Pricing of Timer Options

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Most of what is presented here is taken from [Cui11].

1 Heston model

Consider the Heston stochastic volatility model for a forward price process S_t, defined by the following stochastic differential equations

$$\begin{cases} \frac{dS_t}{S_t} = \mu dt + \sqrt{V_t} dW_t^1, \\ dV_t = (a - bV_t)\, dt + \sigma\sqrt{V_t} dW^2, \ d\langle W^1, W^2\rangle_t = \rho dt, \end{cases} \tag{1.1}$$

A standard timer call option can simply be viewed as a call option with random maturity which depends on the time needed for a pre-specified variance budget to be fully consumed. The buyer of a timer call option specifies an investment horizon and a target volatility. A variance budget is then calculated as the target volatility squared, multiplied by the target maturity. Once the variance budget is consumed, the option expires. Let us now summarize practical details about timer options obtained from a presentation of the Société Générale (2007).

Consider a timer call option written on the underlying asset S with strike K. Its maturity date is linked to the accumulated variance of S. Let us denote by $\mathbb{V}$ the "variance budget" that is chosen by the investor. The stock price evolves in a continuous time framework, we define the realized variance in continuous time at time u as

$$\int_0^u V_s ds$$

corresponding to the realized variance consumption. Denote by τ the random maturity time of the option. It is defined as the first hitting time of the realized variance to the variance budget $\mathbb{V}$

$$\tau = \inf \left\{ u \geq 0 : \int_0^u V_s ds = \mathbb{V} \right\}$$

The payoff of a timer call option is paid at time τ and is $(S_\tau - K)_+$. The price of the timer call option is then given by

$$C_0 = \mathbb{E}^Q\left[e^{-r\tau}(S_\tau - K)_+\right]$$

According to [Cui11], C_0 is given by

$$C_0 = S_0\mathbb{E}\left[e^{(\mu-r)\tau}\ e^{\rho c(\tau,X_\tau)-\frac{1}{2}\rho^2\mathbb{V}}N(d_1(\tau,X_\tau))\right] - K\mathbb{E}\left[e^{-r\tau}N(d_2(\tau,X_\tau))\right] \tag{1.2}$$

where

$$d_1(\tau,x) = \frac{\ln(S_0/K)+\mu\tau+\rho c(\tau,x)+(\frac{1}{2}-\rho^2)\mathbb{V}}{\sqrt{(1-\rho^2)\mathbb{V}}},\quad d_2 = d_2 - \sqrt{(1-\rho^2)\mathbb{V}}$$

$c(\tau,x) = \frac{X_\tau - V_0 - \kappa\theta\tau + \kappa\mathbb{V}}{\sigma}$, $\tau = \int_0^{\mathbb{V}} \frac{ds}{X_s}$ and X is the unique solution of

$$dX_t = \left(\frac{\kappa\theta}{X_t} - \kappa\right)dt + \sigma dW_t^2,\quad X_0 = V_0.$$

2 Hull-White model

The forward price process S_t under Hull-WHite moldel is defined by the following stochastic differential equations

$$\begin{cases} \frac{dS_t}{S_t} = \mu dt + \sqrt{V_t}dW_t^1, \\ dV_t = aV_t dt + \nu V_t dW^2,\ d\langle W^1, W^2\rangle_t = \rho dt, \end{cases} \tag{2.1}$$

The price of Timer-Call under this model is given as

$$C_0 = S_0\mathbb{E}\left[e^{(\mu-r)\tau}\ e^{c_\tau+\frac{1-\rho^2}{2}\mathbb{V}}N(d_1(\tau,X_\tau))\right] - K\mathbb{E}\left[e^{-r\tau}N(d_2(\tau,X_\tau))\right] \tag{2.2}$$

where $c_\tau = \frac{2\rho}{\nu}\left(\sqrt{V_\tau} - \sqrt{V_0}\right) - \rho H_\tau - \frac{1}{2}\mathbb{V}$,

$$d_1(\tau,x) = \frac{\ln(S_0/K)+\mu\tau+c_\tau+(1-\rho^2)\mathbb{V}}{\sqrt{(1-\rho^2)\mathbb{V}}},\quad d_2 = d_2 - \sqrt{(1-\rho^2)\mathbb{V}}$$

and where

$$\tau = \frac{4}{\nu^2}\int_0^{\mathbb{V}}\frac{ds}{X_s^2},\quad \mathbb{V}_\tau = \frac{4}{\nu^2}X_{\mathbb{V}}^2,\quad H_\tau = (\frac{2a}{\nu^2} - \frac{1}{2})\int_0^{\mathbb{V}}\frac{ds}{X_s}$$

with the process X defined as the unique solution of

$$dX_t = \left(\frac{2a}{\nu^2} - \frac{1}{2}\right)\frac{1}{X_t}dt + dW_t^2,\quad X_0 = \frac{2}{\nu}\sqrt{V_0}$$

References

[Cui11] C. Bernard Z. Cui. Pricing of timer options. *Journal of Computational Finance*, 2011. 1, 2