

Analytical formulas for local volatility model with stochastic rates

Ahmed KEBAIER

Premia 18

1 Model specification

This paper presents a new approximation formulae of European options in local volatility model with stochastic interest rates. More precisely, we consider the one factor Hull and white model for interest rates $(r_t)_{t \geq 0}$, the CEV diffusion for the spot process $(S_t)_{t \geq 0}$ and constant correlation ρ . The dynamics of $(S_t, r_t)_{t \geq 0}$ are given by

$$\begin{cases} S_t &= S_0 + \int_0^t r_u S_u du + \int_0^t \nu S_u^\beta dW_u \\ r_t &= f(0, t) - \int_0^t \gamma(u, t) \Gamma(u, t) du + \int_0^t \gamma(u, t) dB_u, \end{cases} \quad (1)$$

where $\gamma(t, T) = \xi e^{-\kappa(T-t)}$ and $\Gamma(t, T) = \frac{\xi}{\kappa} (e^{-\kappa(T-t)} - 1)$.

2 Second order approximation formula for pricing European call

In the following, we introduce the log discounted process $(X_t)_{t \geq 0}$ given by $X_t = \log(S_t) - \int_0^t r_s ds$. We use the smart expansion introduced by Benhamou et al [1] to give an analytical accurate approximation of a call european price, written as the expected value under the risk neutral probability measure of the payoff function $h(x) = (e^x - K)_+$ evaluated at the maturity time T :

$$Call Price = \mathbb{E} \left[e^{-\int_0^T r_s ds} h \left(\int_0^T r_s ds + X_T \right) \right].$$

More precisely, we use the second order approximation price formula, given by Theorem 3.6 of [1], to write

$$Call Price = B(0, T) \left\{ \mathbb{E}_T \left[h \left(\int_0^T r_s ds + X_T^B \right) \right] + \sum_{i=1}^3 \alpha_{i,T} \text{Greek}_i^h \left(\int_0^T r_s ds + X_T^B \right) + Resid_2 \right\} \quad (2)$$

where \mathbb{E}_T is the expectation under the forward neutral probability \mathbb{Q}^T , $B(0, T)$ is the zero coupon bond paying 1 Euro at time T . The process X_T^B is given by

$$X_T^B = \log(S_0) + \int_0^T \sigma_t dW_t - \frac{1}{2} \int_0^T \sigma_t^2 dt, \quad \sigma_t \equiv \nu S_0^\beta.$$

The leading order in this approximation is given by the quantity $A = B(0, T) \mathbb{E}_T \left[h \left(\int_0^T r_s ds + X_T^B \right) \right]$ which is given by the Black formula

$$\begin{aligned} A = S_0 \mathcal{N} \left(\frac{1}{\sigma^{\text{Black}} \sqrt{T}} \log \left(\frac{S_0}{B(0, T) K} \right) + \frac{1}{2} \sigma^{\text{Black}} \sqrt{T} \right) \\ - K B(0, T) \mathcal{N} \left(\frac{1}{\sigma^{\text{Black}} \sqrt{T}} \log \left(\frac{S_0}{B(0, T) K} \right) - \frac{1}{2} \sigma^{\text{Black}} \sqrt{T} \right). \end{aligned}$$

The quantity $Greek_i^h \left(\int_0^T r_s ds + X_T^B \right)$ is the i th derivative of the leading term w.r.t. the initial value $x_0 = \log(S_0)$ and the error term $Resid_2$ is well analyzed in [1]. In the particular case of model (1) the coefficients $\alpha_{i,T}$ are explicit.

$$\begin{aligned}\alpha_{1,T} &= \frac{e^{-2\kappa T}(\beta-1)\nu^2 S_0^{2(\beta-1)}}{4\kappa^4} [2\rho^2 \xi^2 + 2e^{\kappa T} \rho(\kappa \nu S_0^{\beta-1}(2\kappa T + 1) + 2\rho(\kappa T - 1)\xi)\xi + \\ &\quad e^{2\kappa T}(\nu^2 S_0^{2(\beta-1)} T^2 \kappa^4 + \rho \nu S_0^{\beta-1}(\kappa T(3\kappa T - 2) - 2)\xi \kappa + 2\rho^2(\kappa T - 1)^2 \xi^2)], \\ \alpha_{2,T} &= -\alpha_{1,T} - \alpha_{3,T}, \\ \alpha_{3,T} &= \frac{e^{-2\kappa T}(\beta-1)\nu^2 S_0^{2(\beta-1)}}{2\kappa^4} [\rho \xi + e^{\kappa T}(\nu S_0^{\beta-1} T \kappa^2 + \rho T \xi \kappa - \rho \xi)]^2.\end{aligned}$$

Finally, the Call price, given by the approximation formula (2), is computed by the Premia code source function.

References

- [1] Benhamou Eric Gobet Emmanuel and Miri Mohammed. Analytical formulas for local volatility model with stochastic rates. *Quantitative Finance, to appear*, 2011. 1, 2