Stein's method and Zero Transformation for CDOs tranche pricing

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The following note on Stein's method and Zero Transformation for CDOs tranche pricing is based on the papers [El karoui and Jiao(2008)] and [El Karoui et al.(2008)El Karoui, Jiao, and Kurtz].

1 CDOs tranche pricing

We consider a synthetic CDO with some given maturity T. This is based upon N CDS with nominals $N_j, j = 1, \ldots, N$ of maturity T. We denote by R_j the *recovery rate* for credit j and by $M_j = (1 - R_j)N_j$ the corresponding *loss given default*.

For the N names in the collateral pool, we consider the associated default times $\tau_1, \ldots, \tau_N$ defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P})$.

The critical issue for CDO pricing is to compute the value $\mathbb{E}(L(t) - K)_+$, where $L(t)$ is the portfolio aggregate loss on the credit portfolio at time t:

$$L(t) = \sum_{j=1}^{N} M_j \mathbf{1}_{\{\tau_j \leq t\}} = \sum_{j=1}^{N} (1 - R_j) N_j \mathbf{1}_{\{\tau_j \leq t\}}$$

which is a pure jump process. This distribution depends on the joint distribution of the default times $\tau_1, \ldots, \tau_N$ modelised by a classical factor approach and Copula functions.

1.1 Factor approach

In the following, we only consider reduced-form models of default times defined by

$$\tau_i = \inf\Big\{u \in \mathbb{R}^+, \int_0^u h_i(v)dv \geq -\log(U_i)\Big\}, \qquad (H_\tau)$$

where the h_i are deterministic and continuous positive functions, the U_i are some uniform random variables.

We denote $F_1, \ldots, F_N$ the *marginal* distribution functions. From (H_τ), we get

$$F_i(t) = \mathbb{P}(\tau_i \leq t) = 1 - \exp(-\int_0^t h_i(v)dv).$$

1.2 Correlation between the default times

In order to value a CDO, we have to define the correlation between the default events $\tau_1, \ldots, \tau_N$. We will assume that conditionnally on a risk factor V, the default indicators of the names are independent. Suppose in addition that R_i's are random variables mutually independent, then $(L(t)-k)_+$ is a function of a sum of conditionally independent random variables and its expectation can be deduced in two steps.

1. We compute the conditional expectation $H(V) := \mathbb{E}((L(t)-K)_+|V)$,
2. We compute $\mathbb{E}(L(t)-K)_+ = \int_{\mathbb{R}} H(v)p_V(v)dv$, where $p_V(v)$ is the probability density of V.

To describe the default correlation structure in the factor model framework, we need to specify the conditional distribution of each default τ_i. We denote $p_t^{i|V} = \mathbb{P}(\tau_i \leq t|V)$.

Example 1 (Gaussian Copula)**.** We consider a standard Gaussian random variable V, and we define the Gaussian vector $(X_1, \ldots, X_n)$ by

$$X_i = \rho V + \sqrt{1-\rho^2}V_i$$

where V_i are independent ($\forall i, j, V_i \perp V_j$ and $\forall i, V_i \perp V$) standard Gaussian random variables. The uniform random variable U_i appearing in (H_τ) is defined by $U_i := 1-\mathcal{N}(X_i)$ where $\mathcal{N}$ is the cumulative distribution function of a standard Gaussian variable. We get

$$p_t^{i|V} = \mathcal{N}\Big(\frac{\mathcal{N}^{-1}(F_i(t)) - \rho V}{\sqrt{1-\rho^2}}\Big).$$

with $F_i(t) = 1 - \exp(-\int_0^t h_i(v)dv)$.

2 Stein's method and Zero Transformation

We define the default correlation by using the factor model where the default events are supposed to be conditionally independent given a common factor V. Hence, conditionally on V, the cumulative loss $L(t)$ can be written as a sum of independent random variables. The sum of independent random variables may converge to Gauss or Poisson distributions. The Poisson approximation is known to be robust for small probabilities in the approximation of binomial laws. One usually asserts that the normal distribution remains robust when $np \geq 10$. If np is small, the binomial law approaches a Poisson law. In our case, the size of the portfolio is fixed : $n \sim 125$. On the other hand, in the credit context, the default probabilities $p(V)$ take values in $(0,1)$ according to the explicit form w.r.t. the factor V. Hence we may encounter both cases. Stein's method is an efficient tool to estimate the approximation errors in the limit theorem problems. By combining Stein'smethod and the zero bias transformation, [El karoui and Jiao(2008)] provide first order correction terms for both Gauss and Poisson approximations. We present their main results in the following sections.

2.1 First order Gaussian correction of conditional Losses

We recall the explicit form of the corrector term in the particular case of the call function. For more details on the general case, we refer to [El karoui and Jiao(2008), Theorem 3.1].

Theorem 1. *Let $X_1, \cdots, X_n$ be independent zero-mean r.v. such that $\mathbb{E}[X_i^4]$ exists. Let $W = X_1 + \cdots + X_n$ and $\sigma_W^2 = Var[W]$. Then, the Gaussian approximation $\mathcal{N}_{\sigma_W}((x-k)_+)$ of $\mathbb{E}[(W-k)_+]$ has corrector*

$$C^{\mathcal{P}} = \frac{\mu_{(3)}}{6\sigma_W^2} k \mathcal{N}_{\sigma_W}(k)$$

where $\mu_{(3)} = \sum_{i=1}^n \mathbb{E}[X_i^3]$, $\mathcal{N}_\sigma(x)$ is the density function of the distribution $\mathcal{N}(0, \sigma^2)$ and $\mathcal{N}_\sigma(h) := \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} h(u) \exp\left(-\frac{u^2}{2\sigma^2}\right) du$. The corrected approximation error is bounded by

$$\left|\mathbb{E}[(W-k)_+] - \mathcal{N}_{\sigma_W}((x-k)_+) - C^{\mathcal{P}}\right| \le \alpha(k, X_1, \cdots, X_n)$$

where α depends on k and on the moments of X_i up to fourth order.

2.2 First order Poisson correction of conditional Losses

We recall the explicit form of the corrector term in the particular case of the call function. For more details on the general case, we refer to [El karoui and Jiao(2008), Theorem 4.2].

Theorem 2. *Let $Y_1, \cdots, Y_n$ be independent r.v. taking non-negative integer values such that $\mathbb{E}[Y_i^3]$ exists. Let $W = Y_1 + \cdots + Y_n$ with expectation $\lambda_W = \mathbb{E}[W]$ and variance $\sigma_W^2 = Var[W]$. Then, the Poisson approximation $\mathcal{P}_{\lambda_W}((x-k)_+)$ of $\mathbb{E}[(W-k)_+]$ has corrector*

$$C^{\mathcal{P}} = \frac{(\sigma_W^2 - \lambda_W)e^{-\lambda_W}}{2} \left((k - \lfloor k \rfloor) \frac{\lambda_W^{\lfloor k \rfloor}}{\lfloor k \rfloor !} + (\lfloor k \rfloor - k + 1) \frac{\lambda_W^{\lfloor k \rfloor - 1}}{(\lfloor k \rfloor - 1)!} \right).$$

where $\mathcal{P}_{\lambda_W}(h) = \mathbb{E}[h(\Lambda)]$ with $\Lambda \sim \mathcal{P}(\lambda)$. The corrected approximation error is bounded by

$$\left|\mathbb{E}[(W-k)_+] - \mathcal{P}_{\lambda_W}((x-k)_+) - C^{\mathcal{P}}\right| \le \beta(k, Y_1, \cdots, Y_n)$$

where β depends on k and on the moments of Y_i up to third order.

3 Application to the computation of the conditional loss

We now combine the Gauss and Poisson approximations proposed above to calculate the conditional expectations. We want to approximate

$$\mathbb{E}[(L(t)-K)^+|V] = \mathbb{E}[(\sum_{j=1}^{N} N_j(1-R_j)\mathbf{1}_{\{\tau_j\le t\}} - K)^+|V].$$

If the R_j are stochastic, we can only use the Gaussian approximation. Since the summand variables in the Poisson approximation take integers values, the recovery rates R_i's are limited to be identical or proportional constants. Let us first deal with the case of deterministic recovery rates.

3.1 Gaussian approximation with deterministic recovery rates

To apply Theorem 1, we first need to centralize the summand variables. We define $\xi_j := N_j(1-R_j)\mathbf{1}_{\{\tau_j\le t\}}$ and μ_j^V and σ_j^V denote its conditional expectation and standard deviation (w.r.t. V) respectively. Let $X_j := \xi_j - \mu_j^V$ and $W = \sum_{j=1}^N X_j$. Clearly, W is of mean zero and its variance is $(\sigma_W^V)^2 = \sum_{j=1}^N (\sigma_j^V)^2$. Moreover, we have $\mu_j^V = N_j(1-R_j)p_t^{j|V}$ and $\sigma_j^V = N_j(1-R_j)\sqrt{p_t^{j|V}(1-p_t^{j|V})}$. Then

$$\mathbb{E}[(L(t)-K)^+|V] = \mathbb{E}[(W-K^V)^+|V],$$

where $K^V = K - \sum_{j=1}^N \mu_j^V$. Theorem 1 yields

$$\mathbb{E}[(L(t)-K)^+|V] \sim \mathcal{N}_{\sigma_W}((x-K^V)_+) + \frac{1}{6\sigma_W^2}\sum_{i=1}^{N}\mathbb{E}[X_j^3|V]K^V\mathcal{N}_{\sigma_W}(K^V), \tag{1}$$

where $\mathbb{E}[X_j^3|V] = N_j^3(1-R_j)^3 p_t^{j|V}(1-p_t^{j|V})(1-2p_t^{j|V})$.

3.2 Poisson approximation in the homogeneous case

Poisson approximation can only be used in the homogeneous case: $R_j = R$ and $N_j = \frac{1}{N}$ for all j. In homogene condition, we get $L(t) = \frac{1-R}{N}\sum_{j=1}^N \mathbf{1}_{\{\tau_j\le t\}}$, and

$$\mathbb{E}[(L(t)-K)^+|V] = \frac{1-R}{N}\mathbb{E}[(W-m)^+|V],$$

where $W = \sum_{j=1}^{N} \mathbf{1}_{\{\tau_j \le t\}}$ and $m = KN/(1-R)$. Theorem 2 yields

$$\mathbb{E}[(L(t)-K)^{+}|V] \sim \mathcal{P}_{\lambda_W^V}((x-k)_+) - C^{\mathcal{P}},$$

where $\mathcal{P}_{\lambda_W^V}((x-k)_+) = \sum_{l=1}^{N} \frac{(\lambda_W^V)^l}{l!} e^{-\lambda_W^V}(l-m)^+$ and $\lambda_W^V = \sum_{j=1}^{N} p_t^{j|V}$. The correction for the Poisson approximation is

$$C^{\mathcal{P}} = \frac{((\sigma_W^V)^2 - \lambda_W^V)e^{-\lambda_W^V}}{2}\left((m-\lfloor m\rfloor)\frac{(\lambda_W^V)^{\lfloor m\rfloor}}{\lfloor m\rfloor !} + (\lfloor m\rfloor - m + 1)\frac{(\lambda_W^V)^{\lfloor m\rfloor - 1}}{(\lfloor m\rfloor - 1)!}\right),$$

where $(\sigma_W^V)^2 = p_t^{j|V}(1-p_t^{j|V})$.

3.3 Gaussian approximation with stochastic recovery rates

In such a case, we assume that R_j is independent of τ_j. We define $\xi_j := \mathbf{1}_{\{\tau_j \le t\}}$ and μ_{R_j}, $\sigma^2_{R_j}$ and $\gamma^3_{R_j}$ are the first three centered moments of the r.v. R_j. We also define $X_j = N^{-1}(1-R_j)\xi_j - \mu_j^V$ where $\mu_j^V = N^{-1}(1-\mu_{R_j})p_t^{j|V}$. We introduce $W = \sum_{j=1}^{N} X_j$. As in the deterministic case, we get the approximation (1), where σ_W and $\mathbb{E}[X_j^3|V]$ depend on the first three centered moments of the r.v. R_j. For more details, we refer to [El Karoui et al.(2008)El Karoui, Jiao, and Kurtz, Section 3.3].

4 Practical Implementation

We recall that the computation of $\mathbb{E}((L(t)-K)_+)$ is done in two steps:

1. We compute the conditional expectation $H(V) := \mathbb{E}((L(t)-K)_+|V)$,
2. We compute $\mathbb{E}(L(t)-K)_+ = \int_{\mathbb{R}} H(v)p_V(v)dv$, where $p_V(v)$ is the probability density of V.

4.1 Methodology

Practically, the methodology is the following

1. One choose a Copula, i.e. we fix the law of the risk factor V. We denote V_{inf} and V_{sup} the numerical lower and upper bounds of V. (For example, if $V \sim \mathcal{N}(0,1)$, we take $V_{inf} = -6$ and $V_{sup} = 6$).
2. For M values of v uniformly distributed on $[V_{inf}, V_{sup}]$ (for example $M = 200$), we compute $H(v)p_V(v)$ (See below for an explanation of the computation of $H(v)$).
3. We use a Riemann summation to get an approximation of $\int_{\mathbb{R}} H(v)p_V(v)dv$.

The computation of $H(v)$ depends on the case we consider

- Inhomogeneous case (i.e. there exist i and j s.t. $R_j \neq R_i$ (or $N_i \neq N_j$)) with deterministic recovery rates: we use the Gaussian approximation of Section 3.1
- Homogeneous or inhomogeneous case with stochastic recovery rates: we use the Gaussian approximation of Section 3.3
- Homogeneous case with deterministic recovery rates: we use either the Gaussian approximation of Section 3.1 or the Poisson approximation of Section 3.2 depending on the value of $\sum_{j=1}^{N} p_t^{j|v}$. We use the Gaussian approximation when $\sum_{j=1}^{N} p_t^{j|v} > 10$, otherwise we use the Poisson approximation.

4.2 Parameters with Nsp

When we load the Nsp software, we have to fill in the values of the parameters. The connection with our notations is the following

- Number of companies : N
- Intensity : h_i function (default value $h_i(v) = 0.01$). To use an inhomogeneous intensity, fill the file `cdo_intensity.dat`
- Nominals : N_j (if homogeneous, $N_j = \frac{1}{N}$, if not, fill the file `cdo_nominal.dat`)
- Type of recovery : concerns R_j (default value $R_j = R = 0.4$). Otherwise one can choose a random recovery.

Concerning the Copula, one can choose one type of the following table.

Type	Parameters (example value)
Gaussian	Correlation ρ (0.03)
Clayton	θ (0.2)
NIG	Correlation ρ (0.06), α (1.2), β (-0.2)
Student	Correlation ρ (0.02), Degree of freedom $t1$ (5)
Double t	Correlation ρ (0.03), Degree of freedom $t1$ (5), Degree of freedom $t2$ (7)

References

[El karoui and Jiao(2008)] N. El karoui and Y. Jiao. Stein's method and zero bias transformation for cdo tranche pricing. *Finance and Stochastics*, 2008. doi: http://dx.doi.org/10.1007/s00780-008-0084-6. URL `http://dx.doi.org/10.1007/s00780-008-0084-6`. 1, 3

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