A NEW WEAK APPROXIMATION METHOD OF SDE

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1. Introduction

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The object of this document is to introduce a new weak approximation scheme of stochastic differential equations which is proposed in [10]. We present the algorithm and provide an example to illustrate the application of it.

Theorem 1.1 and Corollary 1.1 give a new implementation method of the new higher order scheme of weak approximation. The point in the algorithm is that the approximation operator can be considered to be composition of solutions of ODEs when ω is given. The concrete ODEs are constructed by Theorem 1.2 and can be approximated by the Runge-Kutta method for ODEs by Theorem 1.3.

We should note that another higher-order weak approximation method, the N-V algorithm, is introduced in [16]. Although the N-V algorithm and the new method are based on the same scheme [7] and [9] and have many common features, algorithms themselves are completely different and the diversity is not trivial.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We define $B^0(t)$ as t and $(B^1(t), \dots, B^d(t))$ as the d-dimensional standard Brownian Motion. $C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$ denotes the set of $\mathbb{R}^N$-valued infinitely differentiable functions defined over $\mathbb{R}^N$ whose derivatives are all bounded. Our interest is in weak approximation, that is to say, approximation of $(P_t f)(x) = E[f(X(1,x))]$ where $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$ and $X(t,x)$ is a solution to the Stratonovich stochastic integral equation

$$X(t,x) = x + \sum_{i=0}^{d} \int_0^t V_i(X(s,x)) \circ dB^i(s), \tag{1.1}$$

where $V_i \in C_b^\infty(\mathbb{R}^N; \mathbb{R}^N)$, $i = 0, \dots, d$. $V_i \in C_b^\infty\left(\mathbb{R}^N; \mathbb{R}^N\right)$ is regarded as a vector field in the following way:

$$V_i f(x) = \sum_{j=1}^{N} V_i^j(x) \frac{\partial f}{\partial x_j}(x), \quad \text{for } f \in C_b^\infty(\mathbb{R}^N; \mathbb{R}).$$

Let $A = \{v_0, v_1, \dots, v_d\}, d \geq 1$ be an alphabet and A^* denote the set of all words consisting of the elements of A. The empty word 1 is the identity of A^*. For $u = v_{i_1} \cdots v_{i_n} \in A^*$, $i_k \in \{0, 1, \dots, d\}$, $|u|$ and $\|u\|$ are defined by $|u| = n$ and $\|u\| = |u| + \text{card}\,(\{k \,|\, i_k = 0\})$, respectively, where card(S) denotes the cardinality of a set S. A_m^* and $A_{\leq m}^*$ denote $\{w \in A^* | \, |w| = m\}$ and $\{w \in A^* | \, |w| \leq m\}$, respectively. Let $\mathbb{R}\langle A \rangle$ be the $\mathbb{R}$-coefficient free algebra with basis A^* and $\mathbb{R}\langle\langle A \rangle\rangle$ be the set of all $\mathbb{R}$-coefficient formal series with basis A^*. Then, $\mathbb{R}\langle A \rangle$ is a sub $\mathbb{R}$-algebra of

$\mathbb{R}\langle\langle A\rangle\rangle$. We call an element of $\mathbb{R}\langle A\rangle$ a non-commutative polynomial. Let $\mathbb{R}\langle A\rangle_m = \{P \in \mathbb{R}\langle A\rangle \mid (P,w) = 0, \text{ if } \|w\| \neq m\}$. $P \in \mathbb{R}\langle\langle A\rangle\rangle$ is written as

$$P = \sum_{w\in A^*} (P,w)\, w \quad \text{or} \quad \sum_{w\in A^*} a_w w,$$

where $(P,w) = a_w \in \mathbb{R}$ denotes the coefficient of w. The algebra structure is defined as usual, that is to say,

$$\left(\sum_{w\in A^*} a_w w\right)\left(\sum_{w\in A^*} b_w w\right) = \sum_{\substack{w=uv\\ w\in A^*}} a_u b_v w.$$

The Lie bracket is defined as $[x,y] = xy - yx$ for $x, y \in \mathbb{R}\langle\langle A\rangle\rangle$. For $w = v_{i_1}\cdots v_{i_n} \in A^*$, $\mathfrak{r}(w)$ denotes $[v_{i_1}, [v_{i_2}, [\ldots, [v_{i_{n-1}}, v_{i_n}]\ldots]]]$. We define $\mathcal{L}_{\mathbb{R}}(A)$ as the set of Lie polynomials in $\mathbb{R}\langle A\rangle$ and $\mathcal{L}_{\mathbb{R}}((A))$ as the set of Lie series. For $m \in \mathbb{Z}_{\geq 0}$, let j_m be a map defined as follows:

$$j_m\left(\sum_{w\in A^*} a_w w\right) = \sum_{\|w\|\leq m} a_w w.$$

For arbitrary $P, Q \in \mathbb{R}\langle A\rangle$, the inner product $\langle P, Q\rangle$ is defined as follows:

$$\langle P, Q\rangle = \sum_{w\in A^*} (P,w)(Q,w).$$

For $P \in \mathbb{R}\langle\langle A\rangle\rangle$ such that $(P,1) = 0$, we can define $\exp(P)$ as $1 + \sum_{k=1}^{\infty} P^k/k!$. Also, $\log(Q)$ can be defined as $\sum_{k=1}^{\infty}(-1)^{k-1}(Q-1)^k/k$ for $Q \in \mathbb{R}\langle\langle A\rangle\rangle$ if $(Q,1) = 1$. The following relations hold:

$$\log(\exp(P)) = P \quad \text{and} \quad \exp(\log(Q)) = Q.$$

Let $\|P\|_2 = (\langle P, P\rangle)^{1/2}$, for $P \in \mathbb{R}\langle A\rangle$.

By the natural identification $\mathbb{R}\langle\langle A\rangle\rangle \approx \mathbb{R}^{\infty}$, we can induce the direct product topology into $\mathbb{R}\langle\langle A\rangle\rangle$. $\mathbb{R}\langle\langle A\rangle\rangle$ becomes a Polish space by the topology. Also we can consider its Borel σ-algebra $\mathcal{B}(\mathbb{R}\langle\langle A\rangle\rangle)$, $\mathbb{R}\langle\langle A\rangle\rangle$-valued random variables, their expectations, and other notions as usual.

Let Φ be a homomorphism between $\mathbb{R}\langle A\rangle$ and the $\mathbb{R}$-algebra which consists of smooth differential operators over $\mathbb{R}^N$ such that

$$\begin{aligned} \Phi(1) &= \mathrm{Id}, \\ \Phi(v_{i_1}\cdots v_{i_n}) &= V_{i_1}\cdots V_{i_n},\ i_1,\ldots,i_n \in \{0,1,\ldots,d\}. \end{aligned} \tag{1.2}$$

Also, for $s \in \mathbb{R}_{>0}$, $\Psi_s : \mathbb{R}\langle\langle A\rangle\rangle \longrightarrow \mathbb{R}\langle\langle A\rangle\rangle$ is defined as follows:

$$\Psi_s\left(\sum_{m=0}^{\infty} P_m\right) = \sum_{m=0}^{\infty} s^{m/2} P_m, \text{ where } P_m \in \mathbb{R}\langle A\rangle_m.$$

For a smooth vector field V, i. e. an element of $C_b^{\infty}\left(\mathbb{R}^N; \mathbb{R}^N\right)$, $\exp(V)(x)$ denotes the solution at time 1 of the ordinary differential equation

$$\frac{dz_t}{dt} = V(z_t), \quad z_0 = x.$$

We also define $\|V\|_{C^n}$ for $V \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ as follows:

$$\begin{aligned}\|V\| &= \sup_x |V(x)| \\ \left\|V^{(n)}\right\| &= \sup_x \left\{\left|V^{(n)}_{(x)}(U_1,U_2,\dots,U_n)\right|;\ \|U_i\|=1, i=1,\dots,n\right\} \\ \|V\|_{C^n} &= \sum_{i=0}^{n}\left\|V^{(i)}\right\|.\end{aligned}$$

Here $V^{(k)}$ denotes the k-th order total differential of V.

Definition 1.1. *A map g from $C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ to the set of all maps from $\mathbb{R}^N$ to $\mathbb{R}^N$ is called an integration scheme of order m if there exists a positive constant C_m such that for all $W \in C_b^\infty(\mathbb{R}^N;\mathbb{R}^N)$ and $x \in \mathbb{R}^N$,*

$$\left|g(W)(x) - \exp(W)(x)\right| \le C_m \|W\|_{C^{m+1}}^{m+1}. \tag{1.3}$$

Here C_m depends only on m and g. Let $\mathcal{IS}(m)$ be the set of all integration schemes of order m.

Notation 1.1. *For $z_1, z_2 \in \mathcal{L}_{\mathbb{R}}((A))$, we define $z_2 \text{H} z_1$ as $\log(\exp(z_2)\exp(z_1))$. Then from the definition, for $z_1, z_2, z_3 \in \mathcal{L}_{\mathbb{R}}((A))$,*

$$(z_1 \text{H} z_2) \text{H} z_3 = \log\left(\exp(z_1)\exp(z_2)\exp(z_3)\right) = z_1 \text{H} (z_2 \text{H} z_3),$$

and so we can write for $z_1,\dots,z_n \in \mathcal{L}_{\mathbb{R}}((A))$

$$z_1 \text{H} z_2 \text{H} \cdots \text{H} z_n = \log\left(\exp(z_1)\cdots\exp(z_n)\right). \tag{1.4}$$

The followings are the main results of our study.

Theorem 1.1. *Suppose that for $m \ge 1$ and $n \ge 2$ there exist $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables $Z_1,\dots,Z_n$ satisfying that*

$$\|Z_i\| \le m \quad \text{for all } i \in \{1,\dots,n\}, \tag{1.5}$$

$$E\left[\left\|j_m Z_i\right\|_2\right] < \infty, \tag{1.6}$$

$$E\left[\exp\left(a\sum_{j=1}^{n}\left\|\overline{Z}_j\right\|_{C^{m+1}}\right)\right] < \infty \quad \text{for any } a > 0, \tag{1.7}$$

where $\overline{y}$ denotes $\Phi(\Psi_s(y))$ for an element $y \in \mathcal{L}_{\mathbb{R}}((A))$. Then for arbitrary $g_1,\dots,g_n \in \mathcal{IS}(m)$, there exists a positive constant C such that

$$\left\|g_1\left(\overline{Z}_1\right)\circ\cdots\circ g_n\left(\overline{Z}_n\right)(x) - \exp\left(\overline{j_m\left(Z_n \text{H}\cdots\text{H} Z_1\right)}\right)(x)\right\|_{L^p} \le C s^{(m+1)/2}. \tag{1.8}$$

Here for functions f and g, $f\circ g(x)$ denotes $f(g(x))$ as usual.

Let S_j^i's be $\mathbb{R}$-valued normally distributed random variables and c_j's and $R_{jj'}$ be constants in $\mathbb{R}$ satisfying that for $j, j' = 1,\dots,n$, and $i, i' = 1,\dots,d$,

$$\sum_{j=1}^{n} c_j = 1, \quad E\left[S_j^i\right] = 0, \quad \text{and} \quad E\left[S_j^i S_{j'}^{i'}\right] = R_{jj'}\delta_{ii'}. \tag{1.9}$$

Here our interest is in finding $Z_j = \sum_{i=1}^{d} S_j^i v_i + c_j v_0$, $j = 1,\dots,n$, such that

$$E\left[j_m\left(\exp(Z_1)\cdots\exp(Z_n)\right)\right] = j_m E\left[\exp\left(v_0 + \frac{1}{2}\sum_{i=1}^{d} v_i^2\right)\right]. \tag{1.10}$$

Theorem 1.2. *For $m = 5$, Z_j's as above can be constructed if we take $n = 2$ with*

$$
\begin{aligned}
c_1 &= \frac{\mp\sqrt{2(2u-1)}}{2}, \quad c_2 = 1 \pm \frac{\sqrt{2(2u-1)}}{2}, \quad R_{11} = u \\
R_{22} &= 1 + u \pm \sqrt{2(2u-1)}, \quad R_{12} = -u \mp \frac{\sqrt{2(2u-1)}}{2}.
\end{aligned} \tag{1.11}
$$

If $u \geq 1/2$, then the positive definiteness is satisfied.

Each $g_i(Z_i)(y)$ in Theorem 1.1 is the approximation of $\exp(Z_i(\omega))(y)$ for a given ω which has the form of a solution of an ODE. From this point of view, we take the m-th order Runge-Kutta method for ODEs as a candidate for each g_i.

The M-stage Runge-Kutta method of order m can be written in the sense of [1] as follows: for $W \in C_b^\infty\left(\mathbb{R}^N; \mathbb{R}^N\right)$,

$$
\begin{aligned}
Y_i(W,s) &= y_0 + s\sum_{j=1}^{M} a_{ij} W\left(Y_j(W,s)\right), \\
Y(y_0; W, s) &= y_0 + s\sum_{i=1}^{M} b_i W\left(Y_i(W,s)\right),
\end{aligned} \tag{1.12}
$$

where $A = \left(a_{ij}\right)_{i,j=1,\ldots,M}$ with $a_{ij} \in \mathbb{R}$ and $b = {}^t(b_1, \ldots, b_M) \in \mathbb{R}^M$ satisfy the conditions described in [1][2]. (1.12) gives the m-th order approximation of an ODE

$$
dy(t) = W(y(t))\, dt, \quad y(0) = y_0. \tag{1.13}
$$

Letting $\mathrm{RK}_{(m)}(W)(y_0)$ be $Y(y_0; W, 1)$, $\mathrm{RK}_{(m)}$ belongs to $\mathcal{IS}(m)$, which is Theorem 1.3.

Corollary 1.1. *Let Z_j's $j = 1, \ldots, n$, be $\mathcal{L}_{\mathbb{R}}((A))$-valued random variables as in Theorem 1.2 and define linear operators $Q_{(s)}$, $s \in (0,1]$ by*

$$
\left(Q_{(s)} f\right)(x) = E\left[f\left(\mathrm{RK}_{(m)}\left(\overline{Z}_1\right) \circ \cdots \circ \mathrm{RK}_{(m)}\left(\overline{Z}_n\right)(x)\right)\right]. \tag{1.14}
$$

Then for $f \in C_b^\infty(\mathbb{R}^N; \mathbb{R})$,

$$
\left\| P_s f - Q_{(s)} f \right\|_\infty \leq C s^{(m+1)/2} \left\| \mathrm{grad}(f) \right\|_\infty. \tag{1.15}
$$

where C is a positive constant and $s \in (0,1]$.

Remark 1.1. *Kusuoka shows the following results in* [9]*:*

(1) For a Lipschitz continuous function f, the inequality (1.15) still holds.

(2) The Romberg extrapolation can be applied to this algorithm.

$\mathrm{RK}_{(m)}$ denotes the m-th order Runge-Kutta method with $s = 1$. The following theorem confirms that the m-th order Runge-Kutta method belongs to $\mathcal{IS}(m)$.

Theorem 1.3. $\mathrm{RK}_{(m)} \in \mathcal{IS}(m)$.

2. The new simulation scheme and Corollary 1.1

The Corollary 1.1 indicates the new implementation method of the new higher order scheme proposed by Kusuoka in [7], [8], and [9]. Corollary 1.1 can be proved by Theorem 1.1 and 1.3 and a theorem in [9].

This implementation method seems to be distinguished mainly for two advantages. One is that the approximation operator can be obtained by numerical calculations if the Runge-Kutta method is applied to calculation of each $\exp(Z_j)$ while the tediousness in symbolical calculations of the operator might be an obstacle for practical application, which can be seen in [11], [15], and [17]. The other is that the partial sampling problem discussed in [11] and [15] can be conquered by using quasi-Monte Carlo methods. More precisely, the following two points make effective the use of the Low-Discrepancy sequences, which are essential to quasi-Monte Carlo methods([14]):

- S_j^i's can be taken to be continuous random variables in this implementation
- the scheme itself is characterized by the need of the much less number of discretization of time, which leads to reduction of the number of dimensions of the numerical integration.

3. Application

We give a numerical example in this section in order to illustrate the implementation method proposed in Corollary 1.1 of the new higher-order scheme, comparing with some existing schemes.

3.1. **Simulation.** Let $X(t,x)$ be a diffusion process defined by (1.1). The most popular scheme of first order is the Euler-Maruyama scheme. It is shown in [6][20] that for an arbitrary C^4 function f

$$\left\| E\left[f\left(X_1^{(\mathrm{EM}),n}\right)\right] - E\left[f\left(X(1,x)\right)\right] \right\| \leq C_f \frac{1}{n} \tag{3.1}$$

where $X_1^{(\mathrm{EM}),n}$ denotes the Euler-Maruyama scheme approximating $X(t,x)$.

Construction of higher order scheme is based on the higher order stochastic Taylor formula ([3][6]). When the vector fields $\{V_i\}_{i=0}^d$ commute, higher-order schemes can be easily simplified to a direct product of one-dimensional problem as seen in [6]. Contrastingly, for non-commutative $\{V_i\}_{i=0}^d$, acquisition of all iterated integrals of Brownian motion is required, which is very demanding. This is done in [7][12][18] [19][11] and generalized as cubature method on Wiener space ([13]).

Simulation approach is to be necessarily followed by numerical calculation of $E\left[f\left(X_1^{(\mathrm{ord}\,p),n}\right)\right]$. However, when $n \times d$ is large, it is practically impossible to proceed the integration by using trapezoidal formula and so we fall back on the Monte Carlo or quasi-Monte Carlo method ([14]). Here we only introduce remarks on each method. For details, see [16].

Remark 3.1. *As long as we use the Monte Carlo method for numerical approximation of $E[f(X(1,x))]$, the number of sample points needed to attain a given accuracy is independent of the number of the dimensions of integration, namely both the number n of partitions and the order p of the approximation scheme.*

Remark 3.2. *In contrast to the Monte Carlo case, the number of sample points needed for the quasi-Monte Carlo method for numerical approximation of $E[f(X(1,x))]$ heavily depends on the number of the dimensions of integration. The smaller the number of the dimensions, the less number of samples are needed.*

3.2. **The algorithm and competitors.**

3.2.1. *The algorighm of the new method.* We take the algorithm which is proposed by Corollary 1.1 with Z_j's $j = 1, 2$, given in Theorem 1.2. From Corollary 1.1, we can implement the second order algorithm with numerical approximation of $\exp(Z_i)$'s of at least 5th-order Runge-Kutta method because the order m for an integration scheme attained by Z_1 and Z_2 is five and so the order of the new implementation method becomes two. As a result of the same argument it can be shown that at least 7th-order explicit Runge-Kutta method has to be applied to approximation of $\exp(Z_i)$'s when we boost the new method to the third order by Romberg extrapolation. Details of these Runge-Kutta algorithms used here are given in Appendix.

3.2.2. *Competitive schemes.* Although there are a lot of studies on acceleration of Monte Carlo methods ([5]), we choose by the following reasons only the crude Euler-Maruyama scheme and the N-V algorithm both with and without Romberg extrapolation as competitors:

(i) Only these two schemes can be recognized to be comparable to the new method in that they are model-independent.
(ii) Almost all variance reduction techniques and dimension reduction techniques which we can apply to the Euler-Maruyama scheme are also applicable to the new method.

3.3. **Numerical results.** We provide an example on financial option pricing in the following part of this paper.

3.3.1. *Asian option under the Heston model.* We consider an Asian call option written on an asset having the price process under the Heston model which is known as a two-factor stochastic volatility model. Comparison with the N-V algorithm is to be given as well from the result shown in [16].

Non-commutativity of this example should be of note here.
Let Y_1 be the price process of an asset following the Heston model:

$$
\begin{aligned}
Y_1(t,x) =& x_1 + \int_0^t \mu Y_1(s,x)\, ds + \int_0^t Y_1(s,x)\sqrt{Y_2(s,x)}\, dB^1(s),\\
Y_2(t,x) =& x_2 + \int_0^t \alpha\left(\theta - Y_2(s,x)\right) ds\\
&+ \int_0^t \beta\sqrt{Y_2(s,x)}\left(\rho\, dB^1(s) + \sqrt{1-\rho^2}\, dB^2(s)\right),
\end{aligned}
\tag{3.2}
$$

where $x = (x_1, x_2) \in (\mathbb{R}_{>0})^2$, $(B^1(t), B^2(t))$ is a two-dimensional standard Brownian motion, $|\rho| \le 1$, and α, θ, μ are some positive coefficients such that $2\alpha\theta - \beta^2 > 0$ to ensure the existence and uniqueness of a solution to the SDE ([4]). Then the payoff of Asian call option on this asset with maturity T and strike K is $\max(Y_3(T,x)/T - K, 0)$ where

$$
Y_3(t,x) = \int_0^t Y_1(s,x)\, ds. \tag{3.3}
$$

Hence, the price of this option becomes $D \times E\left[\max\left(Y_3(T,x)/T - K,\, 0\right)\right]$ where D is an appropriate discount factor that we do not focus on in this experiment. We set

$T = 1$, $K = 1.05$, $\mu = 0.05$, $\alpha = 2.0$, $\beta = 0.1$, $\theta = 0.09$, $\rho = 0$, and $(x_1, x_2) = (1.0, 0.09)$ and take

$$E\left[\max\left(Y_3(T,x)/T - K,\, 0\right)\right] = 6.0473534496 \times 10^{-2}$$

which is obtained by the new method with Romberg extrapolation and the quasi-Monte Carlo with $n = 96 + 48$, and $M = 8 \times 10^8$ where M denotes the number of sample points.

Let $Y(t,x) = {}^t(Y_1(t,x), Y_2(t,x), Y_3(t,x))$. Transformation of the SDEs (3.2) and (3.3) gives the following Stratonovich-form SDEs:

$$Y(t,x) = \sum_{i=0}^{2} \int_0^t V_i(Y(s,x)) \circ dB^i(s), \tag{3.4}$$

where

$$\begin{aligned} V_0\left({}^t(y_1,y_2,y_3)\right) &= {}^t\left(y_1\left(\mu - \frac{y_2}{2} - \frac{\rho\beta}{4}\right),\, \alpha(\theta - y_2) - \frac{\beta^2}{4},\, y_1\right) \\ V_1\left({}^t(y_1,y_2,y_3)\right) &= {}^t\left(y_1\sqrt{y_2},\, \rho\beta\sqrt{y_2},\, 0\right) \\ V_2\left({}^t(y_1,y_2,y_3)\right) &= {}^t\left(0,\, \beta\sqrt{(1-\rho^2)\,y_2},\, 0\right). \end{aligned} \tag{3.5}$$

3.3.2. *Discretization Error.* The relation between discretization error and the number of partitions of each algorithm is plotted in Figure 3.1.

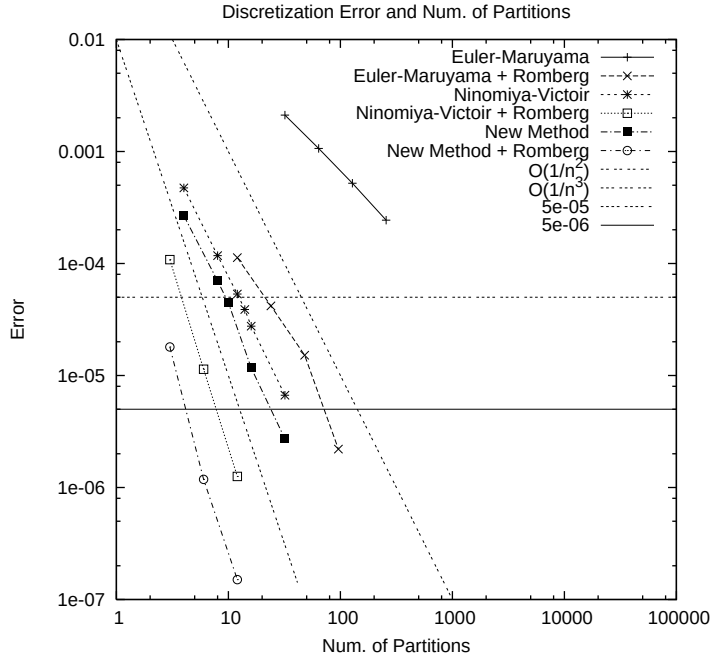

FIGURE 3.1. Error coming from the discretization

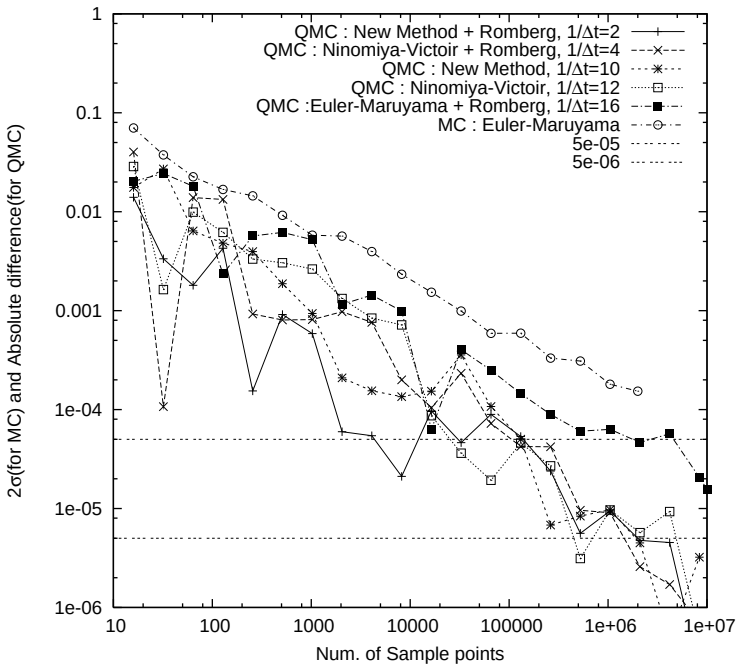

FIGURE 3.2. Convergence Error from quasi-Monte Carlo and Monte Carlo

3.3.3. *Integration Error.* Looking at Figure 3.2, we can compare convergence errors of respective methods for each number of sample points, M. For Monte Carlo case, 2σ of 10 batches is taken as convergence error while for the quasi-Monte Carlo method, absolute difference from the value to be convergent is considered. For 10^{-4} accuracy with 95% confidence level (2σ), $M = 10^8$ is taken for the Monte Carlo method. On the other hand, if we apply the quasi-Monte Carlo method instead, the new method and the N-V method require $M = 2 \times 10^5$ sample points, though $M = 5 \times 10^6$ has to be taken for the Euler-Maruyama scheme.

TABLE 1. #Partitions, #Samples, Dimension, and CPU time required for accuracy of 10^{-4}.

Method	#Part.	Dim.	#Samples	CPU time (sec)
E-M + MC	2000	4000	10^8	1.72×10^5
E-M + Romb. + QMC	16 + 8	48	5×10^6	1.27×10^2
N-V + QMC	16	32 + 16	2×10^5	4.38
N-V + Romb. + QMC	4 + 2	12 + 6	2×10^5	1.76
New Method + QMC	10	40	2×10^5	3.4
New Method + Romb. + QMC	2 + 1	12	2×10^5	1.2

3.3.4. *Overall performance comparison.* The number of partitions, the number of samples, and the amount of computation time required for 10^{-4} accuracy for each

method are summarized in Table 1. CPU used in this experiment is Athlon 64 3800+ by AMD.

From this table, we can see that the speed of the new method is approximately 100 times faster than that of the Euler-Maruyama scheme when Romberg extrapolation and quasi-Monte Carlo are applied to each. Even when the extrapolation is not applied, the new method dose more or less 37 times faster calculation than the Euler-Maruyama scheme with Romberg extrapolation and quasi-Monte Carlo does.

Lastly, Remarks 3.1 and 3.2 should be emphasized to recall that the advantage of the new method is deeply related to the property of the quasi-Monte Carlo method.

Appendix: The fifth-order and the seventh-order Runge-Kutta algorithms

We give the concrete algorithms of the explicit fifth and seventh order Runge-Kutta methods applied in subsection 3.2. The fifth order method is taken from [1] as follows:

$$a_{21} = \frac{2}{5}, \quad a_{31} = \frac{11}{64}, \quad a_{32} = \frac{5}{64}, \quad a_{43} = \frac{1}{2}, \quad a_{51} = \frac{3}{64}, \quad a_{52} = -\frac{15}{64},$$
$$a_{53} = \frac{3}{8}, \quad a_{54} = \frac{9}{16}, \quad a_{62} = \frac{5}{7}, \quad a_{63} = \frac{6}{7}, \quad a_{64} = -\frac{12}{7}, \quad a_{65} = \frac{8}{7},$$
$$a_{ij} = 0 \quad \text{otherwise},$$
$$b = \left(\frac{7}{90} \quad 0 \quad \frac{32}{90} \quad \frac{12}{90} \quad \frac{32}{90} \quad \frac{7}{90}\right).$$

The seventh order method is taken from [2] as follows:

$$a_{21} = \frac{1}{6}, \quad a_{32} = \frac{1}{3}, \quad a_{41} = \frac{1}{8}, \quad a_{43} = \frac{3}{8}, \quad a_{51} = \frac{148}{1331}, \quad a_{53} = \frac{150}{1331}, \quad a_{54} = -\frac{56}{1331},$$
$$a_{61} = -\frac{404}{243}, \quad a_{63} = -\frac{170}{27}, \quad a_{64} = \frac{4024}{1701}, \quad a_{65} = \frac{10648}{1701}, \quad a_{71} = \frac{2466}{2401}, \quad a_{73} = \frac{1242}{343},$$
$$a_{74} = -\frac{19176}{16807}, \quad a_{75} = -\frac{51909}{16807}, \quad a_{76} = \frac{1053}{2401}, \quad a_{81} = \frac{5}{154}, \quad a_{84} = \frac{96}{539}, \quad a_{85} = -\frac{1815}{20384},$$
$$a_{86} = -\frac{405}{2464}, \quad a_{87} = \frac{49}{1144}, \quad a_{91} = -\frac{113}{32}, \quad a_{93} = -\frac{195}{22}, \quad a_{94} = \frac{32}{7}, \quad a_{95} = \frac{29403}{3584},$$
$$a_{96} = -\frac{729}{512}, \quad a_{97} = \frac{1029}{1408}, \quad a_{98} = \frac{21}{16}, \quad a_{ij} = 0 \quad \text{otherwise},$$
$$b = \left(0 \quad 0 \quad 0 \quad \frac{32}{105} \quad \frac{1771561}{6289920} \quad \frac{243}{1560} \quad \frac{16807}{74880} \quad \frac{77}{1440} \quad \frac{11}{70}\right).$$

References

1. John C. Butcher, *The Numerical Analysis of Ordinary Differential Equations*, John Wiley & Sons, Chichester, 1987. 4, 9
2. John C. Butcher, *Numerical Methods for Ordinary Differential Equations*, John Wiley & Sons, Chichester, 2003. 4, 9
3. F. Castell, *Asymptotic expansion of stochastic flows*, Probability theory and related fields **96(2)** (1993), 225–239. 5
4. William Feller, *Two singular diffusion problems*, Annals of Mathematics **54** (1951), 173–182. 6
5. Paul Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer-Verlag, 2004. 6
6. Peter E. Kloeden and Eckhard Platen, *Numerical Solution of Stochastic Differential Equations*, Springer, 1999. 5

7. Shigeo Kusuoka, *Approximation of Expectation of Diffusion Process and Mathematical Finance*, Advanced Studies in Pure Mathematics, Proceedings of Final Taniguchi Symposium, Nara 1998 (T. Sunada, ed.), vol. 31, 2001, pp. 147–165. 1, 4, 5
8. ———, *Approximation of Expactation of Diffusion Processes based on Lie Algebra and Malliavin Calculs*, Advances in Mathematical Economics **6** (2004), 69–83. 4
9. ———, *Kusuoka Scheme and Gaussian type approximation*, Presentation at "Mathematical Finance Seminar in Graduate School of Mathematical Sciences The University of Tokyo (1/June/2005)" (2005). 1, 4
10. Shigeo Kusuoka, Mariko Ninomiya, and Syoiti Ninomiya, *A new weak approximation scheme of stochastic differential equations by using the Runge-Kutta method*, Preprint (2007). 1
11. Shigeo Kusuoka and Syoiti Ninomiya, *A new simulation method of diffusion processes applied to Finance*, Stochastic processes and application to mathematical finance, Proceedings of the Ritsumeikan International Symposium (J. Akahori, S. Ogawa, and S. Watanabe, eds.), World Scientific, 2004, pp. 233–253. 5
12. X. Q. Liu and C. W. Li, *Weak approximation and extrapolations of stochastic differential equations with jumps*, SIAM Journal on Numerical Analysis **37** (2000), 1747–1767. 5
13. Terry Lyons and Nicolas Victoir, *Cubature on Wiener Space*, Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **460** (2004), 169–198. 5
14. Harald Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM, 1992. 5
15. Syoiti Ninomiya, *A partial sampling method applied to the Kusuoka approximation*, Monte Carlo Methods and Applications **9** (2003), 27–38. 5
16. Syoiti Ninomiya and Nicolas Victoir, *Weak Approximation of Stochastic Differential Equations and Application to Derivative Pricing*, preprint: `arXiv:math.PR/0605361` (2006). 1, 5, 6
17. Mariko Shimizu, *Application of the Kusuoka approximation with Tree Based Branching Algorithm to pricing interest-rate derivatives with the HJM model*, Master thesis Imperial College of Scienece, Technology, and Medicine (2002). 5
18. Denis Talay, *Second-order discretization schemes of stochastic differential systems for the computation of the invariant law*, Stochastics and Stochastics Reports **29** (1990), 13–36. 5
19. ———, *Simulation of Stochastic Differential Systems*, Probabilistic Methods in Applied Physics, LNP 451 (P. Kree and W. Wedig, eds.), Springer-Verlag, 1995, pp. 54–96. 5
20. Denis Talay and Luciano Tubaro, *Expansion of the global error for numerical schemes solving Stochastic Differential Equations*, Stochastic Analysis and Applications **8** (1990), 483–509. 5

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