Gap option pricing in exponential Lévy models

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Premia 18

1 Exponential Lévy models

We consider the following model for the stock price:

$$S_t = S_0 e^{rt+X_t},$$

where $\{X_t\}_{t\geq 0}$ is a *Lévy process*. The characteristic function $\phi_t(z) = \mathbb{E}[\exp(izX_t)]$ of a Lévy process has the form $\phi_t(z) = \exp\{t\psi(z)\}$ with

$$\psi(z) = \frac{\sigma^2 z^2}{2} + i\gamma z + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx1_{|x|\leq 1})\nu(\mathrm{d}x).$$

It is determined by three parameters called the *Lévy triplet* of X:

- $\sigma \geq 0$: volatility of the diffusion part
- $\gamma \in \mathbb{R}$: drift
- $\nu(dx)$: positive measure on $\mathbb{R} \setminus \{0\}$ (*Lévy measure*)

The Lévy measure has to satisfy the following integrability conditions:

$$\int_{-1}^{+1} x^2 \nu(\mathrm{d}x) < \infty, \qquad \int_{|x|>1} \nu(\mathrm{d}x) < \infty.$$

In the context of option pricing, due to the martingale condition on the discounted price $e^{-rt}S_t$, ν satisfies in addition

$$\int_{|x|>1} e^x \nu(\mathrm{d}x) < \infty,$$

and the drift is determined by the other parameters:

$$\gamma = -\frac{\sigma^2}{2} - \int_{-\infty}^{\infty} (e^x - 1 - x1_{|x|\leq 1})\nu(\mathrm{d}x).$$

One distinguishes two types of exponential Lévy models: the so-called *jump-diffusion models*, with $\sigma > 0$ and $\nu(\mathbb{R}) < \infty$, and *pure jump models*, where $\sigma = 0$ and $\nu(\mathbb{R}) = \infty$. We give below examples of such models used in financial literature.

1.1 Jump-diffusion models

Merton model
The Lévy measure in this model has a gaussian density:

$$\nu(x) = \lambda \frac{e^{-(x-\mu)^2/2\delta^2}}{\sqrt{2\pi\delta^2}}.$$

Here λ is the jump intensity, μ the average jump size, and δ the standard variation of jump sizes.

Double exponential model (Kou)
In this model, the jumps have an asymmetric exponential distribution:

$$\nu_0(x) = p\lambda\lambda_+ e^{-\lambda_+ x} 1_{x>0} + (1-p)\lambda\lambda_- e^{-\lambda_-|x|} 1_{x<0}.$$

Here λ is the jump intensity, parameters $\lambda_- > 0$ and $\lambda_+ > 1$ control the decrease of the distribution tails of, respectively, negative and positive jumps, and p is the probability of a positive jump.

1.2 Pure jump models

Variance Gamma model
The Lévy measure of a Variance Gamma process X_t has a density given by:

$$\nu(x) = \frac{1}{\kappa|x|} e^{Ax - B|x|} \qquad \text{with} \quad A = \frac{\theta}{\sigma^2} \quad \text{and} \quad B = \frac{\sqrt{\theta^2 + 2\sigma^2/\kappa}}{\sigma^2}.$$

The characteristic function of X_t is equal to

$$\phi_t(u) = e^{itu\gamma}(1 + \frac{u^2\sigma^2\kappa}{2} - i\theta\kappa u)^{-\frac{t}{\kappa}}, \qquad \text{with} \quad \gamma = \frac{1}{\kappa}\log(1 - \frac{\sigma^2\kappa}{2} - \theta\kappa).$$

Normal inverse gaussian models (NIG)
The Lévy density in this model is given by

$$\nu(x) = \frac{C}{|x|} e^{Ax} K_1(B|x|)$$

with

$$C = \frac{\sqrt{\theta^2 + \sigma^2/\kappa}}{\pi\sigma\sqrt{\kappa}}, \qquad A = \frac{\theta}{\sigma^2}, \qquad \text{and} \quad B = \frac{\sqrt{\theta^2 + \sigma^2/\kappa}}{\sigma^2},$$

where K_1 is the modified Bessel function of second kind. Note that the asymptotic behaviour of K_1 in zero implies that

$$\nu(x) \sim \frac{1}{|x|^2} \qquad \text{as} \quad x \to 0.$$

Let us make a computational remark: if Ax is large, the exponential in the expression of ν may lead to an overflow. To avoid this, we use the asymptotic behaviour of K_1 for large arguments:

$$K_1(x) \approx \frac{\pi}{\sqrt{2\pi x}} e^{-x} \qquad \text{as} \quad x >> 1.$$

We then obtain the following approximation[1] of the Lévy density:

$$\nu(x) \approx \frac{C}{|x|} \frac{\pi}{\sqrt{2\pi B|x|}} e^{Ax - B|x|}.$$

Tempered stable models
These models are also known as CGMY or KoBoL. The Lévy density of a tempered stable process has the following expression:

$$\nu(x) = \frac{c_-}{|x|^{1+\alpha_-}} e^{-\lambda_-|x|} 1_{x<0} + \frac{c_+}{|x|^{1+\alpha_+}} e^{-\lambda_+ x} 1_{x>0},$$

with $c_\pm > 0$, $\lambda_- > 0$, $\lambda_+ > 1$, and $0 < \alpha_\pm < 2$.

A detailed presentation of exponential Lévy models and their properties can be found in [2]. The moments of X_1 can be easily calculated in all models mentioned.

$$E_+ = \begin{cases} \Gamma(-\alpha_+)\lambda_+^{\alpha_+} c_+ \left(\left(1 - \frac{1}{\lambda_+}\right)^{\alpha_+} - 1 + \frac{\alpha_+}{\lambda_+} \right), & \text{si } \alpha_+ \neq 1, \\ c_+ \left[(\lambda_+ - 1) \log\left(1 - \frac{1}{\lambda_+}\right) + 1 \right], & \text{si } \alpha_+ = 1, \end{cases} \tag{1}$$

$$E_- = \begin{cases} \Gamma(-\alpha_-)\lambda_-^{\alpha_-} c_- \left(\left(1 + \frac{1}{\lambda_-}\right)^{\alpha_-} - 1 - \frac{\alpha_-}{\lambda_-} \right), & \text{si } \alpha_- \neq 1, \\ c_- \left[(\lambda_- + 1) \log\left(1 + \frac{1}{\lambda_-}\right) - 1 \right], & \text{si } \alpha_- = 1. \end{cases} \tag{2}$$

2 Gap options

We consider a gap option (described in [5]) with maturity T subdivided onto N periods of length Δ (e.g. days): $T = N\Delta$. The return of the k-th period will be denoted by $R_k^\Delta = \frac{S_{k\Delta}}{S_{(k-1)\Delta}}$. The interest rate is deterministic and equal to r.

[1] In practice, we use it when $Ax > 600$.

Table 1: The average and the variance of X_1 in some exponential Lévy models.

model	$\mathbb{E}X_1$	$VarX_1$
Merton	$-\sigma^2/2-\lambda(\exp(\mu+\delta^2/2)-1-\mu)$	$\sigma^2+\lambda(\delta^2+\mu^2)$
Kou	$-\sigma^2/2-\lambda\left(\frac{p}{\lambda_+(\lambda_+-1)}+\frac{1-p}{\lambda_-(\lambda_-+1)}\right)$	$\sigma^2+\lambda\left(\frac{p}{\lambda_+^2}+\frac{1-p}{\lambda_-^2}\right)$
VG	$\theta+\log(1-\sigma^2\kappa/2-\theta\kappa)/\kappa$	$\sigma^2+\theta^2\kappa$
NIG	$\theta+(\sqrt{1-\sigma^2\kappa-2\theta\kappa}-1)/\kappa$	$\sigma^2+\theta^2\kappa$
Temp. Stable	$-E_+-E_-$, see (1)–(2) below	$\frac{c_+\Gamma(2-\alpha_+)}{\lambda_+^{2-\alpha_+}}+\frac{c_-\Gamma(2-\alpha_-)}{\lambda_-^{2-\alpha_-}}$

Definition 1 (Gap Option)**.** *Let α denote the return level which triggers the gap event and k^* be the time of first gap expressed in the units of Δ : $k^* := \inf\{k : R_k^\Delta \le \alpha\}$. The gap option is an option which pays to its holder the amount $f(R_{k^*}^\Delta)$ at time Δk^*, if $k^* \le N$ and nothing otherwise.*

Proposition 1. *Let the log-returns $(R_k^\Delta)_{k=1}^N$ be i.i.d. and denote the distribution of $\log(R_1^\Delta)$ by $p_\Delta(\mathrm{d}x)$. Then the price of a gap option as of previous definition is given by*

$$G_\Delta = e^{-r\Delta}\left(\int_{-\infty}^{\beta} f(e^x)p_\Delta(\mathrm{d}x)\right)\frac{1-e^{-rT}\left(\int_\beta^{+\infty}p_\Delta(\mathrm{d}x)\right)^N}{1-e^{-r\Delta}\int_\beta^{+\infty}p_\Delta(\mathrm{d}x)}, \tag{3}$$

with $\beta=\log(\alpha)<0$.

Numerical evaluation of prices formula (3) allows to compute gap option prices by Fourier inversion. For this, we need to be able to evaluate the cumulative distribution function

$$F_\Delta(x)=\int_{-\infty}^{x}p_\Delta(\mathrm{d}\xi) \tag{4}$$

and the integral

$$\int_{-\infty}^{\beta} f(e^x)p_\Delta(\mathrm{d}x). \tag{5}$$

Let ϕ_Δ be the characteristic function of p_Δ, and suppose that p_Δ satisfies

$$\int_{-\infty}^{+\infty}|x|p_\Delta(\mathrm{d}x)<+\infty \quad \text{and} \quad \int_{-\infty}^{+\infty}\frac{|\phi_\Delta(u)|}{1+|u|}\mathrm{d}u<+\infty. \tag{6}$$

Let F_G^Σ be the CDF and ϕ_G^Σ the characteristic function of a Gaussian random variable with zero mean and standard deviation $\Sigma>0$. Then

$$F_\Delta(x)=F_G^\Sigma(x)+\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{-iux}\frac{\phi_G^\Sigma(u)-\phi_\Delta(u)}{iu}\mathrm{d}u. \tag{7}$$

The Gaussian random variable is only needed to obtain an integrable expression in the right hand side and can be replaced by any other well-behaved random variable. The integral (5) is nothing but the price of a European option with payoff function f and maturity Δ. For arbitrary f it can be evaluated using the Fourier transform method proposed by Lewis. However, in practice, the pay-off of a gap option is either a put option or a put spread. Therefore, for most practical purposes it is sufficient to compute this integral for $f(x) = (K - x)^+$, in which case a simpler method can be used. The price of such a put option with log forward moneyness $k = \log(K/S_0) - r\Delta$ is given by

$$P_\Delta = P_{BS}^{\Sigma}(k) + \frac{S_0}{2\pi}\int_{-\infty}^{+\infty} e^{-ivk}\zeta_\Delta(v)\mathrm{d}v, \tag{8}$$

where

$$\zeta_\Delta(v) = \frac{\phi_\Delta(v-i) - \phi_{BS}^{\Sigma}(v-i)}{iv(1+iv)}, \quad \phi_{BS}^{\Sigma}(v) = \exp\left(-\frac{\Sigma^2 T}{2}(v^2 + iv)\right) \tag{9}$$

and $P_{BS}^{\Sigma}(k)$ is the price of a put option with log-moneyness k and time to maturity Δ in the Black-Scholes model with volatility $\Sigma > 0$.

2.1 Approximation of small jumps

In the case $\nu(\mathbb{R}) = \infty$, the small jumps of a Lévy process generate a behavior similar to that of a brownian motion. This led to the idea to replace the jumps smaller than some $\varepsilon > 0$ by a Wiener process with the same variance:

$$\sigma^2(\varepsilon) = \int_{-\varepsilon}^{\varepsilon} x^2\nu(\mathrm{d}x). \tag{10}$$

Remark that in models with jumps of infinite activity ($\nu(\mathbb{R}) = \infty$) we have $\sigma = 0$. By approximating the small jumps, we get a non-zero diffusion component $\sigma^2(\varepsilon)$ which has a regularizing effect on the solution. It makes the numerical solution easier. Clearly, if the model is of jump-diffusion type (e.g. Merton or Kou), this approximation is not needed.

2.2 Truncation of large jumps

We cannot calculate numerically an integral on the infinite range $(-\infty, \infty)$, so we have to truncate this domain to a bounded interval (B_l, B_r). In terms of the process, this corresponds to truncate the large jumps. Usually, the tails of ν decrease exponentially, so the probability of large jumps is very small. Therefore, we don't change much the solution by truncating the tails of ν. The rigorous proof of the validity of such approximation is given in [6, 3]. In practice, we fix some level of tolerance (e.g. 10^{-5}) and truncate the values of ν which are smaller than this level ($\nu(x) < 10^{-5}$).

2.3 Pricing via Fourier transform

One deterministic approach to pricing European options in exponential Lévy models was proposed by Carr and Madan [1]. They use Fourier transform and, in particular, the Fast Fourier transform algorithm. We present here a slightly improved version of their method proposed in [4, 2].

Let $\{X_t\}_{t\geq 0}$ be a Lévy process. To compute the price of a call option

$$C(k) = e^{-rT}E[(e^{rT+X_T} - e^k)^+],$$

we would like to express its Fourier transform in log strike in terms of the characteristic function $\Phi_T(v)$ of X_T and then find the prices for a range of strikes by Fourier inversion. However we cannot do this directly because $C(k)$ is not integrable (it tends to 1 as k goes to $-\infty$). The key idea is to instead compute the Fourier transform of the (modified) time value of the option, that is, the function

$$z_T(k) = e^{-rT}E[(e^{rT+X_T} - e^k)^+] - (1 - e^{k-rT})^+. \tag{11}$$

Proposition 2 (Carr and Madan [1])**.** *Let $\{X_t\}_{t\geq 0}$ be a real-valued Lévy process satisfying the martingale condition, such that*

$$E[e^{(1+\alpha)X_t}] < \infty \tag{12}$$

for some $\alpha > 0$. Then the Fourier transform in log-strike k of the time value of a call option is given by:

$$\zeta_T(v) := \int_{-\infty}^{+\infty} e^{ivk} z_T(k)dk = e^{ivrT}\frac{\Phi_T(v-i) - 1}{iv(1+iv)}. \tag{13}$$

Remark 1. *Since typically $\Phi_T(z) \to 0$ as $\Re z \to \infty$, $\zeta_T(v)$ will behave like $|v|^{-2}$ at infinity which means that the truncation error in the numerical evaluation of the inverse Fourier transform will be large. The reason of such a slow convergence is that the time value (11) is not smooth; therefore its Fourier transform does not decay sufficiently fast at infinity. For most models the convergence can be improved by replacing the time value with a smooth function of strike. Instead of subtracting the (non-differentiable) intrinsic value of the option from its price, we suggest to subtract the Black-Scholes call price with a non-zero volatility (which is a smooth function). The resulting function will be both integrable and smooth. Suppose that hypothesis (12) is satisfied and denote*

$$\tilde{z}_T(k) = e^{-rT}E[(e^{rT+X_T} - e^k)^+] - C^{\Sigma}_{BS}(k),$$

where $C^{\Sigma}_{BS}(k)$ is the Black-Scholes price of a call option with volatility Σ and log-strike k for the same underlying value and the same interest rate. Proposition 2 then implies that the Fourier transform of $\tilde{z}_T(k)$, denoted by $\tilde{\zeta}_T(v)$, satisfies

$$\tilde{\zeta}_T(v) = e^{ivrT}\frac{\Phi_T(v-i) - \Phi^{\Sigma}_T(v-i)}{iv(1+iv)}, \tag{14}$$

where $\Phi_T^\Sigma(v) = \exp(-\frac{\Sigma^2 T}{2}(v^2+iv))$. *Since for most exp-Lévy models found in the literature (except Variance Gamma) the characteristic function decays faster than every power of its argument at infinity, this means that the expression (14) will also decay faster than every power of* v *as* $\Re v \to \infty$, *and the truncation error in the numerical evaluation of the inverse Fourier transform will be very small for* every $\Sigma > 0$.[2]

Numerical Fourier inversion. Option prices can be computed by evaluating numerically the inverse Fourier transform of $\tilde{\zeta}_T$:

$$\tilde{z}_T(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ivk}\tilde{\zeta}_T(v)dv. \tag{15}$$

This integral can be efficiently computed for a range of strikes using the Fast Fourier Transform. Recall that this algorithm allows to calculate the discrete Fourier transform $\mathrm{DFT}[f]_{n=0}^{N-1}$, defined by,

$$\mathrm{DFT}[f]_n := \sum_{k=0}^{N-1} f_k e^{-2\pi i nk/N}, \quad n = 0\ldots N-1, \tag{16}$$

using only $O(N\log N)$ operations.

To approximate option prices, we truncate and discretize the integral (15) as follows:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\tilde{\zeta}_T(v)dv &= \frac{1}{2\pi}\int_{-A/2}^{A/2} e^{-ivk}\tilde{\zeta}_T(v)dv + \varepsilon_T \\ &= \frac{L}{2\pi(N-1)}\sum_{m=0}^{N-1} w_m\tilde{\zeta}_T(v_m)e^{-ikv_m} + \varepsilon_T + \varepsilon_D, \end{aligned}\tag{17}$$

where ε_T is the truncation error, ε_D is the discretization error, $v_m = -A/2 + mh$, $h = A/(N-1)$ is the discretization step and w_m are weights, corresponding to the chosen integration rule (for instance, for the Simpson's rule $w_0 = 1/3$, and for $k = 1,\ldots,N/2$, $w_{2k-1} = 4/3$ and $w_{2k} = 2/3$).[3] Now, choosing $k_n = k_0 + \frac{2\pi n}{Nh}$ we see that the sum in the last term becomes a discrete Fourier transform:

$$\begin{aligned}&\frac{A}{2\pi(N-1)}e^{ik_nL/2}\sum_{m=0}^{N-1} w_m\tilde{\zeta}_T(k_m)e^{-ik_0mh}e^{-2\pi inm/N} \\ &\qquad = \frac{A}{2\pi(N-1)}e^{ik_nL/2}\mathrm{DFT}_n[w_m\tilde{\zeta}_T(k_m)e^{-ik_0mh}]\end{aligned}$$

Therefore, the DFT algorithm allows to compute $\tilde{z}_T$ and option prices for the log strikes $k_n = k_0 + \frac{2\pi n}{Nh}$. The log strikes are thus equidistant with the step d

[2] The convergence of $\tilde{\zeta}_T$ to zero is faster than exponential for all values of Σ and it is particularly good for the value of Σ for which $\tilde{\zeta}(0) = 0$.

[3] We use the FFT with $N = 2^p$, so N is even.

satisfying

$$dh = \frac{2\pi}{N}.$$

This relationship implies that if we want to compute option prices on a fine grid of strikes, and at the same time keep the discretization error low, we must use a large number of points.

3 Description of the algorithm

3.1 Variables

We give here a list of the variables used in the algorithm with their default values.

```
// Parameters for the asset.
double S0 = 100.0; // Asset price
double Rate = 10;
double r = log(1+Rate/100.0);
double divid = 0.0; // Dividend

// Parameters for the gap option.
double T = 1.; // Maturity
int numberperiod = 252;
double deltagap = T/numberperiod;
double strike = 90; // Strike. It should be less than S0.
double alphagap = 0.9; // strike/S0. It should be between 0 and 1.
double betagap = log(alphagap);
```

Then we give a list of parameters used for the Lévy processes.

```
// Parameters for the Merton model.
double sigmaMerton = 0.20;
double lambdaMerton = 0.1;
double muMerton = 0.0;
double deltaMerton = 0.16;

// Parameters for the Kou model.
double sigmaKou = 0.3;
double lambdaKou = 7.0;
double lambdapKou = 50.0;
double lambdamKou = 25.0;
double probaKou = 0.6;

// Parameters for the VG model.
double sigmaVG = 0.12;
```

```
double thetaVG = -0.33;
double kappaVG = 0.16;

// Parameters for the NIG model.
double sigmaNIG = 0.12;
double thetaNIG = -0.33;
double kappaNIG = 0.16;
```

Moreover in the pricing algorithm, we need to truncate and discretize the integrals (7) and (8). In this aim, we use the following parameters.

```
int Nlimit =  2048;  // Number of integral discretization steps.
double logstrikestep = 0.01;
// log strikes are equidistant with the step d=logstrikestep.
double k0 = log(strike/S0);
double h = 2*M_PI/Nlimit/logstrikestep; // Integral discretization step.
double A = (Nlimit-1)*h; // Integration domain is (-A/2,A/2).
```

where `M_PI` is the constant π.

3.2 Functions

All the functions have the same form, except for the parameters of the associated Lévy process. The output price and delta are obviously `*ptprice` and `*ptdelta`. The parameter `*p` contains the parameters of the payoff in the gap option (i.e. the parameter `strike`).

```
int Ap_Gap_Merton(
double S0,NumFunc_1 *p,
int numberperiod,double alpha_gap,double T,double r,double divid,
double sigma,double lambda,double mu,double deltaVol,
double *ptprice,double *ptdelta) :

int Ap_Gap_Kou(
double S0,NumFunc_1 *p,
int numberperiod,double alpha_gap,double T,double r,double divid,
double sigma,double lambda,double mu,double deltaVol,
double *ptprice,double *ptdelta) :

int Ap_Gap_VG(
double S0,NumFunc_1 *p,
int numberperiod,double alpha_gap,double T,double r,double divid,
double theta, double sigma, double kappa,
double *ptprice,double *ptdelta) :

int Ap_Gap_NIG(
double S0,NumFunc_1 *p,
```

```
int numberperiod,double alpha_gap,double T,double r,double divid,
double theta, double sigma, double kappa,
double *ptprice,double *ptdelta) :
```

3.3 Complete algorithm

Here is a complete description of the algorithm. We use the Lévy measures defined in `levy.h` and the DFT implementation in `fft.h`.

- We declare all the variables used in the algorithm.
- We compute the Black-Scholes part used for regularization.
- We construct a Lévy measure object which contains a closed form of the characteristic function.
- We perform three integration loops with Simpson's rule and we define corresponding arrays.
- We perform the discrete fast Fourier transform of arrays previously defined.
- We compute (4), (5) and the delta of (5) by adding the Black-Scholes part.
- We compute the price of the gap option and its delta.

4 Conclusion

The computation of a price is extremely fast thanks to the discrete fast Fourier transform when the parameter `Nlimit` is of order 10^3. The parameter `Nlimit` could be certainly chosen larger, but in practice 2048 is enough. Nevertheless, for some Lévy parameters, the integral could be miscomputed. In particular, this is the case when Δ tends to zero, because the characteristic function ϕ_Δ decays slowly at infinity, which means that A must be sufficiently big. Thus `Nlimit` must be big and the computation of the integrals will be costly. In the Kou model, an approximated formula would be performed as soon as $r\Delta < 10^{-4}$.

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