Libor with stochastic volatility

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Contents

Abstract

We price caps on libor rates for a stochastic volatility. For this, we use the paper of Leif Andersen and Rupert Brotherton-Ratcliffe[1].

1 Model set up and pricing caps

1.1 model set up

Let $0 = T_0 < T_1 < ... < T_{n+1}$ be a maturity structure and n(t) be a function defined by $T_{n(t)-1} < t \leq T_{n(t)}$. The Libor forward rates on the maturity stucture is defined by:

$$F_k(t) = \frac{1}{\delta_k}\Big(\frac{P(t,T_k)}{P(t,T_{k+1})} - 1\Big)$$

where P(t,T) is the time t of a zero-coupon bond maturing at time T and $\delta_k = T_{k+1} - T_k$.

Under the spot measure Q, the dynamics of F_k is given by:

$$dF_k(t) = \varphi(F_k(t))\lambda_k(t)^T\Big[\mu_k(t)dt + dW(t)\Big] \quad with \quad \mu_k(t) = \sum_{j=n(t)}^{k} \frac{\delta_j\varphi(F_j(t))\lambda_j(t)}{1+\delta_j F_j(t)}.$$

Here, we suppose that $\varphi(x) = x^\alpha$.

Then, we want to generate non-monotonic volatility smiles. For this, we are going to introduce stochastic volatility.
Let V be a process defined by:

$$dV(t) = K(\theta - V(t))dt + \varepsilon\psi(V(t))dZ(t)$$

where K,θ and ε are positiv constants, Z(t) is a brownian motion under Q and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a well-behaved function such that $\psi(0) = 0$.
So we put:

$$dF_k(t) = \varphi(F_k(t))\sqrt{V(t)}\lambda_k(t)^T\Big[\sqrt{V(t)}\mu_k(t)dt + dW(t)\Big]$$

with

$$diag(Z(t)).dW(t) = 0.$$

In using the measure Q^{k+1} defined by

$$\frac{dQ^{k+1}}{dQ}|\mathcal{F}_t = \frac{P(t,T_k)/P(0,T_k)}{B(t)}$$

we obtain:

$$dF_k(t) = \varphi(F_k(t))\sqrt{V(t)}\lambda_k(t)^T dW_{k+1}(t)$$

where W_{k+1} is a brownian motion under Q^{k+1}.

1.2 pricing caps

The time-t of a caplet which pays at time T_k $\delta_K\Big(F_k(T_k) - X\Big)^+$ is:

$$P(t,T^{k+1})\delta_k E_t[(F_k - X)^+] = P(t,T_{k+1})\delta_k G(t, F_k(t), V(t))$$

where $G(t, F_k, V)$ verifies the following PDE

$$\frac{dG}{dt} + K(\theta - V)\frac{dG}{dV} + \frac{1}{2}\varepsilon^2\psi(V)^2\frac{d^2G}{dV^2} + \frac{1}{2}\varphi(F_k)^2 V||\lambda_k(t)||^2\frac{d^2G}{dF_k^2} = 0.$$

1.2.1 constant volatility

Here, we suppose that V=1 and $\lambda_k(t)$ is a constant and we put c=λ_k^2. Then, we get:

$$G(t,F_k(t),1) = g(t,F_k,c) = F\Delta(d_+) - X\Delta(d_-)$$

where

$$d_+ = \frac{\ln(\frac{F_k}{X}) + \frac{1}{2}\Omega(t,F_k,c)^2}{\Omega(t,F_k,c)}, \quad d_- = d_+ - \Omega(t,F_k,c)$$

and

$$\Omega(t,F_k,c) = \Omega_0(F_k)c^{\frac{1}{2}}(T_k-t)^{\frac{1}{2}} + \Omega_1(F_k)c^{\frac{3}{2}}(T_k-t)^{\frac{3}{2}} + O((T_k-t)^{\frac{5}{2}})$$

with

$$\Omega_0(F_k) = \frac{\ln(\frac{F_k}{X})}{\int_X^{F_k}\varphi(u)^{-1}du}$$

$$\Omega_1(F_k) = -\frac{\Omega_0(F_k)}{\Big(\int_X^{F_k}\varphi(u)^{-1}du\Big)^2}\ln\bigg(\Omega_0(F_k)\Big(\frac{F_kX}{\varphi(F_k)\varphi(X)}\Big)^{\frac{1}{2}}\bigg).$$

1.2.2 stochastic volatility

For this case, we have:

$$G(t,F_k,V) = g(t,F_k,c^*(t,V)$$

where

$$c^*(t,V) = \bar{c}(t,V) + \alpha_0\varepsilon^2 + \alpha_1\varepsilon^2Y^2 + O(\varepsilon^4)$$

or

$$c^*(t,V) = \bar{c}(t,V) + \alpha_0\varepsilon^2 + \beta_0\varepsilon^4 + (\alpha_1\varepsilon^2 + \beta_1\varepsilon^4)Y^2 + \beta_2\varepsilon^4Y^4\exp(-\Delta\varepsilon^2Y^2) + O(\varepsilon^6)$$

with

$$\bar{c}(t,V) = (T_k-t)^{-1}\int_t^{T_k}\lambda_k^2\Big(\theta + (V(t)-\theta)\exp(-K(u-T_k))\Big)du.$$

In the appendix C of Leif Andersen's paper [1], we can find the expressions of $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$.

2 Manual program

In this program, we study eight cases:

1. The stochastic volatility factor is assumed constant at V=1.
2. The volatility is stochastic. We calculate c^* to order $\varepsilon^4 (Expansion O2)$ and we put:
   - $V(0) = \theta = 1$,
   - K=1,
   - $\alpha = 1$,
   - $\psi(x) = x^{\frac{3}{4}}$.
3. The volatility is stochastic. We calculate c^* to order $\varepsilon^6 (Expansion O4)$ and we put:
   - $V(0) = \theta = 1$,
   - K=1,
   - $\alpha = 1$,
   - $\psi(x) = x^{\frac{3}{4}}$,
   - $\Lambda = 1$.
4. The volatility is stochastic. We calculate c^* to order $\varepsilon^6 (Expansion O4)$ and we put:
   - $V(0) = \theta = 1$,
   - K=0.5,
   - $\alpha = 1$,
   - $\psi(x) = x^{\frac{3}{4}}$,
   - $\Lambda = 0$.
5. The volatility is stochastic. We calculate c^* to order $\varepsilon^4 (Expansion O2)$ and we put:
   - $V(0) = \theta = 1$,
   - K=1,
   - $0 < \alpha < 1$,
   - $\psi(x) = x^{\frac{3}{4}}$.

6. The volatility is stochastic. We calculate c^* to order $\varepsilon^6 (Expansion O4)$ and we put:
   - $V(0) = \theta = 1$,
   - K=1,
   - $0 < \alpha < 1$,
   - $\psi(x) = x^{\frac{3}{4}}$,
   - $\Lambda = 1$.
7. The volatility is stochastic. We calculate c^* to order $\varepsilon^6 (Expansion O4)$ and we put:
   - $V(0) = \theta = 1$,
   - K=0,
   - $\alpha = 1$,
   - $\psi(x) = x^{\frac{3}{4}}$,
   - $\Lambda = 1$.
8. The volatility is stochastic. We calculate c^* to order $\varepsilon^4 (Expansion O2)$ and we put:
   - $V(0) = \theta = 1$,
   - K=0,
   - $\alpha = 1$,
   - $\psi(x) = x^{\frac{3}{4}}$.

When we run the program, we must choose between this eight cases:

1. For the first case, we must enter 0.
2. For the first case, we must enter 1.
3. For the second case, we must enter 2.
4. For the third case, we must enter 3.
5. For the fourth case, we must enter 4.
6. For the fifth case, we must enter 5.
7. For the sixth case, we must enter 6.

8. For the seventh case, we must enter 7.

Then, the program ask us to enter:

- the strike X,
- the constant ε (exept for the first case),
- the value of λ_k (in this program, we assume that λ_k is constant),
- for the case 1, 5 and 6, the program ask us the value of α.
- the dates T_k and T_{k+1}.

Finally, we have to calculate bond prices. For this, we have two methods:

- in the first method, we use this formula $B(0,t) = \exp(-rt)$.
- in the second method, the previous formula is wrong. Thus, we calculate bond prices by interpolation in using the values and dates of bond prices already known.

Thus, we have to do:

- if we choose the first method to calculate bond prices, the program ask us to enter the constant interest rate r.
- if we choose the second method to calculate bond prices, the program ask us to enter the number of bond prices that we know, their dates and their values.

Then, the program returns:

- $\Omega_0\sqrt{c'(0,V)} + \Omega_1 c'(0,V)^{3/2}T_k$ with $c'(0,V) = c$ for the first case and $c'(0,V) = c^*(0,V)$ for the other cases (implied Black-Sholes volatility),
- the time-0 of a caplet which pays at time T_k the amount $\delta_K\Big(F_k(T_k) - X\Big)^+$.

Moreover, we have nine examples where all the parameters exept the stike have already entered. This example makes possible to calculate $\Omega_0\sqrt{c'(0,V)}+\Omega_1 c'(0,V)^{3/2}T_k$ according to the Moneyless X/F(0). This examples are studied in the the paper of Leif Andersen and Ruper Brotherton-Ratcliffe [1].

1. For the first example, we put: $T_k = 10$, F(0)=0.06, V=1, $\alpha = 0.1$, $\lambda_k = 0.0159$.

2. For the second example, we put: $T_k = 10$, F(0)=0.06, V=1, $\alpha = 0.5$, $\lambda_k = 0.049$.

3. For the third example, we put: $T_k = 1.5$, F(0)=0.06, $\varepsilon = 1.5$, V(0)=θ=1, K=1, α=1, $\lambda_k = 0.2$, $\psi(x) = x^{\frac{3}{4}}$. We calculate c^* to order ε^4.

4. For the fourth example, we put: $T_k = 1.5$, F(0)=0.06, $\varepsilon = 1.5$, V(0)=θ=1, K=1, α=1, $\lambda_k = 0.2$, $\psi(x) = x^{\frac{3}{4}}$, $\Lambda = 1$. We calculate c^* to order ε^6.

5. For the fifth example, we put: $T_k = 4$, F(0)=0.06, $\varepsilon = 1.5$, V(0)=θ=1, K=0.5, α=1, $\lambda_k = 0.2$, $\psi(x) = x^{\frac{3}{4}}, \Lambda = 0$. We calculate c^* to order ε^6.

6. For the sixth example, we put: $T_k = 1.5$, F(0)=0.06, $\varepsilon = 1.5$, V(0)=θ=1, K=1, α=0.6, $\lambda_k = 0.0649$, $\psi(x) = x^{\frac{3}{4}}$. We calculate c^* to order ε^4.

7. For the seven example, we put: $T_k = 1.5$, F(0)=0.06, $\varepsilon = 1.5$, V(0)=θ=1, K=1, α=0.6, $\lambda_k = 0.0649$, $\psi(x) = x^{\frac{3}{4}}$, $\Lambda = 1$. We calculate c^* to order ε^6.

8. For the eighth example, For the seven case, we put: $T_k = 1.5$, F(0)=0.06, $\varepsilon = 1.5$,V(0)=θ=1, K=0, α=1, $\lambda_k = 0.2$, $\psi(x) = x^{\frac{3}{4}}$, $\Lambda_k = 1$. We calculate c^* to order ε^6.

9. For the ninth example, For the seven case, we put: $T_k = 1.5$, F(0)=0.06, $\varepsilon = 1.5$, V(0)=θ=1, K=0, α=1, $\lambda_k = 0.2$, $\psi(x) = x^{\frac{3}{4}}$. We calculate c^* to order ε^4.

References

[1] Leif Andersen and Rupert Brotherton-Ratcliffe. Extended libor market models with stochastic volatility. *Journal of Computational Finance*, 9(1), 2005.