Importance Sampling and Statistical Romberg method for Lévy processes

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1 Input parameters

- First and second Monte Carlo samples N_1 and N_2.
- Generator type
- Spot S_0, the annual interest rate r, the dividend $divid$, ε and ε_{beta}.
- The CGMY parameters: $C > 0$, $G \geq 0$, $M \geq 0$ and $Y < 2$.
- Maturity T
- Strike K

2 Output parameters

- Price
- Length of the Price confidence interval
- Variance of the first and second Monte Carlo

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3 Description

3.1 CGMY model

The CGMY is a pure jump process with a Lévy density given by:

$$\nu(dx) = C\frac{e^{-Mx}}{x^{1+Y}}\mathbf{1}_{x>0}dx + \frac{Ce^{-G|x|}}{|x|^{1+Y}}\mathbf{1}_{x<0}dx.$$

where $C > 0, G \geq 0, M \geq 0$ and $Y < 2$. The condition $Y < 2$ is induced by the requirement that lévy density integrate x^2 in the neighborhood of 0.
The characterictic exponent is given by

$$\psi(u) \quad = \quad iu\gamma_c \,+\, \Gamma(-Y)M^Y C\left\{(1-\frac{iu}{M})^Y - 1 + \frac{iuY}{M}\right\} \,+\, \Gamma(-Y)G^Y C\left\{(1+\frac{iu}{G})^Y - 1 - \frac{iuY}{G}\right\}.$$

We use the Statistical Romberg method to compute the prices of European call options in an exponential Lévy model driven by a CGMY process. The stock price is given by:

$$S_T = S_0 e^{rT+L_T}.$$

where L_T is a Lévy process.

3.2 Importance sampling

The optimal θ is given by:

$$\theta_n^* = \underset{\theta \in \mathbb{R}^2}{\arg\min}\ \mathbb{E}\left[\left(\psi^2(L_T) + (\nabla\psi(L_T).W_T)^2\right) e^{-\theta L_T + T\kappa(\theta)}\right].$$

To obtain the optimal θ, the idea is to make use a constrained Robbins Monro algorithm which is decribed as follows: Let $(L_{T,i})_{i\geq 1}$ be an i.i.d copies of the stochastic process (L_T). Let B be a connected set in $\mathbb{R}^d$ with $\{0\} \in B$. For $\theta_0 \in B$, we construct recursively the sequence of random variables $(\theta_i^\varepsilon)_{i\in\mathbb{N}}$ in $\mathbb{R}^d$ defined by

$$\theta_{i+1}^\varepsilon = \Pi_B\left[\theta_i^\varepsilon - \gamma_{i+1} H(\theta_i^\varepsilon, L_{T,i+1}^\varepsilon)\right] \tag{1}$$

where Π_B is the Euclidean projection onto the constraint set B and the gain sequence $(\gamma_i)_{i\geq 1}$ is a decreasing sequence of positive real numbers satistfying

$$\sum_{i=1}^{\infty} \gamma_i = \infty \text{ and } \sum_{i=1}^{\infty} \gamma_i^2 < \infty \tag{2}$$

The function H is given by

$$H(\theta, L_T) = (T\nabla\kappa(\theta) - L_T)(\psi^2(L_T) + (\nabla\psi(L_T).W_T)^2)\exp(-\theta.L_T + T\kappa(\theta)). \tag{3}$$

3.3 Calculus of the Statistical Romberg price

By rebalancing the optimal θ_n^*, we compute the price of the considered option using the Statistical Romberg method with importance sampling (SR + IS). Hence, we approximate our initial quantity of interest $\mathbb{E}\psi(L_T)$ ($N_1 = \frac{(2-Y)^2}{4C^2\varepsilon^{4-2Y}}$ and $N_2 = \frac{(2-Y)}{2C}\varepsilon^{(2-Y)(\beta-2)}$)

$$V_\varepsilon^\theta = \frac{1}{N_1}\sum_{i=1}^{N_1} \psi(L_{T,i+1}^{\varepsilon^\beta,\theta_i^{\varepsilon^\beta}}) e^{-\theta_i^{\varepsilon^\beta}.L_{T,i+1}^{\varepsilon^\beta,\theta_i^{\varepsilon^\beta}} + T\kappa(\theta_i^{\varepsilon^\beta})} + \frac{1}{N_2}\sum_{i=1}^{N_2}\left(\psi(L_{T,i+1}^{\varepsilon,\theta_i^\varepsilon}) - \psi(L_{T,i+1}^{\varepsilon^\beta,\theta_i^\varepsilon})\right) e^{-\theta_i^\varepsilon.L_{T,i+1}^{\varepsilon,\theta_i^\varepsilon} + T\kappa(\theta_i^\varepsilon)}$$

where with κ is the cummulant generating function of the marginal law at unit time of $\{L_t; t\geq 0\}$ under the probability measure $\mathbb{P}$, that is, $\kappa_t(\theta) := \ln\mathbb{E}\left[e^{\theta.L_t}\right]$.
We compute the length of a 95% confidence interval and the CPU time of our method.

3.4 Complexity Analysis

The optimal average complexity of the Monte Carlo method is given by

$$\mathbb{E}[C_{MC}] = \mathbb{E}[N_T^\varepsilon] \times N$$

where $\mathbb{E}[N_T^\varepsilon] = \nu([\varepsilon, +\infty[) = O(\varepsilon^{-Y})$
Hence,

$$\mathbb{E}[C_{MC}] \backsimeq C \times \frac{1}{\varepsilon^{4-Y}}$$

The optimal average complexity of the Statistical Romberg method is given by

$$\mathbb{E}[C_{SR}] = \mathbb{E}[N_T^{\varepsilon^\beta}] \times N_1 + (\mathbb{E}[N_T^\varepsilon] + \mathbb{E}[N_T^{\varepsilon^\beta}]) \times N_2$$

where $\mathbb{E}\left[N_T^{\varepsilon^\beta}\right] = \nu\left([\varepsilon^\beta, +\infty[\right) = O(\varepsilon^{-\beta Y})$
Hence,

$$\mathbb{E}[C_{SR}] \backsimeq C \times \mathbb{E}[C_{MC}]\,\varepsilon^{Y(1-\beta)}\left(1 + \varepsilon^{(2-Y)\beta} + \varepsilon^{2\beta-Y}\right)$$

Since the only restriction on β is that $\beta \in (0,1)$, we will choose the optimal β^* minimizing the average complexity of the Statistical Romberg algorithm. Numerically, we verify easily that for $\beta = \frac{Y}{2}$, the average complexity of the Statistical Romberg method will be minimal in comparison to the Monte Carlo one.