

A Finite Volume Method for Pricing American options on two stocks.

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Introduction

The valuation of American options on two stocks, also called two-colours Rainbow options by practitioners, is an important problem in financial economics since a wide variety of contracts that are traded in the O.T.C. market involve such options (Exchange options, Best-of options). Unlike European options, American options cannot be valued by closed-form formulae, even in the Black-Scholes model, and require the use of numerical methods.

1 American Options on Two Stocks

The price at time 0 of an American option on two stocks in the Black-Scholes setting is given by

$$P_A(0, s_1, s_2) = \sup_{\tau \in \mathcal{T}_{0,T}} E \left[e^{-r\tau} \psi(S_\tau^1, S_\tau^2) \right].$$

This price can be formulated, after a logarithm change of variable, in terms of the solution u to the following variational inequality (see e.g. [10]),

$$\begin{cases} \max \left(\psi - u, \frac{\partial u}{\partial t} + \frac{\sigma_1^2}{2} \frac{\partial^2 u}{\partial x_1^2} + \frac{\sigma_2^2}{2} \frac{\partial^2 u}{\partial x_2^2} + \alpha_1 \frac{\partial u}{\partial x_1} + \alpha_2 \frac{\partial u}{\partial x_2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 u}{\partial x_1 \partial x_2} - ru \right) = 0, & (t, x_1, x_2) \text{ in } [0, T] \times \mathbb{R}^2 \\ u(T, x_1, x_2) = \psi(e^{x_1}, e^{x_2}) \end{cases} \quad (1)$$

by $P_A(t, s_1, s_2) = u(t, \ln s_1, \ln s_2)$.

With the time change of variable $t' = T - t$ and the following geometrical transformation :

$$(x, y) \longmapsto (X, Y) = \left(x * \cos(\theta) + y * \sin(\theta), \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}} (y * \cos(\theta) - x * \sin(\theta)) \right)$$

with, if $\sigma_1^2 - \sigma_2^2 \neq 0$

$$\begin{cases} \tan(2\theta) = \frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}, \\ \alpha = \frac{(\sigma_1^2 + \sigma_2^2) \cos(2\theta) + \sigma_1^2 - \sigma_2^2}{4 \cos(2\theta)}, \\ \beta = \frac{(\sigma_1^2 + \sigma_2^2) \cos(2\theta) + \sigma_2^2 - \sigma_1^2}{4 \cos(2\theta)}. \end{cases}$$

if, $\sigma_1^2 - \sigma_2^2 = 0$,

$$\begin{cases} \theta = \frac{\pi\rho}{4|\rho|}, \\ \alpha = \frac{\sigma_1^2}{2}(1 + |\rho|), \\ \beta = \frac{\sigma_2^2}{2}(1 - |\rho|). \end{cases}$$

one obtains,

$$\begin{cases} \min \left(\psi - u, \frac{\partial u}{\partial t} - \alpha \Delta u - \text{grad}(\vec{v}u) + ru \right) = 0, & (t, x_1, x_2) \text{ in } [0, T[\times \mathbb{R}^2 \\ u(0, x_1, x_2) = \psi(e^{x_1}, e^{x_2}) \end{cases} \quad (2)$$

with,

$$\vec{v} = ((r - \lambda_1 - \frac{\sigma_1^2}{2}) \cos(\theta) + (r - \lambda_2 - \frac{\sigma_2^2}{2}) \sin(\theta), \left((r - \lambda_2 - \frac{\sigma_2^2}{2}) \cos(\theta) - (r - \lambda_1 - \frac{\sigma_1^2}{2}) \sin(\theta) \right) \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}}).$$

2 The finite volume schemes

Let us consider the problem :

$$\frac{\partial u}{\partial t}(x, t) + \text{div}(u(x, t)\vec{v}) - \alpha \Delta u(x, t) + ru(x, t) \geq 0, \quad (x, t) \in \Omega \times]0, T[\quad (3)$$

$$u(x, t) \geq \psi(x), \quad (x, t) \in \Omega \times]0, T[\quad (4)$$

$$\left(\frac{\partial u}{\partial t}(x, t) + \text{div}(u(x, t)\vec{v}) - \alpha \Delta u(x, t) + ru(x, t) \right) (\psi(x) - u(x, t)) = 0, \quad (x, t) \in \Omega \times]0, T[\quad (5)$$

$$u(x, 0) = \psi(x), \quad x \in \Omega \quad (6)$$

under the following assumptions

Assumption 1. 1. $d \in \mathbb{N}^*$,

2. $\Omega \subset \mathbb{R}^d$ is a bounded open polygonal,

3. $\psi \in H_0^1(\Omega) \cap C^2(\bar{\Omega})$,

4. $\psi \geq 0$ a.e on Ω ,

5. $T > 0$.

A weak for of the problem (3)-(6) yields the following variational inequality :

$$\begin{cases} u \in L^2(0, T; H_0^1(\Omega)), \frac{\partial u}{\partial t} \in L^2(\Omega \times]0, T[), u(x, 0) = \psi(\Omega), \text{ p.p. } x \in \Omega, \text{ satisfying :} \\ \int_{\Omega} \left(\frac{\partial u}{\partial t}(x, t) + ru(x, t) + \text{div}(u(x, t)\vec{v}) \right) (v(x, t) - u(x, t)) + \alpha \nabla u(x, t) \nabla (v(x, t) - u(x, t)) dx \geq 0 \\ \text{p.p } t \in]0, T[, \forall v \in H^1(\Omega), v \geq \psi. \end{cases} \quad (7)$$

By [12], there exists a unique solution of (7).

In order to obtain a numerical approximation of the solution of (7), let us now describe the space and time discretization of $\Omega \times]0, T[$.

Definition 1 (Admissible meshes). *An admissible mesh of Ω is given by a set τ of open bounded polygonal convex subsets of Ω called control volumes, a family ε of subsets of $\bar{\Omega}$ contained in hyperplanes of \mathbb{R}^d with strictly positive measure, and a family of point $(x_K)_{K \in \tau}$ (the "centers" of control volumes) satisfying the following properties:*

- (i) *The closure of the union of all control volumes is $\bar{\Omega}$.*
- (ii) *For any $K \in \tau$, there exists a subset ε_K of ε such that $\partial K = \cup_{\sigma \in \varepsilon_K} \bar{\sigma}$. Furthermore, $\varepsilon = \cup_{K \in \tau} \varepsilon_K$.*
- (iii) *For any $(K, L) \in \tau^2$ with $K \neq L$, either the "length" (i.e. the $(d-1)$ Lebesgue measure) of $\bar{K} \cap \bar{L}$ is 0 or $\bar{K} \cap \bar{L} = \bar{\sigma}$ for some $\sigma \in \varepsilon$. In the latter case, we shall write $\sigma = K|L$ and $\varepsilon_{int} = \sigma \in \varepsilon, \exists(K|L) \in \tau^2, \sigma = K|L$. For any $K \in \tau$, we shall denote by \mathcal{N}_K the set of boundary control volumes of K , i.e. $\mathcal{N}_K = \{L \in \tau, K|L \in \varepsilon_K\}$.*
- (iv) *The family of points $(x_K)_{K \in \tau}$ is such that $x_K \in K$ (for all $K \in \tau$) and, if $\sigma = K|L$, it is assumed that the straight line (x_K, x_L) is orthogonal to σ .*

For a control volume $K \in \tau$, we will denote by $m(K)$ its measure and $\varepsilon_{ext,K}$ the subset of the edges of K included in the boundary $\partial\Omega$. If $L \in \mathcal{N}_K$, $m(K|L)$ will denote the measure of the edge between K and L , $\tau_{K|L}$ the "transmissibility" through $K|L$, defined by $\tau_{K|L} = \frac{m(K|L)}{d(x_K, x_L)}$. Similarly, if $\sigma \in \varepsilon_{ext,K}$, we will denote by $m(\sigma)$ its measure and τ_σ the "transmissibility" through σ , defined by $\tau_\sigma = \frac{m(\sigma)}{d(x_K, \sigma)}$. One denotes $\varepsilon_{ext} = \cup_{K \in \tau} \varepsilon_{ext,K}$ and for $\sigma \in \varepsilon_{ext}$, one denotes by K_σ the control volume K such that $\sigma \in \varepsilon_{ext,K}$. The size of the mesh τ is defined by

$$size(\tau) = \max_{K \in \tau} diam(K), \quad (8)$$

and a geometrical factor, linked with the regularity of the mesh, is defined by

$$reg(\tau) = \max_{K \in \tau} (card \varepsilon_K, \max_{\sigma \in \varepsilon_K} \frac{diam(K)}{d(x_K, \sigma)}). \quad (9)$$

Definition 2 (Time discretization of $(0, T)$). *A time discretization of $(0, T)$ is given by an integer value N and by an increasing sequence of real values $(t^n)_{n \in [0, N+1]}$ with $t^0 = 0$ and $t^{N+1} = T$. The time step is uniform and defined by $\delta t = t^{n+1} - t^n$, for $n \in [0, N]$.*

Definition 3 (Space-time discretization of $\Omega \times (0, T)$). *A finite volume discretization \mathcal{D} of $\Omega \times (0, T)$ is a family $\mathcal{D} = (\tau, \varepsilon, (x_K)_{K \in \tau}, N, (t^n)_{n \in [0, N]})$, where $\tau, \varepsilon, (x_K)_{K \in \tau}$ is an admissible mesh of Ω in the sense of Definition 1 and $N, (t^n)_{n \in [0, N+1]}$ is a time discretization of $(0, T)$ in the sense of Definition 2. For a given mesh \mathcal{D} , one defines :*

$$\begin{aligned} size(\mathcal{D}) &= \max(size(\tau), \delta t), \\ reg(\mathcal{D}) &= reg(\tau). \end{aligned}$$

Let us now introduce the space of piecewise constant functions associated with an admissible mesh and some "discrete H_0^1 " norm for this space. This discrete norm will be used to obtain some estimates on the approximate solution given by a finite volume scheme.

Definition 4. *Let Ω be an open bounded polygonal subset of \mathbb{R}^d , and τ an admissible mesh. Define $X(\tau)$ as the set of functions from Ω to \mathbb{R} which are constant over each control volume of the mesh.*

Definition 5 (Discrete norms). Let Ω be an open bounded polygonal subset of \mathbb{R}^d , and τ an admissible finite volume mesh in the sense of Definition 1. For $u, v \in X(\tau)$ we define a scalar product by

$$[u, v]_{1, \tau} = \sum_{\sigma \in \varepsilon} T_{\sigma} D_{\sigma} u D_{\sigma} v = \sum_{\substack{\sigma \in \varepsilon_{int} \\ \sigma = K|L}} T_{KL} (u_L - u_K) (v_L - v_K) + \sum_{\sigma \in \varepsilon_{ext}} T_{K\sigma} u_{K\sigma} v_{K\sigma} \quad (10)$$

where, for any $\sigma \in \varepsilon$, $T_{\sigma} = \frac{m(\sigma)}{d_{\sigma}}$ and

$$D_{\sigma} u = |u_K u_L| \text{ if } \sigma \in \varepsilon_{int}, \sigma = K|L,$$

$$D_{\sigma} u = |u_K| \text{ if } \sigma \in \varepsilon_{K, ext},$$

where u_K denotes the value taken by u on the control volume K and the sets $\varepsilon, \varepsilon_{int}, \varepsilon_{ext}, \varepsilon_{K, ext}$ are defined in definition 1. We note $\|\cdot\|_{1, \tau}$ the discrete H_0^1 norm associated.

The schemes :

Let \mathcal{D} be a finite volume discretization of $\Omega \times (0, T)$. Let us now define an implicit upwind finite volume scheme, the discrete unknowns are $u = (u_K^{n+1})_{K \in \tau, n \in [0, T]}$ and $\tilde{u} = (\tilde{u}_K^{n+1})_{K \in \tau, n \in [0, T]}$ and verify :

$$u_K^0 = \psi(x_K) = \psi_K, \quad (11)$$

$$u_K^{n+1} = \max \left(\tilde{u}_K^{n+1}, u_K^0 \right), \quad (12)$$

$$m_K \left(\tilde{u}_K^{n+1} - u_K^n \right) + \Delta t \sum_{\sigma \in \varepsilon_K} v_{K, \sigma} u_{\sigma, +}^{n+1} + \alpha \Delta t [u^{n+1}, 1_K]_{1, \tau} + r \Delta t m_K u_K^{n+1} = 0, \quad (13)$$

with,

$$\vec{v} \in \mathbb{R}^d,$$

$$v_{K, \sigma} = - \int_{\sigma} \vec{v} \cdot \vec{n}_{K\sigma} d\gamma(x) = - \vec{v} \cdot \vec{n}_{K\sigma} m(\sigma).$$

$$u_{\sigma, +}^n = \begin{cases} u_K^n & \text{si } \sigma \in \varepsilon_{int}, \sigma = K|L, v_{K, \sigma} \geq 0 \\ u_L^n & \text{si } \sigma \in \varepsilon_{int}, \sigma = K|L, v_{K, \sigma} < 0 \\ u_K^n & \text{si } \sigma \in \varepsilon_{K, ext}, v_{K, \sigma} \geq 0 \\ 0 & \text{si } \sigma \in \varepsilon_{K, ext}, v_{K, \sigma} < 0 \end{cases}$$

Remark 1. we can also consider a implicit central finite volume scheme :

$$u_K^0 = \psi(x_K) = \psi_K, \quad (14)$$

$$u_K^{n+1} = \max \left(\tilde{u}_K^{n+1}, u_K^0 \right), \quad (15)$$

$$m_K \left(\tilde{u}_K^{n+1} - u_K^n \right) + \Delta t \left[\sum_{\substack{\sigma \in \varepsilon_{K, int} \\ \sigma = K|L}} v_{KL} \frac{u_K^{n+1} + u_L^{n+1}}{2} + \sum_{\sigma \in \varepsilon_{K, ext}} v_{K\sigma} \frac{u_{K\sigma}^{n+1}}{2} \right] + \alpha \Delta t [u^{n+1}, 1_K]_{1, \tau} + r \Delta t m_K u_K^{n+1} = 0. \quad (16)$$

Definition 6 (Approximate solution). Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of definition 3. The approximate solution (C^∞ in time on $\Omega \times (0, T)$) of (3) – (6) associated to the discretization \mathcal{D} is defined almost everywhere in $\Omega \times (0, T)$ by :

$$u_{\mathcal{D}}(x, t) = \frac{t - n\Delta t}{\Delta t} u_K^{n+1} + \frac{(n+1)\Delta t - t}{\Delta t} u_K^n, \forall (x, t) \in K \times [n\Delta t, (n+1)\Delta t], \forall n = 0 \dots N, \forall K \in \tau.$$

Thanks to this Definition, one gets almost everywhere in $\Omega \times (0, T)$:

$$\frac{\partial u_{\mathcal{D}}(x, t)}{\partial t} = \frac{u_K^{n+1} - u_K^n}{\Delta t}, \forall t \in [n\Delta t, (n+1)\Delta t], \forall x \in K, \forall n = 0 \dots N, \forall K \in \tau.$$

3 Existence of the solution and stability results for the implicit schemes

Lemma 1. *Under Assumptions 1, let \mathcal{D} be a discretization of $\Omega \times (0, T)$ in the sense of Definition 3. If $(u_K^n)_{\substack{K \in \tau \\ n \in \mathbb{N}}}$ is a solution of the implicit upwind finite volume scheme (11) - (13), then there exists a sequence $(\theta_K^n)_{\substack{n=0 \dots N \\ K \in \tau}} \in [0, 1]$ such that :*

$$u_K^{n+1} - u_K^n = \theta_K^n (\tilde{u}_K^{n+1} - u_K^n), \forall K \in \tau, \forall n = 0 \dots N.$$

Lemma 2 (Existence and uniqueness). *Under Assumptions 1, let \mathcal{D} be a discretization of $\Omega \times (0, T)$ in the sense of Definition 3. Then there exists a unique solution $(u_K^n)_{\substack{K \in \tau \\ n \in \mathbb{N}}}$ to the system of equations (11) - (13).*

Proposition 1 (L^∞ and L^2 estimate). *Under Assumptions 1, let \mathcal{D} be a discretization of $\Omega \times (0, T)$ in the sense of Definition 3 and let $(u_K^n)_{\substack{K \in \tau \\ n \in [0, N+1]}}$ be the unique solution of the scheme (11) - (13). Then,*

$$|u_K^n| \leq \|\psi\|_{L^\infty(\Omega)}, \forall K \in \tau, \forall n \in [0, N+1],$$

and

$$\begin{aligned} \frac{1}{2} \sum_{K \in \tau} m_K (u_K^{l+1})^2 + \sum_{K \in \tau} m_K \sum_{n=0}^l (u_K^{n+1} - u_K^n)^2 + \alpha \sum_{n=0}^l \Delta t [u^{n+1}, u^{n+1}]_{1, \tau} \leq \\ \|\psi\|_{L^\infty(\Omega)}^2 m(\Omega) + \alpha T [u^0, u^0]_{1, \tau}, \forall l \leq N \end{aligned}$$

4 Estimate

Corollary 1. *Under Assumptions 1, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be sequence of discretization of $\Omega \times (0, T)$ in the sense of Definition 3 such that $\Delta t_m \xrightarrow{m \rightarrow +\infty} 0$, $\text{size}(\tau_m) \xrightarrow{m \rightarrow +\infty} 0$, $\zeta \in \mathbb{R}$ such that $\zeta \geq \text{reg}(D_m) \forall m \in \mathbb{N}$, and let $(u_K^n)_{\substack{K \in \tau \\ n \in [0, N+1]}}$ be the unique solution of the scheme (11) - (13). Then, there exists $U \in L^2(\Omega \times]0, T[)$, and a subsequence noted $(u_{D_m})_{m \in \mathbb{N}}$ such that $u_{D_m} \xrightarrow{m \rightarrow +\infty} U$ for the weak topology of $L^2(\Omega \times]0, T[)$.*

Proposition 2. *Under Assumptions 1, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be sequence of discretization of $\Omega \times (0, T)$ in the sense of Definition 3, $\zeta \in \mathbb{R}$ such that $\zeta \geq \text{reg}(D)$, and let $(u_K^n)_{\substack{K \in \tau \\ n \in [0, N+1]}}$ be the unique solution of the scheme (11) - (13). Then, there exists $C > 0$ only depending on ψ , α , Ω , T , \vec{v} , r , such that :*

$$\alpha [u^{N+1}, u^{N+1}]_{1, \tau} + \sum_{n=0}^N \Delta t \sum_{K \in \tau} m_K \left(\frac{u_K^{n+1} - u_K^n}{\Delta t} \right)^2 \leq C(u^0, \psi, \alpha, \Omega, T, \vec{v}, r).$$

Corollary 2. Under Assumptions 1, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be sequence of discretization of $\Omega \times (0, T)$ in the sense of Definition 3 such that $\text{size}(\tau_m) \xrightarrow{m \rightarrow +\infty} 0$, $\zeta \in \mathbb{R}$ such that $\zeta \geq \text{reg}(D)$, and let

$(u_K^n)_{\substack{K \in \tau \\ n \in [0, N+1]}}$ be the unique solution of the scheme (11) - (13). Then, the set $\{\frac{\partial u_D}{\partial t}\}_{N, \tau}$ is borned in $L^2(\Omega \times]0, T[)$ and so, there exists $Z \in L^2(\Omega \times]0, T[)$ such that, op to a subsequence, $\{\frac{\partial u_{D_m}}{\partial t}\}_{m \in \mathbb{N}}$ tends to Z in the weak topology of $L^2(\Omega \times]0, T[)$ as $m \rightarrow +\infty$.

Corollary 3 (Space-translate and Time-Translate estimate). Under Assumptions 1, let \mathcal{D} be a discretization of $\Omega \times (0, T)$ in the sense of Definition 3, $\zeta \in \mathbb{R}$ such that $\zeta \geq \text{reg}(D)$ and let u_D the approximate solution in the sense of Definition 6 be prolonged by zero on $\mathbb{R}^{d+1} \setminus \Omega \times]0, T[$. Then there exists C_2 only depending on T, ψ, α, τ, d and C_3 only depending on $T, \psi, \alpha, d, r, \vec{v}$ such that :

$$\|u_D(\cdot + \eta, \cdot) - u_D(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq C_2 |\eta| (|\eta| + 4\text{size}(\tau)), \forall \eta \in \mathbb{R}^d.$$

and

$$\|u_D(\cdot, \cdot + \lambda) - u_D(\cdot, \cdot)\|_{L^2(\mathbb{R}^{d+1})}^2 \leq \lambda C_3, \forall \lambda \in]0, T[.$$

With this preceding estimates, one can apply the Riesz-Frechet-Kolmogorov compactness criterion.

5 Compactness

Corollary 4. Under Assumptions 1, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be sequence of discretization of $\Omega \times (0, T)$ in the sense of Definition 3 such that $\Delta t_m \xrightarrow{m \rightarrow +\infty} 0$, $\text{size}(\tau_m) \xrightarrow{m \rightarrow +\infty} 0$, $\zeta \in \mathbb{R}$ such that $\zeta \geq \text{reg}(D_m) \forall m \in \mathbb{N}$, and let $(u_K^n)_{\substack{K \in \tau \\ n \in [0, N+1]}}$ be the unique solution of the scheme (11) - (13). Then, up to a subsequence,

$$u_{D_m} \xrightarrow{m \rightarrow +\infty} U \text{ dans } L^2(\Omega \times]0, T[).$$

where U is defined by the Corollary 1 and checks $U \in L^2(0, T; H^1(\Omega))$, $U(t, \cdot) = 0$ a.e. on $\partial\Omega$, a.e. $t \in]0, T[$, and $\frac{\partial U}{\partial t} = Z$ a.e. on $]0, T[\times \Omega$.

6 Convergence

Proposition 3. Under Assumptions 1, let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be sequence of discretization of $\Omega \times (0, T)$ in the sense of Definition 3 such that $\Delta t_m \xrightarrow{m \rightarrow +\infty} 0$, $\text{size}(\tau_m) \xrightarrow{m \rightarrow +\infty} 0$, $\zeta \in \mathbb{R}$ such that $\zeta \geq \text{reg}(D_m) \forall m \in \mathbb{N}$, and let $(u_K^n)_{\substack{K \in \tau \\ n \in [0, N+1]}}$ be the unique solution of the scheme (11) - (13). Let $(u_{D_m})_{m \in \mathbb{N}}$ the sequence of approximate solution in the sense of the Definition ??, and let U the limit of a subsequence $(u_{D_m})_{m \in \mathbb{N}}$ thanks to Corollary 4. Then, for all function $w \in L^2(0, T; H_0^1(\Omega))$, such that $w(t, \cdot) \geq \psi$ a.e. on Ω , the following inequality holds :

$$\int_{\Omega} \frac{\partial U}{\partial t}(x, t)(w(x, t) - U(x, t)) + \alpha \nabla U(x, t) \nabla (w(x, t) - U(x, t)) + r U(x, t)(w(x, t) - U(x, t)) + (w(x, t) - U(x, t)) \text{div}(U(x, t) \vec{v}) dx \geq 0 \text{ p.p. } t \in]0, T[.$$

7 Numerical Results

7.1 Localization and Stability

We define three localizations.

Lemma 3 (Localization). *We consider,*

$$P_{Aloc}(0, s_1, s_2) = \sup_{\tau \in \mathcal{T}_{0,T}} E \left[e^{-r\tau \wedge T_{loc}^{0,s}} \psi(S_{\tau \wedge T_{loc}^{0,s}}^1, S_{\tau \wedge T_{loc}^{0,s}}^2) \right],$$

with $T_{loc}^{0,s} = \inf\{t > 0, |S_t^1 - s^1| < loc, |S_t^2 - s^2| < loc\}$.

Then, if $\psi(s^1, s^2) = (K - \min(s^1, s^2))^+$,

$$|P_A(0, s_1, s_2) - P_{Aloc}(0, s_1, s_2)| \leq 4K \left[2 - N\left(\frac{loc - a1}{\sqrt{T}\sigma_1}\right) - N\left(\frac{loc - a2}{\sqrt{T}\sigma_2}\right) \right]$$

if $\psi(s^1, s^2) = (\max(s^1, s^2) - K)^+$,

$$|P_A(0, s_1, s_2) - P_{Aloc}(0, s_1, s_2)| \leq$$

$$8 \left[\exp\left(s^1 + T(|r - \lambda_1| + \frac{\sigma_1^2}{2})\right) + \exp\left(s^2 + T(|r - \lambda_2| + \frac{\sigma_2^2}{2})\right) \right] \sqrt{(2 - N(\frac{loc - a1}{\sqrt{T}\sigma_1}) - N(\frac{loc - a2}{\sqrt{T}\sigma_2})}$$

if $\psi(s^1, s^2) = (s^1 - \mu s^2)^+$,

$$|P_A(0, s_1, s_2) - P_{Aloc}(0, s_1, s_2)| \leq 8 \exp\left(s^1 + T(|r - \lambda_1| + \frac{\sigma_1^2}{2})\right) \sqrt{(2 - N(\frac{loc - a1}{\sqrt{T}\sigma_1}) - N(\frac{loc - a2}{\sqrt{T}\sigma_2})}$$

where, $ai = T|r_i - \lambda_i - \frac{\sigma_i^2}{2}|$ and N is the repartition function of a standard normal distribution.

One uses the following polynomial approximation for the repartition function of a standard normal distribution :

$$N(x) \approx 1 - \frac{1}{\sqrt{2\Pi}} \exp\left(-\frac{x^2}{2}\right) (b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5), \text{ si } x > 0,$$

with

$$\begin{aligned} p &= 0.2316419 \\ b_1 &= 0.319381530 \\ b_2 &= -0.356563782 \\ b_3 &= 1.781477937 \\ b_4 &= -1.821255978 \\ b_5 &= 1.330274429 \\ t &= \frac{1}{1+px}. \end{aligned}$$

Explicit central finit volume scheme :

$$u_K^0 = \psi_K, \tag{17}$$

$$u_K^{n+1} = \max\left(\tilde{u}_K^{n+1}, u_K^0\right), \tag{18}$$

$$\begin{aligned} m_K \left(\tilde{u}_K^{n+1} - u_K^n \right) + \Delta t \sum_{L \in N_K} \left(\frac{u_K^n + u_L^n}{2} \right) \underbrace{\int_{K|L} -\vec{v} \cdot \vec{n}_{KL} d\gamma(x)}_{v_{KL}} + \Delta t \sum_{\sigma \in \varepsilon_{K,ext}} \underbrace{\frac{u_K^n}{2} \int_{\sigma} -\vec{v} \cdot \vec{n}_{K\sigma} d\gamma(x)}_{k_{K\sigma}} + \\ \alpha \Delta t [u^n, 1_K]_{1,\tau} + r \Delta t m_K u_K^n = 0. \end{aligned} \tag{19}$$

Explicit upwind finite volume scheme

$$u_K^0 = \psi_K, \quad (20)$$

$$u_K^{n+1} = \max \left(\tilde{u}_K^{n+1}, u_K^0 \right), \quad (21)$$

$$m_K \left(\tilde{u}_K^{n+1} - u_K^n \right) + \Delta t \sum_{\sigma \in \varepsilon_K} v_{K\sigma} u_{\sigma,+}^n + \alpha \Delta t [u^n, 1_K]_{1,\tau} + r \Delta t m_K u_K^n = 0. \quad (22)$$

with,

$$\vec{v} = \left((r - \lambda_1 - \frac{\sigma_1^2}{2}) \cos(\theta) + (r - \lambda_2 - \frac{\sigma_2^2}{2}) \sin(\theta), \left((r - \lambda_2 - \frac{\sigma_2^2}{2}) \cos(\theta) - (r - \lambda_1 - \frac{\sigma_1^2}{2}) \sin(\theta) \right) \left(\frac{\alpha}{\beta} \right)^{\frac{1}{2}} \right).$$

Lemma 4 (L^∞ stability). *Let \mathcal{D} be an admissible discretization of $\Omega \times (0, T)$ in the sense of definition 3.*

Let $(u_K^n)_{K,n}$ be the unique solution of the explicit central scheme (17) - (19).

If, $\Delta t \leq \frac{m_K}{m_K \frac{r}{2} + \frac{\alpha}{2} \sum_{\sigma \in \varepsilon_K, ext} T_{K\sigma} + \sum_{L \in N_K} \left(\alpha T_{KL} + \frac{|v_{KL}|}{4} \right)}, \forall K \in \tau$ and if $T_{K\sigma} \geq \frac{1}{2\alpha} \left| \int_{\sigma} -\vec{v} \cdot n_{K\sigma} d\gamma(x) \right|, \forall K \in \tau, \forall \sigma \in \varepsilon_K$, then :

$$\|u_\tau^{n+1}\|_{L^\infty} \leq \|u_\tau^0\|_{L^\infty}, \forall n = 0 \dots N.$$

Let $(u_K^n)_{K,n}$ be the unique solution of the explicit upwind scheme (20) - (22).

If $\Delta t \leq \frac{m_K}{rm_K + (\sum_{\sigma \in \varepsilon_K} (v_{K\sigma})^+ + \alpha T_{K\sigma})}, \forall K \in \tau$, then :

$$\|u_\tau^{n+1}\|_{L^\infty} \leq \|u_\tau^0\|_{L^\infty}, \forall n = 0 \dots N.$$

7.2 Pratical implementation and results

We choose to evaluate the American Put option on the minimum of two underlying assets with payoff $\psi(S^1, S^2) = \left(K - \min(S^1, S^2) \right)^+$. We assume that the initial values of the stock prices are $s^1 = 100, s^2 = 100$, the volatility $\sigma_1 = 0.2, \sigma_2 = 0.2$, the interest rate $r = \log(1.05)$, the continuous dividend rates $\delta_1 = 0, \delta_2 = 0$, the exercise price $K = 100$, the maturity $T = 1$ and the correlation $\rho = 0$. We take as the "true" reference price, the one issued of the multinomial BEG tree-method with 3000 step and compare it with the following algorithm :

1. the explicit DPEXP algorithm
2. the DPADI algorithm
3. the BEG algorithm
4. the explicit finite volume algorithm
5. the explicit finite volume algorithm with a smaller time step

For the last algorithm, we multiply by 0.6 the time step obtained by the stability condition. All computation was performed on a PC Pentium IV 2.4 GH computer with 512 Mb of RAM. The "centers" of control volumes are defined as follow :

$$x_K = x[i * N + j] = (v1 - loc + i * h, v2 - loc + j * g), \forall (i, j) \in [0, N]^2$$

$N \times M$	DP-EXP	DP-ADI	BEG	FV-EXP	FV-EXPdt*0.6	TRUE
100×100	10.3065;1s	10.2947;1s	10.2974;1s	10.3196;1s	10.3081;1s	10.3080
200×200	10.3054;6s	10.3031;2s	10.3030;1s	10.3108;3s	10.3095;4s	10.3080
300×300	10.3073;32s	10.3050;7s	10.3048;1s	10.3098;11s	10.3084;19s	10.3080
400×400	10.3082;100s	10.3058;18s	10.3057;2s	10.3085;35s	10.3082;59s	10.3080

Table 1: American Put option on the minimum of two underlying assets

where the space steps are defined by $h = \frac{2*loc}{N}$ and $g = \frac{2*loc*\sqrt{\frac{\alpha}{\beta}}}{N}$. The control volume K is the rectangle centered in x_K with the measure $m(K) = hg$.

It appears that the numerical FV-EXP method are finally faster than DP-EXP and slower than DP-ADI or BEG.

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