Second order discretization schemes for Wishart processes and applications to option prices

Camilla Pisani*

February 18, 2016

Abstract

In this work we present some simple simulation algorithms for CIR and more general Wishart processes, the main idea being to split the generator and to reduce the problem to the simulation of the square of a matrix valued Ornstein-Uhlenbeck process to be added to a deterministic process. In this way we provide weak second order schemes that require only the simulation of i.i.d. Gaussian r.v.'s.

Premia 18

This work introduces some simple algorithms to simulate a Wishart process. They work only under some assumptions on the parameters, but they are quite simple to implement and work well. The Wishart process was introduced in a simpler form in Bru [4] and then generalized as the solution of

$$X_t^x = x + \int_0^t (\alpha\, a^T a + bX_s^x + X_s^x b^T)ds + \int_0^t \sqrt{X_s^x}dW_s a + a^T dW_s^T \sqrt{X_s^x} \tag{1}$$

where $(W_t)_{t\geq 0}$ denotes a $d \times d$ square matrix of independent standard Brownian motions, $\alpha \in \mathbb{R}^+$, x is symmetric positive semidefinite and a and b are general $d \times d$ matrices. Under the assumption $\alpha \geq d-1$, that we will assume hereafter, existence of a unique weak solution of the previous equation has been proved (see Bru [4], Cuchiero, Filipović, Mayerhofer and Teichmann [6] e.g.). A problem analogous to that arising when applying classical simulation schemes to the square root process appears also for the Wishart. In particular being the Wishart a process defined on the cone of positive semidefinite matrices it may reach the boundary of the cone where the coefficients are not Lipschitz continuous. As a consequence strong existence of the solution is not granted by the usual existence theorems and simulation methods such as the Euler-Maruyama scheme are not applicable. For this reason, it is important to find alternative simulation schemes.

The main reference for the simulation of Wishart processes is Ahdida and Alfonsi [2] where a simulation method working for every value of $\alpha > d-1$ is developed. The methods we propose here are much simpler but work only under the more restrictive assumption $\alpha \geq d$.

*Aarhus University, Department of Economics and Business, Fuglesangs Allé 4, 8210 Aarhus V, Denmark. cpisani@econ.au.dk. Supported by the People Programme (Marie Curie Actions) of the European Union's Seventh Framework Programme FP7/2007-2013/ under REA grant agreement $n°$ 289032.

The rule of composition of schemes

If $p^{(1)}$, $p^{(2)}$ are second order transition probabilities of simulation schemes for diffusion processes with generators L_1 and L_2 respectively, then the schemes

$$q(t) = p^{(1)}(\tfrac{t}{2}) \circ p^{(2)}(t) \circ p^{(1)}(\tfrac{t}{2}) \tag{2}$$

and

$$q(t) = \tfrac{1}{2}\left(p^{(1)}(t) \circ p^{(2)}(t) + p^{(2)}(t) \circ p^{(1)}(t)\right) \tag{3}$$

are both second order schemes for the diffusion with generator $L_1 + L_2$ (see Theorem 1.17 in Alfonsi [1] which extends ideas of Ninomiya and Victoir [9]).

Let us apply this composition rule to the case of a Wishart process that is solution of (1). Its generator can be decomposed

$$\begin{aligned} L &= \operatorname{tr}[(\alpha a^T a + bx + xb^T)D] + 2\operatorname{tr}(xDa^T aD) = \\ &= \underbrace{\operatorname{tr}[(na^T a + bx + xb^T)D] + 2\operatorname{tr}(xDa^T aD)}_{:=L_2} + \underbrace{\operatorname{tr}[(\alpha - n)a^T aD]}_{:=L_1} \end{aligned}$$

where $n = \lfloor \alpha \rfloor$. A second order transition probability for L_1 (which is the generator of a deterministic motion) is simply the translation

$$X_{t_{i+1}} = X_{t_i} + (\alpha - n)a^T a\, t$$

As for L_2 it is the generator of the square of a $n \times d$ Ornstein-Uhlenbeck process. Denoting $h = t_{i+1} - t_i$,

$$C = \int_0^h e^{ub} a^T a e^{ub^T}\, ds$$

an exact scheme for L_2 is given by

$$X_{t_{i+1}} = \left(Y_{t_i} e^{hb^T} + WC^{1/2}\right)^T \left(Y_{t_i} e^{hb^T} + WC^{1/2}\right)$$

where $W \sim N(0_d, I_{n\times d})$ and Y_{t_i} is any $n \times d$ matrix such that $Y_{t_i}^T Y_{t_i} = X_{t_i}$ possibly obtained by taking the square root of the positive defined $d \times d$ matrix X_{t_i} and then adding $n - d$ rows of zeros.

Assume $\alpha \geq d$. Thanks to the composition rule all the following are second order schemes for the Wishart process:

$$q_1(t) = p^{(1)}(\tfrac{t}{2}) \circ p^{(2)}(t) \circ p^{(1)}(\tfrac{t}{2}) \tag{4}$$

$$q_2(t) = p^{(2)}(\tfrac{t}{2}) \circ p^{(1)}(t) \circ p^{(2)}(\tfrac{t}{2}) \tag{5}$$

$$q_3(t) = \tfrac{1}{2}\left(p^{(1)}(t) \circ p^{(2)}(t) + p^{(2)}(t) \circ p^{(1)}(t)\right) \tag{6}$$

and also

$$q_4(t) = p^{(1)}(\tfrac{t}{4}) \circ p^{(2)}(\tfrac{t}{2}) \circ p^{(1)}(\tfrac{t}{2}) \circ p^{(2)}(\tfrac{t}{2}) \circ p^{(1)}(\tfrac{t}{4}) \tag{7}$$

which turns out to be a second order scheme by application of the composition rule twice.

The condition $\alpha \geq d$ is necessary in order to ensure that the matrix produced by the iteration $p^{(1)}$ is positive semi-definite. Note that the schemes above are exact if α is an integer larger that d and are therefore expected to have a very small bias when $\alpha - \lfloor \alpha \rfloor$ is small. This fact is confirmed by the numerical experiments.

A special subcase: the square root process

The square root or CIR process is the one-dimensional subcase of the Wishart process and was introduced in finance by Cox, Ingersoll and Ross [5] to model short interest rates. It is the solution of

$$dv_t = (a - \kappa v_t)dt + \sigma\sqrt{v_t}dB_t \tag{8}$$

with $v_0, \sigma, a \geq 0, \kappa \in \mathbb{R}$, $(B_t)_t$ being a standard Brownian motion. Observe that the previous equation can be written equivalently as

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dB_t \tag{9}$$

where κ is the mean reversion parameter, θ the long run and σ is the volatility of variance. Whenever $v_0 \geq 0$ and $a \geq \frac{\sigma^2}{2}$ the process is always positive.

Define $n := \lfloor \frac{4a}{\sigma^2} \rfloor$ and split the generator as

$$L = (a - \kappa x)\frac{d}{dx} + \frac{1}{2}\sigma^2 x\,\frac{d^2}{dx^2} = \underbrace{\left(n\frac{\sigma^2}{4} - \kappa x\right)\frac{d}{dx} + \frac{1}{2}\sigma^2 x\,\frac{d^2}{dx^2}}_{:=\widetilde{L}_2} + \underbrace{\left(a - n\frac{\sigma^2}{4}\right)\frac{d}{dx}}_{:=\widetilde{L}_1} \cdot \tag{10}$$

where $\widetilde{L}_1$ is the generator associated to a deterministic process

$$v_{t_{i+1}} = v_{t_i} + \left(a - n\,\frac{\sigma^2}{4}\right)h$$

and $\widetilde{L}_2$ is the generator of the square of a $n \times 1$ Ornstein-Uhlenbeck process. An exact scheme for it with step h is

$$v_{t_{i+1}} = \left(Y_{t_i}e^{-\frac{\kappa}{2}h} + \frac{\sigma}{2}\sqrt{\psi_k(h)}\,W\right)^T\left(Y_{t_i}e^{-\frac{\kappa}{2}h} + \frac{\sigma}{2}\sqrt{\psi_k(h)}\,W\right)$$

where W is a $n \times 1$ matrix whose entries are independent and $N(0,1)$ distributed and Y_{t_i} denotes any $n \times 1$ matrix such that $Y_{t_i}^T Y_{t_i} = v_{t_i}$ (see [3] for further details). By using one of the rules (4)–(7) we obtain again weak second order simulation schemes for the square root process. The schemes obtained in this way are exact if $\lfloor \frac{4a}{\sigma^2} \rfloor$ is an integer number and can be expected to perform better than the schemes described in the previous paragraph if $\lfloor \frac{4a}{\sigma^2} \rfloor$ is large.

Application to the Heston model

The Heston stochastic volatility model [8] is defined as the solution of

$$\begin{aligned} dS_t &= rS_t dt + \sqrt{v_t}dW_t \\ dv_t &= \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dB \end{aligned} \tag{11}$$

where the processes W and B are standard Brownian motions with correlation parameter ρ. We can simulate the volatility v_t using one of our schemes, as far as $a \geq \frac{\sigma^2}{4}$ and then simulate the correspondent asset price by a simple Euler- Maruyama scheme or better re-adapting the scheme suggested in Alfonsi [1], paragraph 4.2.

Application to the Gourieroux Sufana model

The Gourieroux and Sufana[7] model is the solution of

$$
\begin{aligned}
dS_t^l &= rS_t^l dt + S_t^l(\sqrt{X_t}dB_t)_l \\
dX_t &= (\alpha a^T a + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}
\end{aligned}
$$

where the processes W and B are independent standard Brownian motions. It can be considered as the multidimensional version of the Heston model [8]. We can simulate the process X_t using one of our schemes, as far as $\alpha \geq d$ and then simulate the correspondent asset prices by a simple Euler-Maruyama scheme or better re-adapting the scheme suggested in Alfonsi [2], paragraph 4.3.

References

[1] Alfonsi, A. (2010). High order discretization schemes for the CIR process: Application to affine term structure and Heston models *Mathematics of Computations, Vol. 79, No.269, pp 209-237* 2, 3

[2] Ahdida, A. and Alfonsi, A. (2013). Exact and high order discretization schemes for Wishart processes and their affine extensions *Ann. Appl. Probab. 23(2013), no.3, 1025-1073* 1, 4

[3] Baldi, P. and Pisani, C. (2013). Simple simulation schemes for CIR and Wishart processes *International Journal of Theoretical and Applied Finance, Vol. 16, No. 8* 3

[4] Bru, M.-F. (1991). Wishart processes *Journal of Theoretical Probability, Vol. 4, No. 4* 1

[5] Cox, J. C. Ingersoll, J.E. and Ross, S.A. (1985). A Theory of the Term Structure of Interest Rates, *Econometrica 53, pp. 385-407* 3

[6] Cuchiero, C. and Filipović, D. and Mayerhofer, E. and Teichmann, J. Affine processes on positive semidefinite matrices *Ann. Appl. Probab., Vol. 21, No. 2* 1

[7] Discrete time Wishart term structure models. *Journal of Economic Dynamics & Control (2011), Vol. 35, No. 6* 4

[8] Heston, S.L. (1993). A Closed-form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies, Vol. 6, No. 2, pp 327-343* 3, 4

[9] Ninomiya, S. and Victoir, N. *Weak approximation of stochastic differential equations and application to derivative pricing, Appl. Math. Finance (2008), pp 107-121* 2