Calibration of the Libor market model - implementation in PREMIA

Nicolas PRIVAULT* Xiao WEI †

INRIA Rocquencourt
Projet MathFi
78153 Le Chesnay Cedex
France

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Abstract

This paper describes the BGM model for Libor interest rates and the C++ implementation in PREMIA of the calibration algorithm of [2] for this model using the market prices of caps and swaptions.

Keywords: Libor market model, BGM model, interest rates, caps, swaps, calibration.

1 Introduction

The calibration of the Brace-Gatarek-Musiela (BGM) and Jamshidian Libor interest rate model to the market values of caps and swaps has proved to involve several numerical stability issues.

In this paper we describe a C++ implementation of the stable algorithm for the joint calibration of [2] for the Libor market model, from the prices of caps and swaptions

*nicolas.privault@inria.fr
†xiao.wet@inria.fr

The outline of this paper is as follows. In Section 2 we recall the basics of Black-Scholes pricing, which will be applied throughout the text to the pricing of interest rate derivatives. In Section 3 we recall the definition of forward rates and contracts, with the arbitrage free modelling of zero coupon bonds in Section 4. In Section 5 we derive the BGM model. The pricing of caps and swaps in this model is described in Section 6. Section 7 is devoted to the calibration algorithm of [2], and in Sections 8 and 9 we present the program user manual and a sample data file.

2 Black-Scholes pricing

The classical Black-Scholes formula is of importance in the pricing of interest rates derivatives since all models we will consider will be essentially based on geometric Brownian motions with deterministic variance parameters.

Consider an asset price $(S_t)_{t\in\mathbb{R}_+}$ modeled by

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dB_t, \qquad t \in \mathbb{R}_+.$$

European contracts: call option with payoff function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$:

$$\phi(S_T) = (S_T - \kappa)^+ = \begin{cases} S_T - \kappa, & S_T \geq \kappa, \\ 0, & S_T \leq \kappa. \end{cases}$$

The fair price of this contract is given by the expectation (Black-Scholes formula):

$$\mathrm{Bl}\,(\kappa, \mathrm{x}, \sigma, \mathrm{r}, \mathrm{T}-\mathrm{t}) \;=\; e^{-\int_t^T r_s ds}\mathbb{E}[\phi(S_T) \mid S_t = x]$$

$$
\begin{aligned}
&= e^{-\int_t^T r_s ds}\mathbb{E}[(S_T-\kappa)^+ \mid S_t = x] \\
&= \kappa\Phi(d_1) - xe^{-r(T-t)}\Phi(d_2),
\end{aligned}
$$

where

$$
d_1 = \frac{\log(\kappa/x) + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},
$$

$$
d_2 = \frac{\log(\kappa/x) + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}},
$$

where

$$
\sigma^2 = \frac{1}{T-t}\int_t^T \sigma_s^2 ds, \qquad r = \frac{1}{T-t}\int_t^T r_s ds.
$$

Recall that the expectation

$$
p(t,x) := \mathrm{Bl}\,(\kappa, \mathrm{x}, \sigma, \mathrm{r}) = \mathrm{e}^{-\int_\mathrm{t}^\mathrm{T} \mathrm{r_s ds}}\mathbb{E}[\phi(\mathrm{S_T}) \mid \mathrm{S_t} = \mathrm{x}]
$$

solves the Black-Scholes PDE

$$
\begin{cases}
\dfrac{\partial p}{\partial t}(t,x) + \dfrac{1}{2}\sigma_t^2\dfrac{\partial^2 p}{\partial x^2}(t,x) + r_t\dfrac{\partial p}{\partial x}(t,x) - r_t p(t,x) = 0 \\
\\
p(T,x) = \phi(x).
\end{cases}
$$

3 Forward rates

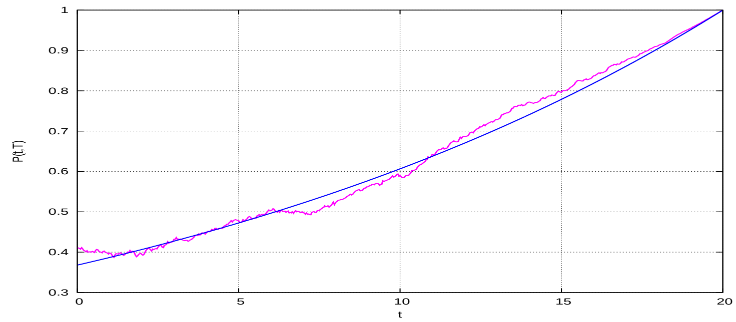
A Zero-coupon bond based on a short term interest rate process $(r_t)_{t\in\mathbb{R}_+}$ given by

$$
dr_t = \mu_t(r_t)dt + \sigma_t(r_t)dB_t, \quad 0 \le t \le T,
$$

is a contract valued

$$
P(t,T) := \mathbb{E}\left[\exp\left(-\int_t^T r_s ds\right)\bigg| r_t\right],
$$

at time $t < T$, to deliver $P(T,T) = 1\$$ at time T. The following graph represents a numerical simulation of $t \mapsto P(t,T)$ in a Vasiçek model.

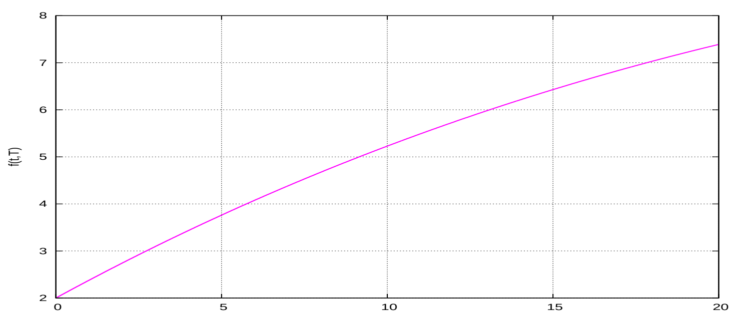


$$t \mapsto P(t, T)$$

The process $(r_t)_{t \in \mathbb{R}_+}$ can be modelled using other short term interest rate models such as the Dothan model. The instantaneous rate process, given by

$$T \mapsto f(t, T) = -\frac{\partial P(t, T)}{\partial T},$$

can be represented as follows.



Instantaneous rate process

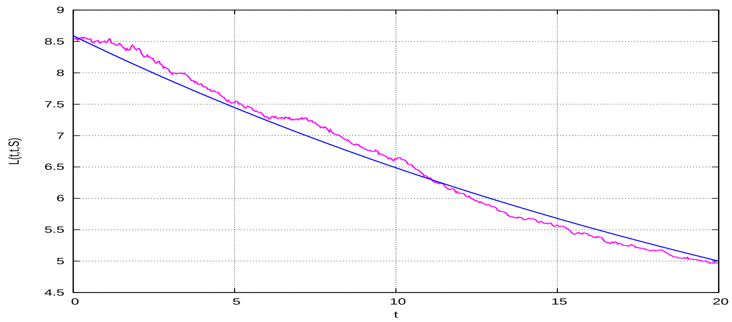
The forward (simply compounded) Libor rate is defined as:

$$F(t; T, S) := \frac{1}{S - T} \left(\frac{P(t, T)}{P(t, S)} - 1 \right), \qquad 0 \leq t < T < S,$$

i.e. the interest rate contracted at time t for a loan over the future period $[T, S]$:

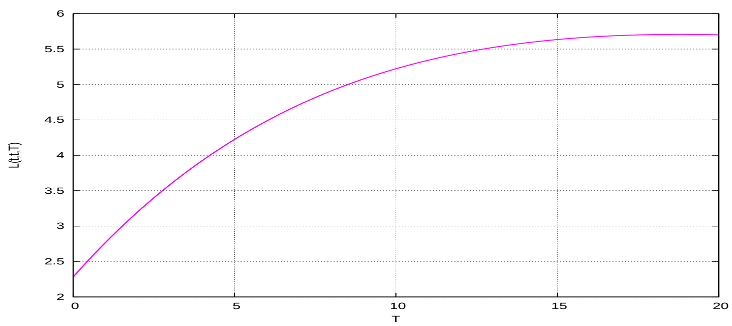
$$P(t, T) - P(t, S) = P(t, S)(S - T)F(t; T, S), \qquad 0 \leq t < T < S.$$

The forward rate agreement at time t gives its holder an interest rate $F(t; T, S)$ on the period $[T, S]$. Next is a simulation of Libor rates using the above model:



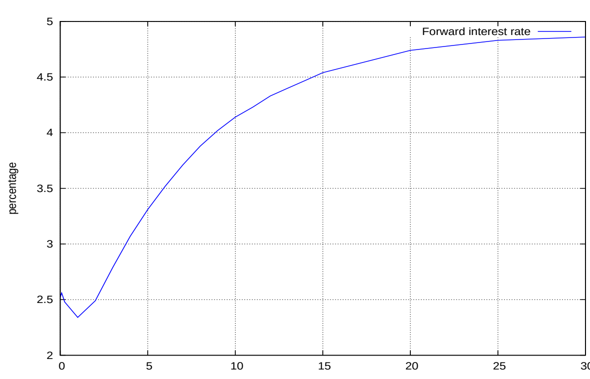
$$t \mapsto L(t, T) = F(t; t, T)$$

The forward curve $T \mapsto F(t; T, T + \delta)$ can be represented as follows:



$$T \mapsto F(t; T, T + \delta)$$

An example of the data used with $t = 07/05/2003$ for the forward interest rate curve is as follows:



$$T \mapsto F(t; T, T + \delta)$$

TimeSerieNb	505
AsOfDate	7-mai-03
2D	2,55
1W	2,53
1M	2,56
2M	2,52
3M	2,48
1Y	2,34
2Y	2,49
3Y	2,79
4Y	3,07
5Y	3,31
6Y	3,52
7Y	3,71
8Y	3,88
9Y	4,02
10Y	4,14
11Y	4,23
12Y	4,33
13Y	4,4
14Y	4,47
15Y	4,54
20Y	4,74
25Y	4,83
30Y	4,86

Note that other models (e.g. Nelson-Siegel) are available to model the actual data of the forward curve.

4 Arbitrage free modelling

We consider the tenor structure

$$0 = T_0 < T_1 < T_2 < \cdots < T_n,$$

with $\delta_i = T_{i+1} - T_i$, $i = 1, \ldots, n-1$. The dynamics of $P(t, T_i)$ is given by

$$\frac{dP(t,T_i)}{P(t,T_i)} = r_t dt + \sigma_i(t) dW(t), \qquad i = 1, \ldots, n,$$

under the assumption (absence of arbitrage condition) that

$$e^{-\int_0^t r_s ds} P(t, T_i)$$

is a martingale, $t \in [0, T_i]$, $i = 1, \ldots, n$.

Definition 4.1. *Let the martingale measure $\mathbb{P}_i$ be defined as*

$$\frac{d\mathbb{P}_i}{d\mathbb{P}} = \frac{1}{P(0,T_i)} e^{-\int_0^{T_i} r_s ds}, \qquad i = 1, \ldots, n-1.$$

In the sequel, the expectation under $\mathbb{P}_i$ will be denoted by E_i.

Proposition 4.2. *The process*

$$\frac{P(t,T_{i+1})}{P(t,T_i)}, \qquad 0 \leq t \leq T_i,$$

is a martingale under $\mathbb{P}_i$, $i = 1, \ldots, n-1$:

Proof. For any bounded and $\mathcal{F}_s$-measurable random variable F we have

$$\begin{aligned}
E_i\left[F E_i\left[\frac{P(t,T_{i+1})}{P(t,T_i)}\Big|\mathcal{F}_s\right]\right] &= E_i\left[F \frac{P(t,T_{i+1})}{P(t,T_i)}\right] \\
&= E\left[F e^{-\int_0^{T_i} r_\tau d\tau} \frac{1}{P(0,T_i)} \frac{P(t,T_{i+1})}{P(t,T_i)}\right] \\
&= E\left[F e^{-\int_0^{T_i} r_\tau d\tau} \frac{1}{P(0,T_i)} P(T_i, T_{i+1})\right]
\end{aligned}$$

$$
\begin{aligned}
&= E\left[F e^{-\int_0^s r_\tau d\tau} \frac{1}{P(0,T_i)} P(s,T_{i+1})\right] \\
&= E\left[F e^{-\int_0^s r_\tau d\tau} \frac{P(s,T_i)}{P(0,T_i)} \frac{P(s,T_{i+1})}{P(s,T_i)}\right] \\
&= E\left[F e^{-\int_0^{T_i} r_\tau d\tau} \frac{P(T_i,T_i)}{P(0,T_i)} \frac{P(s,T_{i+1})}{P(s,T_i)}\right] \\
&= E\left[F e^{-\int_0^{T_i} r_\tau d\tau} \frac{1}{P(0,T_i)} \frac{P(s,T_{i+1})}{P(s,T_i)}\right] \\
&= E_i\left[F \frac{P(s,T_{i+1})}{P(s,T_i)}\right],
\end{aligned}
$$

which shows that

$$
E_i\left[\frac{P(t,T_{i+1})}{P(t,T_i)}\Big|\mathcal{F}_s\right] = \frac{P(s,T_{i+1})}{P(s,T_i)}, \qquad 0 \le s \le\le t \le T_i.
$$

$\square$

5 Derivation of the BGM model

The aim of this section is to establish the BGM model, and to shows that

$$
\frac{dF(t,T_i,T_{i+1})}{F(t,T_i,T_{i+1})} = -\sum_{j=i+1}^{n-1} \frac{\delta_j F(t,T_j,T_{j+1})}{1+\delta_j F(t,T_j,T_{j+1})} \gamma_i^\dagger(t)\gamma_j(t)dt + \gamma_i^\dagger(t)dB_t,
$$

$0 \le t \le T_i$, is a martingale under $\mathbb{P}_{i+1}$, where $\gamma_i(t)$ is a deterministic function, $i = 1, \ldots, n-1$.

By Itô's calculus we have

$$
d\left(\frac{P(t,T_{i+1})}{P(t,T_i)}\right) = (\sigma_{i+1} - \sigma_i)\frac{P(t,T_{i+1})}{P(t,T_i)}(dW_t - \sigma_i dt),
$$

hence

$$
dW^i := dW - \sigma_{i+1} dt
$$

is a standard Brownian motion under $\mathbb{P}_i$, $i = 1, \ldots, n-1$.

Now we have

$$
F(t,T_i,T_{i+1}) = \frac{1}{\delta_i}\left(\frac{P(t,T_i)}{P(t,T_{i+1})} - 1\right),
$$

hence

$$\begin{aligned}
dF(t,T_i,T_{i+1}) &= \frac{1}{\delta_i} d\left(\frac{P(t,T_i)}{P(t,T_{i+1})}\right) \\
&= \frac{\sigma_i - \sigma_{i+1}}{\delta_i} \frac{P(t,T_i)}{P(t,T_{i+1})}(dW_t - \sigma_{i+1}dt) \\
&= \frac{\sigma_i - \sigma_{i+1}}{\delta_i} \left(1 + \delta_i F(t,T_i,T_{i+1})\right)(dW_t - \sigma_{i+1}dt) \\
&= \frac{\sigma_i - \sigma_{i+1}}{\delta_i} \left(1 + \delta_i F(t,T_i,T_{i+1})\right) dW_t^{i+1} \\
&= \frac{\sigma_i - \sigma_{i+1}}{\delta_i} \left(1 + \delta_i F(t,T_i,T_{i+1})\right) dW_t^{i+1}.
\end{aligned}$$

Assuming the existence of a deterministic function γ_i such that

$$(\sigma_i - \sigma_{i+1})(1 + \delta_i F(t,T_i,T_{i+1})) = \delta_i \gamma_i(t) F(t,T_i,T_{i+1}), \qquad i = 0,\ldots,n-1,$$

we get

$$\begin{aligned}
\frac{dF(t,T_i,T_{i+1})}{F(t,T_i,T_{i+1})} &= \gamma_i(t) dW_t^{i+1} \\
&= \gamma_i(t)\left((\sigma_{i+1} - \sigma_k)dt + dW_t^n\right) \\
&= \gamma_i(t)\left(\sum_{j=i+1}^{k-1} (\sigma_j - \sigma_{j+1})dt + dW_t^k\right) \\
&= \gamma_i(t)\left(\sum_{j=i+1}^{k-1} \frac{\delta_j \gamma_j(t) F(t,T_j,T_{j+1})}{1 + \delta_j F(t,T_j,T_{j+1})} dt + dW_t^k\right),
\end{aligned}$$

$0 \leq t \leq T_i$, $k = i+1,\ldots,n$. In particular, for $k = n$ we have

$$\frac{dF(t,T_i,T_{i+1})}{F(t,T_i,T_{i+1})} = \gamma_i(t)\left(\sum_{j=i+1}^{n-1} \frac{\delta_j \gamma_j(t) F(t,T_j,T_{j+1})}{1 + \delta_j F(t,T_j,T_{j+1})} dt + dW_t^n\right),$$

$0 \leq t \leq T_i$, which is a martingale under $\mathbb{P}_{i+1}$, $i = 1,\ldots,n-1$.

6 Pricing of caps and swaps

The caplet

$$e^{-\int_0^{T_{i+1}} r_s ds}(L(T_i,T_{i+1}) - \kappa)^+$$

is priced as

$$E\left[e^{-\int_0^{T_{i+1}} r_s ds}(L(T_i,T_{i+1}) - \kappa)^+\right] = P(0,T_{i+1}) E_{i+1}\left[(L(T_i,T_{i+1}) - \kappa)^+\right]$$

$$= \mathrm{Bl}\left(\kappa, \mathrm{F}(0, \mathrm{T_i}, \mathrm{T_{i+1}}), \sigma_{\mathrm{i}}^{\mathrm{B}}(\mathrm{t}), 0, \mathrm{T_i}\right),$$

with

$$(\sigma_i^B(t))^2 = \frac{1}{T_i - t} \int_t^{T_i} \|\gamma_i(s)\|^2 ds.$$

This formula can be used to recover caplet volatilities σ_i^B from market data as in the following table, where the time to maturity $T_i - t$ is in abscissa and the period $T_j - T_i$ is in ordinate.

764	AsOfDate	11-mai-04								
		Vol Cap At the Money								
	1M	3M	6M	12M	2Y	3Y	4Y	5Y	7Y	10Y
2D	9,25	9	9	23,9	27,8	25,3	23,4	22	18,2	14,6
1M	10	9,75	9,65	23,9	27,8	25,37	23,47	22,07	18,26	14,64
2M	10,75	10,5	10,4	23,65	27,26	24,99	22,94	21,49	18,02	14,64
3M	14,25	14	13,9	23,4	26,71	24,61	22,41	20,91	17,78	14,63
6M	20	19,75	19,65	25,4	25,12	23,12	21,31	19,92	17,36	14,66
9M	24,25	24	23,9	25,55	24,18	22,58	20,87	19,53	17,1	14,6
1Y	26	25,75	25,65	25,7	23,24	22,04	20,42	19,14	16,83	14,53
2Y	24	23,75	22,95	21,9	21,1	20,03	18,87	17,5	15,64	13,99
3Y	20,35	20,1	19,6	18,5	19,43	18,33	17,38	16,17	14,79	13,67
4Y	17,95	17,7	17,4	16,8	17,63	16,71	15,84	14,78	13,84	13,06
5Y	17,4	16,2	15,9	15,2	15,66	14,92	14,19	13,57	12,98	12,52
6Y	15,9	15	14,75	14,15	15,14	14,49	13,83	13,26	12,74	12,37
7Y	14,7	13,9	13,6	13,1	14,39	13,76	13,2	12,75	12,38	12,13
8Y	13,75	13,15	12,95	12,53	13,91	13,37	12,88	12,49	12,15	11,97
9Y	12,85	12,45	12,3	11,97	13,33	12,94	12,51	12,14	11,86	11,74
10Y	12	11,9	11,75	11,4	12,73	12,42	12,05	11,76	11,52	11,48
12Y	11,1	10,95	10,8	10,44	12,23	11,92	11,93	11,73	11,33	11,32
15Y	9,85	9,7	9,6	9,2	11,57	11,57	11,48	11,12	11,04	10,86
20Y	9,55	9,15	9,05	8,7	11,74	11,48	11,32	10,97	10,76	10,55
25Y	9,6	9,25	9,15	8,94	12,33	12,12	11,9	11,55	11,24	10,83
30Y	9,7	9,5	9,4	9,13	12,24	12	11,83	11,62	11,11	10,48

In this table we actually only make use of the data of a single column giving volatilities for a period δ equal to the fixed tenor value. The pricing of caplets can be extended to caps of the form

$$\sum_{j=i+1}^{k} e^{-\int_0^{T_j} r_s ds} (L(T_{j-1}, T_j) - \kappa)^+$$

since they can be decomposed into a sum of caplets, and priced (at $t = 0$) as

$$\sum_{j=i+1}^{k} e^{-\int_0^{T_j} r_s ds} \mathrm{Bl}\left(\kappa, \mathrm{F}(0, \mathrm{T_{j-1}}, \mathrm{T_j}), \sigma_{\mathrm{i}}^{\mathrm{B}}, 0, \mathrm{T_i}\right).$$

Concerning the swap

$$e^{-\int_0^{T_i} r_s ds} \left(\sum_{j=i+1}^{k} \delta_j P(T_i, T_j)(F(T_{j-1}, T_{j-1}, T_i) - \kappa \right)^+,$$

the positive part can not be taken out of the sum, and in general the price of the swap is smaller than the value of the corresponding cap.

As an example, the swap

$$e^{-\int_0^{T_i} r_s ds} \left(P(T_i, T_{i+1})\delta_i(L(T_i, T_{i+1}) - \kappa) + P(T_i, T_{i+2})\delta_{i+1}(L(T_i, T_{i+2}) - \kappa)\right)^+$$

is priced as

$$\begin{aligned} &E\left[e^{-\int_0^{T_i} r_s ds} \left(P(T_i, T_{i+1})(F(t, T_i, T_{i+1}) - \kappa) + P(T_i, T_{i+2})(F(t, T_{i+1}, T_{i+2}) - \kappa)\right)^+\right] \\ &= \; P(0, T_1)E_i\left[\left(P(T_i, T_{i+1})\delta_i(F(t, T_i, T_{i+1}) - \kappa) + P(T_i, T_{i+2})\delta_{i+1}(F(t, T_{i+1}, T_{i+2}) - \kappa)\right)^+\right], \end{aligned}$$

which can be computed using the dynamics of $F(t, T_i, T_{i+1})$ and $F(t, T_{i+1}, T_{i+2})$ under $\mathbb{P}_i$.

The market practice is to price swaps (at time $t = 0$) by the swaption approximation formula

$$P(t; T_i, T_k)\mathrm{Bl}\,(\kappa, \mathrm{S_{i,j}}(0), \sigma^{\mathrm{B}}_{\mathrm{i,j}}(0), 0, \mathrm{T_k}),$$

where

$$S_{i,k}(t) = \frac{P(t, T_i) - P(t, T_k)}{P(t; T_i, T_k)},$$

is the forward swap rate process defined from

$$0 = P(t, T_k) - P(t, T_i) + S_{i,k}(t) \sum_{j=i+1}^{k} \delta_j P(t, T_j),$$

$P(t; T_i, T_k)$ defined as

$$P(t; T_i, T_k) = \sum_{j=i+1}^{k} \delta_{j-1} P(t, T_j)$$

is the annuity numeraire,

$$\sigma^B_{i,j}(t) = \sum_{l,l'=1}^{j-1} \frac{v^{i,j}_l(t) v^{i,j}_{l'}(t) F(t; t, T_l) F(t; t, T_{l'})}{S^2_{i,j}(t)} \int_t^{T_j} \langle \gamma_l(s), \gamma_{l'}(s) \rangle ds,$$

and

$$v^{i,j}_l(t) = \delta_l \frac{P(t, T_{l+1})}{P(t; T_i, T_j)}.$$

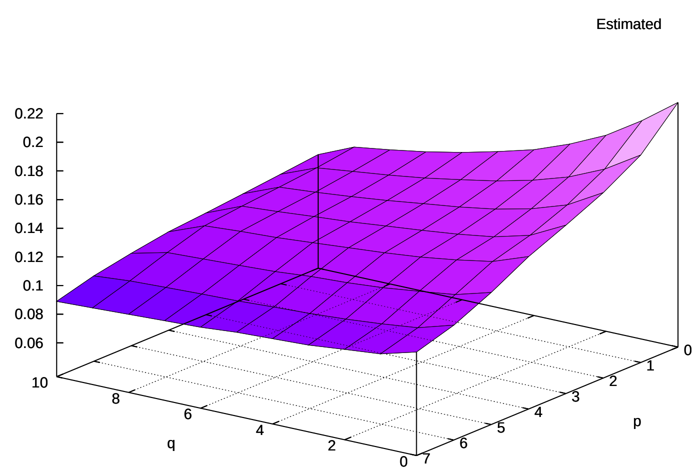
The following figure shows an example of market data expressed in terms of swaption volatilities.

764	AsOfDate	11-mai-04

Vol Swaption At The Money

	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y	25Y
2D	23,9	27,8	25,3	23,4	22	19,9	18,2	16,8	15,6	14,6	10
1M	23,9	27,8	25,3	23,4	22	19,9	18,2	16,8	15,6	14,6	10
2M	23,65	27,25	24,85	22,8	21,35	19,5	17,9	16,6	15,5	14,55	10,05
3M	23,4	26,7	24,4	22,2	20,7	19,1	17,6	16,4	15,4	14,5	10,1
6M	25,4	25,1	22,7	20,9	19,5	18,2	17	16	15,1	14,4	10,6
9M	25,55	24,15	21,95	20,25	18,9	17,65	16,55	15,65	14,85	14,2	10,6
1Y	25,7	23,2	21,2	19,6	18,3	17,1	16,1	15,3	14,6	14	10,6
2Y	21,9	19,8	18,4	17,2	15,9	15	14,3	13,8	13,4	13	10,4
3Y	18,5	17,3	16,2	15,3	14,2	13,6	13,1	12,8	12,5	12,3	10,2
4Y	16,8	15,7	14,7	13,9	13	12,6	12,2	12	11,8	11,6	9,8
5Y	15,2	14,3	13,5	12,8	12,2	11,8	11,6	11,4	11,2	11,1	9,5
6Y	14,15	13,4	12,7	12,1	11,6	11,3	11,1	10,95	10,8	10,7	9,2
7Y	13,1	12,5	11,9	11,4	11	10,8	10,6	10,5	10,4	10,3	8,9
8Y	12,53	12	11,47	11	10,63	10,47	10,27	10,2	10,1	10	8,7
9Y	11,97	11,5	11,03	10,6	10,27	10,13	9,93	9,9	9,8	9,7	8,53
10Y	11,4	11	10,6	10,2	9,9	9,8	9,6	9,6	9,5	9,4	8,3
12Y	10,44	10,18	9,77	9,64	9,43	9,3	9,08	9,03	9	8,99	7,96
15Y	9,2	9,1	9	8,9	8,6	8,6	8,5	8,5	8,4	8,4	7,5
20Y	8,7	8,6	8,4	8,3	8,2	8,2	8,1	8,2	8,1	8,1	7,3
25Y	8,94	8,8	8,61	8,48	8,33	8,29	8,24	8,2	8,14	8,14	7,23
30Y	9,13	8,54	8,38	8,28	8,19	8,12	8,02	7,93	7,88	7,82	6,99

Here, the time to maturity $T_i - t$ is in abscissa and the period $T_j - T_i$ is in ordinate. This data can be also expressed in the form of a graph, where the index q refers to the time maturity $T_q - t$ and the index p refers to the period $T_p - T_q$:



7 LIBOR calibration

The goal of calibration is to estimate the volatility processes

$$\gamma_i(t) \in \mathbb{R}^d, \qquad 1 \le i \le n,$$

appearing in the BGM model from the data of caps and swaps prices observed on the market. This involves several computational and stability issues. Let

$$g_i(t) = \|\gamma_i(t)\|, \qquad i = 1, \ldots, n,$$

and

$$\rho_{i,j}(t)\frac{\langle\gamma_i(t),\gamma_j(t)\rangle}{\|\gamma_i(t)\|\|\gamma_j(t)\|},\qquad i,j=1,\dots,n.$$

We use the Rebonato parametrization:

$$g(t)=g_\infty+(1-g_\infty)e^{-bt},\qquad a,b,g_\infty>0,$$

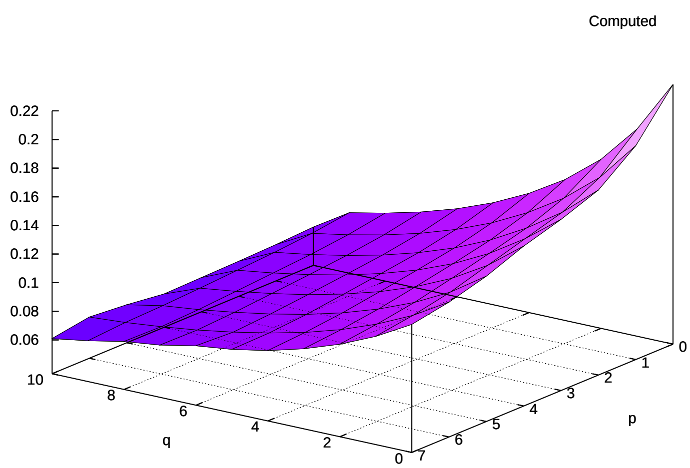
where the correlation coefficients ρ_{ij} are parametrized by η_1, η_2 and ρ_∞. This yields an expression

$$\sigma_{i,j}(b,g_\infty,\eta_1,\eta_2,\rho_\infty)$$

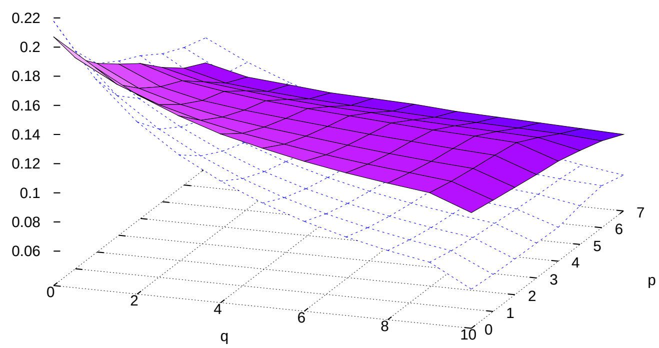
of $\sigma_{i,j}$ as a function of $b,g_\infty,\eta_1,\eta_2,\rho_\infty$, cf. [2]. Following [2] we minimize the mean square distance

$$\mathrm{RMS}(b,g_\infty,\eta_1,\eta_2,\rho_\infty):=\sqrt{\frac{2}{(n-1)(n-2)}\sum_{1\le i\le j-2,\ j\le n}\left(\frac{\sigma^B_{i,j}-\sigma_{i,j}}{\sigma^B_{i,j}}\right)^2},$$

and for this we use the Broyden-Fletcher-Goldfarb-Shanno (BFGS) gradient descent method. The volatilities computed in this way are given by the following graph, the index q refers to T_q-t and p refers to T_p-T_q:



The next graph allows us to compare the estimated and computed volatilities.



The program also makes use the smile data as available by shifting the volatility data according to the following table, for swaps:

Vol Cap CMS Smile **2Y** **Shift compared to ATM**

	-5	-2	-1	-0,5	-0,25	0	0,25	0,5	1	2	5	10	20
1Y	17,65	3,75	1,25	0,25	-0,25	-0,35	-0,3	-0,1	0,45	1,55	5,65	11,4	16,65
5Y	14,4	1,9	-0,5	-1,6	-1,9	-2,1	-2,1	-2	-1,95	-1,2	1,8	5,9	10,1
10Y	9,6	0,4	-1,7	-2,6	-2,75	-2,95	-2,9	-2,85	-2,8	-2	0,35	3,55	7,4
15Y	8,1	-0,3	-1,9	-2,6	-2,7	-2,85	-2,85	-2,8	-2,7	-2,05	0,15	3,1	6,5
20Y	6,63	-0,78	-1,88	-2,38	-2,48	-2,58	-2,58	-2,53	-2,48	-2,13	0,03	2,73	5,73
30Y	5,05	-1,3	-2,05	-2,15	-2,25	-2,25	-2,25	-2,25	-2,25	-2,2	-0,25	2,15	4,8

10Y

	-5	-2	-1	-0,5	-0,25	0	0,25	0,5	1	2	5	10	20
1Y	17,65	3,75	1,25	0,25	-0,25	-0,35	-0,3	-0,1	0,45	1,75	5,9	11,9	17,3
5Y	13,5	1,9	-0,7	-1,55	-1,9	-2	-2	-1,85	-1,55	-0,5	2,75	6,95	11,05
10Y	8,9	0,1	-1,6	-2,3	-2,35	-2,4	-2,4	-2,3	-1,95	-1	1,7	5,25	8,8
15Y	6,8	-0,85	-2,1	-2,4	-2,55	-2,65	-2,5	-2,45	-2,05	-1,2	1,4	4,6	7,75
20Y	5,4	-1,05	-2,1	-2,25	-2,3	-2,35	-2,3	-2,2	-1,8	-1,15	1,3	4,2	7
30Y	3,8	-1,55	-2,05	-2,05	-2,05	-2,05	-2	-1,9	-1,7	-1,05	1,1	3,6	6

and for caps:

Vol Cap Libor Smile **1M** **Shift compared to ATM**

	-5	-2	-1	-0,5	-0,25	0	0,25	0,5	1	2	5	10	20
1Y	20	6,5	2,75	1	0,5	0	0,05	0,1	0,15	0,5	3,5	7	10,5
5Y	17,75	4,7	1,75	0,6	0,15	0	0,05	0,05	0,1	0,4	3,2	7	10,5
10Y	15,5	3,75	0,9	0,3	0,05	0	0,03	0,05	0,05	0,25	2,3	5,5	8
30Y	9	1,75	0,25	0,15	0	0	0	0	0,05	0,1	1,5	4	6

3M

	-5	-2	-1	-0,5	-0,25	0	0,25	0,5	1	2	5	10	20
1Y	20	6,5	2,75	1	0,5	0	0,05	0,1	0,15	0,5	3,5	7	10,5
5Y	17,75	4,7	1,75	0,6	0,15	0	0,05	0,05	0,1	0,4	3,2	7	10,5
10Y	15,5	3,75	0,9	0,3	0,05	0	0,03	0,05	0,05	0,25	2,3	5,5	8
30Y	9	1,75	0,25	0,15	0	0	0	0	0,05	0,1	1,5	4	6

8 User manual

Program files

The program directory contains 13 files:

this documentation file.

the source program file: ”calibrate_libor.cpp” written in C++

a binary executable file ”calibrate.out” generated by the compilation of ”calibrate_libor.cpp”

an output file of calibration results: ”cali_result”

an example of input market data: ”market_data.xml”

header and link files (altogether 8 files): ”min.h” ”min.cpp” ”routines.c” ”routines.cpp” ”routines.o” ”f2c.h” ”libf2c.a” ”iterate”

Compilation

Compilation command under Linux:

g++ calibrate_libor.cpp min.cpp routines.cpp libf2c.a -o calibrate

generates the executable binary file ”calibrate”.

Data input

Before running the calibration routine, the user should prepare the data according to the format described below, or use the existing example ”market_data.xml” which is extracted from [2].

Execution

Type ./calibrate to run the calibration program.
Five parameters can be calibrated in this program:

b: $0 \leq b \leq 10$, initial value 5.01, when b = 0, i.e. swaption volatility norm is constant 1, no meaning for calibrate ginf.

ginf: $0 \leq \text{ginf} \leq 1$, initial value 0.56.

eta1: $0 \leq \text{eta1} \leq 2$, initial value 1.22.

eta2: $0 \leq \text{eta2} \leq 1$, initial value 0.001.

rhoinf: $0 \leq \text{rhoinf} \leq 1$, initial value 0.29

Each of the five parameters b, ginf, eta1, eta2, rhoinf can be calibrated separately.

1. The user is allowed to choose the parameters to be calibrated.
2. If a parameter is calibrated, upper and lower bounds and initial search condition should be provided for this parameter. Default values are suggested by the program, but they can be changed under the condition lower bound $<$ initial value $<$ upper bound.
3. If a parameter is not calibrated, its value should be fixed by the user. Default values are suggested by the program.

The program computes the maximum swaption maturity MS available in the market data file (in years) and the user is asked to provide the number of swaption maturities to be used for calibration (in years, ranging from 1 to MS).

Answering "yes" to a question sets the value of a parameter to its suggested default value.

Output of results

Results are pressented at the end of program and in the result file "cali_result"

XML file structure

In this section we describe in detail the format used to structure the input data in XML.

The header of the file consists in the following information:

1. <date> calendar date </date>
2. <delta delta="0.5"> unit of tenor structure is 0.5 year </delta>
3. <maxnumber> maximum number of units in tenor structure +1 </maxnumber>

The data is then formatted in three main sections:

Discount factors

<disfact> This data section is relative to discount factors.

<dismaturity> list of discount factor maturities </dismaturity>

<discountfactor> corresponding discount factors </discountfactor>

<srmaturity> list of swap rate maturities </srmaturity>

<swaprate> corresponding swap rates </swaprate>

</disfact>

The user can provide discount factors or alternatively swap rates which can be used to derive the discount factors.
In case discount factors are provided, the data of discount factors and corresponding maturity should be put in <dismaturity> and <discountfactor> and <srmaturity> and <swaprate> should be filled with zeros. Otherwise the swap rate and corresponding maturities should be put in <srmaturity> and <swaprate> and <dismaturity> and <discountfactor> should be filled with zeros.

Caplet volatilities

<capvola> This data section containes the caplet volatilities.

<atmcapmaturity> list of ATM caplet maturities </atmcapmaturity>

<atmcapvolatility> ATM caplet volatilities </atmcapvolatility>

<itmotmcapmaturity> ITM or OTM caplet maturities </itmotmcapmaturity>

<itmotmcapshiftmaturity> list of ITM or OTM caplet volatility shifts </itmotmcapshiftmaturity>

<itmotmcapshift> ITM or OTM caplet volatility shifts </itmotmcapshift>

<capshiftmaturity> list of caplet shift maturities </capshiftmaturity>

<capshift> caplet volatility shifts </capshift>

<capsmile> caplet volatility smiles </capsmile>

</capvola>
The user can provide ATM caplet volatilities or ITM or OTM caplet volatilities as well as caplet volatility smiles if available from the data. In case ATM data is provided, put zeros inside <itmotmcapmaturity>. Otherwise, put zeros inside <atmcapmaturity>.

Swaption volatilities

<swapvola> This data section contains the swaption volatilities.

<atmswapmaturity> list of ATM swaption maturities </atmswapmaturity>

<atmswapexpiry> list of ATM swaption expiry dates </atmswapexpiry>

<atmswapvolatility> corresponding ATM swaption volatilities </atmswapvolatility>

<itmotmswapmaturity> list of ITM or OTM swaption maturities </itmotmswapmaturity>

<itmotmswapexpiry> list of ITM or OTM swaption expiry dates </itmotmswapexpiry>

<itmotmswapvolatility> corresponding ITM or OTM swaption volatilities </itmotmswapvolatility>

<itmotmswapshiftmaturity> list of ITM or OTM swaption shift maturities </itmotmswapshiftmaturity>

<itmotmswapshift> ITM or OTM swaption volatility shifts </itmotmswapshift>

<swapshiftmaturity> list of swaption shift maturities </swapshiftmaturity>

<swapshift> swaption volatility shifts </swapshift>

<swapsmile> swaption volatility smiles </swapsmile>

</swapvola>

9 Sample data file

In this section we show how the data of [2] is encoded in our format.


```
<?xml version="1.0" encoding="ISO-8859-1"?>
<!DOCTYPE  data [
        <!ELEMENT date (#PCDATA)>
<!ELEMENT delta (#PCDATA)>
<!ELEMENT maxnumber (#PCDATA)>
<!ELEMENT disfact (dismaturity, discountfactor,srmaturity,swaprate)>
<!ELEMENT capvol (atmcapmaturity,atmcapvolatility,itmotmcapmaturity,itmotmcapvolatility,
                  itmotmshiftmaturity,itmotmshift,shiftmaturity,shift_to_atm,capsmile)>
<!ELEMENT swapvola (atmswapmaturity,atmswapexpiry,atmswapvolatility,itmotmswapmaturity,
                  itmotmswapexpiry,itmotmswapvolatility,itmotmswapshiftmaturity,
                  itmotmswapshift,swapshiftmaturity,swapshift,swapsmile)>
]>

<data>

<date> 18/10/2001 </date>
<delta  delta="0.5"> </delta>
<maxnumber> 42 </maxnumber>

<disfact>
<dismaturity>
1  2  3  4  5  6  7  8  9  10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41
</dismaturity>
<discountfactor>
0.98260 0.96675 0.94967 0.93160 0.91248 0.89262 0.87222 0.85132 0.83017 0.80875
0.78748 0.76618 0.74526 0.72449 0.70415 0.68409 0.66450 0.64527 0.62656 0.60826
0.59043 0.57295 0.55574 0.53894 0.52280 0.50712 0.49174 0.47666 0.46189 0.44767
0.43434 0.42161 0.40917 0.39704 0.38519 0.37383 0.36255 0.35136 0.34063 0.33033
0.32064
</discountfactor>
<srmaturity>
0
</srmaturity>
<swaprate>
0
</swaprate>
</disfact>

<capvola>
```

```
<atmcapmaturity>
0
</atmcapmaturity>
<atmcapvolatility>
23.25  22.97  21.50  20.03  19.06  17.95  17.16  16.38  15.89  15.40
14.90  14.41  14.09  13.77  13.47  13.16  12.95  12.74  12.57  12.40
12.325 12.25  12.175 12.10  12.048 11.99  11.94  11.89  11.84  11.79
11.751 11.712 11.673 11.634 11.595 11.556 11.517 11.478 11.439 11.40
</atmcapvolatility>
<itmotmcapmaturity>
1 2 3 4 5 6 10 15 20 30
</itmotmcapmaturity>
<itmotmcapvolatility>
23.25  22.97  21.50  20.03  19.06 17.95 15.40 13.47 12.40 11.79
</itmotmcapvolatility>
<itmotmcapshiftmaturity>
1 2 3 5 7 10 15 20
</itmotmcapshiftmaturity>
<itmotmcapshift>
-2 -2 3 1.5 4 -2 0.5 2
</itmotmcapshift>
<capshiftmaturity>
1 5 10 30
</capshiftmaturity>
<capshift>
-5 -2 -1 -0.5 -0.25 0 0.25 0.5 1 2 5 10 20
</capshift>
<capsmile>
20 6.5 2.75 1 0.5 0 0.05 0.1 0.15 0.5 3.5 7 10.5
17.75 4.7 1.75 0.6 0.15 0 0.05 0.05 0.1 0.4 3.2 7 10.5
15.5 3.75 0.9 0.3 0.05 0 0.03 0.05 0.05 0.25 2.3 5.5 8
9 1.75 0.25 0.15 0 0 0 0 0.05 0.1 1.5 4 6
</capsmile>
</capvola>

<swapvola>
<atmswapmaturity>
2 4 6 8 10 12 14 16 18 20 30
</atmswapmaturity>
<atmswapexpiry>
2 4 6 8 10 12 14 16 18 20 30
</atmswapexpiry>
<atmswapvolatility>
20.71 18.89 17.32 16.16 15.21 14.53 13.92 13.42 13.01 12.65  11.57
18.12 16.59 15.49 14.71 14.11 13.65 13.22 12.87 12.58 12.28  11.28
16.58 15.17 14.35 13.78 13.38 13.06 12.73 12.45 12.21 12.01  11.01
15.39 14.13 13.48 13.11 12.83 12.58 12.33 12.14 11.94 11.77  10.74
14.28 13.39 12.95 12.60 12.35 12.15 11.95 11.76 11.64 11.48  10.51
12.86 12.16 11.84 11.54 11.34 11.22 11.02 10.90 10.80 10.69  0
11.66 10.93 10.65 10.43 10.28 10.17 10.05 9.98  9.89  9.80   0
10.87 10.19 9.95  9.70  9.60  0     0     0     0     0      0
</atmswapvolatility>
<itmotmswapmaturity>
0
</itmotmswapmaturity>
<itmotmswapexpiry>
2 4 6 8 10 12 14 16 18 20 30
</itmotmswapexpiry>
<itmotmswapvolatility>
20.71 18.89 17.32 16.16 15.21 14.53 13.92 13.42 13.01 12.65  11.57
18.12 16.59 15.49 14.71 14.11 13.65 13.22 12.87 12.58 12.28  11.28
16.58 15.17 14.35 13.78 13.38 13.06 12.73 12.45 12.21 12.01  11.01
15.39 14.13 13.48 13.11 12.83 12.58 12.33 12.14 11.94 11.77  10.74
14.28 13.39 12.95 12.60 12.35 12.15 11.95 11.76 11.64 11.48  10.51
12.86 12.16 11.84 11.54 11.34 11.22 11.02 10.90 10.80 10.69  0
```

```
11.66 10.93 10.65 10.43 10.28 10.17 10.05 9.98  9.89  9.80   0
10.87 10.19 9.95  9.70  9.60  0     0     0     0     0      0
&lt;/itmotmswapvolatility&gt;
&lt;itmotmswapshiftmaturity&gt;
1 2 3 5 7 10 15 20
&lt;/itmotmswapshiftmaturity&gt;
&lt;itmotmswapshift&gt;
-2 -2 3 1.5 4 -2 0.5 2
&lt;/itmotmswapshift&gt;
&lt;swapshiftmaturity&gt;
2 10 20 40 60
&lt;/swapshiftmaturity&gt;
&lt;swapshift&gt;
-5 -2 -1 -0.5 -0.25 0 0.25 0.5 1 2 5 8 10 12 20
&lt;/swapshift&gt;
&lt;swapsmile&gt;
18 4.5 1.5 0.5 0.25 0 0.05 0.3 0.8 2 6 9.5 11.5 13 17
16.5 4.6 1.6 0.7 0.25 0 -0.1 -0.1 0 0.8 3.5 6 7.35 8.3 11.5
12.75 3.8 1.45 0.6 0.2 0 -0.1 -0.1 -0.1 0.7 2.75 4.55 5.85 6.85 9.5
9.5 2.3 0.8 0.3 0.05 0 -0.1 -0.1 -0.05 0.2 2 3.55 4.55 5.35 7.65
8 1.4 0.5 0.2 0.05 0 -0.1 -0.05 -0.05 0.1 1.6 3.05 3.85 4.5 6.6
&lt;/swapsmile&gt;
&lt;/swapvola&gt;

&lt;/data&gt;
```

References

[1] D. Brigo and F. Mercurio. *Interest rate models—theory and practice*. Springer Finance. Springer-Verlag, Berlin, second edition, 2006.

[2] J. Schoenmakers. Calibration of LIBOR models to caps and swaptions: a way around intrinsic instabilities via parsimonious structures and a collateral market criterion. WIAS Preprint No 740, Berlin, 2002.

[3] J. Schoenmakers. *Robust Libor modelling and pricing of derivative products*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2005.