Efficient Pricing of Asian Options under Lévy Processes Based on Fourier Cosine Expansions: Implementation in PREMIA

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Abstract: We apply an efficient pricing method for arithmetic European-style Asian options with discretely monitored version, based on Fourier Cosine expansion and Clenshaw-Curtis quadrature proposed by Zhang and Oosterlee (2011). The dynamics of underlying asset price is assumed to follow Lévy processes, we specified the model as CGMY models. The main idea of this method and the manual of its implementation are provided here.

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1 Introduction

Fourier Cosine expansion method for option pricing was proposed by Fang and Oosterlee (2009), it was firstly applied to pricing European option. The idea of Fourier Cosine expansion is to recover the transitional density function in the risk-neutral formula in terms of the conditional characteristic function, by a Fourier Cosine expansion which composes of the characteristic function of the underlying asset price dynamics. Under the assumption that the underlying asset price follows Lévy process, the characteristic function of the underlying asset is typically available in closed form, thus it is feasible to use this method. Furthermore this method was applied in pricing American options and other path-dependent options. This method was proved to be efficient with low computational complexity in European options and

American options.

Zhang and Oosterlee (2013)[4] developed a method with Fourier-Cosine expansion for pricing the European-style arithmetic Asian options, now we apply this method and implement it into PREMIA. In this method, the Fourier Cosine expansions are not only used in the risk-neutral pricing formula, but also to recursively recover the characteristic function of the log-return of the sum of the underlying asset prices. And to derive the characteristic functions of the arithmetic sum of the underlying asset, which is necessary for the Fourier Cosine expansions, the Clenshaw-Curtis quadrature rule is used as well.

We will introduce the pricing methodology applying Fourier Cosine expansion and Clenshw-Curtis quadrature here, but the error analysis will not be concerned, the readers who are interested in the error analysis please refer to [4]. In the rest of this document is arranged as follows. We will introduce the model and the notations of Asian options in Section 2, then we provide the Fourier-Cosine expansion method, the recovery of characteristic function by Clenshaw-Curtis quadrature, the put-call parity for pricing a call Asian options, and how to determine the truncated range for Fourier-Cosine expansion in Section 3, the program manual will be presented in Section 4.

2 Model and product description

Assume the price of the underlying asset $S(t), t \geq 0$ follows a Lévy model, especially we are interested in the CGMY model. For CGMY model, the characteristic exponent of the log-increments is given in a closed form:

$$\varphi_{CGMY}(u,t) = \exp(iu\mu t) \cdot \exp\left(tC\Gamma(-Y)(M-iu)^Y - M^Y + (G+iu)^Y - G^Y\right),$$

where C, G, M, Y are the parameters of this model and i is the imaginary unit.

We consider the European-style arithmetic Asian option maturity at time T with M monitored-dates and strike K, then the time step in between monitored-dates is $\triangle t = T/M$. Denote $S_j := S(j\triangle t), \text{for} \quad j = 0, 1, \cdots, M$ for the underlying asset price at the jth monitored-date, The payoff function of this Asian option is

$$v(S,T) = \begin{cases} \max\left(\frac{1}{M}\sum_{j=0}^{M} S_j - K, 0\right) & \text{for a call,} \\ \max\left(K - \frac{1}{M}\sum_{j=0}^{M} S_j, 0\right) & \text{for a put.} \end{cases} \tag{2.1}$$

To derive the price of the Asian option with payoff (2.1), we need to know the conditional characteristic function of the arithmetic average of $S_j, j = 0, \cdots, M$, which can be rewriteen as

$$\frac{1}{M}\sum_{j=0}^{M} S_j = \frac{S_0[1+\exp(Y_M)]}{M+1}. \tag{2.2}$$

The Y_M in the above equation (2.2) can be derived recursively as

$$\begin{aligned} Y_1 &:= R_M, \\ Y_j &:= R_{M+1-j} + Z_{j-1} = \log\left(\frac{S_{M-j+1}}{M-j} + \frac{S_{M-j+2}}{S_{M-j}} + \cdots + \frac{S_M}{S_{M-j}}\right), j = 2, 3, \cdots, M \end{aligned}$$

where

$$Z_j := \log(1 + \exp(Y_j)), \tag{2.3}$$

and $R_j, j = 1, 2, \cdots, M$ is the log-increment of $S_j, j = 0, 1, \cdots, M$ given by

$$R_j := \log\left(\frac{S_j}{S_{j-1}}\right).$$

For Lévy processes, the increments $R_j, j = 1, 2, \cdots, M$ are identically and independently distributed, so that $R_j =^d R$.

With the rewritten form (2.2) by Y_M, the payoff function (2.1) can be regarded as the payoff function of a plain vanilla European option with the underlying asset Y_M, i.e.

$$v(S,T) = v'(Y_M,T) = \begin{cases} \max\left(\frac{S_0[1+\exp(Y_M)]}{M+1} - K, 0\right) & \text{for a call,} \\ \max\left(K - \frac{S_0[1+\exp(Y_M)]}{M+1}, 0\right) & \text{for a put.} \end{cases} \tag{2.4}$$

Thus we can apply the Fourier Cosine expansion method for plain vanilla European option with payoff (2.4) to price the arithmetic Asian option with payoff (2.1) given the conditional characteristic function of Y_M.

3 The pricing Method by Fourier Cosine Expansion and Clenshw-Curtis quadrature

In this section, we introduce how to obtain the option price by Fourier Cosine expansion given the conditional characteristic function of Y_M, then present how to recover the conditional characteristic function of Y_M recursively from the characteristic functions of log-increment $R_j, j = 1, \cdots, M.$, at last we propose the Clenshaw-Curtis quadrature in the derivation of the characteristic function of Y_M.

3.1 Fourier Cosine Expansion

Denote the riskless interest rate by r, the risk-neutral option pricing formula for plain vanilla European options is

$$v'(x,t_0) = \mathrm{e}^{-r(T-t_0)}\mathbb{E}^{t_0}\left[v'(Y_M,T)\right] = \mathrm{e}^{-r(T-t_0)}\int_{-\infty}^{+\infty} v'(y,T)f(y \mid x)\mathrm{d}y, \tag{3.1}$$

where $f(y|x)$ is the transitional density function of Y_M given the initial value of $S_0 = x$. $f(y|x)$ typically doesn't have a closed form, it is approximated on a truncated domain $[a,b]$, by a truncated Fourier Cosine series expansion, with N terms, based on its conditional characteristic function, as follows:

$$f(y|x) \approx \frac{2}{b-a} \sum_{k=0}^{N-1}{}' \mathrm{Re}\left[\varphi_{Y_M}\left(\frac{k\pi}{b-a};x\right)\exp\left(-i\frac{ak\pi}{b-a}\right)\right]\cos\left(k\pi\frac{y-a}{b-a}\right), \qquad (3.2)$$

where $\varphi_{Y_M}(u;x)$ the conditional characteristic function of $f(y|x)$ given the initial value of $S_0 = x$, Re means taking the real part of the argument, and the prime at the sum symbol indicates the first term in the expansion is multiplied by one-half. The appropriate size of the integration interval, i.e. $[a,b]$ can be determined with the help of the cumulants as in [1].

Replacing $f(y|x)$ by its approximation (3.2) in equation (3.1) and interchanging integration and summation gives the Fourier Cosine formula for computing of plain vanilla European options:

$$v'(x,t_0) \approx \mathrm{e}^{-r(T-t_0)} \sum_{k=0}^{N-1}{}' \mathrm{Re}\left[\varphi_{Y_M}\left(\frac{k\pi}{b-a};x\right)\exp\left(-i\frac{ak\pi}{b-a}\right)\right] V_k, \qquad (3.3)$$

where

$$V_k = \frac{2}{b-a}\int_a^b v'(y,T)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y, \quad k = 0,\cdots,N-1$$

are the Fourier-Cosine coefficients of $v'(y,M)$, it can be derived analytically as

$$V_k = \begin{cases} \frac{2}{b-a}\left[\frac{S_0}{M+1}\chi_k(x',b) + \left(\frac{S_0}{M+1} - K)\phi_k(x',b)\right)\right], & \text{for a call} \\ \frac{2}{b-a}\left[-\frac{S_0}{M+1}\chi_k(a,x') + \left(K - \frac{S_0}{M+1})\phi_k(a,x')\right)\right], & \text{for a put} \end{cases}, \qquad (3.4)$$

where $x' = \log\left(\frac{K(M+1)}{S_0} - 1\right)$, functions $\chi_k(l^*,u^*)$ and $\phi_k(l^*,u^*)$ for all $l^* \le u^*$ are as follows:

$$\begin{aligned} \chi_k(l^*,u^*) &:= \int_{l^*}^{u^*} \mathrm{e}^x \cos\left(n\pi\frac{x-a}{b-a}\right)\mathrm{d}x \\ = \quad & \frac{1}{1+\left(\frac{n\pi}{b-a}\right)^2}\left[\cos\left(n\pi\frac{u-a}{b-a}\right)\mathrm{e}^u - \cos\left(n\pi\frac{l-a}{b-a}\right)\mathrm{e}^l \right. \\ & \left. + \frac{n\pi}{b-a}\sin\left(n\pi\frac{u-a}{b-a}\right)\mathrm{e}^u - \frac{n\pi}{b-a}\sin\left(n\pi\frac{l-a}{b-a}\right)\mathrm{e}^l\right], \end{aligned}$$

$$\begin{aligned} \psi_k(l^*,u^*) &:= \int_{l^*}^{u^*} \cos\left(n\pi\frac{x-a}{b-a}\right)\mathrm{d}x \\ = & \begin{cases} \left[\sin\left(n\pi\frac{u-a}{b-a}\right) - \sin\left(n\pi\frac{l-a}{b-a}\right)\right]\frac{b-a}{n\pi}, & n \neq 0 \\ u-l, & n = 0. \end{cases} \end{aligned}$$

3.2 Recovery of Characteristic Function of Y_M

In the following, we will use the simplified notation $\varphi_X(u)$ for the conditional characteristic function of any random variable X, $\varphi_X(u;x)$, for convenience without confusion. To recover the conditional characteristic function of Y_M, i.e. $\varphi_{Y_M}(\frac{k\pi}{b-a})$ for $k=0,\cdots,N-1$, we start with Y_1 with characteristic function of closed form:

$$\varphi_{Y_1}(u)=\varphi_R(u). \tag{3.5}$$

Then, by definition of Y_j (2.3), $\varphi_{Y_j}(u)$ can be recovered in terms of $\varphi_{Y_{j-1}}(u)$ and the characteristic function of the log-increment R_{M+1-j} for $j=2,\cdots,M$. Since the log-increment $R_j, j=1,\cdots,M$ are independent, then R_{M+1-j} and $Z_{j-1}=\log(1+\exp(Y_{j-1}))$ are independent, which gives:

$$\varphi_{Y_j}(u)=\varphi_R(u)\varphi_{Z_{j-1}}(u). \tag{3.6}$$

Apply once again the Fourier Cosine series expansion to approximate $\varphi_{Z_{j-1}}$, we have

$$\begin{aligned}
\varphi_{Z_{j-1}}(u) &= \mathbb{E}\left[\exp(iu\log(1+\exp(Y_{j-1})))\right]\\
&= \int_{-\infty}^{+\infty}(\exp(x)+1)^{iu} f_{Y_{j-1}}(x)\mathrm{d}x\\
&\approx \frac{2}{b-a}\sum_{l=0}^{N-1}{}'\mathrm{Re}\left[\hat{\varphi}_{Y_{j-1}}\left(\frac{l\pi}{b-a}\right)\exp\left(-ia\frac{l\pi}{b-a}\right)\right]\\
&\quad\times\int_a^b(\exp(x)+1)^{iu}\cos\left((x-a)\frac{l\pi}{b-a}\right)\mathrm{d}x,
\end{aligned} \tag{3.7}$$

where $\hat{\varphi}_{Y_{j-1}}(u)$ is approximation of $\varphi_{Y_{j-1}}(u)$.

Since we need $\varphi_{Y_j}(\frac{k\pi}{b-a})$ for $k=0,\cdots,N-1$, it needs $\varphi_{Z_{j-1}}(\frac{k\pi}{b-a}), k=0,\cdots,N-1$ as well, which forms a vector

$$\Phi_{j-1}=[\Phi_{j-1}(k)]_{k=0}^{N-1}=\left[\varphi_{Z_{j-1}}\left(\frac{k\pi}{b-a}\right)\right]_{k=0}^{N-1},$$

thus the equation (3.7) can be written as matrix-vector form as

$$\Phi_{j-1}=\mathcal{M}A_{j-1}, \tag{3.8}$$

where

$$\mathcal{M}=[\mathcal{M}(k,l)]_{k,l=0}^{N-1},\quad \mathcal{M}(k,l)=\int_a^b(\exp(x)+1)^i u_k\cos((x-a)u_l)\mathrm{d}x,$$
$$A_j=(A_j(l))_{l=0}^{N-1},\quad A_j(l)=\frac{2}{b-a}\mathrm{Re}\left(\hat{\varphi}_{Y_{j-1}}(u_l)\exp(-iau_l)\right).$$

The elements in the matrix $\mathcal{M}$ will be derived by the Clenshaw-Curtis quadrature as it is shown in the next subsection.

3.3 Clenshaw-Curtis Quadrature

By variable subsititution, the elements of matrix $\mathcal{M}$ are given as

$$\begin{aligned}\mathcal{M}(k,l) &= \int_a^b (\exp(x)+1)^{iu_k}\cos((x-a)u_l)\mathrm{d}x, \\ &= \int_{-1}^{1}\frac{b-a}{2}\left[\exp\left(\frac{b-a}{2}x+\frac{a+b}{2}\right)+1\right]^{iu_k}\cos\left[\left(\frac{b-a}{2}x+\frac{a+b}{2}-a\right)u_l\right]\mathrm{d}x.\end{aligned} \tag{3.9}$$

By Clenshaw-Curtis Quadrature, the above integral can be approximated as follows

$$\begin{aligned}&\mathcal{M}(k,l)\\ \approx\ & \sum_{n=0}^{n_q/2} D_{0,n}y_n + \sum_{m=1}^{n_q/2-1}\frac{2}{1-(2m)^2}\left(\sum_{n=0}^{n_q/2} D_{m,n}y_n\right) + \frac{1}{1-n_q^2}\sum_{n=0}^{n_q/2} D_{n_q/2,n}y_n,\\ =\ & \sum_{m=0}^{n_q/2} d_m\left(\sum_{n=0}^{n_q/2} D_{m,n}y_n\right) = \sum_{n=0}^{n_q/2} y_n\sum_{m=0}^{n_q/2}(d_m D_{m,n}).\end{aligned} \tag{3.10}$$

where

$$D_{m,n} = \frac{2}{n_q}\cos\left(\frac{mn\pi}{n_q/2}\right)\cdot\begin{cases}1/2, & \text{if} \quad n=0, n_q/2,\\ 1, & \text{otherwise,}\end{cases} \qquad m,n = 0,\cdots,n_q/2, \tag{3.11}$$

d and y are $n_q/2+1$-vectors defined respectively as:

$$d := (d_m)_{m=0}^{n_q/2} = \left[1, \frac{2}{1-4}, \frac{2}{1-16}, \cdots, \frac{2}{1-(2k)^2}, \cdots, \frac{1}{1-n_q^2}\right]^T, \tag{3.12}$$

$$y := (y_n)_{n=0}^{n_q/2} = \left(f\left[\cos\left(\frac{n\pi}{n_q}\right)\right] + f\left[-\cos\left(\frac{n\pi}{n_q}\right)\right]\right)_{n=0}^{n_q/2}, \tag{3.13}$$

with

$$f(x) = \frac{b-a}{2}\left[\exp\left(\frac{b-a}{2}x+\frac{a+b}{2}\right)+1\right]^{iu_k}\cos\left[\left(\frac{b-a}{2}x+\frac{a+b}{2}-a\right)u_l\right].$$

The second summation in the last term of (3.10) is a type I Discrete Cosine Transform(DCT), it can be computed by Discrete Fourier Transform (DFT) of n_q-element vector $\bar{d} := (\bar{d}_m)_{m=0}^{n_q-1}$ generated from vector d, which reads

$$\bar{d}_m = \begin{cases} 2 = 2d_m, & m=0;\\ d_m, & m = 1,\cdots,n_q/2-1\\ \frac{2}{1-n_q^2} = 2d_m, & m = n_q/2;\\ d_{n_q-m}, & m = n_q/2+1,\cdots,n_q-1.\end{cases} \tag{3.14}$$

In fact, the DFT of $\overline{d}$ is

$$\begin{aligned}
\hat{d}_n &= \sum_{m=0}^{n_q-1} \overline{d}_m \exp\left(-i\frac{2\pi mn}{n_q}\right) \\
&= \sum_{m=0}^{n_q-1} \overline{d}_m \left[\cos\left(\frac{\pi mn}{n_q/2}\right) - i\sin\left(\frac{\pi mn}{n_q/2}\right)\right] \\
&= 2\sum_{m=0}^{n_q/2} d_m \cos\left(\frac{\pi mn}{n_q/2}\right) - i\sum_{m=0}^{n_q-1} \overline{d}_m \sin\left(\frac{\pi mn}{n_q/2}\right) \\
&= n_q \sum_{m=0}^{n_q/2} d_m D_{m,n} - i\sum_{m=0}^{n_q-1} \overline{d}_m \sin\left(\frac{\pi mn}{n_q/2}\right).
\end{aligned}$$

Thus by taking the real part of the DFT of $\overline{d}$ and divided by n_q, we have the value the second summation in the last term of (3.10), hence the computation complexity of (3.10) can be reduced from $O((n_q/2+1)^2)$ to $O(n_q \log_2(n_q))$.

3.4 Integration Range

Now we explain how to determine integration range $[a, b]$ so that the error of pricing can be controlled. The integration range for each $Y_j, j = 1, 2, \cdots, M$ can be determined by means of its cumulants as:

$$[\zeta_1(Y_j) - L\sqrt{\zeta_2(Y_j) + \sqrt{\zeta_4(Y)}}, \zeta_1(Y_j) + L\sqrt{\zeta_2(Y_j) + \sqrt{\zeta_4(Y)}}] \tag{3.15}$$

where $\zeta_1(Y_j), \zeta_2(Y_j), \zeta_4(Y_j)$ the first, second and fourth cumulants of Y_j, respectively.

Since it is hard to find the close formular to calculate the cumulants, then a different integration range is proposed:

$$\left[\zeta_1\left(\log\left(j\frac{S_{M-j+1}}{S_{M-j}}\right)\right) - L\sqrt{\zeta_2\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right) + \sqrt{\zeta_4\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right)}},\right.$$
$$\left.\zeta_1\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right) + L\sqrt{\zeta_2\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right) + \sqrt{\zeta_4\left(\log\left(j\frac{S_M}{S_{M-j}}\right)\right)}}\right].$$

And it can be simplified as $[a_j, b_j]$ with

$$a_j = \log(j) + \zeta_1(R) - L\sqrt{\zeta_2(R) + \sqrt{\zeta_4(R)}}, \tag{3.16}$$

$$b_j = \log(j) + j\zeta_1(R) + L\sqrt{j\zeta_2(R) + \sqrt{j\zeta_4(R)}}, \tag{3.17}$$

since

$$\zeta_1\left(\log\left(j\frac{S_{M-j+1}}{S_{M_j}}\right)\right)=\log(j)+\zeta_1(R),\quad\text{and}\quad\forall n\geq 2,\zeta_n\left(\log\left(j\frac{S_{M-j+1}}{S_{M-j}}\right)\right)=\zeta_n(R),$$

$$\zeta_1\left(\log\left(j\frac{S_{M}}{S_{M_j}}\right)\right)=\log(j)+j\zeta_1(R),\quad\text{and}\quad\forall n\geq 2,\zeta_n\left(\log\left(j\frac{S_{M-j+1}}{S_{M-j}}\right)\right)=j\zeta_n(R),$$

with R is the increment of an exponential Lévy process, between any two consecutive steps.

In order to compute the integration range only once, we adope the following integration range:

$$\begin{aligned}[a,b] &:= [\min_{j=1,2,\cdots,M} a_j, \max_{j=1,2,\cdots,M} b_j]\\ &= \left[\zeta_1(R)-L\sqrt{\zeta_2(R)+\sqrt{\zeta_4(R)}},\log(M)+M\zeta_1(R)+L\sqrt{M\zeta_2(R)+\sqrt{M\zeta_4(R)}}\right]\end{aligned}$$

for all time steps, where the cumulants of R are as follows:

$$\begin{aligned}\zeta_1(R) &= (r-q)t+Ct\Gamma(-Y)Y(G^{Y-1}-M^{Y-1}),\\ \zeta_2(R) &= \sigma^2t+Ct\Gamma(2-Y)(M^{Y-2}-G^{Y-2}),\\ \zeta_4(R) &= Ct\Gamma(4-Y)(M^{Y-4}-G^{Y-4}).\end{aligned}$$

3.5 Put-Call Parity for Asian Option

Since for a call Asian option, the payoff is unbounded, which lead to large errors when truncating the integration range of the risk-neutral formula. So we use the put-call parity when pricing a call Asian option.

Assuming that no divident is paid, and denoting the Asian and put options prices by $c(S_0,t_0)$ and $p(S_0,t_0)$ respectively. Using the risk-neutral valuation formula gives us, for $t_0<T$,

$$c(S_0,t_0)-p(S_0,t_0)=\mathrm{e}^{-rT}\mathbf{E}\left(\frac{1}{M+1}\sum_{j=0}^{M}S_j-K|\mathcal{F}_0\right)=\frac{S_0\mathrm{e}^{-rT}}{M+1}\sum_{j=0}^{M}\mathrm{e}^{rj\triangle t}-K\mathrm{e}^{-rT}. \tag{3.18}$$

4 Program manual

We implement the pricing of the Asian options by Fourier Cosine expansion. The program HAS TO work with the pnl library.

Parameters of the product:
S0: the initial value of stock price.
K0: strike K of the arithmetic Asian option.
T: the number of the monitored dates of this option.
r: the discount interest rate.
divid: the payout dividend.
Mn: the number of the monitoring dates.

Model Parameters:
C, G, M, Y: the four parameters in the CGMY model.

Parameters for Fourier-Cosine method and Clenshaw-Curtis quadrature:
N: number of Fourier-Cosine series, N in (3.2).
nq: numbers of terms in Clenshaw-Curtis quadrature.

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