# Two Alternatives to SABR

PREMIA Documentation

The article [3] contains two new stochastic–volatility models ALSABR1 and ALSABR2 in which option prices for European plain vanilla options have closed–form expressions. The models are motivated by the well-known SABR model, see [1], but use modified dynamics of the underlying asset.

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## 1 First Model: ALSABR1

The discounted asset price process is given by $S_t = Y_t^{\frac{1}{\gamma}} = \left(\sigma_t^2 z_t\right)^{\frac{1}{\gamma}}$, where $z$ and $\sigma$ are diffusions satisfying

$$dz = (a_1 - a_2 z)dt + 2\sqrt{z}dW, \qquad d\sigma = \eta\sigma dB,$$

where $0 < \eta$ and $0 < \gamma < 2$ are constants and $W$ and $B$ are two independent Brownian motions. The constants $a_1$ and $a_2$ are given by $a_1 = 2(\gamma - 1)/\gamma$, $a_2 = (2-\gamma)\eta^2/\gamma$, values which make $S$ a martingale. The process $z$ is a CIR process. If $a_1 < 0$ the process $z$ will hit 0 almost surely. In that case the stopped process is considered where the stopping time is the first hitting time of 0. The transition density of the process $z$ is well-known. For $t < T$, we define $c := 2a_2(4(1 - \exp(-a_2(T-t))))^{-1}$, $u := cz_t \exp(-a_2(T-t))$, $v := cz_T$, $q := a_1/2 - 1$. Then given $z_t$, $z_T$ is distributed as $\frac{1}{2c}$ times a noncentral $\chi^2$ random variable with $a_1$ degrees of freedom and noncentrality parameter $2u$. For $a_1 > 0$ the transition density from $z_t$ to $z_T$ is given by

$$p(z_t, z_T) = c\exp(-u-v)\left(\frac{v}{u}\right)^{q/2} I_q(2\sqrt{uv}),$$

where $I_q(\cdot)$ denotes the modified Bessel function of the first kind of order $q$. If $a_1 < 0$ the CIR process will hit 0 a.s. and it is assumed that it is then absorbed at 0. Hence the distribution of $z$ has point mass $\Delta := 1 - \int_0^\infty p(z_t, z)dz > 0$ at zero. For $a_1 < 0$ the transition density is given by

$$p(z_t, z_T) = c\exp(-u-v)\left(\frac{v}{u}\right)^{q/2} I_{|q|}(2\sqrt{uv}).$$

Suppose $S_t = \sigma_t^{\frac{2}{\gamma}} z_t^{\frac{1}{\gamma}}$. Let $r$ denote the interest rate and $\tilde{S}_t := e^{rt}S_t$ is the underlying asset price. Then the time–0–prices of a European put option $P^{SV1}$

and of a European call option $C^{SV1}$ with expiry $T$ and strike price $K$ are given by

$$P^{SV1}(S_0,T,K,r,\eta,z_0,\gamma) = \mathbb{E}\left[(e^{-rT}K - S_T)^+\right] = \int_0^\infty h_1(z)p_T(z)dz + \mathbb{I}_{\{a_1<0\}}\Delta e^{-rT}K,$$
$$C^{SV1}(S_0,T,K,r,\eta,z_0,\gamma) = \mathbb{E}\left[(S_T - e^{-rT}K)^+\right] = \int_0^\infty h_2(z)p_T(z)dz,$$

where

$$h_1(z) := e^{-rT}K\Phi(-d_2) - \sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}} \exp\left(\frac{\eta^2 T}{\gamma}\left(\frac{2}{\gamma}-1\right)\right)\Phi\left(-d_1\right),$$
$$h_2(z) := \sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}} \exp\left(\frac{\eta^2 T}{\gamma}\left(\frac{2}{\gamma}-1\right)\right)\Phi\left(d_1\right) - e^{-rT}K\Phi(d_2).$$

Here

$$d_1 := d_2 + \frac{2\eta}{\gamma}\sqrt{T}, \quad d_2 := \frac{\gamma}{2\eta\sqrt{T}}\left(\log\left(\frac{\sigma_0^{\frac{2}{\gamma}} z^{\frac{1}{\gamma}}}{e^{-rT}K}\right) - \frac{\eta^2}{\gamma}T\right),$$

$\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and $p_T(z) := p(z_0, z_T)$ is the probability density function of the non–central $\chi^2$ distribution as specified before.

## 2 Second Model: ALSABR2

The discounted stock price $S$ is a product of a geometric Brownian motion and a *general* function of a CIR process: $S_t = \sigma_t g(z_t)$, where $\sigma$ is a geometric Brownian motion and $z$ is a CIR–process, i.e.

$$d\sigma = \sigma(\mu dt + \eta dB), \qquad dz = (a_1 - a_2 z)dt + 2\sqrt{z}dW.$$

The two Brownian motions $B$ and $W$ are assumed to be independent. The function $g$ solves the following second order ODE

$$2zg''(z) + (a_1 - a_2 z)g'(z) + \mu g(z) = 0. \tag{1}$$

The ODE (1) can be solved in terms of the Kummer functions $M(\cdot,\cdot,\cdot)$ and $U(\cdot,\cdot,\cdot)$, see [2]. A solution $g$ to the ODE (1) is given by

$$g(z,a_1,a_2,\mu) = C_1 M\left(-\frac{\mu}{a_2}, \frac{a_1}{2}, \frac{a_2 z}{2}\right) + C_2 U\left(-\frac{\mu}{a_2}, \frac{a_1}{2}, \frac{a_2 z}{2}\right), \tag{2}$$

for some constants $C_1, C_2$. It is required that $\mu < 0$ and $a_1 > 2, a_2 > 0$. With this choice of parameters the CIR–process $z$ stays away from zero. For $\mu < 0$ the Kummer function $M$ stays positive. In the empirical analysis one needs to choose the constants $C_1, C_2$ such that the asset price stays positive. With this choice of $g$ the process $S_t = \sigma_t g(z_t)$ is a martingale. Let $r$ denote the interest rate and $\tilde{S}_t := e^{rt}S_t$ is the underlying asset price process. Then the time–0–prices of a European put option $P^{SV2}$ and of a European call option $C^{SV2}$ with expiry $T$ and strike price $K$ are given by

$$P^{SV2}(S_0, T, K, r, a_1, a_2, z_0, \mu, \eta) = \mathbb{E}\left[(e^{-rT}K - S_T)^+\right] = \int_0^\infty \tilde{h}_1(z) p_T(z) dz$$

$$C^{SV2}(S_0, T, K, r, a_1, a_2, z_0, \mu, \eta) = \mathbb{E}\left[(S_T - e^{-rT}K)^+\right] = \int_0^\infty \tilde{h}_2(z) p_T(z) dz,$$

where

$$\tilde{h}_1(z) := e^{-rT} K \Phi(-\tilde{d}_2) - \sigma_0 g(z) e^{\mu T} \Phi\left(-\tilde{d}_1\right),$$
$$\tilde{h}_2(z) := \sigma_0 g(z) e^{\mu T} \Phi\left(\tilde{d}_1\right) - e^{-rT} K \Phi(\tilde{d}_2).$$

Here

$$\tilde{d}_1 = \frac{1}{\eta\sqrt{T}}\left(\log\left(\frac{\sigma_0 g(z) e^{\mu T}}{K}\right) + \left(r + \frac{\eta^2}{2}\right) T\right), \qquad \tilde{d}_2 = \tilde{d}_1 - \eta\sqrt{T},$$

and $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution and $p_T(z) := p(z_0, z_T)$ is again the probability density function of the non–central $\chi^2$ distribution.

## References

[1] Patrick S. Hagan, Deep Kumar, Andrew S. Lesniewski, and Diana E. Woodward. Managing smile risk. *Wilmott Magazine*, 2002. 1

[2] M.ABRAMOWITZ I.A.STEGUN, editor. *Handbook of Mathematical Functions*. Dover, 9th edition, 1970. 2

[3] L. C. G. Rogers and L. A. M. Veraart. A stochastic volatility alternative to SABR. *Journal of Applied Probability*, 45(4):1071–1085, 2008. 1