Saddlepoint methods for option pricing

V.M. Zherder

February 18, 2016

1 Introduction

Premia 18

Saddlepoint methods are successfully applied in probability, complex analysis and mathematical physics for more than fifty years.

The idea of the method is to change the path of integration in a given integral of the Laplace or Fourier type to obtain better convergence. It occurs that such an optimal path should pass through the so called saddlepoint of the integrand. As soon as the optimal path is found one can either write approximate expansions or apply some numerical integration routine efficiently.

In Premia, we apply saddlepoint method to approximate European call/put prices in the Heston model (Lugannani-Rice formula, see Section 2.2, formula (12)).We also demonstrate how the saddlepoint itself can be approximated using formulae by Lieberman or Glasserman and Kim (see subsection 2.3).

In some cases (like the NIG is) the global parametrization of the optimal path can be found. In this case we come to an integral with smooth real-valued non-oscilating fast decaying integrand (see subsection 3.4, formula (39)). It provides very fast and accurate computation.

This text is organized as follows.

In Section 2 we show how the Lugannani-Rice formula may be derived and applied to European call/put pricing. To use it for the Heston model we need just to substitute Heston's cumulant generating function. We also present Lieberman's and Glasserman's formulae to approximate saddlepoint. Both formulae perform well for Heston model.

In Section 3 we discuss briefly the general framework of the saddlepoint method, then we consider the European put pricing formula in the NIG model and rewrite it in a form which is convinient for the saddlepoint method to be applied. Next, we find saddlepoint and global parametrization of the optimal path. All results in this section are specific for NIG.

2 Saddlepoint approximation

2.1 Option pricing formula

Consider european put option with strike price X, maturity T, riskless rate r, written on an asset with price S. Its price at the moment t may be written as

$$\begin{aligned} P_t &= e^{-r\tau}\mathbf{E}\left[\max(0, X - S_T)\right] \\ &= e^{\ln X - r\tau} Pr(S_T < X) - e^{-r\tau}\mathbf{E}\left[S_T | S_T < X\right] Pr(S_T < X), \end{aligned} \tag{1}$$

where $\tau = T - t$ is the time to maturity, and the expected value and probability are both conditioned on the current value S_t of the underlying.

Let $Y_t = \ln S_t$ be a diffusion process and let K denote the conditional cumulant generating function of Y_t, that is,

$$K(\tau, u|x) = \ln \mathbf{E}\left[\exp(uY_\tau)|Y_0 = x\right].$$

We assume that for each (τ, x), K exists in some interval $(-c, d)$ with $c \geq 0, d \geq 0$ and $c + d > 0$.

Notice that $\mathbf{E}\left[S_T | S_T < X\right] Pr(S_T < X) = e^{K(1)} \tilde{Pr}(S_T < X)$ where the probability $\tilde{Pr}$ is defined by

$$\tilde{\mathbf{E}}\left[\exp(uY_T)\right] = \mathbf{E}\left[\exp((u+1)Y_T)\right] \exp(-K(\tau, 1|x)).$$

The cumulant generating function of $\tilde{Pr}$ is defined by

$$\tilde{K}(u) = K(u+1) - K(1). \tag{2}$$

Therefore, provided that the function K is known we can express the option price as

$$P_t = e^{\ln X - r\tau} Pr(Y_T < \ln X) - e^{-r\tau + K(1)} \tilde{Pr}(Y_T < \ln X) \tag{3}$$

and all that remains to be done is to approximate the cumulant probabilities $Pr(Y_T < \ln X)$ and $\tilde{Pr}(Y_T < \ln X)$.

To do that, we express probabilities by the Fourier inversion formula

$$\begin{aligned} Pr(Y_T > y | Y_0 = x) &\equiv P(\tau, y|x) \\ &= (2\pi i)^{-1} \int_{\hat{u} - i\infty}^{\hat{u} + i\infty} \exp\left(K(\tau, u|x) - uy\right) \frac{du}{u}. \end{aligned} \tag{4}$$

2.2 Saddlepoint approximation. Lugannani-Rice formula.

An explicit saddlepoint approximation for the cumulative distribution function can be constructed using a method proposed by Lugannani and Rice. The idea is to find the best possible value of $\hat{u}$ in (4). Consider the following choice:

Set $\hat{u} = \hat{u}(\tau, y|x) \in \mathbf{R}$ as a solution in u of the equation

$$K'(\tau, u|x) = y, \tag{5}$$

where $K'(\tau, u|x)$ denotes the derivative of the function K with respect to u. Such a solution exists and is unique because of the convexity in u of the function K.

Write Taylor expansion of the function $u \mapsto (K(\tau, u|x) - uy)$ around its minimum $\hat{u}$:

$$K(\tau, u|x) - uy = K(\tau, \hat{u}|x) - \hat{u}y + \frac{1}{2}(u - \hat{u})^2 K''(\tau, \hat{u}|x) + O(|u - \hat{u}|^3). \tag{6}$$

On the path of integration we have $u = \hat{u} + iv$, $v \in \mathbf{R}$, hence, $u - \hat{u}$ is a purely imaginary. Thus

$$K(\tau, u|x) - uy = K(\tau, \hat{u}|x) - \hat{u}y - \frac{1}{2}v^2 K''(\tau, \hat{u}|x) + O(v^3). \tag{7}$$

Let us approximate the quadratic behavior of $K(\tau, u|x) - uy$ near $\hat{u}$ given by (6) by the same quadratic behavior of w near some $\hat{w}$. That is,

$$\{K(\tau, u|x) - uy\} - \{K(\tau, \hat{u}|x) - \hat{u}y\} = \frac{1}{2}(w - \hat{w})^2$$

for an arbitrary real $\hat{w}$. In particular, choose $\hat{w}$ to satisfy

$$K(\tau, \hat{u}|x) - \hat{u}y = -\frac{1}{2}\hat{w}^2 \tag{8}$$

from which we derive that

$$\hat{w} = \{2\,(\hat{u}y - K(\tau, \hat{u}|x))\}^{1/2}\mathrm{sgn}(\hat{u}),$$

where $\mathrm{sgn}(\hat{u})$ equals -1, 0 or 1 depending on whether $\hat{u}$ is negative, zero or positive, correspondingly. Then

$$K(\tau, \hat{u}|x) - \hat{u}y = \frac{1}{2}w^2 - w\hat{w}. \tag{9}$$

Applying this change of variable to (4) yields

$$P(\tau, y|x) = (2\pi i)^{-1} \int_{\hat{w}-i\infty}^{\hat{w}+i\infty} \exp\left(\frac{1}{2}w^2 - w\hat{w}\right) \frac{1}{u(w)} \frac{du(w)}{dw} dw \tag{10}$$

and the term $du(w)/dw$ can now be approximated near $w = \hat{w}$ as

$$\frac{du(w)}{dw} = \frac{du(\hat{w})}{dw} + O(w - \hat{w}).$$

From (9) one may find that

$$\frac{du(\hat{w})}{dw} = (K''(\tau, \hat{u}|x))^{-1/2}.$$

To avoid singularity at $u = 0$ we process (10) as follows:

$$\begin{aligned} P(\tau, y|x) &= (2\pi i)^{-1} \int_{\hat{w}-i\infty}^{\hat{w}+i\infty} \exp\left(\frac{1}{2}w^2 - w\hat{w}\right) \frac{dw}{w} \\ &+ (2\pi i)^{-1} \int_{\hat{w}-i\infty}^{\hat{w}+i\infty} \exp\left(\frac{1}{2}w^2 - w\hat{w}\right) \left(\frac{1}{u(w)} \frac{du(w)}{dw} - \frac{1}{w}\right) dw. \end{aligned} \tag{11}$$

When $y \neq K'(\tau, 0|x)$

$$\begin{aligned} \frac{1}{u(w)} \frac{du(w)}{dw} - \frac{1}{w} &= \frac{1}{\hat{u}} \frac{du(\hat{w})}{dw} - \frac{1}{\hat{w}} + O(w - \hat{w}) \\ &= \frac{1}{\hat{u}} (K''(\tau, \hat{u}|x))^{-1/2} - \frac{1}{\hat{w}} + O(w - \hat{w}), \end{aligned}$$

and the leading term of (11) is

$$\begin{aligned} P^{(0)}(\tau, y|x) &= 1 - \Phi(\hat{w}) \\ &+ (2\pi i)^{-1} \int_{\hat{w}-i\infty}^{\hat{w}+i\infty} \exp\left(\frac{1}{2}w^2 - w\hat{w}\right) dw \{\frac{1}{\hat{u}} (K''(\tau, \hat{u}|x))^{-1/2} - \frac{1}{\hat{w}}\} \\ &= 1 - \Phi(\hat{w}) + \phi(\hat{w})\{\frac{1}{\hat{u}} (K''(\tau, \hat{u}|x))^{-1/2} - \frac{1}{\hat{w}}\} \end{aligned} \tag{12}$$

where Φ is the standard normal cumulative distribution function, and ϕ is the standard normal density function.

When $y = K'(\tau, 0|x)$ we have $\hat{u} = \hat{w} = 0$ and

$$\frac{1}{u(w)} \frac{du(w)}{dw} - \frac{1}{w} = -\frac{K'''(\tau, 0|x)}{6\,[K''(\tau, 0|x)]^{3/2}} + O(w)$$

so that

$$\begin{aligned} P^{(0)}(\tau, y|x) &= 1 - \Phi(0) + \phi(0)\{-\frac{K'''(\tau, 0|x)}{6[K''(\tau, 0|x)]^{3/2}}\} \\ &= \frac{1}{2} - \frac{1}{6\sqrt{2\pi}} \frac{K'''(\tau, 0|x)}{[K''(\tau, 0|x)]^{3/2}}. \end{aligned} \tag{13}$$

Expansions of higher order may be used as well.

2.3 Approximating the saddlepoint

The key step of saddlepoint method is solving the saddlepoint equation (5). Numerical solution of the equation may require many iterations and demands time-consuming evaluations of the cumulant generating function and its derivatives op to the fourth order (and this is the casee for the option pricing in the Heston model). To avoid these difficulties one may use approximation $\tilde{u}$ instead of the true saddlepoint $\hat{u}$.

For example, Lieberman approximates saddlepoint $\hat{u}$ as a power seriesusing first four cumulants $\kappa_i,\ i = 1..4$:

$$\tilde{u}_L = \frac{y-\kappa_1}{\kappa_2} - \frac{\kappa_3}{2\kappa_2}\left(\frac{y-\kappa_1}{\kappa_2}\right)^2 + \left(\frac{\kappa_3^2}{2\kappa_2^2} - \frac{\kappa_4}{6\kappa_2}\right)\left(\frac{y-\kappa_1}{\kappa_2}\right)^3. \tag{14}$$

Glasserman and Kim propose an improvement that proceeds one more step. They expand $K'(u)$ around $u = \tilde{u}_L$ (rather than $u = 0$) to get

$$\begin{aligned} \tilde{u}_G &= \tilde{u}_L + \frac{y - K'(\tilde{u}_L)}{K''(\tilde{u}_L)} - \frac{K'''(\tilde{u}_L)}{2K''(\tilde{u}_L)}\left(\frac{y-K'(\tilde{u}_L)}{K''(\tilde{u}_L)}\right)^2 \\ &+ \left(\frac{K'''(\tilde{u}_L)^2}{2K''(\tilde{u}_L)^2} - \frac{K^{(4)}(\tilde{u}_L)}{6K''(\tilde{u}_L)}\right)\left(\frac{y-K'(\tilde{u}_L)}{K''(\tilde{u}_L)}\right)^3. \end{aligned} \tag{15}$$

One may also use $\tilde{u}_L$ or $\tilde{u}_G$ as a starting point in numerical solving the saddlepoint equation (5).

3 Saddlepoint integration. Case of the NIG model

3.1 Saddlepoint and the path of integration.

The saddlepoint method is mainly applied to the integrals of Laplace type

$$F(\lambda) = \int_\gamma e^{\lambda S(u)} g(u)\, du \tag{16}$$

where λ is a large parameter, and γ is a path in the complex plane; the functions g and S are holomorphic in the neighborhood of γ. Saddlepoint method gives a procedure for the deformation of the contour γ into a new contour γ_* which is more convenient for computation. We start with the following definitions.

Definition 3.1. *$\hat{u} \in \mathbf{C}$ is called a saddlepoint of the function $S(u)$ if $S'(\hat{u}) = 0$. A number $n \geq 1$ is called the order of the saddle point $\hat{u}$ if*

$$S'(\hat{u}) = ... = S^{(n)}(\hat{u}) = 0, \quad S^{(n+1)}(\hat{u}) \neq 0.$$

A saddle point of order 1 is called a simple saddle point.

Definition 3.2. *A simple path γ starting at the point $\hat{u}$ is called a path of steepest descent for the function $S(u)$, if for all $u \neq \hat{u}$*

$$\Im S(u) = const; \quad and \quad \Re S(u) < \Re S(\hat{u}). \tag{17}$$

Definition 3.3. *Let $F(\lambda)$ be an integral defined by (16).*

Two contours γ' and γ are said to be equivalent iff integrals $F(\lambda)$ along these contours are equal:

$$\int_{\gamma} e^{\lambda S(u)} g(u) du = \int_{\gamma'} e^{\lambda S(u)} g(u) du$$

for any $\lambda > 0$.

It is easy to show that there is the only path of steepest descent starting at the regular (not saddle) point u, and there are $n+1$ curves of steepest descent starting at the saddle point $\hat{u}$ of order n.

Example 3.1. In the simplest case $S(u) = u^2$, the surface $z = \Re S(u)$ in the space $(x = \Re u, y = \Im u, z)$ is a hyperbolic paraboloid $z = x^2 - y^2$. On this surface, there is only one point $\hat{u}$ such that $S'(\hat{u}) = 0$, and this point is a saddle point of order 1. There are two paths of steepest descent that start at $\hat{u}$ and go down different "slopes".

So, our aim is to find a new contour that satisfies the conditions (17). From all such contours the best one for our purpose is the minimizer of

$$\min_{\gamma \in \Gamma} \max_{u \in \gamma} \Re S(u). \tag{18}$$

We call the minimizer by γ_*. It can be proved (see [4]) that

(i) γ_* passes through saddle points, and

(ii) the set of saddle points and endpoints of γ_* (if any) contains all the points of maximum of $\Re S(u)$.

If the parametrization $u = u(s)$ of the contour γ_* is known, then from (17) we conclude that the integral (16) can be written as

$$F(\lambda) = e^{i\lambda \Im S(\hat{u})} \int_{-\infty}^{+\infty} e^{\lambda \phi(s)} g(u(s))\ u'(s)\ ds, \tag{19}$$

where $\phi(s)$ is a real-valued function, and $\hat{u}$ is the point of maximum of $\Re S$.

3.2 Option pricing formula in the Lèvy-based model

Assume that $Y_t = \ln S_t$ is a Lévy process under some equivalent martingale measure (EMM) $\mathbf{Q}$.

Let ψ be the characteristic exponent of Y under $\mathbf{Q}$ defined by the equation:

$$E[e^{i\xi Y_t}] = e^{-t\psi(\xi)}$$

. Since $\mathbf{Q}$ is an EMM, both the bond and the stock must be priced under $\mathbf{Q}$, and therefore ψ must admit the analytic continuation into a strip $\Im\xi \in (0,1)$, and continuous continuation into the closed strip $\Im\xi \in [0,1]$. Further, the following condition must hold

$$r + \psi(-i) = 0. \tag{20}$$

Consider a contingent claim of the European type, with the payoff $g(Y_T)$ at the expiry date T, then its price $P(Y_t, t)$ at time $t < T$ is given by

$$P(x,t) = (2\pi)^{-1}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \exp[ix\xi - \tau(r+\psi(\xi))]\hat{g}(\xi)d\xi, \tag{21}$$

where $\tau = T - t$ is the time to expiry, and

$$\hat{g}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} g(x)dx \tag{22}$$

is the Fourier transform of g. The σ is chosen so that the integral (22) absolutely converges for $\Im\xi = \sigma$, and the integral (21) absolutely converges, too.

Consider the European put option with the strike price X and the terminal date T. The payoff at the expiry date equals $\max\{X - S_t, 0\}$. Then we may write the terminal condition as follows

$$f(Y_T, T) = g(Y_T) = X(1 - e^{Y_T - \ln X})_+. \tag{23}$$

Clearly,

$$\begin{aligned}\hat{g}(\xi) &= X\int_{-\infty}^{+\infty} e^{-ix\xi}(1-e^x)_+ dx \\ &= X\frac{1}{(-i\xi)(-i\xi+1)}.\end{aligned}$$

is well-defined in the half-plane $\Im\xi = \sigma > 0$ and admits the meromorphic extension into the complex plane with two poles at $\xi = 0$ and $\xi = -i$. If

$\Re\psi(\xi)$ is bounded from below on the line $\Im\xi = \sigma$, then the integral in (21) converges. In the case of NIG, this condition is satisfied for any $\sigma \in (0, \alpha+\beta)$.

Similarly, for the European call option, we have to take $\sigma < -1$, and for the integral in (21) to be defined, we need to take $\sigma > -\alpha + \beta$. Thus, (21) is applicable if $\alpha > \beta + 1$, with any $\sigma \in (-\alpha + \beta, -1)$.

Assume that Y is a NIG, take $\sigma \in (0, \alpha + \beta)$, and write $P(x, t)$ with $x = Y_t - \ln X$ as

$$\begin{aligned} P(x,t) &= \frac{X}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \exp(ix\xi - \tau(r + \psi(\xi)))\hat{g}(\xi)d\xi \\ &= \frac{X}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[ix\xi - \tau(r + \psi(\xi))]}{-i\xi(-i\xi + 1)}d\xi \end{aligned}$$

3.3 NIG pricing formula

The characteristic exponent of NIG is given by

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{24}$$

where $\alpha > |\beta| > 0, \delta > 0, \mu \in \mathbf{R}$. Notice that ψ is holomorphic in the complex plane with cuts $[i(\alpha + \beta), +i\infty)$ and $(-i\infty, -i(\alpha - \beta)]$, and in order that the bond and the stock be priced, we need $\alpha - \beta \geq 1$. The equation (20) assumes the form

$$r - \mu + \delta[(\alpha^2 - (\beta + 1)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}] = 0. \tag{25}$$

Assume that Y is a NIG, take $\sigma \in (0, \alpha + \beta)$, and express $P(x, t)$ as

$$\begin{aligned} P(x,t) &= \frac{X}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \exp(ix\xi - \tau(r + \psi(\xi)))\hat{g}(\xi)d\xi \\ &= \frac{X}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[ix\xi - \tau(r + \psi(\xi))]}{-i\xi(-i\xi + 1)}d\xi \\ &= \frac{X}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[ix\xi - \tau r + i\tau\mu\xi - \tau\delta\sqrt{\alpha^2 - (\beta + i\xi)^2} + \tau\delta\sqrt{\alpha^2 - \beta^2}]}{-i\xi(-i\xi + 1)}d\xi. \end{aligned}$$

Thus,

$$P(x,t) = \frac{R_0}{2\pi}\int_{-\infty+i\sigma}^{+\infty+i\sigma} \frac{\exp[i\xi(x + \tau\mu) - \tau\delta\sqrt{\alpha^2 - (\beta + i\xi)^2}]}{-i\xi(-i\xi + 1)}d\xi, \tag{26}$$

with

$$R_0 = X\exp[-\tau r + \tau\delta\sqrt{\alpha^2 - \beta^2}] \tag{27}$$

By changing the variable $\xi \to \alpha\xi$, we obtain

$$P(x,t) = \frac{R_0}{2\pi\alpha}\int_{-\infty+i\sigma/\alpha}^{+\infty+i\sigma/\alpha} \frac{\exp[i\xi\alpha(x+\tau\mu) - \alpha\tau\delta\sqrt{1-(\beta/\alpha+i\xi)^2}]}{-i\xi(-i\xi+1/\alpha)}d\xi.$$

Make the further change of variable $\xi \to \xi + i\beta/\alpha$:

$$P(x,t) = \frac{R_0 e^{-\beta(x+\tau\mu)}}{2\pi\alpha}\int_{-\infty+i(\sigma-\beta)/\alpha}^{+\infty+i(\sigma-\beta)/\alpha} \frac{\exp[i\alpha(x+\tau\mu)\xi - \alpha\tau\delta\sqrt{1+\xi^2}]}{(-i\xi+\beta/\alpha)(-i\xi+(\beta+1)/\alpha)}d\xi.$$

Introduce new parameters:

$$\begin{aligned} U &= \alpha(x+\tau\mu), \\ V &= \alpha\tau\delta, \\ \rho_1 &= \beta/\alpha, \\ \rho_2 &= 1/\alpha+\rho_1, \\ \sigma_+ &= (\sigma-\beta)/\alpha \\ R_1 &= \frac{R_0 e^{-\beta(x+\tau\mu)}}{2\pi\alpha}. \end{aligned} \tag{28}$$

We have $U \in \mathbf{R},\ V > 0,\ |\rho_1| < 1,\ \sigma_+ \in (-\rho_1, 1)$,

$$R_1 = \frac{X\exp(V\sqrt{1-\rho_1^2} - U\rho_1 - \tau r)}{2\pi\alpha}, \tag{29}$$

and

$$P(x,t) = R_1\int_{-\infty+i\sigma_+}^{+\infty+i\sigma_+} \frac{\exp\left[iU\xi - V\sqrt{1+\xi^2}\right]}{(-i\xi+\rho_1)(-i\xi+\rho_2)}d\xi. \tag{30}$$

As $\xi = i\sigma + \eta$ tends to ∞ on the line $\Im\xi = \sigma_+$, the expression under the exponential sign admits the representation

$$iU\xi - V\sqrt{1+\xi^2} = -V|\eta| - U\sigma_+ + iU\eta - iV\sigma_+\eta/|\eta| + O(|\eta|^{-1}), \tag{31}$$

therefore the integral (30) converges faster if $U\sigma_+ > 0$, and the convergence can be enhanced by a good choice of σ_+. If $U > 0$, then one is tempted to choose $\sigma_+ \in (-\rho_1, 1)$.

If $U < 0$, then the integrand in (30) is meromorphic in the strip $\Im\xi \in (-1,1)$ with two poles at $\xi = -i\rho_1$ and $\xi = -i\rho_2$, therefore we can use the residue theorem and transform the line of integration into the line $\Im\xi = \sigma_-$, where $\sigma_- \in (-1, -\rho_2)$, and obtain

$$P(x,t) = X(e^{-r\tau} - e^x) + R_1\int_{-\infty+i\sigma_-}^{+\infty+i\sigma_-} \frac{\exp\left[iU\xi - V\sqrt{1+\xi^2}\right]}{(-i\xi+\rho_1)(-i\xi+\rho_2)}d\xi. \tag{32}$$

Introduce new parameters

$$W = \sqrt{U^2 + V^2},$$

$$z = \frac{U}{W}, \quad v = \frac{V}{W},$$

and functions

$$S(\xi) = iz\xi - v\sqrt{1+\xi^2},$$

$$g(\xi) = (-i\xi + \rho_1)^{-1}(-i\xi + \rho_2)^{-1}.$$

We have $z^2 + v^2 = 1$, $|z| < 1$, $0 < v < 1$ and the branch of the square root is determined by the condition

$$\Re\sqrt{1+\xi^2} > 0. \tag{33}$$

. Further, the S is holomorphic in the complex plane with cuts $(-i\infty, -i]$ and $[i, +i\infty)$, and g is meromorphic with simple poles at $\xi = -i\rho_1$, $\xi = -i\rho_2$. In the new notation,

$$P(x,t) = R_1 \int_{-\infty+i\sigma_+}^{+\infty+i\sigma_+} e^{WS(\xi)} g(\xi) d\xi. \tag{34}$$

The integral in the r. h. s. of (34) is of the Laplace type so the saddlepoint technique can be applied.

3.4 Optimal path of integration

A saddle point $\hat{u}$ solves the equation $S'(\xi) = 0$. Since

$$S'(\xi) = iz - v\frac{\xi}{\sqrt{1+\xi^2}},$$

we obtain the equation

$$iz\sqrt{1+\xi^2} = v\xi. \tag{35}$$

From (35) and (33), it follows that

$$sgn\ \Im\xi = sgn\ z. \tag{36}$$

Using the equality $z^2 + v^2 = 1$ and the condition (36), we find from (35) the unique solution

$$\hat{u} = iz.$$

We have

$$\begin{aligned} S(\hat{u}) &= iz\, iz - v\sqrt{1-z^2} \\ &= -z^2 - v^2 = -1, \end{aligned}$$

and

$$S''(\hat{u}) = -\frac{1}{(1+\hat{u}^2)^{3/2}} = -\frac{1}{v^3} \neq 0,$$

hence $\hat{u}$ is a simple saddle point.

We can find the contour of steepest descent by using the following trick. Introduce the new variable s, $s \in \mathbf{R}$, as a solution to the equation

$$S(\xi) = S(\hat{u}) - s^2,$$

this choice ensures that (17) and (18) hold and allows us to find the parametric equation for the contour of steepest descent γ_*. We have

$$i\xi z - v\sqrt{1+\xi^2} = -1 - s^2,$$

whence we find the parametrization of the optimal contour γ_*:

$$\xi(s) = v\, s\sqrt{2+s^2} + i\, z(1+s^2). \tag{37}$$

Denote by η and σ the real and imaginary parts of ξ. Then from (37) and (36), we obtain

$$v^2\,\sigma^2 \;-\; z^2\,\eta^2 \;=\; z^2\, v^2$$

with the condition

$$sgn\,\sigma \;=\; sgn\, z.$$

Thus, for $z \neq 0$ the optimal contour γ_* is a branch of the hyperbola with asymptotes $\sigma = \pm\frac{z}{v}\eta$, lying in the upper half-plane if $z > 0$ and in the lower half-plane if $z < 0$; if $z = 0$ then the optimal contour is the real axis.

Notice that in the process of the transformation the contour never reaches the cuts (since $|z| < 1$) but may cross the poles of the integrand, and consider the following cases:

1) if $-\rho_1 < z < 1$ then in the process of the deformation, the contour does not cross the poles, and we have

$$P(x,t) = R_1 e^{-W} \mathcal{I}_{\gamma_*}(W, z, v, \rho_1, \rho_2), \tag{38}$$

where

$$\mathcal{I}_{\gamma_*}(W, z, v, \rho_1, \rho_2) = \int_{-\infty}^{+\infty} \frac{\exp[-W\, s^2]}{(-i\xi(s) + \rho_1)(-i\xi(s) + \rho_2)} \xi'(s)\, ds. \tag{39}$$

2) if $z = -\rho_1$, then the optimal contour passes through the pole $\xi = -i\rho_1$, and we obtain

$$P(x,t) = 0.5Xe^{-r\tau} + R_1 e^{-W} \mathcal{I}_{\gamma_*}(W, z, v, \rho_1, \rho_2). \tag{40}$$

Here the integral $\mathcal{I}_{\gamma_*}(W, z, v, \rho_1, \rho_2)$ is understood in the sense of the principal value.

3) if $-\rho_2 < z < -\rho_1$ then in the process of the deformation, the contour crosses the pole $\xi = -i\rho_1$, and by applying the residue theorem, we obtain

$$f(x,t) = Xe^{-\tau r} + R_1 e^{-W} \mathcal{I}_{\gamma_*}(W, z, v, \rho_1, \rho_2). \tag{41}$$

4) if $-\rho_2 = z$ then in the process of the deformation, the contour crosses the pole $\xi = -i\rho_1$, and the optimal contour passes through the second pole; hence, and by applying the residue theorem, we obtain

$$P(x,t) = X(e^{-\tau r} - 0.5e^x) + R_1 e^{-W} \mathcal{I}_{\gamma_*}(W, z, v, \rho_1, \rho_2), \tag{42}$$

where the integral is understood in the sense of the principal value.

5) if $-1 < z < -\rho_2$ then in the process of the deformation, the contour crosses the both poles $\xi = -i\rho_1$ and $\xi = -i\rho_2$, and we obtain

$$P(x,t) = X(e^{-\tau r} - e^x) + R_1 e^{-W} \mathcal{I}_{\gamma_*}(W, z, v, \rho_1, \rho_2). \tag{43}$$

To compute numerically the integrals in (38)– (43), we choose an appropriate neighborhood $|s| \geq A$ of the infinity. If W is large, we integrate by part and calculate explicitly the leading term of the integral over $|x| \geq A$, up to the desired error; in the case of a small W, we obtain an explicit upper bound and show that the integral over the neighborhood of infinity can be included in the error term of the approximate computation procedure.

References

[1] Y. Aït-Sahalia and J. Yu, *Saddlepoint approximations for Continuous Time Markov Processes*, Journal of Econometrics, 134 (2006), 507–551.

[2] O.E. Barndorff-Nielsen, *Processes of Normal Inverse Gaussian Type*, Finance and Stochastics, 2 (1998b), 41–68.

[3] P. Carr and D. Madan, *Saddlepoint methods for option pricing*, Journal of Computational Finance, Volume 13 No 1, (2009), 49–61.

[4] M.V. Fedoryuk, *The saddle point method*, Moscow, Nauka 1988 (In Russian) 6

[5] P.Glasserman and K.K.Kim, *Saddlepoint approximations for Affine Jump-Diffusion Models*, Journal of Economic Dynamics and Control, vol 33 (2009), 37-52.

[6] F.W.J. Olver *Asymptotics and Special Functions.* New York, London, Academic Press (1974)

[7] L.C.G. Rogers and O. Zane, *Saddlepoint approximations to option prices*, Annals of applied probability, 9 (1999), 493–503.