Source | Model | Option
| Model_Option | Help on mc methods | Archived Tests

mc_fixedasian_kemnavorst

Input parameters:

- Time StepNumber M
- Generator_Type
- Number of iterations N
- Scheme
- Confidence Value α
- Delta relative increment

Output parameters:

- Price P
- Error Price σ_P
- Delta δ
- Error delta σ_δ
- Price Confidence Interval: $IC_P =$[Inf Price, Sup Price]
- Delta Confidence Interval: $IC_\delta =$[Inf Delta, Sup Delta]

Description:
Computation for a Asian Call or Put Fixed European Option of its Price and its Delta with the Control Variable of Kemna-Vorst Monte Carlo or Quasi-Monte Carlo simulation. In the case of Monte Carlo simulation, the method also provides an estimation for the integration error and a confidence interval.

For a best understanding of Asian option and a detailed description of the notations, we refer the reader to the general part about options ?????????.
Simulation of an Asian option is not obvious because we need to generate the mean of the underlying asset over a given period. Explanations about this point are described in the next points. You can read the part on simulation of random variables????????/ for a more complete presentation about simulation of Brownian trajectory.
Quasi Monte Carlo simulation is available for this options, but some restrictions appear: we need multidimensional low-discrepancy sequences and for some of them (like Sobol for instance) we are limited in practice with their dimension. See the implemented part for low-discrepancy sequences.

The underlying asset price evolves according to the Black and Scholes model, that is:

$$dS_u = S_u((r-d)du + \sigma dB_u), \quad S_{T_0} = s$$

then

$$S_T = s\exp\left((r-d-\frac{\sigma^2}{2})(T-T_0)\right)\exp(\sigma B_{T-T_0})$$

where S_T denotes the spot at maturity T, s is the initial spot, T_0 is the initial time.

The Price of a Fixed Asian option at t is:

$$P_t = e^{-r(T-t)}E\left[f(K, A(t_0, T))\right]$$

where f denotes the payoff of the option, K the strike and $A(t_0, T)$ the mean of the price of the underlying asset over a given period $[t_0, T]$.
We have

$$A(t_0, T) = \frac{1}{T-t_0}\int_{t_0}^{T} S_u du$$

The Delta is given by:

$$\delta = e^{-r(T-t)}\frac{\partial}{\partial s}E[f(K, A(t_0, T))]$$

Estimators are expressed as:

$$\widetilde{P} = \frac{1}{N}e^{-r(T-t)}\sum_{i=1}^{N} P(i)$$

$$\widetilde{\delta} = \frac{1}{N}e^{-r(T-t)}\sum_{i=1}^{N}\frac{\partial}{\partial s}P(i) = \frac{1}{N}e^{-r(T-t)}\sum_{i=1}^{N}\delta(i)$$

The values for $P(i)$ and $\delta(i)$ are detailed for each option by using the Kemna-Vorst simulation method.

- **Fixed Asian Call**: The payoff is $(A(t_0,T)-K)^+$.
  - Case $t_0 \leq T_0$:
  We decompose $A(t_0,T)$ over $[t_0,T_0]$ and $[T_0,T]$. Then we have:
  $$E\left[(A(t_0,T)-K)^+\right] = E\left[(A'(T_0,T)-K')^+\right]$$
  with $K' = K - \frac{T_0-t_0}{T-t_0}A(t_0,T_0)$
  and $A'(T_0,T) = \frac{1}{T-T_0}\int_{T_0}^{T} S'_u du$ with $S'_u = \frac{T-T_0}{T-t_0}S_u$.
  K' is named the pseudo strike and S' the pseudo spot.

  If $K' \leq 0$ then we obtain:
  $$E\left[(A(t_0,T)(i)-K)^+\right] = E\left[A'(T_0,T)-K'\right]$$
  for which we know an analytic formula.
  In this case we have:
  $$P_t = e^{-r(T-t)}\left[\frac{s}{(r-d)(T-t_0)}(e^{(r-d)(T-T_0)}-1)-K'\right]$$
  and
  $$\delta = \frac{e^{-r(T-t)}}{(r-d)(T-t_0)}(e^{(r-d)(T-T_0)}-1)$$
  For the particular case $r=d$, we have:
  $$P_t = e^{-r(T-t)}(s'-K')$$
  $$\delta = e^{-r(T-t)}\frac{T-T_0}{T-t_0}$$
  Else we cannot suppress the $(.)^+$ and we have to realize a Monte Carlo Simulation to compute the price. The simulation method deals with control variate, due to Kemna and Vorst ([2]).

- **Fixed Asian Put**: The payoff is $(K-A(t_0,T))^+$.
  - Case $t_0 \leq T_0$:
  With the same decomposition as for a Call, we find the following expression:
  $$E\left[(K-A(t_0,T))^+\right] = E\left[(K'-A'(T_0,T))^+\right]$$

If $K' \leq 0$ then we obtain:

$$E\left[(K - A(t_0,T)(i))^+\right] = E\left[0\right]$$

In this case we have:

$$P_t = 0$$

$$\delta = 0$$

Else we cannot suppress the $(.)^+$ and we realize a Monte Carlo Simulation with Kemna and Vorst method ([2]).

Kemna & Vorst Method [2], [1].

• Call case

We detail the presentation for a Call option. Case of the Put is similar.

Kemna-Vorst method uses a control variate $C'(T_0,T)$ in the expression of the mean of the option. We define

$$C'(T_0,T) = \exp\left(\frac{1}{T-T_0}\int_{T_0}^{T}\log(S'_u)du\right)$$

Then we can write:

$$\begin{aligned} P_t &= e^{-r(T-t)}E\left[(A'(T_0,T)-K')^+ - (C'(T_0,T)-K')^+\right] \\ &\quad + e^{-r(T-t)}E\left[(C'(T_0,T)-K')^+\right] \\ &= e^{-r(T-t)}(R_t + Q_t) \end{aligned}$$

- For the term Q_t there is an explicit formula expressed as a Black-Scholes formula.

$$Q_t = s'\exp\left((r-d-\frac{\sigma^2}{6})\frac{T-T_0}{2}\right)N(d_1) - K'N(d_2)$$

with

$$d_2 = \frac{1}{\sigma\sqrt{\frac{T-T_0}{3}}}\left(\log(\frac{s'}{K'}) + (r-d-\frac{\sigma^2}{2})\frac{T-T_0}{2}\right)$$

and

$$d_1 = d_2 + \sigma\sqrt{\frac{T-T_0}{3}}$$

Proof:

$$\begin{aligned} Q_t &= E\left[(C'(T_0,T)-K')^+\right] \\ &= E\left[C'(T_0,T)1_{\{C'>K'\}}\right] - K'P(C'(T_0,T)>K') \end{aligned}$$

$C'(T_0,T)$ can be expressed as:

$$C'(T_0,T) = s' \exp\left((r-d-\frac{\sigma^2}{2})\frac{T-T_0}{2}\right)\exp\left(\frac{\sigma}{T-T_0}\int_0^{T-T_0} B_u du\right)$$

then we obtain:

$$P(C'(T_0,T)>K') = N(d_2)$$

If we note $X_{T-T_0} = \frac{\sqrt{3}}{T-T_0}\int_0^{T-T_0} B_u du$, X_{T-T_0} follows the $N(0,T-T_0)$ law. And under the probability P^* such that

$$\frac{dP^*}{dP} = \exp\left(\frac{\sigma}{\sqrt{3}}X_{T-T_0} - \frac{\sigma^2}{6}(T-T_0)\right)$$

we can calculate

$$\begin{aligned} E\left[C'(T_0,T)1_{\{C'>K'\}}\right] &= E^*\left[s'\exp\left((r-d-\tfrac{\sigma^2}{6})\tfrac{T-T_0}{2}\right)1_{\{C'>K'\}}\right] \\ &= s'\exp\left((r-d-\tfrac{\sigma^2}{6})\tfrac{T-T_0}{2}\right)P^*(C'>K') \\ &= s'\exp\left((r-d-\tfrac{\sigma^2}{6})\tfrac{T-T_0}{2}\right)N(d_1) \end{aligned}$$

Also we have an analytic formula for $\frac{\partial Q_t}{\partial s}$:

$$\frac{\partial Q_t}{\partial s} = \frac{T_0-t_0}{T-t_0}\exp\left((r-d-\frac{\sigma^2}{6})\frac{T-T_0}{2}\right)N(d_1)$$

This expression will be used for the Delta estimation.

- For the term R_t there is no explicit formula. This term is estimated by standard Monte Carlo simulation.
At each step of the simulation, we compute

$$\begin{aligned} R_t(i) &= R_1(i) - R_2(i) \\ &= (A'(T_0,T)(i)-K')^+ - (C'(T_0,T)(i)-K')^+ \end{aligned}$$

from a generation of $A'(T_0,T)(i)$ and $C'(T_0,T)(i)$.
Procedure to simulate these parameters is described in a next point (A' simulation and C' simulation).

- Estimations

For the Price and the Delta we have:

$$\widetilde{P}_t = \frac{1}{N} e^{-r(T-t)} \sum_{i=1}^{N} R_t(i) + e^{-r(T-t)} Q_t$$

$$\widetilde{\delta} = \frac{1}{N} e^{-r(T-t)} \sum_{i=1}^{N} \widetilde{\delta}(i) + e^{-r(T-t)} \frac{\partial Q_t}{\partial s}$$

where $\widetilde{\delta}(i) = \frac{\partial R_1(i)}{\partial s} - \frac{\partial R_2(i)}{\partial s}$

$$\frac{\partial R_1(i)}{\partial s} = \begin{cases} \frac{\partial A'(T_0,T)(i)}{\partial s} & \text{if} \quad R_1(i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial R_2(i)}{\partial s} = \begin{cases} \frac{\partial C'(T_0,T)(i)}{\partial s} & \text{if} \quad R_2(i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

- **Put case**

In the case of a Put, we obtain the following formula:
The explicit price for Q_t is given by

$$Q = K' N(-d_2) - s' \exp\left((r - d - \frac{\sigma^2}{6}) \frac{T - T_0}{2} \right) N(-d_1)$$

At each step of the simulation, we compute

$$\begin{aligned} R_t(i) &= R_1(i) - R_2(i) \\ &= (K' - A'(T_0,T)(i))^+ - (K' - C'(T_0,T)(i))^+ \end{aligned}$$

For the Delta, we have: $\widetilde{\delta}(i) = \frac{\partial R_1(i)}{\partial s} - \frac{\partial R_2(i)}{\partial s}$

$$\frac{\partial R_1(i)}{\partial s} = \begin{cases} -\frac{\partial A'(T_0,T)(i)}{\partial s} & \text{if} \quad R_1(i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\frac{\partial R_2(i)}{\partial s} = \begin{cases} -\frac{\partial C'(T_0,T)(i)}{\partial s} & \text{if} \quad R_2(i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Simulation of the mean $A'(T_0, T)$

The simulation is done with one of the three schemes: Rieman sums, Trapezoidal method or Brownian Bridge method. Description of the three schemes.

Simulation of $C'(T_0, T)$

As for the mean $A'(T_0, T)$ we use one of the three proposed schemes to simulate the variable $C'(T_0, T)$. To improve the efficiency of the variable control method, we have to generate the same brownian motions as for $A'(T_0, T)$. In this case we obtain higher correlations between $A'(T_0, T)$ and $C'(T_0, T)$. It leads to a better variance reduction.

$$\begin{aligned} C'(T_0,T) &= \exp\left(\tfrac{1}{T-T_0}\int_{T_0}^{T}\log(S'_u)du\right) \\ &= s'\exp\left((r-d-\tfrac{\sigma^2}{2})\tfrac{T-T_0}{2}\right)\exp\left(\tfrac{\sigma}{T-T_0}\int_0^{T-T_0}B_u du\right) \end{aligned}$$

We need to simulate

$$Y = \exp\left(\frac{\sigma}{T-T_0}\int_0^{T-T_0}B_u du\right)$$

We obtain the following approximation for the three schemes

- Scheme 1: Rieman sums

$$\widetilde{Y} = \exp\left(\frac{\sigma}{M}\sum_{k=1}^{M}B_{t_k}\right)$$

- Scheme 2: Trapezoidal method

$$\widetilde{Y} = \exp\left(\frac{\sigma}{M}\sum_{k=1}^{M}\frac{B_{t_k}+B_{t_{k+1}}}{2}\right)$$

- Scheme 3: Brownian bridge method

$$\widetilde{Y} = \exp\left(\frac{\sigma}{M}\sum_{k=1}^{M}X_k\right)$$

  where X_k is generated according the law of $\frac{1}{h}\int_{t_k}^{t_{k+1}}B_u du\backslash B_{t_k}, B_{t_{k+1}}$, that is $N(\frac{B_{t_k}+B_{t_{k+1}}}{2}, \frac{h}{6})$.

Finally, we have

$$C'(T_0,T)(i) = s'\exp\left((r-d-\frac{\sigma^2}{2})\frac{T-T_0}{2}\right)\widetilde{Y}$$

Algorithm:

♣ /* Function : AnalyticKemnaVorst */
Analytic formula for an asian option if $K' \leq 0$.
- /* Put Case */

- /* Call case */
/* Case $r = d$ */
/* Case $r \neq d$ */

♣ /* Function : SimulStockAndAverageKemnaVorst */
Computation of the averages $A'(t_0, T)(i)$ and $C'(t_0, T)(i)$ according to the selected scheme for each step of the Monte Carlo simulation with the same brownian motions.
This function is called from the "FixedAsianKemanVorst" function.
/*Initialisation*/

- /* Average Computation */
For the 3 schemes, we need values of B_{t_k} for each of the M times t_k defined on $[T_0, T]$.
We compute $B_{t_{k+1}}(i) = B_{t_k}(i) + \sqrt{h} g_k(i)$, where $g_k(i)$ are independant standard gaussian variables.
Call to the appropriate function to generate independent standard gaussian variables. See the part about simulation of random variables for explanations on this point. We just recall that for a MC simulation, we use the Gauss-Abramovitz algorithm, and for a QMC simulation we use an inverse method and a M or $2M$-dimensional low-discrepancy sequence.
We have $S_{t_k}(i) = s \exp\left((r - d - \frac{\sigma^2}{2})(t_k - T_0)\right) \exp(\sigma B_{t_k}(i))$
And then we use the specific formula for each scheme to estimate $A(t_0, T)(i)$ and $C'(t_0, T)(i)$.
/* Scheme 1 : Rieman sums */
/* Scheme 2 : Trapezoidal method */
/* Simulation of M gaussian variables according to the generator type, that is Monte Carlo or Quasi Monte Carlo. */
For the two first schemes, we need to generate M independent gaussian variables. We keep them in a table.
/* Gaussian value from the table Gaussians */
At each step, we take the next gaussian value in the table.

/* Scheme 3 : Brownian Bridge method */
/* Simulation of $2M$ gaussian variables according to the generator type, that is Monte Carlo or Quasi Monte Carlo. */
/* Gaussian value from the table Gaussians */
For the third scheme, we need to generate $2M$ independent gaussian variables.
In fact at each step, a second standard gaussian variable $g'_k(i)$ is required to

simulate $\left(\frac{1}{h}\int_{t_k}^{t_{k+1}} B_u du \backslash B_{t_k}, B_{t_{k+1}}\right)$ as $\left(\frac{B_{t_k}+B_{t_{k+1}}}{2} + \sqrt{\frac{h}{6}} g_k'(i)\right)$.

- /* Final average $A'(T_0, T)$ */

- /* Final average $C'(T_0, T)$ */

♣ /* Function : FixedAsianKemnaVorst */
Main function to realize the Monte Carlo simulation with Kemna and Vorst method for an Asian option.
Parameters s et K are pseudo-spot and pseudo-strike. */
/* Value to construct the confidence interval */
For example if the confidence value is equal to 95% then the value z_α used to construct the confidence interval is 1.96. This parameter is taken into account only for MC simulation and not for QMC simulation.
/*Initialisation*/
/* Size of the random vector we need in the simulation */
For each of the three schemes, we need a vector of size M (or $2M$ for the third scheme) of independent gaussian variables to simulate the Brownian trajectory. In case of QMC simulation, it involves that we need a M or $2M$-dimensional low-discrepancy sequence.

• /* Computation of the price and the delta for term Q with the control variate */
Black and Scholes formula
/* Put case */
/* Call case */

• /*MC sampling*/
/* Test after initialization for the generator */
Test if the dimension of the simulation is compatible with the selected generator. In this case, we need a vector of size M or $2M$. Some low discrepancy sequences don't work with a so large dimension. See the part on the implementation of low-discrepancy sequences, you can find for each sequence the maximal dimension allowed in our implementation.
Definition of a parameter which exprimes if we realize a MC or QMC simulation. Some differences then appear in the algorithm for simulation of a gaussian variable and in results in the simulation.

/* Begin N iterations */

- /*Price*/

At the iteration i, we obtain $A'(t_0,T)(i)$ and $C'(t_0,T)(i)$ from the function 'SimulStockAndAverage'. And we compute:

$$\begin{aligned} R(i) &= \text{Payoff}(A'(t_0,T)(i)+K') - \text{Payoff}(C'(t_0,T)(i)+K') \\ &= R_1(i) - R_2(i) \end{aligned}$$

In order to obtain the delta, we compute the price for the stock incremented $s*(1+inc)$ and $s*(1-inc)$. Therefore the partial derivative will be the difference divided by the increment.

- /*Delta*/

Calculation of Delta δ_i with formula for a Call:

$$\begin{aligned} \delta(i) = & \frac{(A'_{s*(1+inc)}(T_0,T) - A'_{s*(1-inc)}(T_0,T))(i)1_{R_1(i)>0}}{2*s*inc} \\ & \qquad - \frac{(C'_{s*(1+inc)}(T_0,T)(i) - C'_{s*(1-inc)}(T_0,T)(i))1_{R_2(i)>0}}{2*s*inc} \end{aligned}$$

/*Sum*/

Computation of the sums $\sum R_i$ and $\sum \delta_i$ for the mean price and the mean delta.

/*Sum of squares*/

Computation of the sums $\sum R_i^2$ and $\sum \delta_i^2$ necessary for the variance price and the variance delta estimations. (finally only used for MC estimation)

/* End N iterations */

- /*Price*/

The price estimator R is:

$$R = \frac{1}{N}\sum_{i=1}^{N} R(i)$$

The error estimator is σ_P with :

$$\sigma_P^2 = \frac{e^{-2r(T-t)}}{N-1}\left(\frac{1}{N}\sum_{i=1}^{N} R(i)^2 - R^2\right)$$

Final price estimator is:

$$P = e^{-r(T-t)}\left[R+Q\right]$$

- /* Price Confidence Interval */

The confidence interval is given as:

$$IC_P = [P - z_\alpha \sigma_P; P + z_\alpha \sigma_P]$$

with z_α computed from the confidence value.

- /*Delta*/

First part of the delta estimator:

$$\delta = e^{-r(T-t)} \frac{1}{N} \sum_{i=1}^{N} \delta(i)$$

/* Put Case */
For a Put, we take $-\delta$.
The error estimator is σ_δ with:

$$\sigma_\delta^2 = \frac{1}{N-1} \left(\frac{1}{N} e^{-2r(T-t)} \sum_{i=1}^{N} \delta(i)^2 - \delta^2 \right)$$

Final delta estimator is:

$$\delta = e^{-r(T-t)} \left[\frac{1}{N} \sum_{i=1}^{N} \delta(i) + \frac{\partial Q}{\partial s} \right]$$

- /* Delta Confidence Interval */

The confidence interval is given as:

$$IC_\delta = [\delta - z_\alpha \sigma_\delta; \delta + z_\alpha \sigma_\delta]$$

with z_α computed from the confidence value.

Confidence intervals are always computed, but for a QMC simulation they don't work, thus they don't appear in the results.

Variance reduction by importance sampling

For each of the three previous schemes (Rieman sums, trapezoidal method or brownian bridge method), one can also use a reduction variance method by importance sampling. Indeed, by Girsanov theorem, we have :

$$E\left[e^{-\theta W_T - \frac{\theta^2}{2}T} f(B_t + \theta t, 0 \le t \le T)\right] = E\left[f(B_t, 0 \le t \le T)\right]$$

for any function f of the trajectory of the Brownian.

The schemes are build by replacing B_t by $B_t + \theta t$ and multiplying the result by the correction factor term $e^{-\theta W_T - \frac{\theta^2}{2}T}$. We choose θ in order to get the mean of the modified asset $E[\tilde{A}(t_0, T)] = \frac{1}{T-t_0}\int_{t_0}^{T} s\exp((r-d-\frac{\sigma^2}{2})(u-t_0)+\sigma\theta u)$ equal to the strike K. We have observed that this leads to a variance reduction if the value of θ obtained is positive. Therefore, the formula we chose for θ is the following :

$$\theta = \max\left(0, \frac{1}{\sigma}\left(\frac{\sigma^2}{2} - (r-d) + 2 * (\frac{K}{s} - \frac{1}{T})\right)\right).$$

References

[1] E.TEMAM. Monte carlo methods for asian options. *preprint*, 98-144 CERMICS, 1998. 4

[2] A.G.Z KEMNA and A.C.F.VORST. A pricing method for options based on average asset values. *J. Banking Finan.*, pages 113–129, March 1990. 3, 4