Premia - Optimal execution in a Hawkes-based price model

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1 The model

The optimal execution model presented here is described in detail in [1].

We consider a single asset and denote by P_t its price at time t. We assume

$$P_t = \underbrace{S_t}_{\text{fundamental price}} + \underbrace{D_t}_{\text{mesoscopic price deviation}}.$$

Typically, these quantities are related to the permanent and the transient impact of the market orders, respectively:

$$\begin{aligned} \mathrm{d}S_t &= \frac{\nu}{q} \underbrace{\mathrm{d}N_t}_{\text{market orders}} \\ \mathrm{d}D_t &= \underbrace{-\rho\, D_t\, \mathrm{d}t}_{\text{market resilience}} + \frac{1-\nu}{q} \underbrace{\mathrm{d}N_t}_{\text{market orders}}, \end{aligned}$$

with $\nu \in [0,1]$ the proportion of permanent impact, $\rho > 0$ the resilience speed of the price. The measure of liquidity $q > 0$ corresponds to the quantity of shares that one should trade to shift the price of one unity (for instance, of one euro if P_t is the price of a European stock), if the order book were block-shaped.

We introduce the MIH (Mixed-market-Impact Hawkes) model for the process N:

$$N_t = N_t^+ - N_t^-,$$

where the process (N^+, N^-) is a symmetric two-dimensional Hawkes process of intensity (κ^+, κ^-), with unpredictable jumps of average size $m_1 > 0$. We place ourselves in the Markovian settings

$$\begin{aligned} \mathrm{d}\kappa_t^+ &= -\beta\, (\kappa_t^+ - \kappa_\infty)\, \mathrm{d}t + \iota_{\mathrm{s}}\, \mathrm{d}J_t^+ + \iota_{\mathrm{c}}\, \mathrm{d}J_t^-, \\ \mathrm{d}\kappa_t^- &= -\beta\, (\kappa_t^- - \kappa_\infty)\, \mathrm{d}t + \iota_{\mathrm{c}}\, \mathrm{d}J_t^+ + \iota_{\mathrm{s}}\, \mathrm{d}J_t^-. \end{aligned} \tag{1}$$

where $\beta > 0$ is the reversion speed of the intensity, $\iota_{\mathrm{s}}, \iota_{\mathrm{c}} > 0$ are the self-excitation and cross-excitation parameters, respectively, $\kappa_\infty > 0$ is the baseline intensity and $J^\pm$ count the jumps of $N^\pm$.

Within this model, we consider a liquidating trader who has a position X_t in the considered asset at time t. The initial position $X_0 = x_0$ is known and should be liquidated at time $T > 0$, thus $X_{T^+} = 0$ is imposed.

The liquidating trader impacts the price similarly to other traders, with a proportion $\epsilon \in [0,1]$ of permanent impact:

$$\begin{aligned} \mathrm{d}S_t &= \frac{1}{q}\left(\nu \mathrm{d}N_t + \epsilon \mathrm{d}X_t\right), \\ \mathrm{d}D_t &= -\rho\ D_t\ \mathrm{d}t + \frac{1}{q}\left((1-\nu)\mathrm{d}N_t + (1-\epsilon)\mathrm{d}X_t\right). \end{aligned} \tag{2}$$

but does not impact the intensities $\kappa^{\pm}$. We make the assumption of a block-shaped Limit Order Book: when the liquidating trader places at time t an order of size $v \in \mathbb{R}$ ($v > 0$ for a buy order and $v < 0$ for a sell order), it has the cost

$$\pi_t(v) = \int_0^v \left[P_t + \frac{1}{q}y\right]\ \mathrm{d}y = \underbrace{P_t\ v}_{\text{cost at the current price}} + \underbrace{\frac{v^2}{2q}}_{\text{impact cost}}. \tag{3}$$

Therefore, the cost of the whole strategy X is given by

$$\begin{aligned} C(X) &= \int_{[0,T)} P_u\ \mathrm{d}X_u\ +\ \frac{1}{2q}\sum_{\tau\in\mathcal{D}_X\cap[0,T)}(\Delta X_\tau)^2\ -\ P_T X_T\ +\ \frac{1}{2q}\ X_T^2 \\ &= \int_{[0,T)} P_u\ \mathrm{d}X_u^c\ +\ \sum_{\tau\in\mathcal{D}_X\cap[0,T)} P_\tau(\Delta X_\tau) + \frac{1}{2q}\sum_{\tau\in\mathcal{D}_X\cap[0,T)}(\Delta X_\tau)^2\ -\ P_T X_T\ +\ \frac{1}{2q}\ X_T^2, \end{aligned}$$

since at time T all the remaining assets have to be liquidated. Here, the sum brings on the countable times of discontinuity $\mathcal{D}_X$ of X, and the jumps $\Delta X_\tau = X_{\tau+} - X_\tau \neq 0$ for $\tau \in \mathcal{D}_X$.

The optimal strategy for this model is obtained as a closed formula in [1]. We introduce the processes

$$\delta_t = \kappa_t^+ - \kappa_t^-, \quad \Sigma_t = \kappa_t^+ + \kappa_t^-, \tag{4}$$

along with the useful quantities

$$\alpha = \iota_{\mathrm{s}} - \iota_{\mathrm{c}}, \quad \eta = \beta - \alpha,$$

and the two continuously differentiable functions $\zeta, \omega : \mathbb{R} \to \mathbb{R}^+$ defined by

$$\begin{aligned} &\zeta(0) = 1 \quad \text{and} \quad \forall y \neq 0, \quad \zeta(y) = \frac{1-\exp(-y)}{y}, \\ &\zeta'(0) = -1/2 \quad \text{and} \quad \forall y \neq 0, \quad \zeta'(y) = \frac{(1+y)\exp(-y)-1}{y^2} = \frac{\exp(-y)-\zeta(y)}{y}, \\ &\omega(0) = 1/2 \quad \text{and} \quad \forall y \neq 0, \quad \omega(y) = \frac{\exp(-y)-1+y}{y^2} = \frac{1-\zeta(y)}{y}, \\ &\omega'(0) = -1/6 \quad \text{and} \quad \forall y \neq 0, \quad \omega'(y) = \frac{2(1-\exp(-y)) - y(1+\exp(-y))}{y^3} = \frac{2\zeta(y)-1-\exp(-y)}{y^2}. \end{aligned}$$

The optimal strategy X^* is characterized by

$$(1-\epsilon)X_t^* = -\ [1+\rho(T-t)]\ D_t^*\ +\ \frac{m_1}{2\rho}\times[2+\rho(T-t)\times\{1+\zeta(\eta(T-t))+\nu\rho(T-t)\ \omega(\eta(T-t))\}]\ \delta_t \tag{5}$$

for $t \in [0,T)$, and

$$X_{T+}^* - X_T^* = -X_T^*,$$

so that $X^*_{T+} = 0$. The expected cost of this strategy has the form

$$\begin{aligned} q \times \mathcal{C}(T, x_0, D_0, S_0, \delta_0, \Sigma_0) \;=\; & -q(S_0 + D_0)x_0 \;+\; \left[\frac{1-\epsilon}{2+\rho T} + \frac{\epsilon}{2}\right] x_0^2 \;+\; \frac{\rho T}{2+\rho T}\left[qd - \;\mathcal{G}_\eta(T)\;\frac{\delta_0 m_1}{\rho}\right] x_0 \\ & - \;\frac{1}{1-\epsilon} \times \frac{\rho T/2}{2+\rho T}\left[qd - \;\mathcal{G}_\eta(T)\;\frac{\delta_0 m_1}{\rho}\right]^2 \;+\; \hat{c}_\eta(T)\;\left(\frac{\delta_0 m_1}{\rho}\right)^2 \\ & + e(T)\;\Sigma_0 \;+\; g(T). \end{aligned} \tag{6}$$

where for $u \in [0, T]$,

$$\begin{aligned} \mathcal{G}_\eta(u) &= \zeta(\eta u) + \nu\rho u\;\omega(\eta u), \\ \hat{c}_\eta(u) &= \frac{1}{1-\epsilon} \times (\eta - \nu\rho)^2 \frac{\rho u^3}{8}\omega'(\eta u)\zeta(\eta u). \end{aligned}$$

The functions e and g are uniquely determined but have cumbersome expressions in general. However, in the case $\iota_c = 0$, $\beta = \alpha = \iota_s$, one has the simpler formulas

$$\begin{aligned} e(u) \;=\; & -\;\frac{(1-\nu)^2}{1-\epsilon} \times \left(m_2 - \frac{\beta m_1^2(2\rho - \beta)}{\rho^2}\right) \times \left[\frac{u}{2} - \frac{1}{\rho}\ln\left(1 + \frac{\rho u}{2}\right)\right] \\ & + \frac{\beta\nu(1-\nu)m_1^2}{4\rho^2(1-\epsilon)} \times \left(1 - \frac{\beta}{\rho}\right) \times \rho^2 u^2 \;-\; \frac{\beta^2\nu^2 m_1^2}{8\rho^3(1-\epsilon)} \times \left[\rho^2 u^2 + \frac{1}{3}\rho^3 u^3 + \frac{1}{24}\rho^4 u^4\right], \end{aligned}$$

$$\begin{aligned} g(u) \;=\; & -2\beta\kappa_\infty \times \frac{(1-\nu)^2}{1-\epsilon} \times \left(m_2 - \frac{\beta m_1^2(2\rho - \beta)}{\rho^2}\right)\left\{\left(u + \frac{2}{\rho}\right)\left[\frac{u}{2} - \frac{1}{\rho}\ln\left(1 + \frac{\rho u}{2}\right)\right] - \frac{u^2}{4}\right\} \\ & + \frac{\beta^2\kappa_\infty\nu(1-\nu)m_1^2}{6\rho^3(1-\epsilon)} \times \left(1 - \frac{\beta}{\rho}\right) \times \rho^3 u^3 \;-\; \frac{\beta^3\kappa_\infty\nu^2 m_1^2}{12\rho^4(1-\epsilon)} \times \left[\rho^3 u^3 + \frac{1}{4}\rho^4 u^4 + \frac{1}{40}\rho^5 u^5\right]. \end{aligned}$$

2 Closed-formula cost

The "Closed-formula cost" functionality implemented in Premia takes the parameters of the model as input and returns

- The evaluation of formula (6) if the condition $\iota_c = 0$, $\beta = \alpha = \iota_s$ is satisfied.
- Zero as a default value otherwise.

3 Monte Carlo cost

The "Monte Carlo cost" functionality needs two additional input values: the number of simulations $n \geq 1$ and the discretization step $h > 0$. It computes the cost of the optimal strategy averaged over all simulations, along with a confidence interval.

For each simulation, we first determine a realization of the bi-dimensional Hawkes process (N^+, N^-) on $[0, T]$ using equation (1). We resort to the thinning method described by the following algorithm:

- Initialize $t = 0$, $\kappa^+ = \kappa_0^+$, $\kappa^- = \kappa_0^-$, $\Sigma = \kappa_0^+ + \kappa_0^-$.

- While $t < T$,
  1. Simulate an exponential variable τ of rate Σ.
  2. Set $t = t + \tau$ and $\kappa^{\pm} = \kappa_{\infty} + (\kappa^{\pm} - \kappa_{\infty})\exp(-\beta\tau)$.
  3. Simulate a uniform variable U on $[0, 1]$.
  4. If $U \leq \kappa^{+}/\Sigma$ and $t \leq T$, record t as a jump time of N^{+} and set $\kappa^{+} = \kappa^{+} + \iota_{\mathrm{s}},\ \kappa^{-} = \kappa^{-} + \iota_{\mathrm{c}}$.
  5. If $\kappa^{+}/\Sigma < U \leq (\kappa^{+}+\kappa^{-})/\Sigma$ and $t \leq T$, record t as a jump time of N^{-} and set $\kappa^{+} = \kappa^{+}+\iota_{\mathrm{c}},\ \kappa^{-} = \kappa^{-} + \iota_{\mathrm{s}}$.
  6. Set $\Sigma = \kappa^{+} + \kappa^{-}$.

Then, we choose a positive probability law μ of expected value m_1 for the amplitudes of the jumps. It should be easy to simulate (for instance, an exponential or a log-normal law are suitable). For each jump time of N^{+} and each jump time of N^{-}, we simulate a variable of law μ independently of the rest and record it as the amplitude of the corresponding jump.

Once the trajectory of (N^{+}, N^{-}) is simulated, we determine the corresponding discretized optimal strategy. The discretization grid $(t_i)_{i\geq 0}$ is built incrementally to have steps smaller than h and to include $t = 0,\ t = T$ and all the jumps of (N^{+}, N^{-}):

- Set $t_0 = 0$ and $i = 0$.
- While $t_i < T - h$,
  1. Define θ as the smallest jump time of (N^{+}, N^{-}) greater than t_i.
  2. If $\theta < t_i + h$, set $t_{i+1} = \theta$. Else, set $t_{i+1} = t_i + h$.
  3. Set $i = i + 1$.
- Set $t_{i+1} = T$ and $i = i + 1$.

For each discretization time t_j (except the last one), we compute D_{t_j} and δ_{t_j} using equations (1), (2) and (4), then we deduce the optimal order from equation (5)

$$\begin{aligned}\Delta X^{*}_{t_j} = &-\frac{x_0}{2+\rho(T-t_j)} \\ &+ \frac{1}{1-\epsilon} \times \frac{2+\rho(T-t_j)[1+\zeta(\eta(T-t_j)) + \nu\rho(T-t_j)\omega(\eta(T-t_j))]}{2+\rho(T-t_j)} \times \frac{\delta_{t_j} m_1}{2\rho} \\ &- \frac{1}{1-\epsilon} \times \frac{1+\rho(T-t_j)}{2+\rho(T-t_j)} \times D_{t_j}.\end{aligned}$$

At time T, one has $\Delta X^{*}_{T} = -X^{*}_{T}$. The cost of each order is given by equation (3), and we sum these costs to obtain the total cost of the strategy for the current simulation.

References

[1] Dynamic optimal execution in a mixed-market-impact Hawkes price model, Alfonsi, Aurélien and Blanc, Pierre *arXiv preprint arXiv:1404.0648v2, forthcoming in Finance and Stochastics*

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