to (3), we compute the coefficients  $\alpha$ ,  $\beta$  and  $\lambda$  corresponding to  $B$ ,  $b$ ,  $c$ ,  $\mu$  and  $V$ . Then, the price of a call option on the  $t$ -delivery forward contract on the  $s$ -bond is given by:

$$\begin{aligned} & \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (P_s(T) - K P_t(T))_+ \right] \\ &= P_s(0) \chi^2(-\alpha - \log(K); \frac{\lambda}{1+2\beta}, \frac{\beta}{1+2\beta}) - K P_t(0) \chi^2(-\alpha - \log(K); \lambda, \beta). \end{aligned}$$

## 4 Closed formulae for futures and european options on futures

For any  $0 \leq T \leq t \leq s$ , we set  $F_{t,s}(T) \equiv \mathbb{E}_T[P_s(t)]$ .

### 4.1. Futures

$$Price : F_{t,s}(0) = \mathbb{E}_0[P_s(t)]$$

Under the risk-neutral measure,  $-\log(P_s(t)) \sim \Omega(B, b, c, \mu, V)$  with  $B = B_t(s)$ ,  $b = b_t(s)$ ,  $c = c_t(s)$ , and

$$\mu = p_t + q_t x(0), \quad V = v_t \quad (5)$$

Recall that  $x(0) = \sqrt{2r_0}$ . Subsection 8.2.1 of the Appendix gives expressions to compute  $B_{t,s}(0)$ ,  $b_{t,s}(0)$ ,  $c_{t,s}(0)$ ,  $p_t$  and  $q_t$ . Then, the price of the  $t$ -delivery futures contract on the  $s$ -bond is given by :

$$F_{t,s}(0) = \mathbb{E}_0[P_s(t)] = \exp \left( -B_{s,t}(0)x(0)^2 + b_{s,t}(0)x(0) + c_{s,t}(0) \right). \quad (6)$$

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<sup>1</sup>Evidently, the option to exchange two bonds as in section 3.4. is equivalent to an option on a bond forward contract, as in section 3.4. We thus have two different formulae for this option which agree under the assumption of the model.

#### 4.2. European call option on futures

$$Price : \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (F_{t,s}(T) - K)_+ \right] = P_t(0) \mathbb{E}_0^t \left[ (F_{t,s}(T) - K)_+ \right]$$

Under the  $T$ -forward risk adjusted measure,  $-\log(F_{t,s}(T)) \sim \Omega(B, b, c, \mu, V)$  with  $B = B_{t,s}(T)$ ,  $b = b_{t,s}(T)$ ,  $c = c_{t,s}(T)$  and  $\mu, V$  given by (4). The coefficients  $B_{t,s}(T)$ ,  $b_{t,s}(T)$ , and  $c_{t,s}(T)$  are computed from  $B_{t,s}(0)$ ,  $b_{t,s}(0)$ , and  $c_{t,s}(0)$ , as described in subsection 6.2.2 of the appendix. Thanks to (3), we compute the coefficient  $\alpha, \beta$  and  $\lambda$  corresponding to  $B, b, c, \mu$  and  $V$ . Then, the price of a  $T$ -maturity european call option on the  $t$ -delivery futures on  $s$ -bond is :

$$\begin{aligned} & \mathbb{E}_0 \left[ e^{-\int_0^T r(u)du} (F_{t,s}(T) - K)_+ \right] \\ &= P_T(0) \mathbb{E}_0^T [F_{t,s}(T)] \chi^2(-\alpha - \log(K); \frac{\lambda}{1+2\beta}, \frac{\beta}{1+2\beta}) \\ & \quad - K P_T(0) \chi^2(-\alpha - \log(K); \lambda, \beta), \end{aligned}$$

with  $\mathbb{E}_0^T [F_{t,s}(0)] = e^{-Fx^2(0)-Gx(0)-H}$ . Formula for  $F, G$  and  $H$  are given in section 8.2.3.

#### 4.3. Delivery option

$$Price : \mathbb{E}_0 [\min(kP_s(T), k'P_{s'}(T))]$$

Under the risk-neutral measure, we have  $-\log(P_s(t)) \sim \Omega(B, b, c, \mu, V)$  and  $-\log(P_{s'}(t)) \sim \Omega(B', b', c', \mu, V)$  with  $B = B_s(t)$ ,  $b = b_s(t)$ ,  $c = c_s(t)$ ,  $B' = B_{s'}(t)$ ,  $b' = b_{s'}(t)$ ,  $c' = c_{s'}(t)$ , and  $\mu, V$  given by (4). Then, the price of the  $T$ -maturity futures contract to deliver the cheapest between  $k$   $s$ -bond or  $k'$   $s'$ -bond is :

$$\begin{aligned} & \mathbb{E}_0 [\min(kP_s(T), k'P_{s'}(T))] \\ &= kF_{t,s}(0) \omega(c - c' + \log(\frac{k'}{k}); B' - B, b' - b, \frac{\mu - bV}{1 + BV}, \frac{V}{1 + BV}) \\ & \quad + k'F_{t,s'}(0) \omega(c' - c - \log(\frac{k'}{k}); B - B', b - b', \frac{\mu - b'V}{1 + B'V}, \frac{V}{1 + B'V}). \end{aligned}$$

Notice that  $F_{s,t}(0)$  and  $F_{s',t}(0)$  are given by (6).

## 5 Monte Carlo methods for european options on futures and bonds

For each option, we know that  $(x(t), t \geq 0)$  is a gaussian process both under the risk-neutral measure and under the  $T$ -forward risk adjusted measure and we have expression for its mean and variance. Therefore, to compute Monte Carlo methods, we simulate the variable  $x(t)$  and we simply use the relationships between  $x(t)$  and the price of zero-coupon bond, the spot interest rate or the price of a futures.

For options on bonds, caplet, and call on futures, we use the distribution of  $x$  under the  $T$ -forward risk adjusted measure. Indeed, we have,  $x \sim \mathcal{N}(\mu, V)$  with  $\mu$  and  $V$  given by (4) and for any pay-off  $X(T)$  at maturity  $T$

$$\mathbb{E}_0 \left[ e^{-\int_0^T r(s)ds} X(T) \right] = P_T(0) \mathbb{E}_0^T [X(T)].$$

Hence, with a Monte Carlo method, we get the desired price.

For futures and for delivery options (section 4.1 and 4.3), we directly get the desired price by a Monte Carlo method using the distribution of  $x$  in the risk-neutral measure :  $x \sim \mathcal{N}(\mu, V)$  is again a gaussian process with  $\mu$ , and  $V$  given by (5) for futures and by (4) for delivery options.

## 6 Algorithms

The functions below are common to all programs :

- `void bond_coeffs(ZCMarketData* ZCMarket, Data *data, double T, double beta, double sigma, double x0);`  
This function computes the coefficients  $B_T(0)$ ,  $b_T(0)$ ,  $c_T(0)$ ,  $\dot{B}_T(0)$ ,  $\dot{b}_T(0)$  and stores them in structure `data` .  
Integrations are done with means of trapezoidal rule.
- `void transport(Omega *om, Data data1, Data data2, double alpha, double beta, double sigma, double x0)`  
This function computes the coefficients of  $P_s(T)$  knowing those of  $P_T(0)$  (contained in `data1`) and  $P_s(0)$  (contained in `data2`).  
Results are stored in `om.B`, `om.b` and `om.c`.
- `void om2chn(Omega om, Chn *chn)`  
Transforms an `Omega` structure into a `Chn` structure using equations (3).

## 7 Results and conclusions

To check the accuracy of the computed prices, we did the following tests.

- First, we checked the put prices computed with closed forms agree with the prices given by Pelsser in [2]. Computed prices are exactly the same.
- Then, to check the efficiency of the quadratic interpolation, we have taken a function for  $P_s(0)$  ( $P_s(0) = \exp(-s(0.08 - 0.05e^{-.18s}))$ ) and we have discretized it successively with a time-step of 0.05 and a time-step of 0.25. Prices for each kind of options are nearly the same : the error is always lower than 1 basis point.
- We also checked the prices using Monte Calo method. We launched the program 1000 times and around 95% of computed 95-per-cent-confidence intervals contain the closed form price.
- We also passed the following test : for  $\alpha$  given and constant, we have computed prices of zero-coupon bond for several maturity. These prices were stored in the file `initialyield.dat`. Then, we launched the program for  $\alpha$  time-dependent and we checked if computed prices with these values for bond were the same than for  $\alpha$  constant. The prices are always the same with an error lower than one basis point.

## 8 Appendix

In all the following equations, we set  $\gamma = \sqrt{\beta^2 + \sigma^2}$ .

### 6.1. Bond coefficients

- **6.1.1. Equations to compute initial values of bond coefficients**

For all  $s$  and  $t$  :

$$B_s(0) = \frac{e^{2\gamma s} - 1}{(\gamma + \beta)e^{2\gamma s} + \gamma - \beta}$$



If  $\alpha$  is constant, we have closed forms for  $b_s(0)$  and  $\dot{c}_s(0)$ <sup>2</sup> :

$$\begin{aligned} h(s) &= ((\gamma + \beta)e^{2\gamma s} + \gamma - \beta)^{-1}, \\ b_s(0) &= \frac{\alpha}{\gamma} h(s) (e^{\gamma s} - 1)^2, \\ \dot{c}_s(0) &= \alpha b_s(0) + \frac{1}{2} \sigma^2 B_s(0) - \frac{1}{2} \sigma^2 b_s^2(0). \end{aligned}$$

Else, for all  $s$ , the forward interest rate at time  $t = 0$  is :  $r_s(0) = -\frac{\partial \log(P_s(0))}{\partial s}$ .  
Then, we have :

$$\dot{b}_s(0) = \dot{B}_s(0)x(0) + \sqrt{\dot{B}_s(0)(2r_s(0) - \frac{1}{2}\sigma^2 B_s(0))}$$

$$\dot{c}_s(0) = \frac{1}{2} \left( \frac{(\dot{b}_s(0))^2}{\dot{B}_s(0)} + \sigma^2 B_s(0) \right)$$

$$\alpha(t) = (\dot{B}_t(0))^{-3/2} (\dot{B}_t(0)\ddot{b}_t(0) - \ddot{B}_t(0)\dot{b}_t(0)).$$

### • 6.1.2. Transport equations for bond coefficients

For all  $T$ , and  $s$ , we have :

$$B_s(T) = \frac{B_s(0) - B_T(0)}{\dot{B}_T(0) - \sigma^2 B_T(0)(B_s(0) - B_T(0))},$$

$$\dot{B}_s(T) = \frac{\dot{B}_s(0)\dot{B}_T(0)B_s^2(T)}{(B_s(0) - B_T(0))^2},$$

$$b_s(T) = B_s(T) \sqrt{\dot{B}_s(T)} \left( \frac{b_s(0) - b_T(0)}{B_s(0) - B_T(0)} - \frac{b_T(0)}{B_T(0)} \right),$$

$$\dot{b}_s(T) = \frac{\dot{b}_s}{\sqrt{\dot{B}_T(0)}} \left( 1 + \sigma^2 B_T(0) B_s(T) \right) + \dot{B}_s(T) \left( \sigma^2 B_T(0) (b_s(0) - b_T(0)) - \dot{b}_T(0) \right),$$

$$c_s(T) = c_s(0) - c_T(0) - \tilde{c}(B_s(T), b_s(T), \dot{b}_T(0)/\sqrt{\dot{B}_T(0)}, \sigma^2 B_T(0)).$$

$$\text{with } \tilde{c}(B, b, a, V) = \frac{1}{2} \left( \log(1 + BV) + \frac{Ba^2 + 2ab - Vb^2}{1 + BV} \right).$$

## 6.2. Futures coefficients

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<sup>2</sup>There is a misprint in the formula given by Jamshidian in [1] for  $\dot{c}_s(0)$  which is corrected here.

• **6.2.1. Equations to compute initial values of futures coefficients**

For all  $s$  and  $t$ , we set:

$$p_t = \int_0^t \alpha(u) e^{-\beta(t-u)} du, \quad q_t = e^{-\beta t}, \quad v_t = \frac{\sigma^2(1 - e^{-2\beta t})}{2\beta}.$$

Then, for all  $s$  and  $t$ , we have :

$$\begin{aligned} B_{t,s}(0) &= \frac{q_t^2 B_s(t)}{1 + B_s(t)v_t}, \\ b_{t,s}(0) &= \frac{q_t(b_s(t) + B_s(t)p_t)}{1 + B_s(t)v_t}, \\ c_{t,s}(0) &= c_s(t) + \frac{1}{2} \log(1 + v_t B_s(t)) + \frac{B_s(t)p_t^2 + 2b_s(t)p_t - v_t p_t^2}{2(1 + v_t B_s(t))}. \end{aligned}$$

• **6.2.2. Transport equations for futures coefficients**

For all  $s$ ,  $t$  and  $T$ , we set :

$$\begin{aligned} B_{t,s}(T) &= \frac{B_{t,s}(0)}{q_T^2 - v_T B_{t,s}(0)}, \\ b_{t,s}(T) &= B_{t,s}(T) \left( \frac{b_{t,s}(0)}{B_{t,s}(0)} q_T - p_T \right), \\ c_{t,s}(T) &= c_{t,s}(0) - \frac{1}{2} \log(1 + v_T B_{t,s}(T)) + \frac{B_{t,s}(T)p_T^2 + 2b_{t,s}(T)p_T - v_T q_T^2}{2(1 + v_T B_{t,s}(T))}. \end{aligned}$$

For all  $t$ ,  $s$  and  $T$ , we set :

• **6.2.3. Other formulas for futures**

For all  $s$ ,  $t$  and  $T$ , we set :

$$p_t = \sqrt{\dot{B}_T(0)} \text{ and } q_t = \frac{\dot{b}_T(0)}{\sqrt{\dot{B}_T(0)}}.$$

Then, we have<sup>3</sup> :  $\mathbb{E}_0^T [F_{t,s}(0)] = e^{-Fx^2(0) - Gx(0) - H}$  with :

$$\begin{aligned} F &= \frac{q_T^2 B_{t,s}(T)}{1 + B_{t,s}(T)v_T}, \\ G &= \frac{q_T(b_{t,s}(T) + B_{t,s}(T)p_T)}{1 + B_{t,s}(T)v_T}, \\ H &= c_{t,s}(T) + \frac{1}{2} \log(1 + v_t B_{t,s}(T)) + \frac{B_{t,s}(T)p_T^2 + 2b_{t,s}(T)p_T - v_T p_T^2}{2(1 + v_t B_{t,s}(T))}. \end{aligned}$$

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<sup>3</sup>We recall that  $x(0) = \sqrt{2r(0)}$ .

## References

- [1] Farshid Jamshidian, *Bond, futures and option evaluation in the quadratic interest rate model*. *Applied Mathematical Finance*, 1996. 9
- [2] Antoon Pelsser, *Efficient Methods for Valuing Interest Rate Derivatives*, Springer Verlag, 2000. 8
- [3] William H. Press, Brian P. Flannery, Saul A. Teukolsky, William T. Vetterling. , *Numerical Recipes in C. The Art of Scientific Computing*, Cambridge University Press, 1988.

## References