EFFICIENT MONTE CARLO METHODS FOR PRICING BARRIER OPTIONS UNDER LEVY PROCESSES

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Introduction

A classical problem in mathematical finance deals with computing $\mathbb{E}[G(X)]$, where $X := \{X_s : s \in [0,t]\}$ is a stochastic process modeling the underlying asset and G is a payoff function which may depend on the historical path of X. This is the typical setting for pricing a wide class of exotic options in finance such as lookback and barrier options.

Recall that a barrier option is a contract which pays the specified amount $G(S_T)$ at the terminal date T, provided during the life-time of the contract, the price of the stock does not cross a specified constant barrier H from above (*down-and-out barrier options*) or from below (*up-and-out barrier options*). When the barrier is crossed, the option expires worthless or the option owner is entitled to some *rebate*.

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. Starting with the work of Madan and Seneta [1], the class of Lévy processes has found prominence and becomes more and more popular among researchers. For an introduction to applications of these models applied to finance, we refer to the books [2]-[5]. More recently Lévy processes have also been extensively used in modern insurance risk theory; see for example Asmussen and Albrecher [6] and Klüppelberg et al. [7]. In insurance mathematics, it is the Lévy process itself which models the surplus wealth of an insurance company until its ruin. There are also extensive applications of Lévy processes in queuing theory, genetics and mathematical biology as well as in stochastic differential equations (see e.g. [8]-[11]). We concentrate on one-factor non-gaussian exponential Lévy models. These models provide a better fit to empirical asset price distributions that typically have fatter tails than Gaussian ones, can reproduce volatility smile phenomena in option prices, and admit jumps in asset prices.

In both insurance and financial frameworks, a key quantity of generic interest is the joint law of the current position and the running extremum of a Lévy process at a fixed time. Consider some examples.

Let T, K, H be the maturity, strike and barrier, and $S_t = e^{X_t}$ the stock price under a chosen risk-neutral measure. The riskless rate is assumed constant. Set $h = \ln H$. Then the payoff at maturity is $\mathbf{1}_{(h,+\infty)}(X_T)G(X_T)$, where $G(x) = (K - e^x)_+$, and the no-arbitrage price of the barrier option at time $t = 0$ and $X_0 = x > h$ is given by the expectation $E^x\left[e^{-rT}\mathbf{1}_{\underline{X}_T>h}G(X_T)\right]$, where $\underline{X}_t = \inf_{0\le s\le t} X_t$ is the infimum process. If we define $\overline{X}_t = \sup_{s\le t} X_s$ then the pricing of 'up-and-in" put is equivalent to evaluating expectations of the form $E^x[e^{-rT}\mathbf{1}_{\overline{X}_T>h}G(X_T)]$ for some barrier $h > 0$.

Assume that the infimum process starts at $x = X_0$. Then the time-0 price of the European floating strike lookback option with payoff $g(X_T, \underline{X}_T)$ at maturity is given by $E^x\left[e^{-rT}g(X_T, \underline{X}_T)\right]$. Indeed if $g(x,y) = e^x - e^y$, then the latter expectation is related to the value of a floating strike lookback call.

In credit risk one is predominantly interested in the quantity $\widehat{\mathbb{P}}(\overline{X}_t < x)$ as a function in x and t, where $\widehat{\mathbb{P}}$ is the law of the dual process $-X$. The latter probabilities help to find prices of credit default swaps or convertible contingencies (CoCos). See e.g. the recent book

of Schoutens and Cariboni [5] as well as Corcuera et al. [12]. One is similarly interested in $\widehat{\mathbb{P}}(\overline{X}_t \geq x)$ in ruin theory, since these probabilities are also equivalent to the finite-time ruin probabilities; cf. Asmussen and Albrecher [6].

Option valuation under Lévy processes has been dealt with by a host of researchers, therefore, an exhaustive list is virtually impossible. However, the pricing of path-dependent options in exponential Lévy models still remains a mathematical and computational challenge (see, e.g., [13]-[15] for recent surveys of the state of the art of exotic option pricing in Lévy models).

The Wiener-Hopf factorization method is a standard tool for pricing path-dependent options. Nguyen-Ngoc and Yor [14] obtained formulas in terms of the Wiener-Hopf factors for the Laplace transform of continuously monitored barrier and lookback options in general Lévy models. The probabilistic approach used in the paper allows, in particular, to recover the results for barrier options derived in [2] using the analytical form of the Wiener-Hopf factorization method. The drawback of the formulas in [2, 14] is the complexity of numerical calculations required, since, in general, numerical n-fold integrals (with $n = 2, 3$) are needed.

In the case of jump diffusions with exponentially distributed Poisson jumps (a double-exponential jump diffusion process (DEJD) and its generalization: a hyper-exponential jump-diffusion model (HEJD)), the Laplace transform of the price w.r.t. time has a relatively simple explicit form. Formulas for DEJD model were obtained by Lipton [16] and Kou [17], and, for double-barrier options, by Sepp [18]; for HEJD case, see [19]-[21]. Note that papers [16, 22] consider continuously monitored barrier and lookback options, whereas the other papers cited above studied barrier options only. They can be represented as functions of the temporal variable t via the Gaver-Stehfest algorithm the Laplace transform is derived from the distribution of the first passage time; the distribution is calculated by applying the Wiener-Hopf factorization method in the form used in probability theory. The Laplace transform of the price having being calculated, one uses a suitable numerical Laplace inversion algorithm to recover the option price. However, the problem of the inversion of the Laplace transform is non-trivial from the computational point of view. We refer the reader to [23] for a description of a general framework for related numerical methods.

Calculation of the Laplace transform of the price under a general Lévy process is non-trivial as well. To simplify calculations, one can approximate the initial process by a DEJD or, more generally, HEJD, and then use the Laplace transform method (see, e.g., [19]-[21]). However, this approximation introduces an additional error, which may be quite sizable near the barrier (see examples in [15]).

Kudryavtsev and Levendorskiĭ [24] developed a fast and accurate numerical method labelled Fast Wiener-Hopf factorization method (FWHF-method) for pricing continuously monitored barrier options under Lévy processes of a wide class. FWHF-method is based on an efficient approximation of the Wiener-Hopf factors in the exact formula for the solution and the Fast Fourier Transform (FFT) algorithm. In contrast to finite difference methods which require a detailed analysis of the underlying Lévy model, the FWHF-method deals with the characteristic exponent of the process.

In [25], Kudryavtsev and Levendorskiĭ derive a general formula which is applicable to barrier options, lookbacks, lookbarriers, barrier-lookbacks, and other similar types of options, using an operator form of the Wiener-Hopf factorization. An efficient numerical realization of the formula for lookback options in KoBoL(CGMY) and Kou models uses Gaver-Stehfest algorithm (see [23]) and the Fast Wiener-Hopf factorization method developed in [24].

In this research report we concentrate on Monte Carlo methods for pricing exotic options in Lévy models. A widely used approach to compute expectations of functions depending on the historical path of a Lévy process over the finite time horizon, say $[0, T]$, is to approximate the trajectory by a random walk with n equidistant time steps,and to perform a Monte Carlo (MC) simulation. Giles [26, 27] suggested an adaptation of the straightforward MC methodology, the multilevel Monte Carlo method (MLMC) in the case that X is a pure diffusion process. Very recently, there has been increasing attention to the MLMC method also in the framework of Lévy processes, see Giles and Xia [28] for the jump diffusion setting, and Dereich [9] and Dereich and Heidenreich [29] for more general Lévy processes. Generally speaking all these methods use a common idea, which consists in constructing an embedded sequence of grids that are made up of a mixture of deterministic and random points. The random points in these grids deal with the large jumps of the Lévy process and the deterministic points deal with the "small movements", that is to say, the diffusive part and/or the small jumps.

In the current research report we consider an alternative approach based entirely on a random grid. In particular we shall introduce an adaptation of the FWHF method based on a recently introduced technique for performing MC simulations that appeals to the so-called Wiener-Hopf factorization for one-dimensional Lévy processes. This last technique for simulating the joint law of the position and running maximum at a fixed time of a general Lévy process is called the Wiener-Hopf Monte Carlo (WHMC) simulation method and it was introduced in Kuznetsov et al. [30]. The coupling of the multilevel Monte Carlo and the Wiener-Hopf Monte Carlo methods was developed in [31], we refer on this technique as (MLWH). Our analysis will focus on the particular setting that X is a one-dimensional Lévy process and the payoff function depends on the value of X and its past supremum (infimum) at a fixed time $t > 0$.

The main result in [31] gives the order of the root mean square error in the MLWH method converges for processes of unbounded variation and bounded variation.

In [32] the Wiener-Hopf Monte Carlo (WHMC) simulation technique for Lévy processes from Kuznetsov et al. [30] was applied to path functionals, in particular first passage times, overshoots, undershoots and the last maximum before the passage time. Such functionals have many applications, for instance in finance (the pricing of exotic options in a Lévy model) and insurance (ruin time, debt at ruin and related quantities for a Lévy insurance risk process). The technique works for any Lévy process whose running extremum evaluated at an independent exponential time allow sampling from. This includes classic examples such as stable processes, subclasses of spectrally one sided Lévy processes and large new families such as meromorphic Lévy processes. Ferreiro-Castilla and Schaik (2015) show that the WHMC simulation technique (provided it is applied) performs much better at approximating first passage times than a 'plain' Monte Carlo simulation technique based on sampling increments of the Lévy process.

Notice, however, that an efficient realization of the methods introduced in [30, 31, 32] for exotic options becomes possible if one is able to sample from the distributions related to the Wiener-Hopf factors. It is the case of explicit Wiener-Hopf factorization in terms of finite products (DEJD or HEJD) or infinite products (β-class constructed by Kuznetsov [33] and hypergeometric family introduced in Kuznetsov et al. [30]). In the report, we expand the WHMC method to the case of more general Lévy models including (KoBoL) CGMY.

The goal of the present work is developing of efficient numerical realizations of the Wiener-Hopf Monte Carlo methods with subsequent implementation into the program platform Premia [34]. The research report relies heavily on the results obtained in [24, 30, 32].

The report is organized as follows. In the first section we will review the general setting for the Wiener-Hopf factorization of Lévy processes and describe the Fast Wiener-Hopf factorization method introduced in [24]. Thereafter, in Section 2, we give the overview of Monte Carlo methods for Lévy models including the WHMC method developed in Kuznetsov et al. [30]. The last Section considers the implementation of the Wiener-Hopf Monte Carlo techniques into the program platform Premia including an adaptation of the WHMC method in the context of the FWHF-method.

1. Lévy processes and the Wiener-Hopf factorization

We begin by briefly reviewing the definition of a one-dimensional Lévy process and related Wiener-Hopf factorization identities. For more details we refer the reader to the monographs of Bertoin [35], Kyprianou [36] or Sato [37]. Note that the Wiener-Hopf factorization only exists for one-dimensional Lévy processes.

1.1. General definitions.

Recall that a one-dimensional Lévy process with law $\mathbb{P}$, henceforth denoted by $X := \{X_t : t \geq 0\}$, is a stochastic process issued from the origin which enjoys the properties of having stationary and independent increments with paths that are almost surely right-continuous with left limits (for general definitions, see, e.g., [37]). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality

$$E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}. \tag{1}$$

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y\mathbf{1}_{[-1;1]}(y))F(dy),$$

where $\sigma^2 \geq 0$ is the variance of the Gaussian component, $\mu \in \mathbb{R}$ is the drift, $\mathbf{1}_A$ is the indicator function of the set A, and the Lévy measure $F(dy)$ concentrated on $\mathbb{R}\backslash\{0\}$ satisfies

$$\int_{\mathbb{R}\backslash\{0\}} \min\{1, y^2\} F(dy) < +\infty.$$

As a consequence of this definition, the law of every Lévy process is characterized through a triplet (μ, σ, F). In financial modeling, the most Lévy measures $F(dy)$ have a Lévy density (we denote it by $\pi(y)$).

Assume that the riskless rate r is constant, and, under a risk-neutral measure chosen by the market, the underlying evolves as $S_t = S_0 e^{X_t}$, where X_t is a Lévy process. Then we must have $E[e^{X_t}] < +\infty$, and, therefore, ψ must admit the analytic continuation into the strip $\operatorname{Im}\xi \in (-1, 0)$ and continuous continuation into the closed strip $\operatorname{Im}\xi \in [-1, 0]$.

Further, if $d \geq 0$ is the constant dividend yield on the underlying asset, then the following condition (the EMM-requirement) must hold: $E[e^{X_t}] = e^{(r-d)t}$. Equivalently,

$$r - d + \psi(-i) = 0, \tag{2}$$

which can be used to express the drift μ via the other parameters of the Lévy process:

$$\mu = r - d - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y\mathbf{1}_{[-1;1]}(y)) F(dy). \tag{3}$$

In empirical studies of financial markets, the following classes of Lévy processes are popular: the Merton model [38], double-exponential jump-diffusion model (DEJD) introduced to finance by Lipton [16] and Kou [39], generalization of DEJD model constructed by Levendorskiĭ [40] and labelled later Hyper-exponential jump-diffusion model (HEJD), Variance Gamma Processes (VGP) introduced to finance by Madan with coauthors (see, e.g., [41]), Hyperbolic processes constructed in [42, 43], Normal Inverse Gaussian processes constructed by Barndorff-Nielsen [44] and generalized in [45], and tempered stable processes or extended Koponen's family introduced in [46, 47] and labelled KoBoL model in [2]. Koponen [48] introduced a symmetric version; Boyarchenko and Levendorskiĭ [46, 47] gave a non-symmetric generalization; later, in [49], a subclass of this model appeared under the name CGMY–model.

Example 2.1. [Tempered stable Lévy processes] The characteristic exponent of a tempered stable Lévy process is given by

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[{\lambda_+}^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_+)[(-\lambda_-)_-^{\nu} - (-\lambda_- - i\xi)^{\nu_-}], \tag{4}$$

where $\nu_\pm \in (0, 2), \nu_\pm \neq 1$, $c_\pm > 0$, $\mu \in \mathbb{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. Formula (4) under the name "KoBoL model" is derived in Boyarchenko and Levendorskiĭ [2] from the Lévy-Khintchine formula with the Lévy density $\pi(y)$, given by

$$\pi(x) = c_+ e^{\lambda_+ x}|x|^{-\nu_+ - 1}\mathbf{1}_{\{x<0\}} + c_- e^{\lambda_- x}|x|^{-\nu_- - 1}\mathbf{1}_{\{x>0\}}. \tag{5}$$

Example 2.2. In a DEJD model (also known as the Kou model), the Lévy density $\pi(y)$, is of the form

$$\pi(x) = (1-p)\lambda\Lambda_- e^{\Lambda_- x}\mathbf{1}_{\{x<0\}} + p\lambda\Lambda_+ e^{-\Lambda_+ x}\mathbf{1}_{\{x>0\}}. \tag{6}$$

where $\Lambda_- > 0, \Lambda_+ > 1$, $p > 0$, λ.

If we set $c_+ = (1-p)\lambda\Lambda_-$, $c_- = p\lambda\Lambda_+$, $\lambda_+ = \Lambda_-$, $\lambda_- = -\Lambda_+$, then the characteristic exponent of Kou model can be written in the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $\sigma > 0$, $\mu \in \mathbb{R}$, $c_\pm > 0$ and $\lambda_- < -1 < 0 < \lambda_+$.

Example 2.3. A normal inverse Gaussian process (NIG) can be described by the characteristic exponent of the form (see Barndorff-Nielsen (1998))

$$\psi(\xi) = -i\mu\xi + \delta[(\alpha^2 - (\beta + i\xi)^2)^{1/2} - (\alpha^2 - \beta^2)^{1/2}], \tag{7}$$

where $\alpha > |\beta| > 0$, $\delta > 0$ and $\mu \in \mathbb{R}$.

Example 2.4. The characteristic exponent of a Variance Gamma process is given by (see Madan et al. (1998))

$$\psi(\xi) = -i\mu\xi + c[\ln(\lambda_+ + i\xi) - \ln\lambda_+ + \ln(-\lambda_- - i\xi) - \ln(-\lambda_-)], \tag{8}$$

where $c > 0$, $\mu \in \mathbb{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Example 2.5. The β-class of Lévy processes, introduced in [33], is a 10-parameter Lévy process which has characteristic exponent

$$\begin{aligned}\psi(\xi) &= a\xi + \frac{1}{2}\sigma^2\xi^2 + \frac{c_1}{\beta_1}\left\{\mathrm{B}(\alpha_1, 1-\lambda_1) - \mathrm{B}\left(\alpha_1 - \frac{i\xi}{\beta_1}, 1-\lambda_1\right)\right\} \\ &\quad + \frac{c_2}{\beta_2}\left\{\mathrm{B}(\alpha_2, 1-\lambda_2) - \mathrm{B}\left(\alpha_2 + \frac{i\xi}{\beta_2}, 1-\lambda_2\right)\right\}\end{aligned}$$

with parameter range $a, \sigma \in \mathbb{R}, c_1, c_2, \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and $\lambda_1, \lambda_2 \in (0,3) \setminus \{1,2\}$. Here $\mathrm{B}(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function. The density of the Lévy measure is given by

$$\pi(x) = c_1 \frac{e^{-\alpha_1\beta_1 x}}{(1-e^{-\beta_1 x})^{\lambda_1}}\mathbf{1}_{\{x>0\}} + c_2 \frac{e^{\alpha_2\beta_2 x}}{(1-e^{\beta_2 x})^{\lambda_2}}\mathbf{1}_{\{x<0\}}.$$

Although ψ takes a seemingly complicated form, this particular family of Lévy processes has a number of very beneficial virtues from the point of view of mathematical finance which are discussed in [50]. Moreover, the large number of parameters also allows one to choose Lévy processes within the β-class that have paths that are both of unbounded variation [when at least one of the conditions $\sigma \neq 0$, $\lambda_1 \in (2,3)$ or $\lambda_2 \in (2,3)$ holds] and bounded variation (when all of the conditions $\sigma = 0$, $\lambda_1 \in (0,2)$ and $\lambda_2 \in (0,2)$ hold) as well as having infinite and finite activity in the jumps component (accordingly as both $\lambda_1, \lambda_2 \in (1,3)$ or not).

1.2. **The Wiener-Hopf factorization.** A property that is common to all Lévy processes is the so-called Wiener-Hopf factorization. There are several forms of the Wiener-Hopf factorization. Suppose that for any $q > 0$, $T_q \sim \mathrm{Exp}\, q$ is an exponentially distributed random variable with mean q^{-1} independent of X. Recall that $\overline{X}_t = \sup_{s\le t} X_s$ and let $\underline{X}_t := \inf_{s\le t} X_s$. The Wiener-Hopf factorization states that the random variables $\overline{X}_{T_q}$ and $\overline{X}_{T_q} - X_{T_q}$ are independent. Thanks to the so-called principle of duality, that is to say the equality in law of the pair $\{X_{(t-s)-} - X_t : 0 \le s \le t\}$ and $\{-X_s : 0 \le s \le t\}$, it follows that $\overline{X}_{T_q} - X_{T_q}$ is equal in distribution to $-\underline{X}_{T_q}$. The Wiener-Hopf factorization formula used in probability reads:

$$E[e^{i\xi X_{T_q}}] = E[e^{i\xi \bar{X}_{T_q}}]E[e^{i\xi \underline{X}_{T_q}}], \quad \forall\, \xi \in \mathbb{R}. \tag{9}$$

Equivalently,

$$X_{T_q} \stackrel{d}{=} S_q + I_q, \tag{10}$$

where S_q and I_q are independent and equal in distribution to $\overline{X}_{T_q}$ and $\underline{X}_{T_q}$, respectively. Here we use the notation $\stackrel{d}{=}$ to mean equality in distribution.

Introducing the notation

$$\begin{aligned}\phi_q^+(\xi) &= qE\left[\int_0^\infty e^{-qt}e^{i\xi\bar{X}_t}dt\right] = E\left[e^{i\xi\bar{X}_{T_q}}\right], \\ \phi_q^-(\xi) &= qE\left[\int_0^\infty e^{-qt}e^{i\xi\underline{X}_t}dt\right] = E\left[e^{i\xi\underline{X}_{T_q}}\right]\end{aligned}$$

we can write (9) as

$$\frac{q}{q+\psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi). \tag{11}$$

1.3. **Fast Wiener-Hopf factorization method (the FWHF–method), [24].** We recall a sufficiently accurate numerical procedure for approximations of the Wiener-Hopf factors constructed in [24].

The first ingredient is the reduction of the factorization problems to symbols of order 0, which stabilize at infinity to 1. The approach in [24] consider the reduction for a wide class of Lévy processes which consists of Variance Gamma processes and RLPE of order $\nu \in (0;2]$. Loosely speaking, a Lévy process X is called a *Regular Lévy Process of Exponential type* (RLPE) if its Lévy density has a polynomial singularity at the origin and decays exponentially at infinity (see details in [2]). An almost equivalent definition is: the characteristic exponent is analytic in a strip $\operatorname{Im}\xi \in (\lambda_-,\lambda_+)$, continuous up to the boundary of the strip, and admits the representation

$$\psi(\xi) = -i\mu\xi + \phi(\xi), \tag{12}$$

where $\phi(\xi)$ stabilizes to a positively homogeneous function at the infinity:

$$\phi(\xi) \sim c_\pm|\xi|^\nu, \quad \text{as } \operatorname{Re}\xi \to \pm\infty, \quad \text{in the strip } \operatorname{Im}\xi \in (\lambda_-,\lambda_+), \tag{13}$$

where $c_\pm > 0$.

Introduce functions

$$\Lambda_-(\xi) = {\lambda_+}^{\nu_+}(\lambda_+ + i\xi)^{-\nu_+}; \tag{14}$$

$$\Lambda_+(\xi) = (-\lambda_-)^{\nu_-}(-\lambda_- - i\xi)^{-\nu_-}; \tag{15}$$

$$\Phi(\xi) = q\Big((q+\psi(\xi))\Lambda_+(\xi)\Lambda_-(\xi)\Big)^{-1}. \tag{16}$$

Choices of ν_+ and ν_- depend on properties of ψ, hence on order ν (see (12)–(13)) and drift μ. We have to consider the following cases.[1].

(1) If X_t – RLPE of order $\nu \in (1,2)$, we set $\nu_+ = \nu_- = \nu/2$.
(2) If X_t – RLPE of order $\nu \in (0,1]$ and drift $\mu = 0$, we set $\nu_+ = \nu_- = \nu/2$.
(3) If X_t – RLPE of order $\nu \in (0,1)$ and drift $\mu > 0$, we set $\nu_+ = 0, \nu_- = 1$.
(4) If X_t – RLPE of order $\nu \in (0,1)$ and drift $\mu < 0$, we set $\nu_+ = 1, \nu_- = 0$.
(5) If X_t – VGP and drift $\mu > 0$, we set $\nu_+ = 0, \nu_- = 1$.
(6) If X_t – VGP and drift $\mu < 0$, we set $\nu_+ = 1, \nu_- = 0$.

Functions $\Lambda_\pm(\xi)$ are analytic and do not vanish in the half-plane $\pm\operatorname{Im}\xi > 0$, continuous up to the boundary. In addition, $\Lambda_\pm(\xi)$ and its reciprocal grow not faster than a polynomial. Therefore, it remains to factorize

$$\Phi(\xi) = \Phi^+(\xi)\Phi^-(\xi), \tag{17}$$

and then set

$$\phi_q^\pm(\xi) = \Lambda_\pm(\xi)\Phi^\pm(\xi). \tag{18}$$

Next, we consider an approximation of symbols $\Phi^\pm(\xi)$. We, first, approximate Φ by a periodic function Φ_d with a large period $2\pi/d$, which is the length of the truncated region in ξ-state, then approximate the latter by a partial sum of the Fourier series, and, finally, use the factorization of the latter instead of the exact one.

We can apply this realization both after the reduction to symbols of order 0 has been made, and without this reduction. In the latter case, we obtain a Poisson type approximation.

It is well-known (see e.g. [51]) that the limit of a sequence of the Poisson type characteristic functions is infinitely divisible characteristic function. The converse is also true. Every infinitely divisible characteristic function can be written as the limit of a sequence of finite products of Poisson type characteristic functions. Since $\psi(\xi)$ is the characteristic exponent of Lévy process, then the function $q/(q+\psi(\xi))$ is infinitely divisible characteristic function.

The second step is straightforward. We impose an additional condition

$$|\Phi'(\xi)| \le C(1+|\xi|)^{-\rho}, \tag{19}$$

where $\rho > 0$; this condition is satisfied by all RLPEs and VGPs (and can be relaxed), which makes the following lemma applicable.

[1]We can reduce the case $\nu \in (0,1]$ and $\mu \neq 0$ to (2) after the elimination of the drift. For driftless VGPs we reduce to the cases (5) or (6) by suitable change of numeraire

Consider a function $f_d \in C^1[-\pi/d, \pi/d]$ depending on a small parameter d, and, for $m > 0$, construct the partial sum

$$S_m(f_d) = \frac{d}{2\pi} \sum_{|k| \le m} \hat{f}_{d,k} e^{idkx}$$

of the Fourier series for f_d.

Lemma 1.1. *Let $f_d(-\pi/d) = f_d(\pi/d)$, and let there exists $C > 0$ such that for all $x \in [-\pi/d, \pi/d]$ and all $d \in (0, 1]$, $|\partial_x f_d(x)| \le C$.*

Then there exists a function $d \mapsto m_0(d)$ such that $\forall\ m \ge m_0(d)$, $x \in [-\pi/d, \pi/d]$ and $d \in (0, 1]$,

$$|f_d(x) - S_m(f_d)(x)| \le \epsilon. \tag{20}$$

For the first and third steps, we fix a small positive d and large even M (on the strength of Lemma 1.1, M should be much larger than $1/d$), set

$$b_k^d = \frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} \ln \Phi(\xi) e^{-i\xi kd} d\xi, \quad k \ne 0, \tag{21}$$

$$b_{d,M}(\xi) = \sum_{k=-M/2+1}^{M/2} b_k^d (\exp(i\xi kd) - 1), \tag{22}$$

$$b_{d,M}^+(\xi) = \sum_{k=1}^{M/2} b_k^d (\exp(i\xi kd) - 1), \tag{23}$$

$$b_{d,M}^-(\xi) = \sum_{k=-M/2+1}^{-1} b_k^d (\exp(i\xi kd) - 1); \tag{24}$$

$$\Phi_{d,M}(\xi) = \exp(b_{d,M}(\xi)), \tag{25}$$

$$\Phi_{d,M}^{\pm}(\xi) = \exp(b_{d,M}^{\pm}(\xi)). \tag{26}$$

Now consider an approximation of $\phi_q^{\pm}(\xi)$ using the Fast Fourier Transform. Let d be the step in x-space, ζ–the step in ξ-space, and $M = 2^m$ the number of the points on the grid. Consider the algorithm (the discrete Fourier transform (DFT)) defined by

$$G_l = DFT[g](l) = \sum_{k=0}^{M-1} g_k e^{2\pi ikl/M}, \quad l = 0, ..., M-1. \tag{27}$$

(It differs in sign in front of i from the algorithm fft in MATLAB). The DFT maps m complex numbers (the g_k's) into m complex numbers (the G_l's). The formula for the inverse DFT which recovers the set of g_k's exactly from G_l's is:

$$g_k = iDFT[G](k) = \frac{1}{M} \sum_{l=0}^{M-1} G_l e^{-2\pi ikl/M}, \quad k = 0, ..., M-1. \tag{28}$$

In our case, the data consist of a real-valued array $\{g_k\}_{k=0}^M$. The resulting transform satisfies $G_{M-l} = \bar{G}_l$. Since this complex-valued array has real values G_0 and $G_{M/2}$, and $M/2 - 1$ other independent complex values $G_1, ..., G_{M/2-1}$, then it has the same "degrees of freedom" as the original real data set. In this case, it is inefficient to use full complex FFT algorithm. The main idea of FFT of real functions is to pack the real input array cleverly, without extra zeros, into a complex array of half of length. Then a complex FFT can be applied to this shorter length; the trick is then to get the required values from this result (see Press, W. et al (1992) [34] for technical details). To distinguish DFT of real functions we will use notation RDFT.

Depending on the type of the option under consideration, we choose real ω, and apply the Fourier transform $F_{x\to\xi}$ and the inverse Fourier transform $F_{\xi\to x}^{-1}$ with x living in $\mathbb{R}$ and ξ living in $\mathbb{R} + i\omega$. Thus, a grid for ξ is the grid $\xi_j = \eta_j + i\omega$ on the line $\operatorname{Im} \xi = \omega$. Then $\psi_1(\eta) = \psi(\eta + i\omega) - \psi(i\omega)$ be also characteristic exponent of infintely divisible distribution. Set $q_1 = q + \psi(i\omega)$, and we will apply FWH-method for factorization of $q_1/(q_1 + \psi_1(\eta))$ with q_1 and $\psi_1(\eta)$ instead q and $\psi(\eta)$, respectively. In the case of down-and-out options, for typical parameters values, we will choose $\omega = -2$; for up-and-out options, $\omega = 1$ is a good choice.

Fix the space step $d > 0$ and number of the space points $M = 2^m$. Define the partitions of normalized log-price domain $[-\frac{Md}{2}; \frac{Md}{2})$ by points $x_k = -\frac{Md}{2} + kd$, $k = 0, ..., m-1$, and frequency domain $[-\frac{\pi}{d} + i\omega; \frac{\pi}{d} + i\omega]$ by points $\xi_l = \frac{2\pi l}{dM} + i\omega$, $l = -M/2, ..., M/2$.

To approximate the characteristic functions $\phi_q^{\pm}(\xi)$, we find an approximation of function $\ln \Phi$ by the Fourier series using the formula (22). The coefficients b_k^d in (22) are defined by (21) and can be efficiently computed by using iRDFT. We have:

$$b_k^d = \frac{d}{2\pi} \int_{-\pi/d}^{\pi/d} \ln \Phi(\xi) e^{-i\xi kd} d\xi \approx iRDFT[\ln \Phi](k). \tag{29}$$

Next, we calculate $b_{d,M}^{\pm}$ in (23)–(24), and then, calculate the approximations to the Wiener-Hopf factors

$$\phi_q^{\pm}(\xi_l) \approx \Lambda_{-\nu/2}^{\pm}(\xi_l) \exp(b_{d,M}^{\pm}(\xi_l)), l = -M/2, ..., 0. \tag{30}$$

2. Wiener-Hopf Monte Carlo methods

2.1. **The overview of Monte Carlo methods.** Due to the independent increments of Lévy processes, the most common approaches to the Monte Carlo simulation of expectations involving the joint law of $(X_t, \overline{X}_t)$ work with the correspondent random walk whose increments have the same law as $X_{\Delta t}$ for some small $\Delta t > 0$. A 'plain' Monte Carlo method simulate paths of X by discretizing time and generating the target process forward in time. However, there are only few examples of Lévy processes X for which the law of $X_{\Delta t}$ is known. This requires one either to be able to simulate the increments of the Lévy process exactly for a fixed time step or to be able to suitably approximate the Lévy process, typically by a jump diffusion process or a numerical Fourier inversion. This introduces numerical inaccuracy and an extra potentially expensive computation step.

It is well known that the convergence of Monte Carlo estimators of quantities involving first passage is rather slow. Hence, a large number of paths is needed to obtain a good convergence. Another drawback is the well known problem that the empirical first passage time distribution suffers from a very significant bias which vanishes only very slowly as Δt vanishes, due to the fact that the random walk approach misses excursions over the barrier within a short time period.

A typical run for the standard Monte Carlo method would require evaluating thousands of time steps per process path. In addition, tens of thousands of Monte Carlo runs (generated paths) are needed to obtain an accurate density estimation. For many applications there are thousands of processes, and it is required to compute the first passage time density for each process (e.g. a loan portfolio with many corporate debt instruments, or a many nodes in a computer network).

There are three main approaches to simulate the trajectory of a Lévy process. The first method relies on the ability to sample from the increments of the Lévy process exactly for a fixed time step and therefore construct a random walk as described at the beginning of the current Subsection. This approach requires knowledge of the probability distribution of the Lévy process at a deterministic time, either through an explicit analytical formula, or via numerical inversion of the characteristic function. General results on the simulation of infinite divisible distributions can be found in Bondesson [52]. As mention earlier, approximating a Lévy process by an embedded random walk may cause significant errors on path functionals of X such as the running extremum (cf. Broadie and Glasserman [53]).

The second approach uses a time-dependent infinite series expansion to approximate the value of the Lévy process at each fixed time. For a general result when an explicit expression for the Lévy measure is known we refer the reader to Rosiński [54]. The indicated series representation converges uniformly and almost surely in any compact set of time. This lends itself better to sampling path functionals of the process than, perhaps the random walk approximation but it might make the numerical analysis difficult.

Finally, the third and most common method is to approximate X via a sum of a linear Brownian motion and an independent compound Poisson process. The approximation can be done using an appropriate linear and/or Gaussian component instead small jumps. The truncation of small jumps ensures that the remaining jumps conform to a compound Poisson structure The method of truncating small jumps, originally due to Asmussen and Rosiński [55], is an obvious approach (cf. Cont and Tankov [3, Sect. 6.3] and the references therein), but may change the regularity properties of the solution to the problem (cf. [24]).

Recently introduced by Kuznetsov et al. [30], the so-called Wiener-Hopf Monte Carlo method is related to the simulation of increments. This technique combines ideas from both Carr [56] and Doney [57]. On the one hand, the method requires the underlying time grid to be random with independent and exponentially distributed spacings (so-called "Carr's randomization"). This idea allows to simulate the trajectories of a variety of Lévy model from the law of $X_{\boldsymbol{\tau}}$ and its extremum where $\boldsymbol{\tau}$ is a random time whose distribution can be concentrated arbitrarily close around t depending on a parameter chosen in the algorithm that controls the resolution of the random grid and thus the amount of work. On the other hand, the approximation uses a simple trick of embedding a random walk into the path of a Lévy process with two types of step distribution determined by the Wiener–Hopf factorization which was used in a different context by Doney [57] in bounding the path of a Lévy process from above and below by a random walk.

According to [31], we formulate the main differences between these three methods and the WHMC simulation scheme are three–fold. Firstly, whilst the the WHMC scheme is restricted to Lévy processes that allow sampling from the distributions of the variables $\overline{X}_{T_q}$ and $\underline{X}_{T_q}$, it is otherwise indifferent to the jump structure of the underlying Lévy process. The approximation of a Lévy process by a compound Poisson process decomposes the process with respect to the endogenous jump structure. In contrast, the Wiener-Hopf factorization is related to the decomposition of the path of a Lévy process according to the distribution of its extremums. Secondly, the WHMC method uses a exponentially distributed time grid. Note for comparison that the approach in [29, 9] requires a fixed time grid interlaced by a random grid capturing the large jumps. Thirdly, the WHMC works on the principle of sampling the Lévy process over the "wrong" time horizon (but in a way which can be made arbitrarily close to the desired time horizon) purely in order to be able to sample from the exact maximum. Note that a similar idea of randomizing the time horizon in a computation involving the expectation of a functional of the path of a linear Brownian motion that appears in the context of pricing American options is described in Carr [56]. Later this approach was applied to pricing options under Lévy processes, see e.g. [2]. The rate of convergence of the WHMC and of the MLWH method can be expressed directly in terms of the rate of convergence of the randomized time horizon to the fixed time t.

In the next subsection we describe the WHMC method in more detail.

2.2. **The overview of Wiener-Hopf Monte Carlo methods,** [30, 31, 32]**.** Let us suppose that $\mathbf{e}_1(1), \mathbf{e}_2(1), \cdots$ are a sequence of i.i.d. exponentially distributed random variables with unit mean. The basis of the WHMC algorithm is the following simple observation, which follows directly from the Strong Law of Large Numbers. For all $t > 0$,

$$\sum_{i=1}^{n} \frac{t}{n} \mathbf{e}_i(1) \to t \text{ as } n \uparrow \infty \tag{31}$$

almost surely. Note that the random variable on the left hand side of (31) can also be written as the sum of n independent random variables with an exponential distribution having mean t/n and therefore is equal in law to a Gamma random variable with parameters n and n/t, henceforth written as $\mathbf{g}(n, n/t)$. For sufficiently large n, Kuznetsov et al. [30] argue that a suitable approximation to $\mathbb{P}(X_t \in \mathrm{d}x, \overline{X}_t \in \mathrm{d}y)$ is $\mathbb{P}(X_{\mathbf{g}(n,n/t)} \in \mathrm{d}x, \overline{X}_{\mathbf{g}(n,n/t)} \in \mathrm{d}y)$. Indeed, it is a triviality to note that, thanks to (31) and the independence of $\mathbf{g}(n, n/t)$ from X, the pair $(\overline{X}_{\mathbf{g}(n,n/t)}, X_{\mathbf{g}(n,n/t)})$ converges almost surely to $(\overline{X}_t, X_t)$ as $n \uparrow \infty$. The rates of convergence computed in [32].

The following theorem is straightforward to prove using (10) together with the stationary and independent increments of the underlying Lévy process.

Theorem 2.1 (Kuznetsov et al. [30, Thm. 1])**.** *Let $\{S^j_{n/t} : j \geq 1\}$ and $\{I^j_{n/t} : j \geq 1\}$ be i.i.d. sequences of random variables with common distribution equal to that of $\overline{X}_{T_{(n/t)}}$ and $\underline{X}_{T_{(n/t)}}$, respectively, where $T_q \sim \operatorname{Exp} q$ is independent of X. Then, for all $n \in \mathbb{N}$,*

$$(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)}) \stackrel{d}{=} (V(n, n/t), J(n, n/t)),$$

where, for any $k \in \mathbb{N}$, *and setting* $V(0,n/t) := 0$ *and* $J(0,n/t) := 0$ *we define*

$$V(k,n/t) = V(k-1,n/t) + \left(S^k_{n/t} + I^k_{n/t}\right), \tag{32}$$

$$J(k,n/t) = \max\left\{J(k-1,n/t), V(j-1,n/t) \ + \ S^j_{n/t}\right\}. \tag{33}$$

Theorem 2.1 suggests that as soon as we are able to simulate i.i.d. copies of the distributions of $S_{n/t}$ and $I_{n/t}$, then by the simple functional transformations given in (35) and (36), we may produce an exact draw from the distribution of $(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)})$. Moreover, for a suitably nice function F, using standard Monte Carlo methods based on the Strong Law of Large Numbers, one may estimate $\mathbb{E}(G(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)}))$ by

$$F^{n,M}_{\mathrm{MC}} := \frac{1}{M}\sum_{i=1}^{M} F^{n,(i)}, \tag{34}$$

where $F^{n,(i)}$ is the i-th sample of

$$F^n := G\left(V(n,n/t), J(n,n/t)\right).$$

Indeed, we have $\lim_{M\uparrow\infty} F^{n,M}_{\mathrm{MC}} = \mathbb{E}\left(G(X_{\mathbf{g}(n,n/t)}, \overline{X}_{\mathbf{g}(n,n/t)})\right)$ almost surely, which in turn converges to $\mathbb{E}(G(X_t, \overline{X}_t))$ as $n \uparrow \infty$.

It follows from the Theorem 2.1 that a similar result for the pair $(\underline{X}_t, X_t)$ is also valid.

Theorem 2.2. *Let* $\{S^j_{n/t} : j \geq 1\}$ *and* $\{I^j_{n/t} : j \geq 1\}$ *be i.i.d. sequences of random variables with common distribution equal to that of* $\overline{X}_{T_{(n/t)}}$ *and* $\underline{X}_{T_{(n/t)}}$, *respectively, where* $T_q \sim \operatorname{Exp} q$ *is independent of* X. *Then, for all* $n \in \mathbb{N}$,

$$(X_{\mathbf{g}(n,n/t)}, \underline{X}_{\mathbf{g}(n,n/t)}) \stackrel{d}{=} (V(n,n/t), J(n,n/t)),$$

where, for any $k \in \mathbb{N}$, *and setting* $V(0,n/t) := 0$ *and* $J(0,n/t) := 0$ *we define*

$$V(k,n/t) = V(k-1,n/t) + \left(S^k_{n/t} + I^k_{n/t}\right), \tag{35}$$

$$J(k,n/t) = \min\left\{J(k-1,n/t), V(j-1,n/t) \ + \ I^j_{n/t}\right\}. \tag{36}$$

It should be mentioned that the WHMC method is numerically feasible only if samples from the distributions of $S_{n/t}$ and $I_{n/t}$ are available. However popular examples of Lévy processes used in finance do not allow to apply this method directly. For example, Merton's jump-diffusion model (see Merton [38]) where the driving Lévy process X is a drifted Brownian motion plus a compound Poisson process with normally distributed jumps. As the Wiener-Hopf factors are not explicitly known in this case, the WHMC cannot be applied while plain Monte Carlo does. The same applies to the popular Variance Gamma (see Madan et al [41]), CGMY (see Carr et al [49]) and NIG (see Barndorff-Nielsen [44]) models.

Until recently, this would have proved to be a significant stumbling block on account of there being few examples for which the aforesaid distributions are known in explicit form. However, on the one side, developments in Wiener-Hopf theory for Lévy processes in the last couple of years (see for example Kuznetsov [33] or Kuznetsov et al. [30]) have provided a rich enough variety of examples for which the necessary distributional sampling can be performed. This family of processes are named meromorphic Lévy processes in Kuznetsov et al. [30]. One large subfamily of such processes is the β-class of Lévy processes, which also conveniently offers all the desirable properties of better known Lévy processes that are used in mathematical finance, such as CGMY processes, VG processes or Meixner processes; see for example the discussions in Ferreiro-Castilla and Schoutens [58] and Schoutens and van Damme [59].

The starting point for the Wiener–Hopf Monte Carlo algorithm is the distribution of $\overline{X}_q$ and $\underline{X}_q$. However, Kuztetsov et al. [30] presented another alternative for extending the application of the Wiener–Hopf Monte Carlo method to a much larger class of Lévy processes than those for which explicit formulas of the Wiener–Hopf factors are known. The importance of Theorem 2.3 is that one may simulate desired paths of any Lévy processes whose Lévy measure can be written as a sum of a Lévy measure from the β-family or hypergeometric family and *any* other measure with finite mass. It should be noted that many Lévy processes necessarily take this form. However, there are some obvious exclusions from this class, for example, cases of Lévy processes with bounded jumps.

Theorem 2.3 (Kuznetsov et al. [30, Thm. 4]). *Let $Y = \{Y_t : t \geq 0\}$ be a sum of a Lévy process X and a compound Poisson process such that for all $t \geq 0$,*

$$Y_t = X_t + \sum_{i=1}^{N_t} \xi_i,$$

where $N = \{N_t : t \geq 0\}$ is a Poisson process with intensity λ, independent of the i.i.d. sequence of random variables, $\{\xi_i : i \geq 1\}$, and X. Define iteratively for $n \geq 1$

$$\begin{aligned} V(n,\lambda) &= V(n-1,\lambda) + S^{(n)}_{n/t+\lambda} + I^{(n)}_{n/t+\lambda} + \xi_n(1-\beta_n), \\ J(n,n/t) &= \max\Big(V(n,n/t), J(n-1,n/t), V(n-1,n/t) + S^{(n)}_{n/t+\lambda}\Big), \end{aligned}$$

where $V(0,n/t) = J(0,n/t) = 0$, sequences $\{S^{(j)}_{n/t+\lambda} : n \geq 1\}$ and $\{I^{(n)}_{n/t+\lambda} : n \geq 1\}$ are defined in Theorem 2.1, and $\{\beta_n : n \geq 1\}$ are an i.i.d. sequence of Bernoulli random variables such that $\mathbb{P}(\beta_n = 1) = \frac{(n/t)}{(n/t)+\lambda}$. Then

$$\left(Y_{\mathbf{g}(n,n/t)}, \overline{Y}_{\mathbf{g}(n,n/t)}\right) \stackrel{d}{=} (V(T_n, n/t), J(T_n, n/t)), \tag{37}$$

where $T_n = \min\{j \geq 1 : \sum_{i=1}^{j} \beta_i = n\}$.

An important example where the use of the above theorem may be efficient is a sum of a linear Brownian motion and an independent compound Poisson process. This would include the following popular jump diffusion processes: the Kou model and the Merton model. In the case that X is a linear Brownian motion, the quantities $\overline{X}_{T_q}$ and $-\underline{X}_{T_q}$ are both exponentially distributed with easily computed intensities.

On the other side, due to the unavailability of the exact law of $\overline{X}_{T_q}$ ($\underline{X}_{T_q}$), one can use approximate formulas not for the actual driving Lévy process, but for the Wiener-Hopf factors directly. For a wide class of Lévy models, it becomes possible by using Fast Wiener-Hopf factorization method developed in [24]. A modification of the WHMC method based on a numerical Wiener-Hopf factorization will be suggested in the Subsection 3.

For such cases a separate study would be helpful to decide whether or not the advantages the WHMC simulation method has over plain Monte Carlo as described in the previous paragraph outweighs the disadvantage that the WHMC method only applies to an approximation of the actual driving Lévy process or to approximates for the Wiener-Hopf factors.

The Wiener-Hopf Monte Carlo (WHMC) simulating technique introduced in Kuznetsov et al. [30] allows to sample from a law that is a good approximation of the law of $(X_T, \overline{X}_T)$, provided that samples can be produced from $\overline{X}_{T_q}$ and $\underline{X}_{T_q}$. This method was extended to a multilevel version and a theoretical analysis was given in Ferreiro-Castilla et al. [32].

In [32] the main idea behind the WHMC simulating technique was used to generate samples from (an approximation of)

$$(\tau_h, X_{\tau_h} - h, h - X_{\tau_h -}, h - \overline{X}_{\tau_h -}), \tag{38}$$

where τ_h is the first passage time of X over a level $h > 0$, i.e.

$$\tau_u := \inf\{t > 0 \,|\, X_t > h\}\ ,$$

$X_{\tau_h} - h$ is referred to as the overshoot, $h - X_{\tau_h -}$ the undershoot and finally $h - \overline{X}_{\tau_h -}$ the last maximum before first passage. Here and throughout we employ the usual notation $\overline{X}_t := \sup_{s \leq t} X_s$ and $\underline{X}_t := \inf_{s \leq t} X_s$ for all $t \geq 0$. Such extension of the WHMC-method allows to calculate quantities of the form

$$\mathbb{E}\left[f(\tau_h, X_{\tau_h} - h, h - X_{\tau_h -}, h - \overline{X}_{\tau_h -})\right]. \tag{39}$$

In fact, not only (38) but any functional of the pair $(X, \overline{X})$ could be handled by the method. Once this observation has been established it is simply a matter of applying the usual setup: generate a large number of such samples, apply the function G to each of them and compute the resulting average to obtain an approximation of (39).

Following [32], we will refer to (38) as *the 4-tuple*. Below we list the main results and ideas of [32] for producing samples from the 4-tuple as well.

Recall that T_q denotes an exponentially distributed random variable with mean $1/q$ independent of X. For any $n \geq 1$, enlarge the probability space on which X lives with an i.i.d. sequence $\{T^i_{n/t}\}_{i\geq 1}$ and define a set of grid points as

$$g(0, n/t) := 0, \quad g(k, n/t) := \sum_{i=1}^{k} T^i_{n/t} \quad \text{for } k \geq 1. \tag{40}$$

For any n the set of random points $\{0 = g(0, n/t) < g(1, n/t) < ...\}$ forms a grid on the time axis, the distance between the grid points forming a sequence of i.i.d. exponentially distributed random variables; equivalently the grid points can be seen as the arrival times of a Poisson process with rate n/t. The idea is quite straightforward: the 'stochastic grid' as defined in (40) does not only satisfy $g(n, n/t) \overset{\text{a.s.}}{\to} t$ as $n \to \infty$, but for any sequence $k(n)$ such that $k(n) \in \{0, \dots, n\}$ and $k(n)t/n \overset{\text{a.s.}}{\to} s \in [0, t]$ as $n \to \infty$, we have again by the law of large numbers $g(k(n), n/t) \overset{\text{a.s.}}{\to} s$.

Consequently, as above, for large n the law of $(V(k(n), (n/t), J(k(n), (n/t))$ provides an approximation of the law of $(X_s, \overline{X}_s)$. In this sense the 'stochastic grid' becomes dense in the interval $[0, t]$. Furthermore, as is obvious from Theorem 2.1, due to the iterative nature of the definitions of V and J, obtaining a sample from the pair $(V(n, (n/t)), J(n, (n/t))$ requires producing a sample from the vector $((V(0, (n/t)), J(0, (n/t)), \dots, (V(n, (n/t)), J(n, (n/t))))$. Hence constructing the approximative law of $(X_t, \overline{X}_t)$ automatically yields an approximative law of the vector $((X_0, \overline{X}_0), (X_{1/n}, \overline{X}_{1/n}), \dots, (X_t, \overline{X}_t))$ — see Proposition 2.4 below — and it is therefore at least intuitively clear that we should also be able to approximate a quantity like (39). This is made rigorous in Theorem 2.5. The difficulty to be overcome is that convergence on the 'stochastic grid' is less obvious than on a traditional deterministic grid.

Proposition 2.4 (Ferreiro-Castilla et al. [32, Pr. 2.2]). *Let X be a Lévy process, $\lambda > 0$ and recall V and J as defined in Theorem 2.1 and the definition of the stochastic grid in (40). Then*

$$\left((X_{g(0,\lambda)}, \overline{X}_{g(0,\lambda)}), \dots, (X_{g(k,\lambda)}, \overline{X}_{g(k,\lambda)})\right) \overset{d}{=} ((V(0, \lambda), J(0, \lambda)), \dots, (V(k, \lambda), J(k, \lambda))) \ . \tag{41}$$

Notice that the random walk produced in Proposition 2.4 should be particularly useful when used to approximate pathwise quantities of X. Consider how the setup introduced in [30] can be used to generate an approximate distribution of the 4-tuple as well. The idea is to approximate τ_h by finding points on our 'stochastic grid' (40) enclosing it, i.e. $k(n) \in \mathbb{N}$ such that $g(k(n) - 1, n/t) \leq \tau_h \leq g(k(n), n/t)$ for all $n \in \mathbb{N}$, and evaluate the functionals involving overshoots and undershoots using these grid points.

Theorem 2.5 (Ferreiro-Castilla et al. [32, Thm. 3.1]). *Let X be any Lévy process. Fix some $t > 0$ and $h > 0$. Recall V and J as defined in Theorem 2.1. Set for all $n \in \mathbb{N}$*

$$\kappa^{(n)}_h := \inf\{k \in \{0, \dots, n\} \,|\, J(k, (n/t)) > h\}$$

(where as usual we understand $\inf \emptyset = \infty$). Then we have as $n \to \infty$

$$\begin{aligned}\Big(\frac{t}{n}(\kappa^{(n)}_h \wedge n), V(\kappa^{(n)}_h \wedge n, (n/t)) - h, h - V((\kappa^{(n)}_h - 1) \wedge n, (n/t)), h - J((\kappa^{(n)}_h - 1) \wedge n, (n/t))\Big) \\ \xrightarrow{d} \left(\tau_h \wedge t, X_{\tau_h \wedge t} - h, h - X_{(\tau_h \wedge t)-}, h - \overline{X}_{(\tau_h \wedge t)-}\right) \ .\end{aligned} \tag{42}$$

All quantities involved in (39) ultimately depend on the first passage time. Now we indicate the results of [32] on a convergence rate for the approximation of the first passage time. n order to define such auxiliary random vector let us first introduce the following quantities:

Notice that $\kappa^{(n)}_h \overset{d}{=} k^{(n)}_X$ where $k^{(n)}_X$ lives on the same probability space as X and has the form:

$$k^{(n)}_X := \inf\{k \in \{0, \dots, n\} \,|\, \overline{X}_{g(k,n/t)} > u\}$$

Theorem 2.6 (Ferreiro-Castilla et al. [32, Thm. 4.1]). *Using the same notation as in Theorem 2.5, we have*

$$\mathbb{E}\left[\left(\frac{t}{n}(k^{(n)}_X \wedge n) - \tau \wedge t\right)^2\right] \leq \frac{2t^2}{n}.$$

3. Implementation of the WHMC methods into the program platform PREMIA

The aim of this section is to provide an example of the application of the WHMC methods to pricing options in chosen Lévy models.

3.1. **General settings.** We consider options, whose payoff at maturity date T depends on $(X_T, \underline{X}_T)$.This implies that the barrier $H = e^h$ may play a role only when it is reached or crossed from above. The definitions and results below admit the straightforward reformulation for the case of options with payoffs depending on $(X_T, \overline{X}_T)$. In this case, the barrier $H = e^h$ would take effect only when it is reached or crossed from below. When the barrier is crossed the option's owner would be entitled to a constant rebate.

Consider

$$F(T,x) = \mathbb{E}^x\Big[e^{-rT} g(X_T, \underline{X}_T)\mathbf{1}_{\{\tau_h^- > T\}}\Big] + \mathbb{E}^x\Big[Re^{-r\tau_h^-}\mathbf{1}_{\{\tau_h^- \le T\}}\Big], \tag{43}$$

where time 0 is the beginning of a period under consideration (so that $X_0 = \underline{X}_0 = x$), T is the final date, h is the absorbing barrier, τ_h^- denotes the first entrance time into $(-\infty, h]$, R is the rebate and $g(X_T, \underline{X}_T)$ is the payoff at time T.

We choose n, the parameter of the MCWH method. Set V and J as defined in Theorem 2.2 (or in Theorem 2.3). Denote for all $n \in \mathbb{N}$

$$\kappa_h^{(n)} := \inf\{k \in \{0, \dots, n\} \mid J(k, (n/T)) < h\}$$

(where we understand $\inf \emptyset$ as ∞). Then according to Theorem 2.2 and Theorem 2.5, we have as $n \to \infty$

$$\left(\frac{T}{n}(\kappa_h^{(n)} \wedge n), V(n, (n/T)), J(n, (n/T))\right) \xrightarrow{d} (\tau_h \wedge T, X_T, \underline{X}_T)\ . \tag{44}$$

Recall S_q^k and I_q^k as defined in Theorem 2.2. Following the algorithm in Theorem 2.2, we need to simulate $S_{n/T}^k$ and $I_{n/T}^k$, $k = 1, \dots, n$, by using samples from the laws $\underline{X}_{T_{n/t}}$ and $\overline{X}_{T_{n/t}}$, respectively. If the probability distributions of $\underline{X}_{T_{n/T}}$ and $\overline{X}_{T_{n/t}}$ are known explicitly (e.g. in the case of Black-Scholes model, Kou model or when X_t belongs to the β-class of Lévy processes, introduced in [33]), then we may sample directly from the laws. In general case, we have to approximate characteristic functions of Wiener-Hopf factors using the FWHF-method described in Subsection 1.3. Then we numerically calculate the correspondent cumulative distribution functions and using their inverses simulate $S_{n/T}^k$ and $I_{n/T}^k$. Let us describe the latter case in details.

Let X be the random infinitely divisible variable, and $F_X(x) = \mathbf{P}(X < x)$, $p_X = \frac{d}{dx}F_X(x)$, $\phi(\xi) = E[e^{i\xi X}]$ stand for the cumulative distribution function (cdf), the probability distribution function (pdf), and the characteristic function (chf) of X, respectively. If $F_X(x)$ is continuous, then

$$F_X(x) = \int_{-\infty}^{x} p_X(y)dy. \tag{45}$$

If the variable X is continuous, then the cumulative distribution function F_X has an inverse $F_X^{-1} : (0,1) \to \mathbb{R}$. If the cdf F_X is known then one may simulate X by using samples from $F_X^{-1}(U)$, where U is a uniform distribution on $(0,1)$. To approximate $F_X^{-1}(U)$ with the pdf supported at a half-line, we choose a uniformly spaced grid $x_0, \dots, x_M \in \mathbb{R}$ subject to the condition $F_X(x_0) < \epsilon$ (if p_X is supported at a subset of $(-\infty, 0]$) or $F_X(x_n) > 1 - \epsilon$ (if p_X is supported at a subset of $[0, +\infty)$), where ϵ is a desired accuracy. Then for arbitrary $u \in (0,1)$, we define $F_X^{-1}(u)$ as follows:

$$F_X^{-1}(u) = \begin{cases} x_0, & u < F(x_0), \\ x_k + d \cdot \frac{u - F(x_k)}{F(x_{k+1}) - F(x_k)}, & F(x_k) \le u < F(x_{k+1}), 0 \le k < M, \\ x_n, & u \ge F(x_M). \end{cases}$$

Notice the formula from above means that we estimate X with the distribution which has atoms in x_0 and x_M. When the cdf is calculated numerically by using Fast Fourier Transform the number of points n should be a power of 2. It follows that for $M = 2^m$ the operation to find x_k satisfying $F(x_k) \le u < F(x_{k+1})$ requires m or less iterations.

If the probability density p_X is known, one can apply a quadrature rule to (45) for computing numerically the cdf F_X. However, in the case of infinitely divisible distributions as a rule explicit analytical formulas for pdf are not available. In order to recover the pdf p_X one can use the characteristic function ϕ_X which is typically known in the closed form. In the general case, p_X can be expressed in terms of the characteristic function $\phi_X(\xi)$, by using the Fourier transform

$$p_X(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{-ix\xi} \phi_X(\xi) d\xi, \tag{46}$$

where the chf $\phi_X(\xi)$ can be written via the characteristic exponent, see (1). The formula 46 can be efficiently realized by means of the Fast Fourier Transform algorithm.

One can also substitute the formula (46) into (45) and express the cumulative distribution function F_X in terms of the Fourier integral (see [60].

$$F_X(x) = \frac{e^{x\rho}}{\pi} \operatorname{Re} \int_0^{\infty} e^{-ix\xi} \frac{\phi_X(\xi + i\rho)}{\rho - i\xi} d\xi, x \in \mathbb{R}, \tag{47}$$

where $\rho > 0$. The difference between results obtained by these two approaches is insignificant. However, the method which uses quadrature rule and (46) is more simpe for a numerical implementation. Thus we have chosen to implement the first approach.

Consider the algorithm for computing the pdf of an infinitely divisible distribution by using Fast Fourier Transform. Let d be the step in x-space, ζ–the step in ξ-space, and $M = 2^m$ the number of the points on the grid. An approximation for the pdf can be efficiently computed by using the Fast Fourier Transform (FFT). Direct and inverse discrete Fourier transform (DFT) defined by (27) and (28), respectively.

In our case, the data (pdf) consist of a real-valued array $\{g_k\}_{k=0}^{M}$. The resulting transform satisfies $G_{M-l} = \bar{G}_l$. As in the FWHF-method, it is more efficient to use the real Fast Fourier Transform.

Fix the space step $d > 0$ and number of the space points $M = 2^m$. Define the partitions of normalized log-price domain $[-\frac{Md}{2}; \frac{Md}{2})$ by points $x_k = -\frac{Md}{2} + kd$, $k = 0, ..., M-1$, and frequency domain $[-\frac{\pi}{d}; \frac{\pi}{d}]$ by points $\xi_l = \frac{2\pi l}{dM}$, $l = -M/2, ..., M/2$.

Using the formula 46 we can approximate the pdf p_X as follows.

$$\begin{aligned} p_X(x_k) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix_k\xi} \phi_X(\xi) d\xi \\ &\approx \frac{1}{2\pi} \int_{-\pi/d}^{\pi/d} e^{ix_k\xi} \phi_X(\xi) d\xi \approx \frac{1}{2\pi} \sum_{l=-M/2+1}^{M/2} e^{ix_k\xi_l} \phi_X(\xi_l) \frac{2\pi}{dM} \\ &\approx \left(\frac{2}{Md} \operatorname{Re} \sum_{l=1}^{M/2-1} e^{2\pi ikl/M} p(\xi_l)(-1)^l + \frac{1}{Md}(1 + \operatorname{Re} \phi_X(\xi_{M/2})) \right). \end{aligned}$$

Finally,

$$p_X(x_k)(x_k) \approx \frac{1}{d} iRDFT[\tilde{\phi_X}](k), \quad k = 0, ..., M-1, \tag{48}$$

where $(\tilde{\phi_X})_l = \phi_X(\xi_l) \cdot (-1)^l$. Note that real-FFT is two times faster than FFT.

The described adaptation of the WHMC method in the context of the FWHF-method we will call the "Fast Wiener-Hopf Factorization Monte Carlo method" (in short – the FWHFMC-method).

3.2. **Implementation of the WHMC-method [30, 32] into the program platform Premia.** We have implemented the WHMC of Kuznetsov et al [30] into program platform Premia for the case of barrier knocked-and-out options with rebate in Kou model on C++ programming language. The algorithm is based on Theorem 4 of Kuznetsov et al and the results of [32], see Theorem 2.3 and Theorem 2.5 of the current technical report. The implementation required a sampling from double exponential, exponential and Bernoilli distributions. This instrument is provided by the PNL (a numerical library for C and C++ programmers, available at http://pnl.gforge.inria.fr). The Premia software currently uses this library. The correspondent procedure has the name Kou_Mc_Out_WHMC and depend on 19 parameters including

- Kou model parameters (see Example 2.2):

  - r – the interest rate r;
  - *divid* – the divident rate d;
  - *sigma* – the diffusion coeffcient σ;
  - *lambda* – λ, the intensity of Poisson process;
  - *lambdap* – Λ_+, the intensity of positive jumps;
  - *lambdam* – Λ_-, the intensity of negative jumps;
  - p – the probability of positive jumps.
- Option parameters:
  - K – the strike price;
  - $S0$ – the spot price;
  - Bar – the barrier;
  - *ifCall* – a flag parameter for the option type ("1" for "Call" and "0" for "Put");
  - *b_type* – a flag parameter for the barrier type ("1" for "Up-and-Out" and "0" for "Down-and-Out");
  - *rebate* – the rebate.
- The Wiener Hopf Monte Carlo method parameters:
  - *n_points* – the number of randomized time steps;
  - *n_paths* – the number of trajectories;
  - *generator* – a random number generator.
- The output parameters:
  - **ptPrice* – the option price;
  - **priceError* – the Monte Carlo error.

3.3. **Implementation of the FWHFMC method into program platform Premia.** We have implemented the 'Fast Wiener-Hopf Factorization Monte Carlo method" into program platform Premia for the case of barrier knocked-and-out options with rebate in the TSL model on C++ programming language. The algorithm is presented in the Subsection 3.1. The implementation required a possibility of sampling from a uniform distribution. This instrument is provided by the PNL (a numerical library for C and C++ programmers, available at http://pnl.gforge.inria.fr). The correspondent procedure has the name TSL_Mc_Out_WHMC and depend on 21 parameters including

- TSL model parameters (see Example 2.1):
  - r – the interest rate r;
  - *divid* – the divident rate d;
  - *cp* – the TSL parameter c_+;
  - *cm* – the TSL parameter c_-;
  - *lambdap* – the TSL parameter λ_+;
  - *lambdam* – the TSL parameter λ_-;
  - p – the probability of positive jumps.
- Option parameters:
  - K – the strike price;
  - $S0$ – the spot price;
  - Bar – the barrier;
  - *ifCall* – a flag parameter for the option type ("1" for "Call" and "0" for "Put");
  - *b_type* – a flag parameter for the barrier type ("1" for "Up-and-Out" and "0" for "Down-and-Out");
  - *rebate* – the rebate.
- The Wiener Hopf Monte Carlo method parameters:
  - h – the space step (the FWHF-method parameter);
  - *er* – the scale parameter L (the FWHF-method parameter);
  - *n_points* – the number of randomized time steps;
  - *n_paths* – the number of trajectories;
  - *generator* – a random number generator.
- The output parameters:
  - **ptPrice* – the option price;
  - **priceError* – the Monte Carlo error.

Note that in the program implemented into Premia one can manage by two parameters of the numerical Wiener-Hopf factorization algorithm: the space step h and the scale of logprice range L. Parameter L controls the size of the truncated region in x-space; it corresponds to the region $(-L\ln(2)/h; L\ln(2)/h)$. The typical values of the parameter are $L = 1$, $L = 2$ and $L = 4$. To improve the results one should decrease h, when L is fixed.

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