Evaluation of Guaranteed Lifelong Withdrawal Benefits

INRIA
PREMIA 16
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Janvier 2014

Premia 18

We present here the implementation of a methodology for the valuation of Guaranteed Lifelong Withdrawal Benefits (GLWB) developed by Forsyth and Vetzal1 [1] for the Black-Scholes model. First we describe the model, then the numerical implementation. Finally we consider the case of the Black-Scholes model with Hull-White stocastic interest rate and the case of the Heston model.

1 GLWB model

Let the mortality function $M(t)$ be defined such that the fraction of the original owners of the GLWB who die in the interval $[t, t + \mathrm{d}t]$ is $M(t)\mathrm{d}t$. The fraction of the original owners still aliveat time t, denoted by $R(t)$ is

$$R(t) = 1 - \int_0^t M(s)\mathrm{d}s$$

Time t is measured in years from the contract inception date. Typically, mortality tables are given in terms of integer ages $\{0, 1, \dots\}$. Specifically, let x and y be integers with

$$\begin{aligned} x &= \text{insured's age at contract inception,} \\ {}_y p_x &= \text{probability that an } x \text{ year old will survive the next } y \text{ years,} \\ q_{x+y} &= \text{probability for an } x+y \text{ year old to die in the next year.} \end{aligned}$$

This gives

$$M(t) = {}_y p_x q_{x+y} \text{ where } t \in [y, y+1).$$

Note that $M(t)$ is assumed constant for $t \in [y, y + 1)$.

Let S be the amount in the investment account (i.e. mutual fund) of any holder of the GLWB-contract still alive at time t. Let A be the guarantee account balance. We suppose that percentage fees based on the value of the investment account S are charged to the policy holder at the annual rate α_{tot} and withdrawn continuously from that account. These fees include mutual fundmanagement fees α_m and a fee charged to fund the guarantee (also known as the rider) α_g, so that $\alpha_{tot} = \alpha_g + \alpha_m$. Let $V(S, A, t)$ be the value of the entire contract (express in backward time) as sum of the no-arbitage value of the guarantee only portion of the contract (the GLWB rider) and the amount in the investment accounts of those remaining alive. We have the following dynamics for $V(S, A, t)$:

$$V_t = \frac{\sigma^2 S^2}{2} V_{SS} + (r - \alpha_{tot})SV_S - rV + \alpha_m R(t)S + M(t)S \tag{1}$$

with $V(S, A, 0) = 0$ which is in fact the terminal condition.

Between two annual dates t_i and t_{i+1}, the contract follows the dynamics. At date t_{i+1} there are jumps depending on the specification of the contract. We assume that the order of event occurring at an event time t_i is ratchet then withdrawal events.

Ratchet Event. If the contract specifies a ratchet (step-up) feature, then the value of the guaranteeaccount A is increased if the investment account has increased. The guarantee account A can neverdecrease, unless the contract is partially or fully surrendered. At a ratchet event time t_i, we then have

$$V(S, A, t_i^+) = V(S, \max(S, A), t_i^-).$$

General Withdrawal Event. The contract will typically specify a withdrawal rate G_r. Givena time interval of $t_i - t_{i-1}$ between withdrawals, the contract withdrawal amount at $t = t_i$ is$G_r(t_i - t_{i-1})A$. At this point we do not make any particular assumptions about the withdrawal-strategy of the policy holder. In general terms, the policy holder's actions at t_i can be representedby a policy parameter γ_i, where $0 \leq \gamma_i \leq 2$. Withdrawals of amounts less than or equal to the contract withdrawal amount $G_r(t_i - t_{i-1})A$ are represented by $\gamma_i \in [0, 1]$. Withdrawals in excessof the contract amount are indicated by $\gamma_i \in (1, 2]$, with $\gamma_i = 2$ corresponding to full surrender.Withdrawal events can be written in the general form

$$V(S, A, t_i^+) = V(S^{\gamma_i}, A^{\gamma_i}, t_i^-) + CashFlow(S, A, t_i, \gamma_i),$$

where S^{γ_i} and A^{γ_i} are particular values depending on the withdrawal event, and where $CashFlow$ is the cash flow from the event depending on the withdrawal event represented by the value γ_i.

Bonus Event ($\gamma_i = 0$. If the contract holder chooses not to withdraw at $t = t_i$, this is indicatedby $\gamma_i = 0$. Let the bonus fraction be denoted by $B(t_i)$. If no bonus is possible at $t = t_i$, then$B(t_i) = 0$. In this case, we have

$$S^{\gamma_i} = S, \quad A^{\gamma_i} = A(1 + B(t_i)) \quad \text{and} \quad CashFlow = 0.$$

Withdrawal not Exceeding Contract Amount ($\gamma_i \in (0, 1]$**).** The withdrawal amount is $\gamma_i G_r(t_i - t_{i-1})A$ and we have

$$S^{\gamma_i} = \max(S - \gamma_i G_r(t_i - t_{i-1})A, 0), \quad A^{\gamma_i} = A \quad \text{and} \quad CashFlow = R(t_i)\gamma_i G_r(t_i - t_{i-1})A.$$

Note that withdrawals at the contract rate (or less) are allowed even if the amount in the investment account $S = 0$.

Partial or Full Surrender ($\gamma_i \in (1, 2]$**).** Next, consider the case of a withdrawal of an amount-greater than the contract amount $G_r(t_i - t_{i-1})A$, i.e. the withdrawal amount is

$$G_r(t_i - t_{i-1})A + (\gamma_i - 1)\max(S - G_r(t_i - t_{i-1})A, 0)(1 - \kappa(t_i)$$

where $\kappa(t_i) \in [0, 1]$ is a penalty for withdrawal above the contract amount. In this case we have

$$S^{\gamma_i} = (2 - \gamma_i)\max(S - \gamma_i G_r(t_i - t_{i-1})A, 0), \quad A^{\gamma_i} = A(2 - \gamma_i)$$

and

$$CashFlow = R(t_i)(G_r(t_i - t_{i-1})A + (\gamma_i - 1)\max(S - G_r(t_i - t_{i-1})A, 0)(1 - \kappa(t_i)).$$

Note that it is assumed that the guarantee account value A is reduced proportionately for any-withdrawal above the contract rate.

Withdrawal strategy :
The risk neutral price is the cost of hedging. If we consider that the insurer should charge a price which ensures that no losses can occur (assuming that the claim is hedged), then the withdrawal strategy is assumed to be

$$\gamma_i = \underset{\gamma \in [0,2]}{argmax} \left(V(S^{\gamma}, A^{\gamma}, t_i^-) + CashFlow(S, A, t_i, \gamma) \right)$$

Assuming such a strategy by policy holders and hedging against it is obviously very conservative from the standpoint of the insurer, since it seeks to provide complete protection against policy holder withdrawal behaviour(given assumptions about parameter values such as volatilities). In other words, if investors follow this strategy, and if the insurer hedges continuously, the balance in the insurer's overall hedged portfolio will be zero. On the other hand, if investors deviate from this strategy, then the insurer's portfolio will have a positive balance.

There is an other strategy which depends on a parameter denoted F referred as suboptimal withdrawal. The assumption of worst case hedging is often referred to as optimal withdrawal. This terminologyis unfortunate, in that any withdrawal strategy different from strategy (4.1) is suboptimal only inthe sense that it does not maximize the cost of hedging. This may have little to do with any givenpolicy holder's economic circumstances. Completely rational actions for a given policy holder maydepart from the previous strategy. As noted by many authors, and particularly in Cramer et al. (2007), this is controversial issue. One possible approach that is quite simple is to assume that the contract holder will follow thedefault strategy of withdrawing at the contract rate at each event time t_i unless the extra valueobtained by withdrawing optimally is greater than $FG_rA(t_i - t_{i-1})$. In this case, $F = 0$ correspondsto withdrawing optimally, while $F = 1$ corresponds to withdrawing at the contract rate.

2 Numerical method

We solve the PDE using second order (as much as possible) finite difference methods in the S directions, while still retaining the positive coefficient condition, and θ-scheme timestepping is used. The PDE is originally posed on the domain $(S, A, t) \in [0, +\infty) \times [0, +\infty) \times [0, T]$. For computational purposes, we need to truncate this domain to $[0, S_{\max}] \times [0, A_{\max}] \times [0, T]$. Moreover we will use a similarity reduction since for any scalar $\eta > 0$, we have

$$V(\eta S, \eta A, t) = \eta V(S, A, t).$$

Therefore, choosing $\eta = A^*/A$ for a fixed value A^*, we obtain

$$V(S, A, t) = \eta^{-1} V(S\eta, A^*, t)$$

which means that we need only solve for this single representative value A^* (which is chosen as S_0 the initial amount in the investment). The domain is now $[0, S_{\max}] \times [0, T]$. At initial time (which is in fact the terminal condition) we impose $V = 0$. At $S = 0$, we solve $V_t = -rV$ and at $S = S_{\max}$ we impose a second order Neumann condition $V_{SS} = 0$. If the domain is chosen sufficiently large (typically $S_{\max} = 100S_0$), this condition does not affect the solution.

Between dates t_{i-1} and t_i we solve the PDE using a classical LU algorithm for a tridiagonal matrix. Then

At date t_i, we need to compute the withdrawal strategy, so we search the maxima by a linear search on values $\gamma \in [0, 2]$. From the experience, in case of $F = 1$, we can say that only values 0, 1 and 2 are selected by this linear search. When withdrawal is chosen, only full withdrawal is selected, and in case of surrender event, only a full surrender event is selected.

The pricing problem now reduces to find the rider fee α_g such that

$$V_{\alpha_g}(S = S_0, A^* = S_0, t = T) = S_0 \tag{2}$$

when V_{α_g} is the solution of the PDE (1). Viewing V_{α_g} as being parametrized by the rider fee α_g, we solve the equation (2) using a classical secant method. Typically, only 5 or 7 iterations are necessary to obtain convergence of the algorithm under a fixed tolerance of 10^{-8}.

2.1 Numerical examples

There is multiple choices for parameters. Most of them can be treated by giving values in fixed arrays or modifying correct lines in the program:

- The surrender fee $\kappa(t)$ can be specified in the array `Kt`. By default it has the value

κ	t
0.05	$0 \leq t \leq 1$
0.04	$1 < t \leq 2$
0.03	$2 < t \leq 3$
0.02	$3 < t \leq 4$
0.01	$4 < t \leq 5$
0.00	$5 < t$

- The bonus can be specified in the array `Bt`. By default it has the value `Bt=0.05` at all dates (every year)
- The withdrawal rate G_r can be specified in the array `Gt`. By default it has the value `Gt=0.05` at all dates.
- The date of ratchet events can be specifiediat every year.
- The values for computing $M(t)$ and $R(t)$ are given by a table (DAV 2004R - 65 years old male) recorded in a file named `mortalityDAV2004R.dat`. The program reads this file then computes values saved in the arrays `Mt` and `Rt`.
- The suboptimal strategy has been implemented but has been commented in the program.
- Then there are the numerical parameters used in the PDE. By default the volatility σ has value 0.15, the interest rate r has value 0.04, expiry time T is 60, initial payment S_0 is 100 and management fee $\alpha_m = 0$.

3 Conclusion

The method is relatively easy to implement, the main difficulty is to understand the specifications of the contract. I hope that this documentation is easier to read than the original article.

An other PDE (with other coefficients given by a logarithmic change of variables on S) has been implemented, but it does not give better results.

4 Black-Scholes HW and Heston Models

In these models we use the same algorithm developped in the Black-Scholes case. For the numerical resolution of the PDE associated to the variables annuities pricing problems we use the method developped in there and there.

References

[1] P.A.FORSYTH K.R.VETZAL. An optimal stochastic control framework for determining the cost of hedging of variable annuities. *Preprint*, 2013. 1