Two Dimensional Fourier Cosine Series Expansion Methods For Pricing Options: Implementation in PREMIA

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Abstract: The 2-dimensional Fourier Cosine expansion methods (in short, $2D-COS$ methods) for pricing rainbow options is introduced by Ruijter and Oosterlee (2012), we will apply this method for pricing European and Bermudan rainbow options under the 2-dimensional Merton's jump-diffusion model and implement the algorithm in PREMIA.

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1 The Model and products

In this work, we will price the European and Bermudan type arithmetic basket call option with the two underlying assets and the payoff function is given as

$$((S_T^1+S_T^2)/2-K)^+,$$

where K is the strike and the underlying asset prices S_t^1 and S_t^2 follow the Merton's jump-diffusion process as follows

$$\mathrm{d}S_t^i=(r-\lambda\varkappa_i)S_t^i t+S_t^i\sigma_i\mathrm{d}Z_t^i+(\mathrm{e}^{J_i}-1)S_t^i\mathrm{d}q_t, i=1,2, \tag{1.1}$$

with $\varkappa_i:=\mathbb{E}[\mathrm{e}^{J_i}-1], q_t$ a Poisson process with mean arrival rate λ, and $\mathbf{J}=(J_1,J_2)$ bivariate normally distributed jumps, with mean $\mu^J=[\mu_1^J,\mu_2^J]'$ and covariance matrix $\sum_{ij}^J=\sigma_i^J\sigma_j^J\rho_{ij}^J$. The log-processes $X_t^i:=\log S_t^i$ read as

$$\mathrm{d}X_t^i=(r-\lambda\varkappa_i-\frac{1}{2}\sigma_i^2)\mathrm{d}t+\sigma_i\mathrm{d}Z_t^i+J_i\mathrm{d}q_t. \tag{1.2}$$

The characteristic function reads as $\varphi(\mathbf{u}|\mathbf{x})=\mathrm{e}^{i\mathbf{x}'\mathbf{u}}\varphi_{levy}(\mathbf{u})$, with

$$\varphi_{levy}(\mathbf{u})=\exp(i\mu'\mathbf{u}-\frac{1}{2}\mathbf{u}'\sum\mathbf{u})\exp\left(\lambda\Delta t(\exp(i\mu'^J\mathbf{u}-\frac{1}{2}\mathbf{u}'\sum^J\mathbf{u})-1)\right), \tag{1.3}$$

where $\mu_i=(r-\lambda\varkappa_i-\frac{1}{2}\sigma_i^2)\Delta t$ and $\sum_{ji}=\sigma_i\sigma_j\rho_{ij}\Delta t$.

2 European rainbow options

In this section, we the sketch main idea of the $2D-COS\ formula$ for approximating discounted expected payoffs of basket option with 2 underlying assets. This method is an extension of the one dimensional Fourier-Cosine method for option pricing by Fang and Oosterlee (2008) [1].

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $T > 0$ be a finite terminal time, and $\mathbb{F} = (\mathcal{F}_S)_{0\leq s\leq T}$ be a filtration satisfying the usual conditions. The process $X_t = (X_t^1, X_t^2)$ denotes a 2D stochastic process on the filtered probability space $(\Omega, \mathcal{F}, P)$, representing the log-asset prices. We assume that the bivariate characteristic function of the stochastic process is known, which is the case, for example, for affine jump-diffusions [**?**]. The value of a European rainbow option, with payoff function $g(\cdot)$, is given by the risk-neutral option valuation formula

$$v(t_0, \mathbf{x}) = \mathrm{e}^{-r\Delta t}\mathbb{E}^{t_0,\mathbf{x}}[g(\mathbf{X}_T)] = \mathrm{e}^{-r\Delta t}\iint_{\mathbb{R}^2} g(\mathbf{y})f(\mathbf{y}|\mathbf{x})\mathrm{d}\mathbf{y}. \tag{2.1}$$

Here, $\mathbf{x} = (x_1, x_2)$ is the current state, $f(y_1, y_2|x_1, x_2)$ is the conditional density function, r is the risk-free rate, and time to expiration is denoted by $\Delta t := T - t_0$. In the derivation of the **COS** formula, we distinguish three different approximation steps.

Step 1. We assume that the integrand is integrable, which is common for the problems we deal with. Because of that, we can, for given $\mathbf{x}$, truncate the infinite integration ranges to some domain $[a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ without losing significant accuracy, This gives the multidimensional Fourier cosine expansion formulation

$$\begin{aligned} v_1(t_0, \mathbf{x}) &= \mathrm{e}^{-r\Delta t}\int_{a_2}^{b_2}\int_{a_1}^{b_1} g(\mathbf{y})f(\mathbf{y}|\mathbf{x})\mathrm{d}y_1\mathrm{d}y_2 \\ &= \mathrm{e}^{-r\Delta t}\int_{a_2}^{b_2}\int_{a_1}^{b_1} g(\mathbf{y})\sum_{k_1=0}^{+\infty}{}'\sum_{k_2=0}^{+\infty}{}' A_{k_1,k_2}(\mathbf{x}) \\ &\quad \cos\left(k_1\pi\frac{y_1 - a_1}{b_1 - a_1}\right)\cos\left(k_2\pi\frac{y_2 - a_2}{b_2 - a_2}\right)\mathrm{d}y_1\mathrm{d}y_2. \end{aligned} \tag{2.2}$$

The notation v_i is used for the different approximations of v and keeps track of the numerical errors that set in from each step. For final approximations we also use the "hat" notation, like $\hat{v}$, $\hat{c}$, etc. In the second line in (2.2), the conditional density is replaced by its Fourier cosine expansion in $\mathbf{y}$ on $[a_1, b_1] \times [a_2, b_2]$, with series coefficients $A_{k_1,k_2}(\mathbf{x})$ defined by

$$A_{k_1,k_2}(\mathbf{x}) := \frac{2}{b_1 - a_1}\frac{2}{b_2 - a_2}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1 - a_1}{b_1 - a_1}\right)\cos\left(k_2\pi\frac{y_2 - a_2}{b_2 - a_2}\right)\mathrm{d}y_1\mathrm{d}y_2. \tag{2.3}$$

$\sum'$ in (2.2) means that the first term of the summation has half weight. We interchange summation and integration and define

$$V_{k_1,k_2}(T) := \frac{2}{b_1 - a_1}\frac{2}{b_2 - a_2}\int_{a_2}^{b_2}\int_{a_1}^{b_1} g(\mathbf{y})\cos\left(k_1\pi\frac{y_1 - a_1}{b_1 - a_1}\right)\cos\left(k_2\pi\frac{y_2 - a_2}{b_2 - a_2}\right)\mathrm{d}y_1\mathrm{d}y_2, \tag{2.4}$$

which are the Fourier cosine series coefficients of $v(T,\mathbf{y}) = g(\mathbf{y})$ on $[a_1,b_1] \times [a_2,b_2]$.

Step 2. Truncation of the series summations gives

$$v_2(t_0,\mathbf{x}) = \frac{b_1-a_1}{2}\frac{b_2-a_2}{2}\mathrm{e}^{-r\Delta t}\sum_{k_1=0}^{N_1-1}{}'\sum_{k_2=0}^{N_2-1}{}' A_{k_1,k_2}(\mathbf{x})V_{k_1,k_2}(T). \tag{2.5}$$

Step 3. Next, the coefficients $A_{k_1,k_2}(\mathbf{x})$ are approximated by

$$F_{k_1,k_2}(\mathbf{x}) := \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)\mathrm{d}y_1\mathrm{d}y_2. \tag{2.6}$$

The 2D-COS formula is based on the following goniometric relation [**?**]:

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta). \tag{2.7}$$

With this we obtain

$$2F_{k_1,k_2}(\mathbf{x}) = F^{+}_{k_1,k_2}(\mathbf{x}) + F^{-}_{k_1,k_2}(\mathbf{x}), \tag{2.8}$$

where

$$F^{\pm}_{k_1,k_2}(\mathbf{x}) := \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1} \pm k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)\mathrm{d}y_1\mathrm{d}y_2. \tag{2.9}$$

Now, the coefficients $F^{\pm}_{k_1,k_2}(\mathbf{x})$ can be calculated by

$$\begin{aligned} &F^{\pm}_{k_1,k_2}(\mathbf{x}) \qquad\qquad (2.10)\\ &= \tfrac{2}{b_1-a_1}\tfrac{2}{b_2-a_2}\quad \mathrm{Re}\Big(\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\exp\left(ik_1\pi\frac{y_1}{b_1-a_1} \pm ik_2\pi\frac{y_2}{b_2-a_2}\right)\mathrm{d}y \\ &\qquad\qquad \exp\left(-ik_1\pi\frac{a_1}{b_1-a_1} \mp ik_2\pi\frac{a_2}{b_2-a_2}\right)\Big) \\ &= \tfrac{2}{b_1-a_1}\tfrac{2}{b_2-a_2}\quad \mathrm{Re}\Big(\varphi\Big(\frac{k_1\pi}{b_1-a_1}, \pm\frac{k_2\pi}{b_2-a_2}\Big|\mathbf{x}\Big)\exp\Big(-ik_1\pi\frac{a_1}{b_1-a_1} \mp ik_2\pi\frac{a_2}{b_2-a_2}\Big)\Big) \\ &= \tfrac{2}{b_1-a_1}\tfrac{2}{b_2-a_2}\quad \mathrm{Re}\Big(\varphi_{levy}\Big(\frac{k_1\pi}{b_1-a_1}, \pm\frac{k_2\pi}{b_2-a_2}\Big)\exp\left(ik_1\pi\frac{x_1-a_1}{b_1-a_1} \pm ik_2\pi\frac{x_2-a_2}{b_2-a_2}\right)\Big). \end{aligned}$$

Re(.) denotes taking the real part of the input argument. $\varphi(.,.|\mathbf{x})$ is the *bivariate conditional characteristic function* of $\mathbf{X}_T$, given $\mathbf{X_{t_0}} = \mathbf{x}$ [**?**]:

$$\varphi(\mathbf{u}|\mathbf{x}) = \mathbb{E}\big[\mathrm{e}^{i\mathbf{u}\cdot\mathbf{X}_T}\big|\mathcal{F}_{t_0}\big] = \iint_{\mathbb{R}^2}\mathrm{e}^{i\mathbf{u}\cdot\mathbf{y}}f(\mathbf{y}|\mathbf{x})\mathrm{d}\mathbf{y}. \tag{2.11}$$

Examples of these characteristic functions can be found in section 6. The last equality in (2.10) holds particularly for Lévy processes, for which $\varphi_{levy}(u_1,u_2) := \varphi(u_1,u_2|0,0)$. Inserting (2.10) into (2.5) gives us the *2D-COS formula* for approximation of $v(t_0, \mathbf{x})$:

$$\hat{v}(t_0, \mathbf{x}) := \frac{b_1 - a_1}{2}\frac{b_2 - a_2}{2}\mathrm{e}^{-r\Delta t}\sum_{k_1=0}^{N_1-1}{}'\sum_{k_2=0}^{N_2-1}{}'\frac{1}{2}\left[F^{+}_{k_1,k_2}(\mathbf{x}) + F^{-}_{k_1,k_2}(\mathbf{x})\right]V_{k_1,k_2}(T) \tag{2.12}$$

$$\begin{aligned} = \ & \mathrm{e}^{-r\Delta t}\sum_{k_1=0}^{N_1-1}{}'\sum_{k_2=0}^{N_2-1}{}'\frac{1}{2}\Big[\mathrm{Re}\Big(\varphi_{levy}\Big(\frac{k_1\pi}{b_1 - a_1}, +\frac{k_2\pi}{b_2 - a_2}\Big) \\ & \exp\Big(ik_1\pi\frac{x_1 - a_1}{b_1 - a_1} + ik_2\pi\frac{x_2 - a_2}{b_2 - a_2}\Big)\Big) \\ & +\mathrm{Re}\Big(\varphi_{levy}\Big(\frac{k_1\pi}{b_1 - a_1}, -\frac{k_2\pi}{b_2 - a_2}\Big) \\ & \exp\Big(ik_1\pi\frac{x_1 - a_1}{b_1 - a_1} - ik_2\pi\frac{x_2 - a_2}{b_2 - a_2}\Big)\Big)\Big]V_{k_1,k_2}(T). \end{aligned}$$

With the multidimensional-COS formula, calculation of the option's Greeks is straightforward, as explained for the 1D case in [1].

Remark 1. Cosine terms facilitate the usage of the characteristic function. Fourier sine expansions may also be used; however, their coefficients decrease at a lower rate for the payoff functions discussed, and because of this the cosine series are preferred. Alternative basis functions, like certain wavelet basis functions, may represent another interesting research direction for option pricing, but this is not yet known and is part of future research.

If the characteristic function is not available directly or not known analytically, it may be approximated. Local volatility models, for example, typically do not yield analytic functions φ, but recent research in [**?**] proposes a second-order approximation formula, so that an approximate characteristic function may be derived.

3 Bermudan rainbow options

We generalize the $2D - COS$ method to pricing Bermudan rainbow options with a 2D underlying log-asset price process, $X_t = (X_t^1, X_t^2)$, that is in the class of Lévy processes. A Bermudan option can be exercised at a fixed set of $\mathcal{M}$ early-exercise times $t_0 < t_1 < ...t_{\mathcal{M}} = T$, with $\Delta t := t_{m+1} - t_m$. The payoff function is denoted by $g(\cdot)$. The problem is solved backwards in time, with

$$\begin{cases} v(t_{\mathcal{M}}, \mathbf{x}) = g(\mathbf{x}), \\ c(t_{m-1}, \mathbf{x}) = \mathrm{e}^{-r\Delta t}\mathbb{E}[v(t_m, \mathbf{X}_{t_m})|\mathbf{X}_{t_{m-1}} = \mathbf{x}], \\ v(t_{m-1}, \mathbf{x}) = \max[g(\mathbf{x}, c(t_{m-1}, \mathbf{x})], \quad 2 \leq m \leq \mathcal{M}, \\ v(t_0, \mathbf{x}_0) = c(t_0, \mathbf{x}_0), \end{cases}$$

Function $c(t_{m-1},\mathbf{x})$ is called the continuation value and is approximated by the 2D-COS formula

$$\hat{c}(t_{m-1},\mathbf{x}) := \frac{b_1-a_1}{2}\frac{b_2-a_2}{2}\mathrm{e}^{-r\Delta t}\sum_{k_1=0}^{N_1-1}{}'\sum_{k_2=0}^{N_2-1}{}'\frac{1}{2}\left[F^+_{k_1,k_2}(\mathbf{x})+F^-_{k_1,k_2}(\mathbf{x})\right]V_{k_1,k_2}(t_m). \tag{3.1}$$

The Fourier coefficients of the value function in (3.1) are given by

$$V_{k_1,k_2}(t_m) := \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\int_{a_2}^{b_2}\int_{a_1}^{b_1} v(t_m,\mathbf{y})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)\mathrm{d}y_1\mathrm{d}y_2. \tag{3.2}$$

The option function is now approximated by $\hat{v}(t_{m-1},\mathbf{x}) := \max[g(\mathbf{x}),\hat{c}(t_{m-1},\mathbf{x})]$.

3.1 Recursion formula for coefficients $V_{k_1,k_2}(t_m)$

In this section, a recursive algorithm for recovering the coefficients $V_{k_1,k_2}(t_m)$, backwards in time, is derived.

In the coefficients $V_{k_1,k_2}(t_{\mathcal{M}})$, the terminal condition $v(t_{\mathcal{M}},\mathbf{y}) = g(\mathbf{y})$ appears. Some payoff functions provide analytic solutions to these coefficients in (3.2); otherwise they can be approximated, as explained in Section 3.2.

For the coefficients that are used to approximate the continuation valued at times $t_0,...,t_{\mathcal{M}-2}$, the value function, $v(t_m,\mathbf{y}) = \max[g(\mathbf{y}),c(t_m,\mathbf{y})]$, appears in the terms $V_{k_1,k_2}(t_m)$ and we need to find an optimal policy for all state values $\mathbf{y}\in[a_1,b_1]\times[a_2,b_2]$ into rectangular subdomains $\mathcal{C}^q$ and $\mathcal{G}^p$, so that approximately for all states $\mathbf{y}\in\mathcal{C}^q$ it is optimal to continue and for all $\mathbf{y}\in\mathcal{G}^p$ it is optimal to exercise the option. We can split the integral in the definition of V_{k_1,k_2} into different parts:

$$\begin{aligned}
V_{k_1,k_2}(t_m) &= \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\sum_p\iint_{\mathcal{G}^p} g(\mathbf{y})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)\mathrm{d}\mathbf{y}\\
&+ \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\sum_q\iint_{\mathcal{C}^q} c(t_m,\mathbf{y})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)\mathrm{d}\mathbf{y}\\
&:= \sum_p G_{k_1,k_2}(\mathcal{G}^p)+\sum_q C_{k_1,k_2}(t_m,\mathcal{C}^q) \qquad (m\neq 0,\mathcal{M}).
\end{aligned} \tag{3.3}$$

We approximate the terms $C_{k_1,k_2}(t_{\mathcal{M}-1},[z_q,z_{q+1}]\times[w_q,w_{q+1}])$ in (3.3), where the variables z_q, z_{q+1}, w_q, and w_{q+1} denote the corner points of the rectangular continuation region $\mathcal{C}^q$. For the integrand of the terms C_{k_1,k_2} we again apply the 2D Fourier cosine expansion by inserting the COS formula for $c(t_{\mathcal{M}-1},\mathbf{y})$, i.e., (3.1). The approximation reads as

$$\begin{aligned}
&\hat{C}_{k_1,k_2}(t_{\mathcal{M}-1},[z_q,z_{q+1}]\times[w_q,w_{q+1}]) \qquad (3.4)\\
:=\ & \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\int_{w_q}^{w_{q+1}}\int_{z_q}^{z_{q+1}} \hat{c}(t_{\mathcal{M}-1},\mathbf{y})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)\mathrm{d}y_1\mathrm{d}y_2\\
=\ & \int_{w_q}^{w_{q+1}}\int_{z_q}^{z_{q+1}} \sum_{j_1=0}^{N_1-1}{}' \sum_{j_2=0}^{N_2-1}{}' \mathrm{e}^{-r\Delta t}\frac{1}{2}\left[F^+_{j_1,j_2}(\mathbf{y})+F^-_{j_1,j_2}(\mathbf{y})\right]\\
& V_{j_1,j_2}(t_M)\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)\mathrm{d}y_1\mathrm{d}y_2\\
=\ & \mathrm{Re}\Bigg(\sum_{j_1=0}^{N_1-1}{}' \sum_{j_2=0}^{N_2-1}{}' \frac{1}{2}\mathrm{e}^{-r\Delta t}\varphi_{levy}\left(\frac{j_1\pi}{b_1-a_1},+\frac{j_2\pi}{b_2-a_2}\right)\\
& V_{j_1,j_2}(t_{\mathcal{M}})M^+_{k_1,j_1}(z_q,z_{q+1},a_1,b_1)M^+_{k_2,j_2}(w_q,w_{q+1},a_2,b_2)\Bigg)\\
+\ & \mathrm{Re}\Bigg(\sum_{j_1=0}^{N_1-1}{}' \sum_{j_2=0}^{N_2-1}{}' \frac{1}{2}\mathrm{e}^{-r\Delta t}\varphi_{levy}\left(\frac{j_1\pi}{b_1-a_1},-\frac{j_2\pi}{b_2-a_2}\right)\\
& V_{j_1,j_2}(t_{\mathcal{M}})M^+_{k_1,j_1}(z_q,z_{q+1},a_1,b_1)M^+_{k_2,j_2}(w_q,w_{q+1},a_2,b_2)\Bigg)
\end{aligned}$$

where the elements of square-matrices M^+ and M^- are given by

$$M^+_{m,n}(u_1,u_2,a,b) := \frac{2}{b-a}\int_{u_1}^{u_2}\mathrm{e}^{in\pi\frac{y-a}{b-a}}\cos\left(m\pi\frac{y-a}{b-a}\right)\mathrm{d}y, \tag{3.5}$$

$$M^-_{m,n}(u_1,u_2,a,b) := \frac{2}{b-a}\int_{u_1}^{u_2}\mathrm{e}^{-in\pi\frac{y-a}{b-a}}\cos\left(m\pi\frac{y-a}{b-a}\right)\mathrm{d}y. \tag{3.6}$$

We thus find

$$\hat{C}_{k_1,k_2}(t_{\mathcal{M}-1},[z_q,z_{q+1}]\times[w_q,w_{q+1}]) = \mathrm{Re}\left(\sum_{j_1=0}^{N_1-1}{}' M^+_{k_1,j_1}(z_q,z_{q+1},a_1,b_1)\mathcal{A}^q_{j_1,k_2}\right), \tag{3.7}$$

where

$$\begin{aligned}
\mathcal{A}^q_{j_1,k_2} &:= \textstyle\sum_{j_2=0}^{N_2-1}{}'\frac{1}{2}\mathrm{e}^{-r\Delta t}\varphi_{levy}\left(\frac{j_1\pi}{b_1-a_1},+\frac{j_2\pi}{b_2-a_2}\right)V_{j_1,j_2}(t_{\mathcal{M}})M^+_{k_2,j_2}(w_q,w_{q+1},a_2,b_2)\\
&\quad+\textstyle\sum_{j_2=0}^{N_2-1}{}'\frac{1}{2}\mathrm{e}^{-r\Delta t}\varphi_{levy}\left(\frac{j_1\pi}{b_1-a_1},-\frac{j_2\pi}{b_2-a_2}\right)V_{j_1,j_2}(t_{\mathcal{M}})M^-_{k_2,j_2}(w_q,w_{q+1},a_2,b_2).
\end{aligned} \tag{3.8}$$

The elements of $(N_1\times N_2)$-matrix $\mathcal{A}^q$ are calculated in a rowwise fashion. The row-vector $\mathcal{A}^q_{j_1,\cdot} = \{\mathcal{A}^q_{j_1,k_2}\}_{k_2=0}^{N_2-1}$ can be written as two matrix-vector multiplications:

$$\mathcal{A}^q_{j_1,\cdot} = M^+(w_q,w_{q+1},a_2,b_2)\mathbf{w}^{q+}_{j_1,\cdot} + M^-(w_q,w_{q+1},a_2,b_2)\mathbf{w}^{q-}_{j_1,\cdot}, \tag{3.9}$$

where

$$\mathbf{w}^{q\pm}_{j_1,\cdot} := \{w^{q\pm}_{j_1,j_2}\}_{j_2=0}^{N_2-1},\ \text{with}\ w^{q\pm}_{j_1,j_2} := \frac{1}{2}\mathrm{e}^{-r\Delta t}\varphi_{levy}\left(\frac{j_1\pi}{b_1-a_1},\pm\frac{j_2\pi}{b_2-a_2}\right)V_{j_1,j_2}(t_{\mathcal{M}}). \tag{3.10}$$

Then, the matrix $\hat{C}_{k_1,k_2}$ is computed in a columnwise fashion. The column-vector $\hat{C}_{,k_2} = \{\hat{C}_{k_1,k_2}\}_{k_1=0}^{N_1-1}$ is calculated by one matrix-vector product,

$$\hat{C}_{,k_2}(t_{\mathcal{M}-1}, [z_q, z_{q+1}] \times [w_q, w_{q+1}]) = \mathrm{Re}\left(M^+(z_q, z_{q+1}, a_1, b_1)\mathcal{A}^q_{\cdot,k_2}\right), \quad (3.11)$$

with column-vector $\mathcal{A}^q_{\cdot,k_2} = \{\mathcal{A}^q_{j_1,k_2}\}_{j_1=0}^{N_1-1}$.

The coefficients $G_{k_1,k_2}([z_p, z_{p+1}] \times [w_p, w_{p+1}]$ are defined by

$$\begin{aligned} &G_{k_1,k_2}(z_p, z_{p+1}] \times [w_p, w_{p+1}]) \qquad (3.12)\\ &= \frac{2}{b_1 - a_1}\frac{2}{b_2 - a_2}\int_{w_p}^{w_p+1}\int_{z_p}^{z_p+1} g(\mathbf{y})\cos\left(k_1\pi\frac{y_1 - a_1}{b_1 - a_1}\right)\cos\left(k_2\pi\frac{y_2 - a_2}{b_2 - a_2}\right)\mathrm{d}y_1\mathrm{d}y_2. \end{aligned}$$

These terms may admit an analytic solution; however, in some practical applications an analytic solution is not present. Methods to approximate these terms are proposed in Section 3.2.

We end up with the approximated coefficients

$$\hat{V}_{k_1,k_2}(t_{\mathcal{M}-1}) := \sum_p G_{k_1,k_2}(\mathcal{G}^p) + \sum_q \hat{C}_{k_1,k_2}(t_{\mathcal{M}-1}, \mathcal{C}^q). \quad (3.13)$$

For the other coefficients $V_{k_1,k_2}(t_m)$, the approximations $\hat{c}(t_m, \mathbf{y})$ and $\hat{V}_{j_1,j_2}(t_{m+1})$ will be used to approximate the terms $C_{k_1,k_2}(t_m, [z_q, z_{q+1}] \times [w_q, w_{q+1}])$, and the elements of the corresponding matrix $\mathcal{A}^q$ are

$$\begin{aligned} \mathcal{A}^q_{j_1,k_2} &= \sum_{j_2=0}^{N_2-1}{}' \frac{1}{2}\mathrm{e}^{-r\Delta t}\varphi_{levy}\left(\frac{j_1\pi}{b_1 - a_1}, +\frac{j_2\pi}{b_2 - a_2}\right)\hat{V}_{j_1,j_2}(t_{m+1})M^+_{k_2,j_2}(w_q, w_{q+1}, a_2, b_2)\\ &+ \sum_{j_2=0}^{N_2-1}{}' \frac{1}{2}\mathrm{e}^{-r\Delta t}\varphi_{levy}\left(\frac{j_1\pi}{b_1 - a_1}, -\frac{j_2\pi}{b_2 - a_2}\right)\hat{V}_{j_1,j_2}(t_{m+1})M^-_{k_2,j_2}(w_q, w_{q+1}, a_2, b_2). \end{aligned} \quad (3.14)$$

FFT. The matrix-vector products $M^+\mathbf{v}$ and $M^-\mathbf{v}$ in the computation of matrices $\mathcal{A}^q$ and $\hat{C}$ can be computed efficiently by a Fourier-based algorithm, as stated in Fang and Oosterlee (2009) [2]. The computation time achieved is $O(N\log_2 N)$, with N the length of the vector.

Algorithm. We can recover the terms $\hat{V}_{k_1,k_2}(t_m)$ recursively, starting with $V_{k_1,k_2}(t_{\mathcal{M}})$. The algorithm for solving the pricing problem backwards in time reads as follows.

Computational complxity. The initialization is of order $O(N_1N_2)$. In the main loop there are $\mathcal{M} - 1$ iterations in which the following computations are performed. The construction of one matrix $\mathcal{A}^q$ costs $O(2N_1N_2\log_2 N_2)$ operations. Computation of $\hat{C}_{k_1,k_2}(t_m, [z_q, z_{q+1}] \times [w_q, w_{q+1}])$ takes $O(N_2N_1\log_2 N_1)$ operations. $G_{k_1,k_2}([z_p, z_{p+1}] \times [w_p, w_{p+1}])$ is of order $O(N_1N_2)$. The computation time is linear in the number of continuation and early-exercise regions. The final step takes $O(N_1N_2)$ operations.

ALGORITHM 1. (2D-COS method for pricing Bermudan rainbow options)
Initialization: *Calculate coefficients* $V_{k_1,k_2}(t_{\mathcal{M}})$.
Main loop to recover $\hat{V}(t_m)$: *For* $m = \mathcal{M} - 1$ *to* 1:
- *Determine optimal continuation regions* $\mathcal{C}^q$ *and early-exercise regions* $\mathcal{G}^p$.
- *Compute* $\hat{V}(t_m)$ *from* (3.3) *with the help of the FFT algorithm.*

Final step: *Compute* $\hat{v}(t_0, \mathbf{x}_0)$ *by inserting* $\hat{V}_{k_1,k_2}(t_1)$ *into* (3.1).

3.2 Approximations for the coefficients $V(T)$ and $G(\mathcal{G}^p)$

In this section, we propose methods for approximating the terminal coefficients $V_{k_1,k_2,\ldots,k_n}(T)$ and the terms $G_{k_1,k_2,\ldots,k_n}(\mathcal{G}^p)$ that are specific for the multidimensional-COS method.

In the 1D pricing problem, the terminal coefficients $V_{k_1}(T)$ admit analytic solutions for several options, like put-and call-based options, digital options, and power options. Besides, in the 1D-COS method for pricing Bermudan options, the terms $G_{k_1}(\mathcal{G}^p)$ are also usually known analytically.

In two dimensions, the payoff functions of, for instance, a geometric basket or a call-on-maximum option provide analytic solutions to the 2D coefficients $V_{k_1,k_2}(T)$, but this is generally an exception. If no exact representation is available, then they can be approximated by using *discrete cosine transforms* (DCTs) or the *ClenshawCurtis quadrature rule*. The usage of DCTs is explained in section 3.2.1. Also, analytic forms for the terms $G_{k_1,k_2,\ldots,k_n}(\mathcal{G}^p)$ are in general not available in the multidimensional version. An approximation method, based on Fourier cosine expansion of the payoff function, is discussed in section 3.2.

3.2.1 DCTs for $V(T)$

In this section, we explain this idea of using DCTs to approximate the terminal coefficients $V_{k_1,k_2}(T)$. For this, we take $Q \geq \max[N_1, N_2]$ grid-points for each spatial dimension and define

$$y_i^{n_i} := a_i + \left(n_i + \frac{1}{2}\right)\frac{b_i - a_i}{Q} \text{ and } \Delta y_i := \frac{b_i - a_i}{Q}, i = 1, 2. \tag{3.15}$$

The midpoint-rule integration gives us

$$\begin{aligned} V_{k_1,k_2}(T) \quad &\approx \quad \sum_{n_1=0}^{Q-1}\sum_{n_2=0}^{Q-1} \frac{2}{b_1 - a_1}\frac{2}{b_2 - a_2} g(y_1^{n_1}, y_2^{n_2}) \cos\left(k_1\pi\frac{y_1^{n_1} - a_1}{b_1 - a_1}\right) \qquad (3.16) \\ &\qquad \cos\left(k_2\pi\frac{y_2^{n_2} - a_2}{b_2 - a_2}\right)\Delta y_1 \Delta y_2 \\ &= \quad \sum_{n_1=0}^{Q-1}\sum_{n_2=0}^{Q-1} g(y_1^{n_1}, y_2^{n_2}) \cos\left(k_1\pi\frac{2n_1 + 1}{2Q}\right)\cos\left(k_2\pi\frac{2n_2 + 1}{2Q}\right)\frac{2}{Q}\frac{2}{Q}. \end{aligned}$$

The above 2D DCT (Type II) can be calculated efficiently by, we will apply the 2D FFT in the implementation. The approximated coefficients are denoted by $V_{k_1,k_2}^{DCT}(T)$, with the corresponding computed European option value $\hat{v}^{DCT}(t_0, \mathbf{x})$.

3.2.2 Approximation methods for $G(\mathcal{G}^p)$

The terms G_{k_1,k_2} are defined by

$$
\begin{aligned}
&G_{k_1,k_2}(([z_p, z_{p+1}] \times [w_p, w_{p+1}]) \qquad (3.17)\\
&= \frac{2}{b_1 - a_1}\frac{2}{b_2 - a_2}\int_{w_p}^{w_p+1}\int_{z_p}^{z_p+1} g(\mathbf{y})\cos\left(k_1\pi\frac{y_1 - a_1}{b_1 - a_1}\right)\cos\left(k_2\pi\frac{y_2 - a_2}{b_2 - a_2}\right)\mathrm{d}y_1\mathrm{d}y_2.
\end{aligned}
$$

These terms may admit an analytic solution; however, in many practical applications the calculation of coefficients $G_{k_1,k_2}(\mathcal{G}^p)$ is time consuming, or an analytic solution is not present. Then, we can use *discrete Fourier transforms* to approximate them, similarly as in section 3.2.1. Another way is the usage of the Fourier cosine expansion of the payoff function. The Fourier cosine expansion of the payoff function can be written as

$$
\begin{aligned}
\hat{g}(\mathbf{y}) &:= \sum_{k_1=0}^{N_1-1}{}' \sum_{k_2=0}^{N_2-1}{}' \cos\left(k_1\pi\frac{y_1 - a_1}{b_1 - a_1}\right)\cos\left(k_2\pi\frac{y_2 - a_2}{b_2 - a_2}\right)V_{k_1,k_2}(T)\\
&= \sum_{k_1=0}^{N_1-1}{}' \sum_{k_2=0}^{N_2-1}{}' \frac{1}{2}\Bigg[\mathrm{Re}\left(\exp\left(ik_1\pi\frac{y_1 - a_1}{b_1 - a_1} + ik_2\pi\frac{y_2 - a_2}{b_2 - a_2}\right)\right)\\
&+ \mathrm{Re}\left(\exp\left(ik_1\pi\frac{y_1 - a_1}{b_1 - a_1} - ik_2\pi\frac{y_2 - a_2}{b_2 - a_2}\right)\right)\Bigg]V_{k_1,k_2}(T). \qquad (3.18)
\end{aligned}
$$

With this, the coefficients G_{k_1,k_2} can by approximated by $\hat{G}_{k_1,k_2}$, similarly as in (3.4):

$$
\begin{aligned}
&\hat{G}_{k_1,k_2}([z_p, z_{p+1}] \times [w_p, w_{p+1}]) \qquad (3.19)\\
&= \mathrm{Re}\left(\sum_{j_1=0}^{N_1-1}{}' \sum_{j_2=0}^{N_2-1}{}' \frac{1}{2}V_{j_1,j_2}(T)M^{+}_{k_1,j_1}(z_p, z_{p+1}, a_1, b_1)M^{+}_{k_2,j_2}(w_p, w_{p+1}, a_2, b_2)\right)\\
&+ \mathrm{Re}\left(\sum_{j_1=0}^{N_1-1}{}' \sum_{j_2=0}^{N_2-1}{}' \frac{1}{2}V_{j_1,j_2}(T)M^{+}_{k_1,j_1}(z_p, z_{p+1}, a_1, b_1)M^{-}_{k_2,j_2}(w_p, w_{p+1}, a_2, b_2)\right).
\end{aligned}
$$

We will apply the above approximation in (3.13).

3.3 Early-exercise Region

For Pricing Bermudan options we need to determine the continuation and early-exercise region. For this, we divide the domain of the second dimension $[a_2, b_2]$ into J subintervals:

$$[a_2, b_2] = [w_0, w_1] \cup [w_1, w_2] \cdots [w_q, w_{q+1}] \cdots [w_{J-1}, w_J].$$

At each subinterval, we determine the value y^* for which the optimal exercise policy changes to the optimal continuation, i.e.,

$$g(y^*, w_q) = c(t_m, y^*, w_q).$$

For the basket put option, the early-exercise region is $\mathcal{G}_q = [a_1, y^*] \times [w_q, w_{q+1}]$ and the continuation region is $\mathcal{C}_q = [y^*, b_1] \times [w_q, w_{q+1}]$, while for the basket call option, the early-exercise region is $\mathcal{G}_q = [y^*, b_1] \times [w_q, w_{q+1}]$ and the continuation region is $\mathcal{C}_q = [a_1, y^*] \times [w_q, w_{q+1}]$.

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