

A Multifactor Volatility Heston Model

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The following method computes the price of call and put options on a single risky asset whose volatility follows a multifactor Wishart process. It is based on the paper [1].

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1 Introduction

This method computes the price of call and put options on a single risky asset whose volatility follows a multifactor Wishart process by using the Fast Fourier Transform as in [2].

2 Theoretical framework

We consider an arbitrage-free frictionless financial market and a risky asset whose price follows

$$\frac{dS_t}{S_t} = r dt + Tr \left(\sqrt{\Sigma_t} dZ_t \right),$$

where r denotes the risk-free interest rate, Tr is the trace operator, $Z_t \in M_n$ (the set of square matrices of size n) is a matrix of Brownian motions (i.e. composed by n^2 independent Brownian motions) under the risk neutral measure and Σ_t belongs to the set of symmetric $n \times n$ positive definite matrices (as well as its square root $\sqrt{\Sigma_t}$). Σ_t is assumed to follow the dynamics

$$d\Sigma_t = (\beta Q^T Q + M \Sigma_t + \Sigma_t M^T) dt + \sqrt{\Sigma_t} dW_t Q + Q^T (dW_t)^T \sqrt{\Sigma_t}, \quad (1)$$

with M and Q in M_n , $W_t \in M_n$ is a matrix Brownian motion and $\beta \geq n - 1$ (T represents the transposition). M is a semi-negative matrix. (1) characterizes the Wishart process introduced by Bru (1991).

We also assume correlation between the noises driving the asset and the noises driving the volatility process :

$$\forall k, \text{Cov}(Z_t^{i,k}, W_t^{k,j}) = R_{i,j}.$$

This leads to a constant matrix $R \in M_n$ which completely describes the correlation structure, i.e. $Z_t := W_t R^T + B_t \sqrt{\mathbb{I} - R R^T}$ (\mathbb{I} represents the identity matrix).

3 The pricing problem solved by FFT

We consider a Call option with payoff $(S_T - K)_+$. In order to solve the pricing problem of such a vanilla option, it is enough to compute the conditional characteristic function of the underlying, or equivalently of the return process $Y_t = \ln(S_t)$, which satisfies

$$dY_t = \left(r - \frac{1}{2} \text{Tr}(\Sigma_t)\right)dt + \text{Tr} \left(\sqrt{\Sigma_t} (dW w_t R^T + dB_t \sqrt{\mathbb{I} - R R^T}) \right).$$

3.1 Characteristic function of the asset returns

From [1, Section 3.1], we get

$$\Phi_{\gamma,0}(T) := \mathbb{E}(\exp(\gamma Y_T)) = \exp(\text{Tr}(A(T)\Sigma_0) + b(T) \ln(S_0) + c(T)),$$

where for all t

$$\begin{aligned} A(t) &= A_{22}(t)^{-1} A_{21}(t), \\ b(t) &= \gamma, \\ c(t) &= -\frac{\beta}{2} \text{Tr} \left(\ln(A_{22}(t)) + (M^T + 2\gamma R Q)T \right) + \gamma r T \end{aligned}$$

and

$$\begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} = e^T \begin{pmatrix} M & -2Q^T Q \\ \frac{\gamma(\gamma-1)}{2} \mathbb{I}_n & -(M^T + 2\gamma R Q) \end{pmatrix}$$

3.2 Fast Fourier Transform

The following method is issued from [2]. For fixed $\alpha > 0$, we consider the scaled price at time 0 as

$$c_T(k) := e^{\alpha k} \mathbb{E}(e^{-rT} (S_T - K)_+) = e^{\alpha k} \mathbb{E}(e^{-rT} (e^{Y_T} - e^k)_+),$$

where $k = \ln(K)$. Its Fourier transform is given by

$$\begin{aligned} \Psi_T(v) &:= \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk, \\ &= e^{-rT} \frac{\Phi_{(\alpha+1)+iv,0}(T)}{(\alpha + iv)(\alpha + 1 + iv)} \end{aligned}$$

The price of a Call option of strike K is given by

$$\begin{aligned} \text{Call}(k) &= \frac{e^{-\alpha k}}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \Phi_T(v) dv, \\ &= \frac{e^{-\alpha k}}{\pi} \int_0^{\infty} e^{-ivk} \Phi_T(v) dv \end{aligned} \tag{2}$$

4 Algorithm

In order to compute (2), we use a Simpson approximation to compute the integral on $[0, a]$

$$Call(k) \sim \frac{e^{-\alpha k}}{\pi} \left(\Psi_T(0) + \Psi_T(a) + \sum_{j=1}^{N-1} e^{-i\eta j k} \Psi_T(j\eta) \frac{\eta}{3} (3 + (-1)^{j+1}) \right)$$

where $\eta := \frac{a}{N}$. In the following, N and a are fixed ($N = 4096$ and $a = 200\pi$). We can also use the FFT method, by writing

$$Call(k_u) \sim \frac{e^{-\alpha k_u}}{\pi} \sum_{j=1}^N e^{-\frac{2i\pi}{N}(j-1)(u-1)} x(j),$$

where $k_u := -\frac{N\pi}{a} + \frac{2\pi}{a}(u-1)$ and $x(j) = e^{i\pi(j-1)} \psi((j-1)\eta) \frac{\eta}{3} c_j$, where $c_1 = 1$, $c_N = 1$, $c_{2j} = 4$ and $c_{2j+1} = 2$.

5 Numerical experiments

We test the algorithm with the following parameters, for different maturities :

r	S_0	K	β	n
0	100	110	3	2

(β should be greater than $n-1$) and

$$M = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}, R = \begin{pmatrix} -0.7 & 0 \\ 0 & -0.7 \end{pmatrix}$$

$$Q = \begin{pmatrix} 0.25 & 0 \\ 0 & 0.25 \end{pmatrix}, \Sigma_0 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}$$

M should be a semi-definite negative matrix.

T	0.25	2
Price	10.554	19.034
Implied Volatility	0.16710	0.228

References

- [1] J. Da Fonseca, M. Grasselli, C. Tebaldi. A Multifactor Volatility Heston Model *Quantitative Finance*, 2008, vol. 8, issue 6, pages 591-604. [1](#), [2](#)
- [2] P. Carr, D.B. Madan Option Valuation using the Fast Fourier Transform *Journal of Computational Finance*, 1992, vol. 2, issue 4. [1](#), [2](#)