Recovering portfolio default intensities implied by CDO quotes

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February 18, 2016

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# 1 Introduction

“Top-down" models for portfolio credit derivatives have been introduced as an alternative to the market standard copula model, that would avoid its most important shortcomings - namely its static character which prevents any model-based assessment of hedging strategies and its proven inability to calibrate to the market values of CDO spreads - while allowing for analytical tractability. Top down models correspond to a “reduced form" of the portfolio loss dynamics, as a jump process whose intensity $\lambda_t$ represents the (conditional) rate of occurrence of the next default and whose jump sizes represent losses given default.

In the aim of calibrating such models in [3], we first asses the information contained in market data. We show a “mimicking theorem" for point processes which states that the marginal distributions of a loss process $L$ with arbitrary stochastic intensity $\lambda$ can be matched using a *Markovian* point process $\tilde{L}$ (the Markovian projection of $L$) with (effective) intensity

$$\lambda_{\text{eff}}(t,l) = E^{\mathbb{Q}}[\lambda_t | L_{t-} = l, \mathcal{F}_0]. \tag{1}$$

The relation between $\lambda$ and $\lambda_{\text{eff}}$ is analogous to the relation between instantaneous and local volatility in diffusion models (see Dupire [6]).

Regarding our application, this implies that values of credit derivatives such as CDOs (and more generally any derivative whose payoff depends continuously on the aggregate loss $L_T$ of the portfolio on a fixed grid of dates), depends in any top down model on the intensity $\lambda$ only through the effective default intensity $\lambda_{\text{eff}}(.,.)$.

## 1.1 Forward equations for expected tranche notionals

Being able to mimick the marginal distribution of the loss processes using a Markovian model allows for considerable simplification of pricing and calibration algorithms. First, for a Markovian jump process the transition probabilities can be computed by solving a Fokker Planck equation. In the sequel, we consider a constant loss given default $\delta$, so if we denote by $N_t$ the number of defaults in the portfolio, we have $L_t = \delta N_t$. The transition probabilities

$q_j(0,T) = \mathbb{Q}(N_T = j|\mathcal{F}_0)$ *also* solve the Fokker-Planck equation corresponding to the effective intensity: for $T \geq 0$,

$$\begin{aligned} \frac{dq_0}{dT}(0,T) &= -\lambda_{\text{eff}}(T,0)q_0(0,T) \\ \frac{dq_j}{dT}(0,T) &= -\lambda_{\text{eff}}(T,j)q_j(0,T) + \lambda_{\text{eff}}(T,j-1)q_{j-1}(0,T) \\ \frac{dq_n}{dT}(0,T) &= \lambda_{\text{eff}}(T,n-1)q_{n-1}(0,T) \quad \text{with initial conditions} \\ q_j(0,0) &= \mathbb{1}_{\{N_0=j\}} \; \forall j = 1,\ldots,n. \end{aligned} \tag{2}$$

Moreover, by analogy with the Dupire equation for diffusion models [6], one can show that the expected tranche notional $P(T,K)$ can be obtained by solving a (single) forward equation [4]:

$$\begin{aligned} \frac{\partial P(T,K)}{\partial T} &- P(T,K-\delta)\lambda_k(T) + \lambda_{k-1}(T)P(T,K) \\ &+ \sum_{j=1}^{k-2}[\lambda_{j+1}(T) - 2\lambda_j(T) + \lambda_{j-1}(T)]\; P(T,j) = 0 \end{aligned} \tag{3}$$

where $\lambda_k(T) = \lambda_{\text{eff}}(T,k\delta)$. This is a bidiagonal system of ODEs which can be solved efficiently in order to compute the expected tranche notionals (and thus the values of CDO tranches) given the local intensity function $\lambda_{\text{eff}}(.,.)$ without Monte Carlo simulation.

## 2 Calibration

Having stated these results, we proceed to solving the ill-posed problem of calibrating to the market spreads the effective default intensity associated to the loss process. We formalize this problem in terms of the minimization of relative entropy with respect to the law of a prior loss process under calibration constraints. We are given the spreads for the $I$ tranches of the portfolio, at $m$ maturities. The payment dates are denoted $(t_j, j = 1,\ldots,J)$. At $t = 0$ we observe the tranche spreads $(S_0(K_i,K_{i+1},T_k), i = 1,\ldots,I-1, k = 1,\ldots,m)$ and the upfront fee $(U_0(K_1,T_k), k = 1,\ldots,m)$ for equity tranches.

**Problem 1** (Calibration via relative entropy minimization)**.** *Given a prior loss process with law $\mathbb{Q}_0$, find a loss process with law $\mathbb{Q}^\lambda$ and default intensity $(\lambda_t)_{t\in[0,T^*]}$ which minimizes*

$$\inf_{\mathbb{Q}^\lambda\in\mathbb{M}} E^{\mathbb{Q}_0}[\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}\ln\frac{d\mathbb{Q}^\lambda}{d\mathbb{Q}_0}] \quad \text{under} \quad E^{\mathbb{Q}^\lambda}[H_{i,k}|\mathcal{F}_0] = 0, \quad i = 0,\ldots,I-1, \qquad k = 1,\ldots,m \tag{4}$$

*where,*

$$
\begin{aligned}
H_{ik} &= S_0(K_i, K_{i+1}, T_k) \sum_{t_j \le T_k} B(0,t_j)(t_j - t_{j-1})[(K_{i+1} - L(t_j))^+ - (K_i - L(t_j))^+] \\
&\quad + \sum_{t_j \le T_k} B(0,t_j)[(K_{i+1} - L(t_j))^+ - (K_i - L(t_j))^+ - (K_{i+1} - L(t_{j-1}))^+ + (K_i - L(t_{j-1}))^+ )\ ] \quad (5) \\
H_{0k} &= K_1 U_0(K_1, T_k) + f \sum_{t_j \le T_k} B(0,t_j)(t_j - t_{j-1})[(K_1 - L(t_j))^+] \\
&\quad + \sum_{t_j \le T_k} B(0,t_j)[(K_1 - L(t_j))^+ - (K_1 - L(t_{j-1}))^+ )\ ]. \quad (6)
\end{aligned}
$$

The primal problem (Problem 1) is an infinite-dimensional constrained optimization problem whose solution does not seem obvious. A key advantage of using the relative entropy as a calibration criterion is that it can be computed explicitly in the case of point processes. The constrained optimization problem (4) can then be simplified by introducing Lagrange multipliers and using convex duality methods [5, 7].

**Proposition 1** (Duality)**.** *The primal problem (4) is equivalent to*

$$
\sup_{\mu \in \mathbb{R}^{m.I}} \inf_{\lambda \in \Lambda} E^{\mathbb{Q}^\lambda}[\int_0^T (\lambda_s \ln \frac{\lambda_s}{\gamma_s} + \gamma_s - \lambda_s) ds - \sum_{i=0}^{I-1} \sum_{k=1}^{m} \mu_{i,k} H_{ik}. \quad (7)
$$

The inner optimization problem

$$
J(\mu) = \mathcal{L}(\lambda^*(\mu), \mu) = \inf_{\lambda \in \Lambda} \mathcal{L}(\lambda, \mu)
$$

is an example of an *intensity control* problem [1, 2]: the optimal choice of the intensity of a jump process in order to minimize a criterion of the type

$$
E^{\mathbb{Q}^\lambda}[\int_0^T \varphi(t, \lambda_t, N_t) dt + \sum_{j=1}^{J} \Phi_j(L_{t_j})], \quad (8)
$$

where $t_j,\ j = 1, \ldots, J$ are the spread payment dates, $\varphi(t, \lambda_t, N_t)$ is a *running cost* and $\Phi_j(L)$ represents a “terminal" cost. In our case

$$
\varphi(t, x, k) = x \ln \frac{x}{g(t,k)} + g(t,k) - x \text{ and } \Phi_j(L) = \sum_{i=1}^{I-1} M_{ij} (K_i - L)^+ \qquad , \quad (9)
$$

where

$$
\begin{aligned}
M_{ij} &= B(0, t_{j+1}) \sum_{T_k \ge t_{j+1}} (\mu_{ik} - \mu_{i-1,k}) + \\
&\quad B(0,t_j) \sum_{T_k \ge t_j} [\mu_{ik}(-1 - \Delta S(K_i, K_{i+1}, T_k)) - \mu_{i-1,k}(1 - \Delta S(K_{i-1}, K_i, T_k)], \quad (10)
\end{aligned}
$$

with $\Delta = t_j - t_{j-1}$ is the interval between payments and $S(K_0, K_1, T_k) = f$.

The solution of an intensity control problem can be obtained using a dynamic programming principle and is characterized in terms of a system of Hamilton-Jacobi equations [2, Ch. VII]. We will now use these properties to solve (8). Once the inner optimization/ intensity control problem has been solved we have to solve the outer problem by optimizing $J(\mu)$ over the Lagrange multipliers $\mu \in \mathbb{R}^{mI}$: the corresponding optimal control $\lambda^*$ then yields precisely the default intensity which calibrates the observations.

### 2.1 Hamilton Jacobi equations

Let us consider the case where $J = 1$ i.e a single time horizon is involved (the general case can be treated similarly). The dual problem is then to minimize

$$\inf_{\lambda \in \Lambda} E^{\mathbb{Q}^\lambda}[\int_0^T \varphi(t, \lambda_t, N_t)dt + \Phi(T, L_T)] \tag{11}$$

where $\Phi(.)$ is of the form (9) (and thus depends on the Lagrange multipliers $\mu$). The solution of the stochastic control problem (7) can be obtained using dynamic programming methods [1, 2]. The idea is to define a family of optimization problems indexed by the initial condition $(t, n)$,

$$V(t, N_t) = \inf_{\lambda \in \Lambda([t,T])} E^{\mathbb{Q}^\lambda}[\int_t^T (\lambda_s \ln \frac{\lambda_s}{\gamma_s} + \gamma_s - \lambda_s)ds + \Phi(T, \delta N_T))|\mathcal{H}_t] \tag{12}$$

where $\delta = (1 - R)/n$ is the loss given a single default and $\Lambda([t, T])$ is the set of restrictions to $[t, T]$ of elements of $\Lambda$. The value function $V(t, k)$ then solves the dynamic programming equation [2]:

$$\frac{\partial V}{\partial t}(t, k) + \inf_{\lambda \geq 0}\{\lambda[V(t, k + 1) - V(t, k)] + \lambda \ln \frac{\lambda}{g(t, k)} - \lambda + g(t, k)\} = 0 \tag{13}$$

$$\text{for} \quad t \in [0, T] \quad \text{and} \quad V(T, k) = \Phi(T, k\delta)). \tag{14}$$

The value function of (11) is then given by $V(0, 0)$ and the optimal intensity control is obtained by maximizing over $\lambda$ in the nonlinear term [2]

**Proposition 2** (Value function)**.** *Consider a function $\Phi$ such that $\Phi(x) = 0$ for $x \geq n\delta$. The solution of (13)-(14) has the probabilistic representation*

$$V(t, k) = -\ln[1 + \sum_{j=0}^{n-k} \mathbb{Q}_0(N_T = k + j|N_t = k)(e^{-\Phi(T,(k+j)\delta)} - 1)]. \tag{15}$$

### 2.2 Calibration algorithm

The above results lead to a non-parametric algorithm for recovering a market-implied portfolio default intensity from CDO spreads. The algorithms consists of the following steps:

1. Solve the dynamic programming equations (13)–(14) for $\mu \in \mathbb{R}^{m.I}$ to compute $V(0, 0, \mu)$.

2. Solve the maximization problem

$$\sup_{\mu \in \mathbb{R}^{m.I}} V(0,0,\mu) + \sum_{k=1}^{m} \mu_{0k} U_0(K_1, T_k)$$

   using a gradient–based method to obtain the Lagrange multipliers $\mu^*$.

3. Compute the calibrated default intensity (optimal control) as follows:

$$\lambda^*(t,k) = \gamma(t,k) e^{V^*(t,k) - V^*(t,k+1)}. \tag{16}$$

4. Compute the term structure of loss probabilities by solving the Fokker-Planck equations (2).

5. The calibrated default intensity $\lambda^*(.,.)$ can then be used to compute CDO spreads for different tranches, forward tranches, etc.: first we compute the expected tranche notionals $P(T,K)$ by solving the forward equation (3) and then use the expected trance notionals to evaluate CDO tranche spreads, mark to market value, etc. In particular the calibrated default intensity can be used to "fill the gaps" in the base correlation surface in an arbitrage-free manner, by first computing the expected tranche loss for all strikes and then computing the base correlation for that strike.

We use convex duality techniques to solve the problem: the dual problem is shown to be an intensity control problem, characterized in terms of a Hamilton-Jacobi system of differential equations which can be analytically solved using a change of variable. Given a set of observed CDO tranche spreads, we have thus proposed a stable method to construct an implied intensity process $\lambda_{eff}(t, L_t)$ calibrated to the market spreads. The intensity of a new default depends steeply on the number of defaults in the portfolio, which leads to contagion effects and clustering in the occurrence of defaults. This is in accordance with properties observed in data series.

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