Pricing Foreign Exchange Options and Inflation Products in Heston model incorporate with stochastic interest rate model: Implementation in PREMIA

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Abstract

We price the foreign exchange rate (FX) options and year on year-on-year (YoY) inflation caps under the Heston type models for FX and inflation index CPI respectively. In combination with these two models, we assume the Hull-White interst rate model for the domestic and the foreign interest rate, and also for the nominal and the real interest rate. Since these models are not affine, the approximations of the non-affine terms are proposed so that the approximations of the characteristic function of the forward FX and the forward inflation index are obtained, then by the Fourier-Cosine method we can efficiently calculated the price of FX options and inflation products.

1 Introduction

We assume both of FX and CPI follow the Heston-type stochastic volatility model and both models incorperate with the Hull-White interest rate models. Under such model settings, we present an efficient pricing method for the FX options and the inflation products by using Fourier-Cosine expansions. We also implement both methods in PREMIA.

The rest of this file is organized as follows. We present the FX models and the FX options description in Section 2, and then introduce the inflation index models and inflation products in Section 3. The sketch of the Fourier-Cosine expansion pricing method will be stated in Section 4. Section 5 is the program manual for the implementations of both pricing methods.

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2 The FX Model and the FX Options

2.1 The Domestic and Foreign Interest Rate Models

We assume that the domestic interest rate $r_d(t)$ and the foreign interest rate $r_f(t)$ are deriven by the Hull-White one-factor model under their spot measures, i.e. $\mathbb{Q}$-domestic and $\mathbb{Z}$-foreign respectively:

$$\mathrm{d}r_d(t) = \lambda_d(\theta_d(t) - r_d(t))\mathrm{d}t + \eta_d \mathrm{d}W_d^{\mathbb{Q}}(t), \tag{1}$$

$$\mathrm{d}r_f(t) = \lambda_f(\theta_f(t) - r_f(t))\mathrm{d}t + \eta_f \mathrm{d}W_f^{\mathbb{Z}}(t), \tag{2}$$

where $W_d^{\mathbb{Q}}(t)$ and $W_f^{\mathbb{Z}}(t)$ are Brownian motions under $\mathbb{Q}$ and $\mathbb{Z}$, respectively. Parameters λ_d, λ_f determine the speed of mean reversion to the time-dependent term structure functions $\theta_d(t), \theta_f(t)$ and parameters η_d, η_f are the volatility coefficients.

Under the domestic and foreign interest rate models (1) and (2), the dynamics of the domestic and foreign zero-coupon bond prices,

$$P_d(t,T) := \mathrm{e}^{-\int_t^T r_d(s)}\mathrm{d}s \quad \text{and} \quad P_f(t,T) := \mathrm{e}^{-\int_t^T r_f(s)}\mathrm{d}s,$$

are given by the stochastic differential equations as follows:

$$\mathrm{d}P_d(t,T) = r_d(t)\mathrm{d}t + \eta_d B_d(t,T)\mathrm{d}W_d^{\mathbb{Q}}(t),$$

$$\mathrm{d}P_f(t,T) = r_f(t)\mathrm{d}t + \eta_f B_f(t,T)\mathrm{d}W_f^{\mathbb{Z}}(t),$$

where

$$B_d(t,T) = \frac{1}{\lambda_d}\left[\mathrm{e}^{-\lambda_d(T-t)} - 1\right], B_f(t,T) = \frac{1}{\lambda_f}\left[\mathrm{e}^{-\lambda_f(T-t)} - 1\right]. \tag{3}$$

2.2 The FX Models

The spot FX $\xi(t)$ expressed in units of domestic currency, per unit of a foreign currency, is assumed to follow the Heston-type dynamics:

$$\mathrm{d}\xi(t)/\xi(t) = [r_d(t) - r_f(t)]\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W_\xi^{\mathbb{Q}}(t), \tag{4}$$

where the volatility $\nu(t)$ follows

$$\mathrm{d}\nu(t) = \kappa[\overline{\nu} - \nu(t)]\mathrm{d}t + \gamma\sqrt{\nu(t)}\mathrm{d}W_\nu^{\mathbb{Q}}(t), \tag{5}$$

where $W_\xi^{\mathbb{Q}}(t), W_\nu^{\mathbb{Q}}(t)$ are Brownian motions under the domestic risk-neutral measure $\mathbb{Q}$, the parameters $\kappa, \overline{\nu}$ determine the speed and the long term mean of the volatility and γ is the volatility of volatility for the FX.

By changing the spot foreign measure $\mathbb{Z}$ to the spot domestic measure $\mathbb{Q}$, we have the dynamics of the foreign short rate $r_f(t)$ as

$$\mathrm{d}r_f(t) = \left[\lambda_f(\theta_f(t) - r_f(t)) - \eta_f\rho_{\xi,f}\sqrt{\nu(t)}\right]\mathrm{d}t + \eta_f \mathrm{d}W_f^{\mathbb{Q}}(t), \tag{6}$$

where $\rho_{\xi,f}$ is the correlation of the $\mathbb{Q}$-measured Brownian motions $W_\xi^{\mathbb{Q}}(t)$ and $W_f^{\mathbb{Q}}(t)$. Then under the $\mathbb{Q}$ measure, the foreign zero-coupon bond price dynamics is

$$\mathrm{d}P_f(t,T) = \left[r_f(t) - \rho_{\xi,f}\eta_f B_f(t,T)\sqrt{\nu}(t)\right]\mathrm{d}t + \eta_d B_f(t,T)\mathrm{d}W_f^{\mathbb{Q}}(t). \tag{7}$$

Under the spot measure $\mathbb{Q}$, we assume a full matrix of correlations between the Brownian motions $W_\xi^{\mathbb{Q}}(t), W_\nu^{\mathbb{Q}}(t), W_d^{\mathbb{Q}}(t), W_f^{\mathbb{Q}}(t)$, i.e. $\rho_{i,j} := \mathrm{d}W_i^{\mathbb{Q}}(t) \cdot \mathrm{d}W_j^{\mathbb{Q}}(t)/\mathrm{d}t \neq 0$, when $i \neq j$ and $i, j \in \{\xi, \nu, d, f\}$.

2.3 The FX Models under Forward Measure

Direct calculation of the price of the FX options under the spot measure will result in solving a 4-dimentional PDE which is infeasible and unstable. To reduce the complexity of the pricing problem, we move from the spot measure to the forward measure where the numéraire is the domestic zero-coupon bond, $P_d(t,T)$.

The forward domestic measure $\mathbb{Q}^T$ is defined by Radon-Nikodym derivative as follows:

$$\Lambda_{\mathbb{Q}}^T(t) := \frac{\mathrm{d}\mathbb{Q}^T}{\mathrm{d}\mathbb{Q}} = \frac{P_d(t,T)}{P_d(0,T)M_d(t)}. \tag{8}$$

Denote

$$FX^T(t) := \xi(t)\frac{P_f(t,T)}{P_d(t,T)} \tag{9}$$

the forward FX under the T-forward measure. By Itô's formula, the dynamics of $FX^T(t)$ under measure $\mathbb{Q}$ is

$$\begin{aligned}
\mathrm{d}FX^T(t) \quad = \quad & \frac{P_f(t,T)}{P_d(t,T)}\mathrm{d}\xi(t) + \frac{\xi(t)}{P_d(t,T)}\mathrm{d}P_f(t,T) - \xi(t)\frac{P_f(t,T)}{P_d^2(t,T)}\mathrm{d}P_d(t,T) \\
& + \xi(t)\frac{P_f(t,T)}{P_d^3(t,T)}(\mathrm{d}P_d(t,T))^2 + \frac{1}{P_d(t,T)}(\mathrm{d}\xi(t)\mathrm{d}P_f(t,T)) \\
& - \frac{P_f(t,T)}{P_f^2(t,T)}(\mathrm{d}P_d(t,T)\mathrm{d}\xi(t)) - \frac{\xi(t)}{P_d^2(t,T)}\mathrm{d}P_d(t,T)\mathrm{d}P_f(t,T).
\end{aligned}$$

Then substitute the SDEs of $\xi(t), P_d(t,T)$ and $P_f(t,T)$, we have

$$\begin{aligned}
\frac{\mathrm{d}FX^T(t)}{FX^T(t)} \quad = \quad & \eta_d B_d(t,T)\left[\eta_d B_d(t,T) - \rho_{\xi,d}\sqrt{\nu(t)} - \rho_{d,f}\eta_f B_f(t,T)\right]\mathrm{d}t \qquad (10) \\
& + \sqrt{\nu(t)}\mathrm{d}W_\xi^{\mathbb{Q}}(t) - \eta_d B_d(t,T)\mathrm{d}W_d^{\mathbb{Q}}(t) + \eta_f B_f(t,T)\mathrm{d}W_f^{\mathbb{Q}}(t),
\end{aligned}$$

where $B_d(t,T)$ and $B_f(t,T)$ as defined in (3).

From the definition of Radon-Nikodym derivative (8), we can redefine the driven Brownian motions for $FX^T(t), \nu(t), r_d(t)$ and $r_f(t)$, then we have the dynamics of $FX^T(t), \nu(t), r_d(t)$ and $r_f(t)$, under the T-forward measure $\mathbb{Q}^T$, as follows:

$$
\begin{aligned}
\frac{\mathrm{d}FX^T(t)}{FX^T(t)} &= \sqrt{\nu(t)}\mathrm{d}W_\xi^T(t) - \eta_d B_d(t,T)\mathrm{d}W_d^T(t) + \eta_f B_f(t,T)\mathrm{d}W_f^T(t) \qquad (11)\\
\mathrm{d}\nu(t) &= \left[\kappa(\overline{\nu} - \nu(t)) + \gamma\rho_{\nu,d}\eta_d B_d(t,T)\sqrt{\nu(t)}\right]\mathrm{d}t + \gamma\sqrt{\nu(t)}\mathrm{d}W_\nu^T(t),\\
\mathrm{d}r_d(t) &= \left[\lambda_d(\theta_d(t) - r_d(t)) + \eta_d^2 B_d(t,T)\right]\mathrm{d}t + \eta_d \mathrm{d}W_d^T(t)\\
\mathrm{d}r_f(t) &= \left[\lambda_f(\theta_f(t) - r_f(t)) - \eta_f\rho_{\xi,f}\sqrt{\nu(t)} + \eta_d\eta_f\rho_{d,f}B_d(t,T)\right]\mathrm{d}t + \eta_f \mathrm{d}W_f^T(t).
\end{aligned}
$$

For more details on the above change of measure, please refer to the Appendix of [1].

2.4 The Payoff Function of FX Options

Denote the time t price of the call option of the spot FX by $V(t, X(t))$, $X(t) := [\xi(t), \nu(t), r_d(t), r_f(t)]^T$:

$$V(t, X(t)) = \mathbb{E}^{\mathbb{Q}}\left(\frac{M_d(t)}{M_d(T)}\max(\xi(T) - K, 0)|\mathcal{F}_t\right), \qquad (12)$$

with

$$M_d(t) = \exp\left(\int_0^t r_d(s)\mathrm{d}s\right).$$

We use the following formula to define the strike K:

$$K(T) = FX^T(0)\exp\left(0.1\delta_n\sqrt{T}\right), \quad \text{with} \quad \delta_n = \{-1.5, -1.0, -0.5, 0, 0.5, 1.0, 1.5\}. \qquad (13)$$

Note that given the information at time t, $M_d(t)$ is deterministic, then

$$V(t, X(t)) = M_d(t)\mathbb{E}^{\mathbb{Q}}\left(\frac{\max(\xi(T) - K, 0)}{M_d(T)}|\mathcal{F}_t\right),$$

denote by $\Pi(t)$ the forward price

$$\Pi(t) = \mathbb{E}^{\mathbb{Q}}\left(\frac{\max(\xi(T) - K, 0)}{M_d(T)}|\mathcal{F}_t\right).$$

2.5 The Characteristic Functions for Logarithm of Forward FX under Forward Measure

By switching from the domestic risk-neutral measure $\mathbb{Q}$ to the domestic T-forward measure $\mathbb{Q}^T$, the discounting will be decoupled from the expectation, i.e.

$$\Pi(t) = P_d(t,T)\mathbb{E}^T\left[\max\left(FX^T(T) - K, 0\right)|\mathcal{F}_t\right], \qquad (14)$$

where $FX^T(t)$ is the forward FX under the T-forward measure as given by (9) and $\mathbb{E}^T[\cdot]$ is the expectation taking under the forward measure $\mathbb{Q}^T$.

We make a log-transform of the forward FX, i.e. $x^T(t) := \log FX^T(t)$, its dymanics can be obtained from (11) as

$$\mathrm{d}x^T(t) = \left[\zeta(t, \sqrt{\nu(t)}) - \frac{1}{2}\nu(t)\right]\mathrm{d}t + \sqrt{\nu(t)}\mathrm{d}W_\xi(t) - \eta_d B_d \mathrm{d}W_d^T(t) + \eta_f B_f \mathrm{d}W_f^T(t), \tag{15}$$

where $B_d := B_d(t,T)$ and $B_f := B_f(t,T)$,

$$\zeta(t, \sqrt{\nu(t)}) = \left[\rho_{x,d}\eta_d B_d - \rho_{x,f}\eta_f B_f\right]\sqrt{\nu(t)} + \rho_{d,f}\eta_d\eta_f B_d B_f - \frac{1}{2}\left(\eta_d^2 B_d^2 + \eta_f^2 B_f^2\right).$$

Applying the Feynman-Kac formula, we obtain the PDE for the characteristic function of the log-transformed forward FX $\phi^T := \phi^T(u, X(t), t, T) = \mathbb{E}^T\left[\mathrm{e}^{iux^T(T)}|\mathcal{F}_t\right]$ as

$$\begin{aligned} -\frac{\partial \phi^T}{\partial t} &= \left[\kappa(\overline{\nu} - \nu) + \rho_{\nu,d}\gamma\eta_d\sqrt{\nu(t)}B_d\right]\frac{\partial \phi^T}{\partial v} + \left[\frac{1}{2}\nu - \zeta(t, \sqrt{\nu(t)})\right]\left(\frac{\partial^2 \phi^T}{\partial x^2} - \frac{\partial \phi^T}{\partial x}\right) \\ &\quad + \left[\phi_{x,\nu}\gamma\nu - \rho_{\nu,d}\gamma\eta_d\sqrt{\nu(t)}B_d + \rho_{\nu,f}\gamma\eta_f\sqrt{\nu(t)}B_f\right]\frac{\partial^2 \phi^T}{\partial x \partial v} + \frac{1}{2}\gamma^2\nu\frac{\partial^2 \phi^T}{\partial v^2}. \end{aligned}$$

Since the above PDE is not affine, it is not easy to find its solution. An approximation of the non-affine term in the PDE is proposed as

$$\sqrt{\nu(t)} \approx \mathbb{E}\left[\sqrt{\nu(t)}\right] \approx \beta_1 + \beta_2\mathrm{e}^{-\beta_3 t} := \varphi(t), \tag{16}$$

where $\beta_1 = \sqrt{\overline{\nu} - \gamma^2/8\kappa}, \beta_2 = \sqrt{\nu(0)} - \beta_1, \beta_3 = -\log[\beta_2^{-1}(\Lambda(1) - \beta_1)]$, and

$$\begin{aligned} \Lambda(t) &= \sqrt{c(t) - [\lambda(t) - 1] + c(t)d + \frac{c(t)d}{2[d + \lambda(t)]}}, \\ c(t) &= \frac{1}{4\kappa}\gamma^2(1 - \mathrm{e}^{-\kappa t}), \quad d = \frac{4\kappa\overline{\nu}}{\gamma^2}, \quad \lambda(t) = \frac{4\kappa\nu(0)\mathrm{e}^{-\kappa t}}{\gamma^2(1 - \mathrm{e}^{-\kappa t})}. \end{aligned}$$

Note that the approximation of $\mathbb{E}\left[\sqrt{\nu(t)}\right]$ in (16) requires

$$\overline{\nu} > \gamma^2/8\kappa. \tag{17}$$

For the cases that don't meet this requirement, one can use the exact formula of $\mathbb{E}\left[\sqrt{\nu(t)}\right]$ as an approximation for $\sqrt{\nu(t)}$. For more details about this approximation, please refer to Section 3.1 of [2].

With the approximation of the non-affine term, the forward characteristic function is of the following form:

$$\phi^T(u, X(t), t, T) = \exp[A(u,\tau) + B(u,\tau)x^T(t) + C(u,\tau)\nu(t)], \tag{18}$$

where $\tau = T - t$, the function $A(\tau) := A(u,\tau), B(\tau) := B(u,\tau), C(\tau) := C(u,\tau)$ are given by

$$\begin{aligned} B'(\tau) &= 0, \\ C'(\tau) &= -\kappa C(\tau) + [B^2(\tau) - B(\tau)]/2 + \rho_{x,\nu}\gamma B(\tau)C(\tau) + \gamma^2 C^2(\tau)/2, \\ A'(\tau) &= \kappa\overline{\nu}C(\tau) + \rho_{\nu,d}\gamma\eta_d\varphi(\tau)B_d(\tau)C(\tau) - \zeta(\tau, \varphi(\tau))[B^2(\tau) - B(\tau)] \\ &\quad + [-\rho_{\nu,d}\eta_d\gamma\varphi(\tau)B_d(\tau) + \phi_{\nu,f}\gamma\eta_f\varphi(\tau)B_f(\tau)]B(\tau)C(\tau), \end{aligned}$$

with $B_i(\tau) = \lambda_i^{-1}[\mathrm{e}^{-\lambda_i \tau} - 1]$ for $i = \{d, f\}$ and the initial conditions $B(0) = iu, C(0) = 0, A(0) = 0$.

The above ODE can be sloved as

$$\begin{aligned}
B(\tau) &= iu, \\
C(\tau) &= \frac{1 - \mathrm{e}^{-d\tau}}{\gamma^2(1 - g\mathrm{e}^{-d\tau})}\left(\kappa - \rho_{x,\nu}\gamma iu - d\right), \\
A(\tau) &= \int_0^\tau [\kappa\overline{\nu} + \rho_{x,\nu}\gamma\eta_d\varphi(s)B_d(s) - \rho_{\nu,d}\eta_d\gamma\varphi(s)B_d(s)iu \\
&\quad +\rho_{\nu,f}\gamma\eta_f\varphi(s)B_f(s)iu]\, C(s)\mathrm{d}s + (u^2 + iu)\int_0^\tau \zeta(s, \varphi(s))\mathrm{d}s,
\end{aligned} \tag{19}$$

with $d = \sqrt{(\rho_{x,\nu}\gamma iu - \kappa)^2 - \gamma^2 iu(iu - 1)}$, $g = \frac{\kappa - \gamma\rho_{x,\nu}iu - d}{\kappa - \gamma\rho_{x,\nu}iu + d}$.

Subsitituting the solution of $A(u, \tau), B(u, \tau), C(u, \tau)$ into (18), we have a closed form of characteristic function of the forward FX, then by Fourier-Cosine method, we can calculate the forward price of the option (14) and then the option price (12) efficiently.

3 The Inflation Models and the YoY Inflation Caps

3.1 The Nominal and Real Interest Rate Models

The nominal and real interst rates, r_n and r_r, under the risk-neutral nominal and real economy measure $\mathbb{Q}_n$ and $\mathbb{Q}_r$, respectively, are modeled by one-factor Hull-White models:

$$\mathrm{d}r_n(t) = (\theta_n(t) - a_n r_n(t))\mathrm{d}t + \eta_n \mathrm{d}W_n^{\mathbb{Q}_n}(t), \tag{20}$$

$$\mathrm{d}r_r(t) = (\theta_r(t) - a_r r_r(t))\mathrm{d}t + \eta_r \mathrm{d}W_r^{\mathbb{Q}_r}(t), \tag{21}$$

where $W_n^{\mathbb{Q}_n}(t)$ and $W_r^{\mathbb{Q}_r}(t)$ are Brownian motios under $\mathbb{Q}_n$ and $\mathbb{Q}_r$, respectively. Parameters a_n, a_r represent the mean reversion speed with the similar meaning as λ_d and λ_f in model (1) and (2) and η_n, η_r are the volatility parameters, the time-dependent function $\theta_n(t)$ and $\theta_r(t)$ determine by the nominal and real initial term structure as observed in the markets.

3.2 The Inflation Index Models

The inflation index, which is also called CPI, denoted by $I(t)$, and its stochastic variance process $v(t)$ are modeled by the Heston model under the nominal economy spot measure, $\mathbb{Q}_n$:

$$\begin{cases} \mathrm{d}I(t) = [r_n(t) - r_r(t)]I(t)\mathrm{d}t + \sqrt{v(t)}I(t)\mathrm{d}W_I^{\mathbb{Q}_n}(t), & I(0) \geq 0, \\ \mathrm{d}v(t) = \kappa[\overline{v} - v(t)]\mathrm{d}t + \sigma_v\sqrt{v(t)}\mathrm{d}W_v^{\mathbb{Q}_n}(t), & v(0) \geq 0. \end{cases} \tag{22}$$

where $W_I^{\mathbb{Q}_n}(t), W_v^{\mathbb{Q}_n}(t)$ are Brownian motions under the nominal economy spot measure $\mathbb{Q}_n$, κ is a mean-reversion parameter, σ_v is the volatility of the volatility and $\overline{v}$ denotes the long-term variance level. The inflation rate is defined as the percentage change of the CPI, i.e. $\frac{I(t)}{I(s)} - 1$, for $0 < s < t$.

To include the stochastic nominal and real interest rate model for inflation products pricing, we need to determine the real interest rate in the nominal economy. By change of measure, we have the real interest rate dynamics under the nominal economy as

$$\mathrm{d}r_r(t) = [\theta_r(t) - \rho_{I,r}\eta_r\sqrt{v(t)} - a_r r_r(t)]\mathrm{d}t + \eta_r \mathrm{d}W_r^{\mathbb{Q}_n}(t). \tag{23}$$

The correlation structure between the Brownian motions

$$\mathrm{d}\mathbf{W}_t := \left(\mathrm{d}W_I^{\mathbb{Q}_n}(t), \mathrm{d}W_v^{\mathbb{Q}_n}(t), \mathrm{d}W_n^{\mathbb{Q}_n}(t), \mathrm{d}W_r^{\mathbb{Q}_n}(t)\right)$$

is defined by the following symmetric instantaneous correlation matrix:

$$\mathrm{d}\mathbf{W}_t(\mathrm{d}\mathbf{W}_t)^T = \begin{pmatrix} 1 & \rho_{I,v} & \rho_{I,n} & \rho_{I,r} \\ \cdot & 1 & \rho_{v,n} & \rho_{v,r} \\ \cdot & \cdot & 1 & \rho_{n,r} \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \mathrm{d}t. \tag{24}$$

We assume a full rank matrix for the correlation.

3.3 The Inflation Models under Forward Measure

To value the inflation-dependent derivatives, it is convenient to use the inflation model under the T-forward nominal economy measure $\mathbb{Q}_n^T$, instead of the spot measure $\mathbb{Q}_n$. The forward nominal economy measure is generated by the nominal zero-coupon bond, $P_n(t,T)$. In other words, under the T-forward measure the forward measure, the forward CPI,

$$I_T(t) := I(t)\frac{P_r(t,T)}{P_n(t,T)} \tag{25}$$

is a martingale.

Under the T-forward measure, the inflation model with a full matrix of correlations is given by:

$$\begin{cases} \mathrm{d}I^T(t) &= I^T(t)\left[\sqrt{v(t)}\mathrm{d}W_I^T(t) - \eta_n B_n(t,T)\mathrm{d}W_n^T(t) + \eta_r B_r(t,T)\mathrm{d}W_r^T(t)\right] \\ \mathrm{d}v(t) &= \left[\kappa(\overline{v} - v(t)) - \sigma_v\rho_{v,n}\eta_n B_n(t,T)\sqrt{v(t)}\right]\mathrm{d}t + \sigma_v\sqrt{v(t)}\mathrm{d}W_v^T(t), \\ \mathrm{d}r_n(t) &= \left[\theta_n(t) - a_n r_n(t) + \eta_n^2 B_n(t,T)\right]\mathrm{d}t + \eta_n \mathrm{d}W_n^T(t) \\ \mathrm{d}r_r(t) &= \left[\theta_r(t) - a_r r_r(t) - \eta_r\rho_{I,r}\sqrt{v(t)} + \eta_n\eta_r\rho_{n,r}B_n(t,T)\right]\mathrm{d}t + \eta_r\mathrm{d}W_r^T(t), \end{cases} \tag{26}$$

where

$$B_l(t,T) := \frac{1}{a_l}\left[1 - \mathrm{e}^{-a_l(T-t)}\right], \quad \text{for} \quad l = \{n, r\}. \tag{27}$$

3.4 The Inflation Products

We will price YoY inflation cap options under the above stated inflation model. In general, a cap option, $\widetilde{\Pi}$, is defined by a series of so-called caplets, $\widehat{\Pi}$, i.e.

$$\widetilde{\Pi}(t,\tau,T,K)=\sum_{k=1}^{n}\widehat{\Pi}(t,T_{k-1},T_k,K) \tag{28}$$

where $\tau := T_k - T_{k-1}$ defines the tenor parameter with $T_0 = t_0 \geq t$ and $T_n = T$. The integer n, defined as $n = \frac{T-t_0}{\tau}$, denotes the number of caplets in the cap option and it depends on the tenor parameter and time to maturity of the cap options. The strike of the cap is K. The pricing of the cap option can be decomposed by pricing a series of caplets.

The time t price of the YoY inflation caplets starting at time T_{k-1} and maturing at time T_k is given by

$$\widehat{\Pi}(t,T_{k-1},T_k,K)=M_n(t)\mathbb{E}^{\mathbb{Q}_n}\left[\frac{\max\left(\frac{I(T_k)}{I(T_{k-1})}-(K+1),0\right)}{M_n(T_k)}|\mathcal{F}_t\right], \tag{29}$$

where $M_n(t) := \exp\left[\int_0^t r_n(s)\mathrm{d}s\right]$ is the nominal money-saving account and $\mathbb{E}^{\mathbb{Q}_n}[\cdot]$ is the expectation taking under the nominal spot economy measure $\mathbb{Q}_n$.

By changing of measure from the nominal spot economy measure $\mathbb{Q}_n$ to the forward nominal measure $\mathbb{Q}^{T_k}$ and using the definition of the T_k-forward CPI $I_{T_k}(t)$ given in (25), we can rewrite the pricing formula of the YoY inflation caplet price as:

$$\widehat{\Pi}(t,T_{k-1},T_k,K)=P_n(t,T_k)\mathbb{E}^{\mathbb{Q}^{T_k}}\left[\left(\frac{P_r(T_{k-1},T_k)}{P_n(T_{k-1},T_k)}\frac{I_{T_k}(T_k)}{I_{T_k}(T_{k-1})}-(K+1)\right)^+|\mathcal{F}_t\right], \tag{30}$$

where $(\cdot)^+$ means $\max(\cdot,0)$, $P_n(s,t)$ and $P_r(s,t)$ for $0<s<t$ are nominal and real zero-coupon bond prices, respectively.

3.5 The Characteristic Functions for the Logarithm of the Forward CPIs under Forward Nominal Measure

Define the log-transformation of the discounted forward CPI as:

$$\begin{aligned} x(T_{k-1},T_k) &:= \log\left(\frac{P_r(T_{k-1},T_k)}{P_n(T_{k-1},T_k)}\frac{I_{T_k}(T_k)}{I_{T_k}(T_{k-1})}\right) \qquad (31)\\ &= \log I_{T_k}(T_k)-\log I_{T_k}(T_{k-1})+\log P_r(T_{k-1},T_k)-\log P_n(T_{k-1},T_k). \end{aligned}$$

To compute the time t price of the YoY inflation caplets, we need to derive the forward condtional characteristic function of $x(T_{k-1},T_k)$,

$$\begin{aligned} \phi_{YoY}(u,t,x(T_{k-1},T_k)) &:= \mathbb{E}^{\mathbb{Q}^{T_k}}\left[\mathrm{e}^{iux(T_k,T_{k-1})}|\mathcal{F}_t\right] \qquad (32)\\ &= \mathbb{E}^{\mathbb{Q}^{T_k}}\left\{\mathrm{e}^{iu\left[\log I_{T_k}(T_k)-\log I_{T_k}(T_{k-1})+\log P_r(T_{k-1},T_k)-\log P_n(T_{k-1},T_k)\right]}|\mathcal{F}_t\right\}, \end{aligned}$$

given the information up to time t. We derive a closed form for $\phi_{YoY}(u,t,x(T_{k-1},T_k))$ by three steps of approximations.

Approximation 1: By taking iterated expectation under $\mathcal{F}_{k-1}$, we have

$$\begin{aligned}
& \phi_{YoY}(u,t,x(T_{k-1},T_k)) \\
= \; & \mathbb{E}^{\mathbb{Q}^{T_k}}\Big\{ \mathrm{e}^{-iu[I_{T_k}(T_{k-1})-\log P_r(T_{k-1},T_k)+\log P_n(T_{k-1},T_k)]}\times \\
& \quad \mathbb{E}^{\mathbb{Q}^{T_k}}\Big[\mathrm{e}^{iu\log I_{T_k}(T_k)}|\mathcal{F}_{k-1}\Big]|\mathcal{F}_t\Big\}, \qquad (33)
\end{aligned}$$

where the inner expectation is the characteristic function for $\log I_{T_k}(T_k)$ given $I_{T_k}(T_{k-1})$ and it can be approximated by

$$\begin{aligned}
& \phi_i(u,\log I_{T_k}(T_k),T_{k-1},T_k) := \mathbb{E}^{\mathbb{Q}^{T_k}}\Big[\mathrm{e}^{iu\log I_{T_k}(T_k)}|\mathcal{F}_{k-1}\Big] \\
\approx \; & \mathrm{e}^{A(u,T_k-T_{k-1})+iu\log I_{T_k}(T_{k-1})+C(u,T_k-T_{k-1})v(T_k-1)}. \qquad (34)
\end{aligned}$$

The function of $A(u,\tau)$ and $C(u,\tau)$ can be derived similarly as (19), for the explicit formulas and theirs derivations, please refer to [4]. By approximation (34), the characteristic function of ϕ_{YoY} defined by (32) can be approximated by

$$\begin{aligned}
& \phi_{YoY}(u,t,x(T_{k-1},T_k)) \\
\approx \; & \mathbb{E}^{\mathbb{Q}^{T_k}}\Big\{\mathrm{e}^{iu[\log P_r(T_{k-1},T_k)-\log P_n(T_{k-1},T_k)]}\mathrm{e}^{A(u,T_k-T_{k-1})+C(u,T_k-T_{k-1})v(T_{k-1})}|\mathcal{F}_t\Big\} \\
:= \; & \phi_{YoY,1}. \qquad (35)
\end{aligned}$$

Approximation 2: The zero-coupon bond price of nominal economy $P_n(T_{k-1},T_k)$ in (35) can be given by, see page 75-78 of [5],

$$P_n(T_{k-1},T_k) = \mathrm{e}^{A_n(T_{k-1},T_k)-B_n(T_{k-1},T_k)r_n(T_{k-1})}, \qquad (36)$$

where $B_n(T_{k-1},T_k)$ as defined in (27) and

$$\begin{aligned}
A_n(T_{k-1},T_k) \; = \; & \log\frac{P_n(0,T_k)}{P_n(0,T_{k-1})}\big[B_n(T_{k-1},T_k)f_n(0,T_{k-1}) \\
& -\frac{\eta_n^2}{4a_n}\left(1-\mathrm{e}^{-2a_nT_{k-1}}\right)B_n(T_{k-1},T_k)^2\Big].
\end{aligned}$$

But the zero-coupon bond price of real economy $P_r(T_{k-1},T_k)$ can not be derived analytically due to the fact that the dynamics of the real interest rate are not affine under the nominal measure. So we propose an approximation of $P_r(T_{k-1},T_k)$ by approximating the variance process under $\mathbb{Q}_n$ by its expectation to make the dynamics of real interst rate affined. The appriximation of $P_r(T_{k-1},T_k)$ is given by

$$P_r(T_{k-1},T_k) \approx \mathrm{e}^{A_r(T_{k-1},T_k)-B_r(T_{k-1},T_k)r_r(T_k)}, \qquad (37)$$

where $B_r(T_{k-1},T_k)$ as defined in (27) and

$$\begin{aligned}
A_r(T_{k-1},T_k) \; = \; & \log\frac{P_r(0,T_k)}{P_r(0,T_{k-1})}\big[B_r(T_{k-1},T_k)f_r(0,T_{k-1})+\Lambda(T_{k-1},T_k) \\
& -\frac{\eta_r^2}{4a_r}\left(1-\mathrm{e}^{-2a_rT_{k-1}}\right)B_r(T_{k-1},T_k)^2\Big],
\end{aligned}$$

with

$$\begin{aligned}\Lambda(T_{k-1},T_k) &= \mathbb{E}\left[\sqrt{v(T_k)}\right]\frac{\rho_{I,r}\eta_r}{a_r}\left[T_k - T_{k-1} - B_r(T_{k-1},T_k) - B_n(T_{k-1},T_k)\right.\\ &\qquad \left.\frac{1}{a_n+a_r}\left(1-\mathrm{e}^{-(a_n+a_r)(T_k-T_{k-1})}\right)\right].\end{aligned}$$

$\mathbb{E}\left[\sqrt{v(T_k)}\right]$ is given as in (16).

By substituting the nominal and the real zero-coupon bond expressions (36) and (37) into (35), we have the approximation of (32) as

$$\begin{aligned} & \phi_{YoY}(u,t,x(T_{k-1},T_k)) \\ \approx\; & \mathrm{e}^{iu[A_r(T_{k-1},T_k)-A_n(T_{k-1},T_k)]+A(u,T_k-T_{k-1})}\times \\ & \mathbb{E}^{\mathbb{Q}^{T_k}}\left[\mathrm{e}^{C(u,T_k-T_{k-1})v(T_{k-1})+iu[B_n(T_{k-1},T_k)r_n(T_{k-1})-B_r(T_{k-1},T_k)r_r(T_{k-1})]}|\mathcal{F}_t\right] \\ :=\; & \phi_{YoY,2}. \end{aligned} \tag{38}$$

Approximation 3: We approximate (38) by spliting the expectation into two terms as follows:

$$\begin{aligned} & \phi_{YoY}(u,t,x(T_{k-1},T_k)) \\ \approx\; & \mathrm{e}^{iu[A_r(T_{k-1},T_k)-A_n(T_{k-1},T_k)]+A(u,T_k-T_{k-1})}\mathbb{E}^{\mathbb{Q}^{T_k}}\left[\mathrm{e}^{C(u,T_k-T_{k-1})v(T_{k-1})}|\mathcal{F}_t\right]\times \\ & \mathbb{E}^{\mathbb{Q}^{T_k}}\left[\mathrm{e}^{iu[B_n(T_{k-1},T_k)r_n(T_{k-1})-B_r(T_{k-1},T_k)r_r(T_{k-1})]}|\mathcal{F}_t\right] \\ :=\; & \phi_{YoY,3}. \end{aligned} \tag{39}$$

We will derive the formulas for the two expectations (39).

For $0\le s\le t\le T$, we have

$$\mathbb{E}^{\mathbb{Q}^T}\left[\mathrm{e}^{C(u,T-t)v(t)}|\mathcal{F}_s\right] = \psi(u,s,t,T)^{\frac{2\kappa\bar{v}}{\sigma_v^2}}\exp\left[\psi(u,s,t,T)\mathrm{e}^{-\kappa(t-s)}C(u,T-t)v(t)\right], \tag{40}$$

provided that

$$\psi(u,s,t,T) := \frac{1}{1-\frac{2\sigma_v^2}{4\kappa}\left[1-\mathrm{e}^{-\kappa(t-s)}\right]C(u,T-t)} \ge 0.$$

For $0\le s\le t\le T$, denote $Y(t,T) := B_n(t,T)r_n(t) - B_r(t,T)r_r(t)$, its conditional characteristic function can be approximated by

$$\mathbb{E}^{\mathbb{Q}^T}\left[\mathrm{e}^{iuY(t,T)}|\mathcal{F}_s\right] \approx \exp\left\{iu\mathbb{E}^{\mathbb{Q}^T}\left[Y(t,T)|\mathcal{F}_s\right] - \frac{1}{2}u^2\mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[Y(t,T)|\mathcal{F}_s\right]\right\}, \tag{41}$$

where

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}^T}\left[Y(t,T)|\mathcal{F}_s\right] &= B_n(t,T)\mathbb{E}^{\mathbb{Q}^T}\left[r_n(t)|\mathcal{F}_s\right] - B_r(t,T)\mathbb{E}^{\mathbb{Q}^T}\left[r_r(t)|\mathcal{F}_s\right],\\ \mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[Y(t,T)|\mathcal{F}_s\right] &= B_n^2(t,T)\mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[r_n(t)|\mathcal{F}_s\right] + B_r^2(t,T)\mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[r_r(t)|\mathcal{F}_s\right]\\ &\quad -2B_n(t,T)B_r(t,T)\mathbb{C}\mathrm{ov}^{\mathbb{Q}^T}\left[r_n(t),r_r(t)|\mathcal{F}_s\right],\end{aligned}$$

with $\mathbb{E}^{\mathbb{Q}^T}\left[r_l(t)|\mathcal{F}_s\right], \mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[r_l(t)|\mathcal{F}_s\right]$, for $l=\{n,r\}$ given in Chap. 3.3 of [5]:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T}\left[r_n(t)|\mathcal{F}_s\right] &= r_n(s)\mathrm{e}^{-a_n(t-s)}-\alpha_n(t)-\alpha_n(s)\mathrm{e}^{-a_n(t-s)} \\
&\quad +\frac{\eta_n^2}{a_n}\left[\frac{1-\mathrm{e}^{-a_n(t-s)}}{a_n}-\frac{\mathrm{e}^{-a_n(T-t)}-\mathrm{e}^{-a_n(T-2s+t)}}{2a_n}\right] \\
\mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[r_n(t)|\mathcal{F}_s\right] &= \frac{\eta_n^2}{2a_n}\left[1-\mathrm{e}^{-2a_n(t-s)}\right], \\
\mathbb{E}^{\mathbb{Q}^T}\left[r_r(t)|\mathcal{F}_s\right] &= r_r(s)\mathrm{e}^{-a_r(t-s)}-\alpha_r(t)-\alpha_r(s)\mathrm{e}^{-a_r(t-s)} \\
&\quad +\int_s^t \mathrm{e}^{-a_r(t-u)}\left\{\eta_n\eta_r\rho_{n,r}B_n(u,T)-\rho_{I,r}\eta_r\mathbb{E}\left[\sqrt{v(u)}\right]\right\}\mathrm{d}u, \\
\mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[r_r(t)|\mathcal{F}_s\right] &= \frac{\eta_r^2}{2a_r}\left[1-\mathrm{e}^{-2a_r(t-s)}\right],
\end{aligned}$$

$$\mathbb{C}\mathrm{ov}^{\mathbb{Q}^T}\left[r_n(t),r_r(t)|\mathcal{F}_s\right]=\rho_{n,r}\sqrt{\mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[r_n(t)|\mathcal{F}_s\right]\mathbb{V}\mathrm{ar}^{\mathbb{Q}^T}\left[r_r(t)|\mathcal{F}_s\right]},$$

$$\alpha_l(t)=r_l(0)+\frac{\eta_l^2}{2a_l^2}\left(1-\mathrm{e}^{-a_l t}\right)^2, \quad \text{for} \quad l=\{n,r\}.$$

Note that using the approximation form of $\mathbb{E}\left[\sqrt{v(u)}\right]$ as given by (16), the integration in $\mathbb{E}^{\mathbb{Q}^T}\left[r_r(t)|\mathcal{F}_s\right]$ can be calculated analytically, then we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^T}\left[r_r(t)|\mathcal{F}_s\right] &\approx r_r(s)\mathrm{e}^{-a_r(t-s)}-\alpha_r(t)-\alpha_r(s)\mathrm{e}^{-a_r(t-s)} \\
&\quad +\frac{\eta_n\eta_r\rho_{n,r}}{a_n}\left[\frac{1-\mathrm{e}^{-a_r(t-s)}}{a_r}-\frac{\mathrm{e}^{-a_n(T-t)}-\mathrm{e}^{-a_r(t-s)-a_n(T-s)}}{a_r+a_n}\right] \\
&\quad -\eta_r\rho_{I,r}\left[\beta_1\frac{1-\mathrm{e}^{-a_r(t-s)}}{a_r}+\beta_2\frac{\mathrm{e}^{-\beta_3 t}-\mathrm{e}^{-a_r(t-s)-\beta_3 s}}{a_r-\beta_3}\right].
\end{aligned}$$

Substituting (40) and (41) into (39), we derive the closed form approximation for the conditional characteristic fucntion of $x(T_{k-1},T_k)$, for $k=1,\cdots,n$, which is defined in (32) and will be used to derive the caplet price in the next section.

4 Pricing FX Options and Inflation Products by Fourier-Cosine Expansion

We present here a sketch of the Fourier-Cosine expansion method for pricing the FX options and inflation YoY cap, for more details about the calculatation of option prices by Fourier-Cosine method, please refer to [3].

The expectation part in the forward price of the FX option, $\Pi(t)$, as given in (14), and in the inflation YoY caplet price, $\widehat{\Pi}(t,T_{k-1},T_k,K)$ in (30), can be expressed as:

$$\mathbb{E}^{T^*}\left[K^*\max(\mathrm{e}^{x(T^*)}-1,0)|\mathcal{F}_t\right]=\mathbb{E}^{T^*}\left[v(y,T^*)|x\right]=\int_{\mathbb{R}}v(y,T^*)f(y|x)dy, \quad (42)$$

where x and y are the state variables of the process $x(\cdot)$ at time t and time T^* and

$$v(y, T^*) := K^* \max(\mathrm{e}^y - 1, 0), \tag{43}$$

$f(y|x)$ is the conditional probability density for $x(T^*) = y$ given $x(t) = x$. For the FX option forward price $\Pi(t)$, $T^* = T, K^* = K, x(T) = \log\left[FX^T(T)/K^*\right]$; for inflation YoY caplet price $\widehat{\Pi}(t, T_{k-1}, T_k, K)$, $T^* = T_k, K^* = K+1, x(T_k) = \log\left\{\left[\frac{P_r(T_{k-1},T_k)}{P_n(T_{k-1},T_k)} \frac{I_{T_k}(T_k)}{I_{T_k}(T_{k-1})}\right]/K^*\right\}$.

Firstly, we truncate the infinite integration range of (42) without loosing significant accuracy to $[a, b] \in \mathbb{R}$, and we obtain its approximation v_1 :

$$v_1(x, t) = \int_a^b v(y, T) f(y|x) dy, \tag{44}$$

where a and b are choosen such that the truncate error is under control.

Secondly, we replace the density by its cosine expansion in y,

$$f(y|x) = \sum_{k=0}^{+\infty} A_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right), \tag{45}$$

where the summation Σ here with the first term is weightedd by one-half and

$$\begin{aligned} A_k(x) &:= \frac{2}{b-a} \int_a^b f(y|x) \cos\left(k\pi \frac{y-a}{b-a}\right) \\ &= \frac{2}{b-a} \mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a}; x\right) \exp\left(-i\frac{ka\pi}{b-a}\right)\right\}, \end{aligned} \tag{46}$$

where the second equation in (46) is obtained by comparing the cosine coefficient $A_k(x)$ of $f(y|x)$ with the definition of conditional characteristic function $\phi\left(\frac{k\pi}{b-a}; x\right)$ and $\phi\left(\frac{k\pi}{b-a}; x\right)$ is the conditional characteristic function of $x(T^*)$ given $x(t) = x$ and the closed forms for the prices of FX options and YoY inflation caplets can be derived from (18) and (39), respectively.

Substituting (45) into (44), we have

$$v_1(x, t_0) = \frac{1}{2}(b-a) \int_a^b \frac{2}{b-a} v(y, T) \sum_{k=0}^{+\infty} A_k(x) \cos(k\pi \frac{y-a}{b-a}) dy. \tag{47}$$

Then interchange the summation and integration, we have

$$v_1(x, t_0) = \frac{1}{2}(b-a) \sum_{k=0}^{+\infty} A_k(x) V_k \approx \frac{1}{2}(b-a) {\sum_{k=0}^{N-1}}' A_k(x) V_k, \tag{48}$$

with

$$V_k := \frac{2}{b-a} \int_a^b v(y, T) \cos(k\pi \frac{y-a}{b-a}) dy. \tag{49}$$

Then replacing (46) of A_k in (48), we have

$$\mathbb{E}^{T^*}[K^* \max(e^{x(T^*)} - 1, 0)|\mathcal{F}_t] \approx \sum_{k=0}^{N-1} \mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi\frac{a}{b-a}} V_k\right\}, \quad (50)$$

which is the COS formula for general underlying processes.

At last, we just need to determine V_k in the above COS formula which can be calculate analytically. By definition of $v(y, T)$ in (43) and definition of V_k in (49), we have

$$V_k = \frac{2}{b-a} K^*[\xi_k(0, b) - \psi_k(0, b)], \quad (51)$$

where

$$\xi_k(c,d) := \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{d-a}{b-a}\right)e^d - cos\left(k\pi\frac{c-a}{b-a}\right)e^c \right. \quad (52)$$

$$\left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{d-a}{b-a}\right)e^d - \frac{k\pi}{b-a}\sin\left(k\pi\frac{c-a}{b-a}\right)e^c\right] \quad (53)$$

$$\psi_k(c,d) := \begin{cases}\left[\sin\left(k\pi\frac{d-a}{b-a}\right) - \sin\left(k\pi\frac{c-a}{b-a}\right)\right]\frac{b-a}{k\pi}, & k \neq 0, \\ (d-c), & k = 0.\end{cases} \quad (54)$$

Please refer to [3] for more details about Fourier-Cosine expansion method.

5 Program Manual

We implement the pricing method of FX options and inflation YoY caps in Heston model incorperated with Hull-White interest rate model by using Fourier Cosine expansions. The number of Fourier-Cosine terms is 2^9 for FX options and 2^{10} for inflation YoY caps. The program HAS TO work with the pnl library.

5.1 Pricing FX options

Model Parameters:
v0: the initial value of the volatility, $\nu(0)$ for dynamics given in (5);
kappa: the speed of the mean reversion of varince, κ as given in (5);
vbar: the long-term variance level, $\overline{\nu}$ as given in (5);
gammav: the volatility of volatility, γ as given in (5);
pd0: the initial domestic interest rate, $r_d(0)$ as given in (1);
lambdad: the mean reversion parameter of domestic interest rate, λ_d as given in (1);
etad: the volatility of domestic interest rate, η_d as given in (1);
pf0: the initial foreign interest rate, $r_f(0)$ as given in (2);
lambdaf: the mean reversion parameter of foreign interest rate, λ_f as given in (2);
etaf: the volatility of foreign interest rate, η_f as given in (2);
rho12, rho13, rho14, rho23, rho24, rho34: the correlation parameters $\rho_{\xi,\nu}, \rho_{\xi,d}, \rho_{\xi,f}, \rho_{\nu,d}, \rho_{\nu,f}, \rho_{d,f}$ respectively, as given in Section 2.2.

Parameters of the product:
csi0: the initial value of exchange rate, $\xi(0)$ as in (4).
delta_n: strike of the FX option, δ_n as in (13);
T: the maturity of FX options, T as in (12).

5.2 Pricing inflation YoY caps

Model Parameters:
v0: the initial value of the volatility, $\nu(0)$ for dynamics given in (22);
kappa: the speed of the mean reversion, κ as given in (22);
vbar: the long-term variance leverl, $\overline{v}$ as given in (22);
sigmav: the volatility of volatility, σ_v as given in (22);
rn0: the initial nominal interest rate, $r_n(0)$ as given in (20);
an: the mean reversion parameter of nominal interest rate, a_n as given in (20);
etan: the volatility of nominal interest rate, η_n as given in (20);
rr0: the initial real interest rate, $r_r(0)$ as given in (21);
ar: the mean reversion parameter of real interest rate, a_r as given in (21);
etar: the volatility of real interest rate, η_r as given in (21);
rho12, rho13, rho14, rho23, rho24, rho34: the correlation parameters $\rho_{I,v}, \rho_{I,n}, \rho_{I,r}, \rho_{v,n}, \rho_{v,r}, \rho_{n,r}$ respectively, as given in (24).

Parameters of the product:
t0: the starting date of the cap, T_0 as in (28);
tau: the tenor parameter τ and defines as $\tau := T_k - T_{k-1}$;
T: the maturity of options, T_n as in (28);
strike: the strike of cap, K as in (28).

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