Hull and White Generalized model

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1 Model Presentation

This part is mostly taken from [3], chapter 13.

Hull and White model is a short-rate model. One of its main characteristics is its ability to match the initial yield curve by using time-varying parameter. A one factor version of this model was first proposed in [1] (already implemented in Premia). It's described by the the sde :

$$dr(t) = [\theta(t) - a\,r(t)\,]\,dt + \sigma\,dW(t)$$

, where the function $\theta(t)$ is used to fit the interest rate term struture.

In this project we implement an extention of this model, by considering the parameter σ time-dependant. Hence we have the following evolution for the short rate $r(t)$:

$$dr(t) = [\theta(t) - a\, r(t)\,]\, dt + \sigma(t)\, dW(t)$$

This function $\sigma(t)$ may be used to fit the market prices of basic interest rate derivatives like caplet/floorlet and/or swaption.

The process $r(t)$ can be writen as : $r(t) = \alpha(t) + x(t)$ where :

$$\alpha(t) = \int_0^t \exp(-a(t-y))\theta(y)dy$$

$$x(t) = \exp(-at)r(0) + \int_0^t \exp(-a(t-y))\sigma(y)\, dW_y$$

Hence, the price of a Zero coupon bond can be computed through the same methodology as in [1], for $0 < t < T$:

$$P(t,T) = \exp\left(-\int_t^T \alpha(u)du - A(t,T) - B(t,T)x(t)\right)$$

Where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a},\ \ A(t,T) = -\frac{1}{2}\int_t^T B(y,T)^2\sigma(y)^2 dy$$

Then the discount bond satisfy :

$$\frac{dP(t,T)}{P(t,T)} = r(t)\, dt - \sigma(t)B(t,T)\, dW_t$$

and the bracket of the forward discount bond $F(t) = \dfrac{P(t,S)}{P(t,T)}$ is :

$$\left\langle \frac{dF(t)}{F(t)} \right\rangle = \sigma(t)^2[B(t,S) - B(t,T)]^2$$

Then, the average volatility from 0 to T of $F(t)$ is :

$$\begin{aligned}
\sigma_{avg} &= \sqrt{\int_0^T \left\langle \frac{dF(t)}{F(t)} \right\rangle dt} \\
&= [B(0,S) - B(0,T)]\sqrt{\frac{1}{T}\int_0^T \sigma(t)^2 e^{2at} dt}
\end{aligned}$$

This quantity can be plugged in the following formula to get the price of a caplet with reset date T, payement date S, caplet rate K and $\tau = S - T$ (cf [3], ch.13, for details on the computation):

$$\text{caplet price} = P(0,T)\,N(-d_2) - (1+K\tau)\,P(0,S)\,N(-d_1)$$

where :

$$\begin{aligned} d_1 &= \frac{\log\left((1+K\tau)\frac{P(0,S)}{P(0,T)}\right) + \frac{1}{2}\sigma_{avg}^2 T}{\sigma_{avg}\sqrt{T}} \\ d_2 &= d_1 - \sigma_{avg}\sqrt{T} \end{aligned}$$

2 Calibration of the model

The use of time-dependant parameters in the model allow more flexibility to fit market data : yield curve and caplet implied volatility.

The function $\alpha(t)$ (similarly $\theta(t)$) is selected so that the model fits the initial term structure. The function $\sigma(t)$ is chosen to fit the market prices of a set of actively traded interest-rate options. In our case we will use only caplet.

2.1 Fitting the initial yield curve

Suppose here that the volatility function $\sigma(t)$ has been chosen and we will compute the function $\alpha(t)$ so that we exaclty fit the market price of zero coupon bonds $P^M(0,T)$ for $T > 0$.

We recall the price of a discount factor :

$$P(t,T) = \exp\left(-\int_t^T \alpha(u)du + B(t,T)\alpha(t) - A(t,T) - B(t,T)r(t)\right)$$

So, to have $P(0,T) = P^M(0,T)$ for all $T > 0$, we should choose :

$$\alpha(t) = -\frac{\partial P^M(0,t)}{\partial T} - \frac{\partial A(0,t)}{\partial T} - \frac{\partial B(0,t)}{\partial T} r(0)$$

Hence, after some computations, we get :

$$P(t,T) = \frac{P^M(0,T)}{P^M(0,t)} \exp\left(B(t,T)f^M(0,t) - B(t,T)r(t) - B(t,T)^2 \int_0^t \frac{e^{-2a(t-u)}}{2}\sigma(u)^2\,du\right)$$

, where $f^M(0,t) = -\frac{\partial P^M(0,t)}{\partial T}$ is the instantaneous forward rate.

2.2 Fitting the caplet implied volatility surface

Suppose we have a vector of implied volatilities, of at-the-money caplets, from the market. Furthermore, we assume that all the caplets have the same time to maturity.

This means that we consider a vector of dates $[T_0, T_1, ..., T_{N-1}]$ of caplet's reset dates and their implied volatilities $[\sigma^0_{impl}, \sigma^1_{impl}, ..., \sigma^{N-1}_{impl}]$. Each caplet is payed at $T_i + \tau$. For exemple we can use the 6-months caplets for maturities from 1year to 20years.

Since the volatility function $\sigma(t)$ intervene direclty in the expression of the average volatility σ_{avg} of the forward discount bond $F(t)$, we first translate the caplet's volatilitites σ^{impl} into average volatilities σ_{avg}, by a biscection method for example :

$$[\sigma^0_{impl}, \sigma^1_{impl}, ..., \sigma^{N-1}_{impl}] \longrightarrow [\sigma^0_{avg}, \sigma^1_{avg}, ..., \sigma^{N-1}_{avg}]$$

Then, the simplest approach is to take the function $\sigma(t)$ to be piecewise constant. We use the same dates T_i to define the volatility function :

$$\forall t \in]T_{i-1}, T_i],\ \sigma(t) = \sigma_i$$

where σ_i is a positive constant. (we set $T_{-1} = 0$).

We recall that $\sigma(t)$ is related to σ_{avg} by :

$$\int_0^{T_i} \sigma(t)^2 e^{2at} dt = T_i \left[\frac{\sigma^i_{avg}}{B(0, T_i + \tau) - B(0, T_i)}\right]^2$$

then :

$$\sum_{j=0}^{i-1} \sigma^2_{j+1} \frac{e^{2aT_{j+1}} - e^{2aT_j}}{2a} = T_i \left[\frac{a\sigma^i_{avg}}{e^{-aT_i}(1 - e^{-a\tau})}\right]^2$$

So, for $i > 1$:

$$\sigma_i^2 = \frac{2a}{e^{2aT_i} - e^{2aT_{i-1}}} \left[T_i \left(\frac{a\sigma_{avg}^i}{e^{-aT_i}(1-e^{-a\tau})} \right)^2 - T_{i-1} \left(\frac{a\sigma_{avg}^{i-1}}{e^{-aT_{i-1}}(1-e^{-a\tau})} \right)^2 \right]$$

and for $i = 1$:

$$\sigma_1 = \frac{a\sigma_{avg}^1}{e^{-aT_1}(1-e^{-a\tau})} \sqrt{\frac{2aT_1}{e^{2aT_1} - 1}}$$

3 Trinomial Tree

The methodology to construct a recombining trinomial tree is almost the same in the case of constant volatility (described in [1]).

Initialy we will assume that the volatility parameters $\sigma(t)$ and a have been chosen in the way described above to match the caplet volatility surface. Then, the only difference between the two cases (constant and time dependant volatility) is in the expression of the second moment of the process $x(t)$ (recall that $r(t) = \alpha(t) + x(t)$).

In fact we use the approximation :

$$\begin{aligned} V_i &= \mathrm{Var}\left[x(t_{i+1}) \,|\, x(t_i)\right] \\ &\simeq \sigma(t_i)^2 (t_{i+1} - t_i) \end{aligned}$$

For further specification about the tree construction, see [2].

4 Example of calibration

In this section, we calibrate the following data : initial discount factor curve and 1-year ATM caplet volatility curve for marturities : 1y, 2y, ..., 19y. [1]

To test our method, we use the calibration procedure of the volatility function σ described in the continous model. Then we price the same caplets using the trinomial tree. The table 1 reports the results of the calibration and tree pricing. *Model Volatility* is the caplet volatility computed by trinomial tree, to be compared with the *Market Volatility*. We used a tree with 100 steps per year. We also report the percent error.

[1]The caplet volatility curve is taken from the book [4].

Maturity	Market Volatility	Model Volatility	Nb Time Steps	Percent Error
1.0	0.180253	0.180078	200	0.10%
2.0	0.191478	0.191554	300	0.04%
3.0	0.186154	0.186398	400	0.13%
4.0	0.177294	0.177573	500	0.16%
5.0	0.167887	0.168095	600	0.12%
6.0	0.158123	0.158167	700	0.03%
7.0	0.152688	0.152835	800	0.10%
8.0	0.148709	0.148911	900	0.14%
9.0	0.144703	0.144877	1000	0.12%
10.0	0.141259	0.141459	1100	0.14%
11.0	0.137982	0.138165	1200	0.13%
12.0	0.134708	0.134834	1300	0.09%
13.0	0.131428	0.131457	1400	0.02%
14.0	0.128148	0.128227	1500	0.06%
15.0	0.127100	0.127235	1600	0.11%
16.0	0.126822	0.127022	1700	0.16%
17.0	0.126539	0.126737	1800	0.16%
18.0	0.126257	0.126243	1900	0.01%
19.0	0.125970	0.126206	2000	0.19%

Table 1: Calibration results.

The results of the calibration are pretty much satisfactory, knowing that the calibration of the volatility function is done the continous model, then the output function is used in the discrete model trinomial tree.

References

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[4] D. Brigo, F. Mercurio , *Interest Rate Models - Theory and Practice: With Smile, Inflation and Credit (Springer Finance)*

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References