CDO Pricing method for affine point Hawkes processes [1]

Short abstract

This work presents a numerical method for the model given by Errais, Goldberg and Giesecke [1] to price credit derivatives. Indeed, they analyze a family of multivariate point process models of correlated event timing whose arrival intensity is driven by an affine jump diffusion. We give then the numerical method based on the inversion of Fourier transform to price CDO. The tractability of the characteristic function of the loss portfolio guarantees a low complexity degree of computation. However, if the Fourier transform does not comply with the theoretical hypothesis, its inversion remains quite difficult, we give then a method that insure the regularity assumptions.

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The work presents the numerical method that have been adopted to price credit derivatives (e.g) for point affine processes self exciting. We give also the estimation of the numerical error and present the exact truncation one to enhance the computation.

Short presentation of point affine processes self exciting

If we consider the loss $(L_t)_{t\geq 0}$ for a given a portfolio, the empirical observation shows that the stochastic properties of its intensity $(\lambda_t)_{t\geq 0}$ have the same behaviour such that the equity context. Indeed, on the one hand the intensity replicates the clustering of default, and on the other hand, mean reversion is also indicated by observation. Moreover, a self-exiting phenomena was observed in the sense that if the portfolio encountered a default, then its intensity grows up substantially; while if default doesn't occur, then the intensity remains in a stable level.

Therefore, the intensity is a solution of the following SDE:

$$d\lambda_t = \kappa(c - \lambda_t)dt + \delta dL_t \tag{1}$$

Moreover, if $(N_t)_{t\geq 0}$ denotes the number of default at time t. Since the process $(L_t, N_t)_{t\geq 0}$ is an affine process, we obtain then that :$\forall u \in \mathbb{C}^2,\ t \leq T$

$$\mathbb{E}[\exp(\langle u, J_t\rangle)|\mathcal{F}_t] = \exp(a(t) + b(t)\lambda_t + \langle u, J_t\rangle)$$

Where $\forall t \in [0,T],\ J_t = (L_t, N_t)$, and the coefficient functions $a(t) = a(t,u,T)$ and $b(t) = b(t,u,T)$ satisfy the following ODE

$$\begin{array}{rcl} \partial_t b(t) & = & \kappa b(t) + 1 - \theta(\delta b(t) + \langle u, (1,0)^T\rangle)\exp(\langle u, (0,1)^T\rangle) \\ \partial_t a(t) & = & -\kappa c b(t). \end{array}$$

with the boundary conditions $a(T) = b(T) = 0$, where θ is the jump transform

$$\theta(c) = \int \exp(cz)d\nu(z);\ c \in \mathbb{C}$$

We stress that the SDE (1) can be extend to the Wiener integration, in the sense that the new process keeps the affinity property. In the present work, we have just implemented the first case, while the estimation of the numerical method errors can be used for the extended model.

Index and tranche Swaps are based on a portfolios whose n constituent securities have a notional 1, maturity T and a premium payments date (t_m). The loss at the default $l_n \in [0,1]$. The swap is specified by a lower attachement point $\underline{K} \in [0,1]$ and the upper attachment point $\overline{K} \in [\underline{K}, 1]$. Th index swap is a special case, in the sense that $\overline{K} = 1$ and $\underline{K} = 0$. The swap notional $K = n(\overline{K} - \underline{K})$. The protection seller cover portfolios losses as they occur, given that the cumulative losses are larger that $\underline{K}$ but do not exceed $\overline{K}$. The cummulative payments at time t, denotes U_t, are given by the call spread

$$U_t = (L_t - \underline{K}n)^+ - (L_t - \overline{K}n)^+$$

the value at time $t \leq T$ of theses payments is given by

$$D_t = \mathbb{E}[\int_t^T \exp(-r(s-t))dU_s|\mathcal{F}_t]$$

Under the hypothesis that the pricing is under the risk neutral probability and that the risk free rate r is a constant, we integrate by part and we obtain that

$$D_t = \exp(-r(T-t))\mathbb{E}[U_T|\mathcal{F}_t] - U_t + r\int_t^T \exp(-r(s-t))\mathbb{E}[U_s|\mathcal{F}_t]ds$$

The protection buyer receives the loss payments, and in return, makes premium payments to the protection seller. Each premium payment has two parts. The first one is the upfront payment F and the second part consists of payments that are proportional to the premium notional $K - U_t$. Let c_m be the day count fraction for the period m, roughly $\frac{1}{4}$ for quarterly payments, and S the running premium rate. The value at time $t \leq T$ of the premium payments is given by

$$KF + S \sum_{t_m \geq t} \exp(-r(t_m - t)) c_m \mathbb{E}[(K - U_{t_m})|\mathcal{F}_t]$$

For a fixed upfront rate F, the running spread S_t at time t is defined as

$$S_t = \frac{\exp(-r(T-t))\mathbb{E}[U_T|\mathcal{F}_t] - U_t + r\int_t^T \exp(-r(s-t))\mathbb{E}[U_s|\mathcal{F}_t]ds - KF}{\sum_{t_m \geq t} \exp(-r(t_m - t)) c_m \mathbb{E}[(K - U_{t_m})|\mathcal{F}_t]} \tag{2}$$

Numerical method

Since in the theoretical point of view, the spread running is a solution of (2), we deduce S_t can be approximated into

$$S_t = \frac{\exp(-r(T-t))\mathbb{E}[U_T|\mathcal{F}_t] - U_t + r\Delta_t \sum_{i=0}^n \exp(-r(n\Delta_t))\mathbb{E}[U_{t+i\Delta_t}|\mathcal{F}_t]ds - KF}{\sum_{t_m \geq t} \exp(-r(t_m - t)) c_m \mathbb{E}[(K - U_{t_m})|\mathcal{F}_t]}$$

We conclude from the previous equality that ti is sufficient to compute the call expected loss on the both attachement points for all maturity : $(\mathbb{E}[(L_{t+i\Delta_t} - \underline{K})^+|\mathcal{F}_t], \mathbb{E}[(L_{t+i\Delta_t} - \overline{K})^+|\mathcal{F}_t])_{i\in\{1,\dots,n\}}$ in order to price the running spread of the tranche CDO.

In the light of this, we use the inversion of the Fourier transform. We consider that x is the strike of a call option on the loss defined as $\mathbb{E}[(L_t - x)^+]$. Since, the characteristic function of the loss portfolio is not necessary a square integrable and as $u \to (u-x)^+$ is not square integrable too, we use then the following regularity function, such that $\rho > 0$

$$\begin{array}{rcl}\mathbb{E}[(L_t - x)^+] & = & \mathbb{E}(\exp(\rho L_t)(L_t - K)^+ \exp(-\rho L_t)) \\ & = & \frac{1}{2\pi}\int_{\mathbb{R}} \mathbb{E}(\exp((\rho - iv)L_t))\hat{f}(v)dv\end{array} \tag{3}$$

Since $\lim_{\rho \to 0^-} \mathbb{E}[\exp(\rho L_t)] = 1$, one has to choose $0 < \rho << 1$ such that $\mathbb{E}[\exp(\rho L_t)] < \infty$. Moreover, the Fubini used in the equation (3) is well defined, because of the choice of ρ and that $u \to \exp(-\rho u)(u-x)^+) \in L^2(\mathbb{R}^+)$. Moreover, after some straightforward calculus we find that

$$\forall v \in \mathbb{R},\ \hat{f}(v) = \frac{\exp((-\rho + iv)x)}{(\rho - iv)^2}$$

We deduce then that

$$\begin{array}{rcccl}\mathbb{E}[(L_t - x)^+] & = & \frac{1}{2\pi}\int_{\mathbb{R}} \phi(t, \rho - iv)\frac{\exp((-\rho+iv)x)}{(\rho-iv)^2}dv & = & \frac{1}{\pi}\int_{\mathbb{R}^+} \mathrm{Real}(\phi(t,\rho - iv)\frac{\exp((-\rho+iv)x)}{(\rho-iv)^2})dv \\ \phi(t, \rho - iv) & \triangleq & \mathbb{E}[\exp((\rho - iv)L_t)] & = & \exp(a(t) + \lambda_0 b(t) + (\rho - iv)L_0)\end{array}$$

We give then the numerical scheme used to compute the expected call loss

$$\mathbb{E}[(L_t - x)^+] \approx \frac{1}{\pi}\sum_{j=0}^{M} \mathrm{Real}(\phi(t, \rho - ij\Delta_v)\frac{\exp((-\rho+ij\Delta_v)x)}{(\rho-ij\Delta_v)^2})\Delta_v$$

An important remark which can be emphasized is that the calculus of the Fourier transform is just done one time by each maturity, because it does not depend on the value of the strike x. The second remark is that under the call payoff, we get an analytical formula for the Fourier transform associated to the function $u \to \exp(-\rho u)(u - x)^+$.

Before talking about the enhancement of the numerical method and the choice of model parameters. Let give some results about the truncation error. Indeed we have to calculate the truncation error $\zeta(M, \rho)$ such that

$$\begin{aligned} \zeta(t, M, \rho) &= \frac{1}{\pi}\int_M^\infty \mathrm{Real}(\phi(t, \rho - iv)\frac{\exp(-\rho v)}{(\rho-iv)^2})dv \\ |\zeta(t, M, \rho)| &\leq \frac{\phi(t,\rho)\exp(-x\rho)}{M}(\frac{\tan(M/\rho)}{M/\rho})^{-1} \\ &\leq \frac{\phi(t,\rho)\exp(-x\rho)}{M} \end{aligned}$$

If we assume that the loss process $(L_t)_{t\geq 0}$ has a bounded trajectory $\hat{L}^{\max}$ (which is the case in the CDO pricing), then the error can be controlled by $\frac{\exp((\hat{L}^{\max}-x)\rho)}{M}$.

In practice, we choose a discret jump law (the same configuration described in the paper [1]). On the one hand, because the calculus of $(b(t), a(t))$ is less complex, and on the other hand, the use of a discret jumps give a flexible interpretation for the behaviour of the model.

To improve the numerical calculus we can study the error due to the discretization of the integral. In this perspective, we can use other scheme e.g Simpson integral to reduce the error to the 4^{th} order. The implementation has just taken into consideration trapezoidal rule.

References

[1] E.Errais, K.Giesecke, L. R. Goldberg. (2009) Affine point processes and portfolio credit risk. 1, 4