

[Source](#) | [Model](#) | [Option](#)
[Model_Option](#) | [Help on mc methods](#) | [Archived Tests](#)

mc_variancereduction

1 Computation of the price by Monte Carlo methods

We want to compute the price at time $t \in [0, T]$ of a “down and out” call option:

$$\Pi_{i=1}^j 1(S_{t_i} > L) \exp(-r(T-t)) E^* \left[\Pi_{i=j+1}^m 1(S_{t_i} > L) (S_T - K)_+ | F_t \right]$$

where

$$E^* [\cdot]$$

denote the expectation under the risk neutral probability and

$$t_j \leq t \leq t_{j+1}$$

In order to simplify the notations, let us place at time 0. Our aim is then to approximate the quantity

$$P = E^* \left[\Pi_{i=1}^m 1(S_{t_i} > L) (S_T - K)_+ | F_t \right]$$

by using Monte Carlo methods, while attaching ourselves to reduce the variance of the estimator.

2 Variance Reduction

First, for an integrable random variable Y and an event A we can write

$$\begin{aligned}
 E[Y 1_A] &= P(A) E \left[Y \frac{1_A}{P(A)} \right] \\
 &= P(A) E^Q[Y]
 \end{aligned}$$

where

$$\frac{dQ}{dP} = \frac{1_A}{P(A)}$$

So, to approximate

$$E[Y1_A]$$

we can either simulate the natural estimator

$$Y1_A$$

under the real probability or simulate

$$P(A)Y$$

under the probability Q, that is equivalent to simulate Y (P(A) is a constant) conditionally to the event A. Let Y₁ be a random variable such that

$$\mathcal{L}(Y_1) \equiv \mathcal{L}(Y|A)$$

It's quite straightforward that:

$$\sigma^2(P(A)Y_1) \leq \sigma^2(Y1_A)$$

indeed we have

$$\sigma^2(P(A)Y_1) = (P(A))^2 (E[Y^2|A] - (E[Y|A])^2)$$

and

$$\sigma^2(Y1_A) = P(A) E[Y^2|A] - (P(A))^2 (E[Y|A])^2$$

but

$$P(A) > (P(A))^2$$

Let us now consider the price P of a “down and out” call option with barrier L and 2 monitoring instant t₁, t₂ :

$$\begin{aligned} P &= E^* \left[1(S_{t_1} > L) 1(S_{t_2} > L) (S_T - K)_+ \right] \\ &= E^* \left[1(S_{t_1} > L) 1(S_{t_2} > L) (S_T - K)_+ | (S_{t_1} > L) \right] P(S_{t_1} > L) \\ &= E^* \left[1(S_{t_1} > L) 1\left(\frac{S_{t_2}}{S_{t_1}} S_{t_1} > L\right) \left(\frac{S_T}{S_{t_2}} \frac{S_{t_2}}{S_{t_1}} S_{t_1} - K\right)_+ | (S_{t_1} > L) \right] P(S_{t_1} > L) \end{aligned}$$

with

$$X_1 = S_{t_1}, \quad X_2 = \frac{S_{t_2}}{S_{t_1}}, \quad X_3 = \frac{S_T}{S_{t_2}}$$

$$P = E^* \left[1 (X_2 X_1 > L) (X_3 X_2 X_1 - K)_+ | (X_1 > L) \right] P(X_1 > L)$$

Denote Y_1 a random variable independant of X_2 and X_3 such that

$$\mathcal{L}(Y_1) \equiv \mathcal{L}(X_1 | X_1 > L)$$

then we can write

$$\begin{aligned} P &= E^* \left[1 (X_2 Y_1 > L) (X_3 X_2 Y_1 - K)_+ \right] P(X_1 > L) \\ &= E^* \left[E^* \left[(X_3 X_2 Y_1 - K)_+ | \left(X_2 > \frac{L}{Y_1} \right) \right] P \left(X_2 > \frac{L}{Y_1} \right) \right] P(X_1 > L) \end{aligned}$$

Denote Y_2 a random variable independant of X_3 such that

$$\mathcal{L}(Y_2) \equiv \mathcal{L} \left(X_2 | X_2 > \frac{L}{Y_1} \right)$$

then we can write

$$P = E^* \left[(X_3 Y_2 Y_1 - K)_+ P \left(X_2 > \frac{L}{Y_1} \right) \right] P(X_1 > L)$$

this leads to the new estimator

$$(X_3 Y_2 Y_1 - K)_+ P \left(X_2 > \frac{L}{Y_1} \right) P(X_1 > L)$$

We generalize this result to the case with n monitoring instant t_1, \dots, t_n and obtain the general form of the new estimator that we denote N

$$N = (X_{n+1} Y_n Y_{n-1} \dots Y_1)_+ P \left(X_{n+1} > \frac{L}{Y_n Y_{n-1} \dots Y_1} \right) \dots P \left(X_2 > \frac{L}{Y_1} \right) P(X_1 > L)$$

where

$$\begin{aligned} X_1 &= S_{t_1} \\ X_i &= \frac{S_{t_i}}{S_{t_{i-1}}} \quad 2 \leq i \leq n \\ X_{n+1} &= \frac{S_T}{S_{t_n}} \\ \mathcal{L}(Y_1) &= \mathcal{L}(X_1 | X_1 > L) \\ \mathcal{L}(Y_i) &= \mathcal{L} \left(X_i | X_i > \frac{L}{Y_{i-1} \dots Y_1} \right) \quad 2 \leq i \leq n \end{aligned}$$

3 Simulation

3.1 Simulation of N

To simulate N we proceed as follows : (1) We simulate Y_1 (2) Knowing the value of $\frac{L}{Y_1}$ we can simulate Y_2 ... (i) Knowing the value of $\frac{L}{Y_{i-1} \dots Y_1}$ we can simulate Y_i for $2 \leq i \leq n$... (n+1) X_{n+1} being independant of Y_1, \dots, Y_n we can simulate it (n+2) Knowing the values of Y_1, \dots, Y_n we can compute $P\left(X_{n+1} > \frac{L}{Y_n Y_{n-1} \dots Y_1}\right) \dots P\left(X_2 > \frac{L}{Y_1}\right) P(X_1 > L)$

3.2 Simulation of Y_n, Y_{n-1}, \dots, Y_1

In our example, the simulation of the random variable Y_n, Y_{n-1}, \dots, Y_1 amounts to the simulation of random variables $(g_i)_{i=1 \dots n}$ normally distributed with mean 0 and variance 1 conditionaly to the events $(g_i \in [a_i, +\infty])_{i=1 \dots n}$ To do this, we use the following method: Let Z be a random variable such that

$$\mathcal{L}(Z) \equiv \mathcal{L}(g_i | g_i \in [a_i, +\infty])$$

and

$$u \geq a_i$$

we can write

$$\begin{aligned} P(Z \leq u) &= P(X \leq u | X \in [a_i, +\infty]) \\ \Leftrightarrow F_Z(u) &= \frac{P(a_i \leq X \leq u)}{P(a_i \leq X)} \\ \Leftrightarrow F_Z(u) &= \frac{F_{g_i}(u) - F_{g_i}(a_i)}{1 - F_{g_i}(a_i)} \end{aligned}$$

it follows then

$$F_Z^{-1}(y) = F_{g_i}^{-1}(F_{g_i}(a_i) + y(1 - F_{g_i}(a_i)))$$

But we know that if the random variable U is uniformly distributed on $[0, 1]$ then $\mathcal{L}(F_Z^{-1}(U)) \equiv \mathcal{L}(Z)$ has the same law than Z . Since we can easily simulate U and compute $F_{g_i}^{-1}(U)$ the simulation of Z doesn't raise any technical problem.

References