

Premia 18

Closed Formula Methods

Contents

1	The Black and Scholes model	2
2	Standard European Options	2
2.1	Call Options	3
2.2	Put Options	3
2.3	Call Spread Options	3
2.4	Digit Options	3
3	European Barrier Options	4
3.1	Down-Out Call Options	5
3.2	Down-Out Put Options	6
3.3	Up-Out Call Options	6
3.4	Up-Out Put Options	7
3.5	Down-In Call Options	7
3.6	Down-In Put Options	8
3.7	Up-In Call Options	8
3.8	Up-In Put Options	9
4	Double Barrier European Option	9
4.1	Knock-Out Call Options	10
4.2	Knock-Out Put Options	11
4.3	Knock-In Call Options	11
4.4	Knock-In Put Options	11
5	Lookback European Options	12
5.1	Fixed Lookback Call Options	12
5.2	Fixed Lookback Put Options	13
5.3	Floating Lookback Call Options	14
5.4	Floating Lookback Put Options	14

6	Standard 2D European Options	15
6.1	Call On the Maximum	15
6.2	Put On the Minimum	16
6.3	Exchange Options	16
6.4	Best Of Option	16

1 The Black and Scholes model

Suppose the underlying asset price $(S_t, t \geq 0)$ evolves according to the Black and Scholes model with continuous yield δ , that is

$$dS_s = S_s ((r - \delta)ds + \sigma dB_s), \quad S_t = x$$

From now on, we denote by

- T = maturity date (Tt)
- $\theta = T - t$
- $b = r - \delta$
- K = strike price

In the following, we state the closed form solutions for the price of options whose payoff is given by $f(S_T)$, being f a suitable function, i.e. for the quantity

$$F(t, x) = E \left[e^{-r\theta} f(S_T(x)) \right].$$

2 Standard European Options

We have the general version of the Black-Sholes Formula [4] to price European options on stocks paying a continuous dividend yield. In this section, we set:

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(b + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} \quad d_2 = d_1 - \sigma\sqrt{\theta}$$

and N as the cumulative normal distribution function:

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-x^2/2} dx.$$

2.1 Call Options

$$\begin{array}{ll}
 \text{PAYOFF} & C_T = (S_T - K)_+ \\
 \text{PRICE} & C(t, x; K) = x e^{-\delta\theta} N(d_1) - K e^{-r\theta} N(d_2) \\
 \text{DELTA} & \frac{\partial C(t, x; K)}{\partial x} = e^{-\delta\theta} N(d_1)
 \end{array}$$

Routine `cf_call.c`

2.2 Put Options

$$\begin{array}{ll}
 \text{PAYOFF} & P_T = (K - S_T)_+ \\
 \text{PRICE} & P(t, x; K) = K e^{-r\theta} N(-d_2) - x e^{-\delta\theta} N(-d_1) \\
 \text{DELTA} & \frac{\partial P(t, x)}{\partial x} = -e^{-\delta\theta} N(-d_1)
 \end{array}$$

Routine `cf_put.c`

2.3 Call Spread Options

$$\begin{array}{ll}
 \text{PAYOFF} & CS_T = (S_T - K_1)_+ - (S_T - K_2)_+ \\
 \text{PRICE} & CS(t, x) = C(t, x; K_1) - C(t, x; K_2) \\
 \text{DELTA} & \frac{\partial CS(t, x)}{\partial x} = \frac{\partial C(t, x; K_1)}{\partial x} - \frac{\partial C(t, x; K_2)}{\partial x}
 \end{array}$$

Routine `cf_callspread.c`

2.4 Digit Options

$$\begin{array}{ll}
 \text{PAYOFF :} & C_T = \begin{cases} K & \text{if } S_T > K, \\ 0 & \text{otherwise} \end{cases} \\
 \text{PRICE :} & C(t, x) = K e^{-r\theta} N(d_1) \\
 \text{DELTA :} & \frac{\partial C(t, x)}{\partial x} = e^{-r\theta} K \frac{e^{-d_1^2/2}}{\sqrt{2\pi\theta} \sigma x}
 \end{array}$$

Routine `cf_digit.c`

3 European Barrier Options

Barrier options are known as *knock-out* if the value of the option nullifies if the underlying asset price reaches a fixed barrier before the maturity date and *knock-in* if it does not. We use the names *up-out* and *up-in* call/put options, or *down-out* and *down-in* call/put options, to stress if the considered barrier is an upper or lower one. Reiner-Rubinstein [3] have developed formulas for pricing standard barrier options with cash rebate R .

We set here:

- $A = \phi x e^{-\delta\theta} N(\phi x_1) - \phi K e^{-r\theta} N(\phi x_1 - \phi\sigma\sqrt{\theta})$
- $B = \phi x e^{-\delta\theta} N(\phi x_2) - \phi K e^{-r\theta} N(\phi x_2 - \phi\sigma\sqrt{\theta})$
- $C = \phi x \left(\frac{L}{x}\right)^{2(\mu+1)} e^{-\delta\theta} N(\eta y_1) - \phi K e^{-r\theta} \left(\frac{L}{x}\right)^{2\mu} N(\eta y_1 - \eta\sigma\sqrt{\theta})$
- $D = \phi x \left(\frac{L}{x}\right)^{2(\mu+1)} e^{-\delta\theta} N(\eta y_2) - \phi K e^{-r\theta} \left(\frac{L}{x}\right)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{\theta})$
- $E = R e^{-r\theta} \left[N(\eta x_2 - \eta\sigma\sqrt{\theta}) - \left(\frac{L}{x}\right)^{2\mu} N(\eta y_2 - \eta\sigma\sqrt{\theta}) \right]$
- $F = R \left(\frac{L}{x}\right)^{\mu+\lambda} N(\eta z) + \left(\frac{L}{x}\right)^{\mu-\lambda} N(\eta z - 2\eta\lambda\sigma\sqrt{\theta})$

and

- $dA = \phi e^{-\delta\theta} N(\phi x_1)$
- $dB = \phi e^{-\delta\theta} N(\phi x_2) + e^{-\delta\theta} \frac{n(x_2)}{\sigma\sqrt{\theta}} \left(1 - \frac{k}{l}\right)$
- $dC = \frac{1}{x} \phi 2\mu \left[\left(\frac{L}{x}\right)^{2\mu} \left(x e^{-\delta\theta} \frac{L^2}{x^2} N(\eta y_1) - K e^{-r\theta} N(\eta y_1 - \eta\sigma\sqrt{\theta}) - \phi \left(\frac{L}{x}\right)^{2(\mu+1)} e^{-b\theta} N(\eta y_1)\right]$
- $dD = -2\mu \frac{\phi}{x} \left[\left(\frac{L}{x}\right)^{2\mu} \left(s e^{-\delta\theta} \frac{L^2}{x^2} N(\eta y_2) - K e^{-r\theta} N(\eta y_2) \times \right.\right.$
 $\left.\times N\left(\eta y_2 - \eta\sigma\sqrt{\theta} - \phi \left(\frac{L}{x}\right)^{2\mu+2} e^{-\delta\theta}\right) N(\eta y_2) - \phi \eta \left(\frac{L}{x}\right)^{2\mu+2} e^{-\delta\theta}\right] \times$
 $\left.\times \frac{n(y_2)}{\sigma\sqrt{\theta}} \left(1 - \frac{k}{l}\right)\right]$
- $dE = 2 \frac{R}{x} e^{-r\theta} \left(\frac{L}{x}\right)^{2\mu} \left[N(\eta y_2 - \eta\sigma\sqrt{\theta}) \mu + \eta \frac{n(\eta y_2 - \sigma\sqrt{\theta})}{\sigma\sqrt{\theta}} \right]$
- $dF = -\frac{R}{x} \left(\frac{L}{x}\right)^{\mu+\lambda} \left[(\mu + \lambda) N(\eta z) + (\mu - \lambda) \left(\frac{L}{x}\right)^{2\lambda} \right] - 2\eta R \left(\frac{L}{x}\right)^{\mu+\lambda} \frac{n(z)}{x\sigma\sqrt{\theta}}$

where

$$\begin{aligned}
\bullet \quad x_1 &= \frac{\log(x/K)}{\sigma\sqrt{\theta}} + (1+\mu)\sigma\sqrt{\theta} & x_2 &= \frac{\log(x/L)}{\sigma\sqrt{\theta}} + (1+\mu)\sigma\sqrt{\theta} \\
\bullet \quad y_1 &= \frac{\log(L^2/xK)}{\sigma\sqrt{\theta}} + (1+\mu)\sigma\sqrt{\theta} & y_2 &= \frac{\log(L/x)}{\sigma\sqrt{\theta}} + (1+\mu)\sigma\sqrt{\theta} \\
\bullet \quad z &= \frac{\log(L/x)}{\sigma\sqrt{\theta}} + \lambda\sigma\sqrt{\theta} \\
\bullet \quad \mu &= \frac{b - \sigma^2/2}{\sigma^2} & \lambda &= \sqrt{\mu^2 + \frac{2r}{\sigma^2}}
\end{aligned}$$

and η, ϕ are suitable numbers belonging to $\{-1, 1\}$ (see next sections for details).

Let

$$M_{t,T} = \sup_{t \leq \tau \leq T} S_\tau \quad \text{and} \quad m_{t,T} = \inf_{t \leq \tau \leq T} S_\tau$$

3.1 Down-Out Call Options

$$\begin{aligned}
\text{PAYOFF} \quad C_T &= \begin{cases} (S_T - K)_+ & \text{if } m_{t,T}L, \\ R & \text{otherwise} \end{cases} \\
\text{PRICE} \quad C(t, x) &= \begin{cases} A - C + F & \text{if } KL \\ B - D + F & \text{otherwise} \end{cases} \\
\text{DELTA} \quad \frac{\partial C(t, x)}{\partial x} &= \begin{cases} dA - dC + dF & \text{if } KL \\ dB - dD + dF & \text{otherwise} \end{cases}
\end{aligned}$$

and take

$$\eta = 1 \quad \phi = 1$$

[Routine cf_calldownout.c](#)

3.2 Down-Out Put Options

$$\begin{array}{ll}
 \text{PAYOFF} & P_T = \begin{cases} (K - S_T)_+ & \text{if } m_{t,T}L, \\ R & \text{otherwise} \end{cases} \\
 \text{PRICE} & P(t, x) = \begin{cases} A - B + C - D + F & \text{if } KL, \\ F & \text{otherwise} \end{cases} \\
 \text{DELTA} & \frac{\partial P(t, x)}{\partial x} = \begin{cases} dA - dB + dC - dD + dF & \text{if } KL, \\ dF & \text{otherwise} \end{cases}
 \end{array}$$

and take

$$\eta = 1 \quad \phi = -1$$

[Routine cf_putdownout.c](#)

3.3 Up-Out Call Options

$$\begin{array}{ll}
 \text{PAYOFF} & C_T = \begin{cases} (S_T - K)_+ & \text{if } M_{t,T} < L, \\ R & \text{otherwise.} \end{cases} \\
 \text{PRICE} & C(t, x) = \begin{cases} F & \text{if } KL, \\ A - B + C - D + F & \text{otherwise} \end{cases} \\
 \text{DELTA} & \frac{\partial C(t, x)}{\partial x} = \begin{cases} dF & \text{if } KL, \\ dA - dB + dC - dD + dF & \text{otherwise} \end{cases}
 \end{array}$$

and take

$$\eta = -1 \quad \phi = 1$$

[Routine cf_callupout.c](#)

3.4 Up-Out Put Options

$$\begin{array}{ll}
 \text{PAYOFF} & P_T = \begin{cases} (K - S_T)_+ & \text{if } M_{t,T} < L, \\ R & \text{otherwise.} \end{cases} \\
 \text{PRICE} & P(t, x) = \begin{cases} B - D + F & \text{if } KL, \\ A - C + F & \text{otherwise} \end{cases} \\
 \text{DELTA} & \frac{\partial P(t, x)}{\partial x} = \begin{cases} dB - dD + dF & \text{if } KL, \\ dA - dC + dF & \text{otherwise} \end{cases}
 \end{array}$$

and take

$$\eta = -1 \quad \phi = -1$$

[Routine cf_putupout.c](#)

3.5 Down-In Call Options

$$\begin{array}{ll}
 \text{PAYOFF} & C_T = \begin{cases} R & \text{if } m_{t,T}L, \\ (S_T - K)_+ & \text{otherwise.} \end{cases} \\
 \text{PRICE} & C(t, x) = \begin{cases} C + E & \text{if } KL, \\ A - B + D + E & \text{otherwise} \end{cases} \\
 \text{DELTA} & \frac{\partial P(t, x)}{\partial x} = \begin{cases} dC + dE & \text{if } KL, \\ dA - dB + dD + dE & \text{otherwise} \end{cases}
 \end{array}$$

and take

$$\eta = 1 \quad \phi = 1$$

[Routine cf_calldownin.c](#)

3.6 Down-In Put Options

$$\begin{array}{ll}
 \text{PAYOFF} & P_T = \begin{cases} R & \text{if } m_{t,T}L, \\ (K - S_T)_+ & \text{otherwise.} \end{cases} \\
 \text{PRICE} & P(t, x) = \begin{cases} B - C + D + E & \text{if } KL, \\ A + E & \text{otherwise} \end{cases} \\
 \text{DELTA} & \frac{\partial P(t, x)}{\partial x} = \begin{cases} dB - dC + dD + dE & \text{if } KL, \\ dA + dE & \text{otherwise} \end{cases}
 \end{array}$$

and take

$$\eta = 1 \quad \phi = -1$$

[Routine cf_putdownin.c](#)

3.7 Up-In Call Options

$$\begin{array}{ll}
 \text{PAYOFF} & C_T = \begin{cases} R & \text{if } M_{t,T} < L, \\ (S_T - K)_+ & \text{otherwise.} \end{cases} \\
 \text{PRICE} & P(t, x) = \begin{cases} A + E & \text{if } KL, \\ B - C + D + E & \text{otherwise} \end{cases} \\
 \text{DELTA} & \frac{\partial C(t, x)}{\partial x} = \begin{cases} dA + dE & \text{if } KL, \\ dB - dC + dD + dE & \text{otherwise} \end{cases}
 \end{array}$$

and take

$$\eta = -1 \quad \phi = 1$$

[Routine cf_callupin.c](#)

3.8 Up-In Put Options

$$\begin{array}{ll}
 \text{PAYOFF} & P_T = \begin{cases} R & \text{if } M_{t,T} < L, \\ (K - S_T)_+ & \text{otherwise.} \end{cases} \\
 \text{PRICE} & P(t, x) = \begin{cases} A - B + D + E & \text{if } KL, \\ C + E & \text{otherwise} \end{cases} \\
 \text{DELTA} & \frac{\partial C(t, x)}{\partial x} = \begin{cases} dA - dB + dD + dE & \text{if } KL, \\ dC + dE & \text{otherwise} \end{cases}
 \end{array}$$

and take

$$\eta = -1 \quad \phi = -1$$

[Routine cf_putupin.c](#)

4 Double Barrier European Option

A double barrier option is knock-in or knock-out if the underlying price reaches or not a lower and/or an upper boundary prior to expiration. The exact value for double barrier call/put knock-out options is given by the Ikeda-Kunitomo formula [5], which allows to compute exactly the price when the boundaries suitably depend on the time variable t . More precisely, set

$$U(s) = Ue^{\delta_1 s} \quad L(s) = Le^{\delta_2 s}$$

where the constants U, L, δ_1, δ_2 are such that $L(s) < U(s)$, for every $s \in [t, T]$. The functions $U(s)$ and $L(s)$ play the role of *upper* and *lower* barrier respectively. δ_1 and δ_2 determine the curvature and the case of $\delta_1 = 0$ and $\delta_2 = 0$ corresponds to two flat boundaries.

The numerical studies suggest that in most cases it suffices to calculate the leading five terms of the series giving the price of the knock-out and knock-in double barrier call options.

Let τ stand for the first time at which the underlying asset price S reaches at least one barrier, i.e.

$$\tau = \inf\{s; S_s \leq L(s) \text{ or } S_s \geq U(s)\}.$$

We define the following coefficients:

- $\mu_1 = 2 \frac{b - \delta_2 - n(\delta_1 - \delta_2)}{\sigma^2} + 1$
- $\mu_2 = 2 \frac{n(\delta_1 - \delta_2)}{\sigma^2}$
- $\mu_3 = 2 \frac{b - \delta_2 + n(\delta_1 - \delta_2)}{\sigma^2} + 1$

4.1 Knock-Out Call Options

$$\text{PAYOFF} \quad C_T = \begin{cases} (S_T - K)_+ & \text{if } \tau T \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \text{PRICE} \quad C(t, x) = & x e^{-\delta \theta} \sum_{n=-\infty}^{+\infty} \left[\left(\frac{U^n}{L^n} \right)^{\mu_1} \left(\frac{L}{x} \right)^{\mu_2} (N(d_1^+) - N(d_2^+)) \right. \\ & \left. - \left(\frac{L^{n+1}}{x U^n} \right)^{\mu_3} (N(d_3^+) - N(d_4^+)) \right] \\ & - K e^{-r \theta} \sum_{n=-\infty}^{+\infty} \left[\left(\frac{U^n}{L^n} \right)^{\mu_1-2} \left(\frac{L}{x} \right)^{\mu_2} (N(d_1^-) - N(d_2^-)) \right. \\ & \left. - \left(\frac{L^{n+1}}{x U^n} \right)^{\mu_3-2} (N(d_3^-) - N(d_4^-)) \right] \end{aligned}$$

where $F = U e^{\delta_1 \theta}$ and

$$\begin{aligned} d_1^\pm &= \frac{\log(x U^{2n} / K L^{2n}) + (b \pm \frac{\sigma^2}{2}) \theta}{\sigma \sqrt{\theta}} & d_2^\pm &= \frac{\log(x U^{2n} / F L^{2n}) + (b \pm \frac{\sigma^2}{2}) \theta}{\sigma \sqrt{\theta}} \\ d_3^\pm &= \frac{\log(L^{2n+2} / K x U^{2n}) + (b \pm \frac{\sigma^2}{2}) \theta}{\sigma \sqrt{\theta}} & d_4^\pm &= \frac{\log(L^{2n+2} / F x U^{2n}) + (b \pm \frac{\sigma^2}{2}) \theta}{\sigma \sqrt{\theta}} \end{aligned}$$

Routine [cf_callout_kunitomoikeda.c](#)

4.2 Knock-Out Put Options

$$\begin{aligned}
 \text{PAYOFF} \quad P_T &= \begin{cases} (K - S_T)_+ & \text{if } \tau T \\ 0 & \text{otherwise} \end{cases} \\
 \text{PRICE} \quad P(t, x) &= Ke^{-r\theta} \sum_{n=-\infty}^{+\infty} \left[\left(\frac{U^n}{L^n} \right)^{\mu_1-2} \left(\frac{L}{x} \right)^{\mu_2} (N(y_1^-) - N(y_2^-)) \right. \\
 &\quad \left. - \left(\frac{L^{n+1}}{xU^n} \right)^{\mu_3-2} (N(y_3^-) - N(y_4^-)) \right] \\
 &\quad - xe^{-\delta\theta} \sum_{n=-\infty}^{+\infty} \left[\left(\frac{U^n}{L^n} \right)^{\mu_1} \left(\frac{L}{x} \right)^{\mu_2} (N(y_1^+) - N(y_2^+)) \right. \\
 &\quad \left. - \left(\frac{L^{n+1}}{xU^n} \right)^{\mu_3} (N(y_3^+) - N(y_4^+)) \right]
 \end{aligned}$$

where $E = Le^{\delta_2\theta}$ and

$$\begin{aligned}
 y_1^\pm &= \frac{\log(xU^{2n}/EL^{2n}) + (b \pm \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}} & y_2^\pm &= \frac{\log(xU^{2n}/KL^{2n}) + (b \pm \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}} \\
 y_3^\pm &= \frac{\log(L^{2n+2}/ExU^{2n}) + (b \pm \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}} & y_4^\pm &= \frac{\log(L^{2n+2}/KxU^{2n}) + (b \pm \frac{\sigma^2}{2})\theta}{\sigma\sqrt{\theta}}
 \end{aligned}$$

[Routine cf_putout_kunitomoikedada.c](#)

4.3 Knock-In Call Options

The double barrier knock-in call option is priced via the no-arbitrage relationship between knock-out and knock-in option:

European “IN” + European “OUT” = European Standard

[Routine cf_callin_kunitomoikedada.c](#)

4.4 Knock-In Put Options

The double barrier knock-in call option is priced via the no-arbitrage relationship between knock-out and knock-in option:

European “IN” + European “OUT” = European Standard

Routine `cf_putin_kunitomoikeda.c`

5 Lookback European Options

Floating strike lookback options can be priced using Goldman-Sosin-Gatto formula [2] while fixed strike lookback options can be priced using Conze-Viswanathen formula[7].

We set, as $0 \leq u \leq v \leq T$,

$$M_{u,v} = \sup_{u \leq \tau \leq v} S_\tau \quad \text{and} \quad m_{u,v} = \inf_{u \leq \tau \leq v} S_\tau$$

and

$$\begin{aligned} \bullet \quad d_1 &= \frac{\log\left(\frac{x}{K}\right) + \left(b + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} & d_2 &= d_1 - \sigma\sqrt{\theta} \\ \bullet \quad e_1 &= \frac{\log\left(\frac{x}{M_{0,t}}\right) + \left(b + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} & e_2 &= e_1 - \sigma\sqrt{\theta} \\ \bullet \quad f_1 &= \frac{\log\left(\frac{x}{M_{0,t}}\right) + \left(b + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}} & f_2 &= f_1 - \sigma\sqrt{\theta} \end{aligned}$$

5.1 Fixed Lookback Call Options

$$\text{PAYOFF} \quad C_T = (M_{t,T} - K)_+$$

Both price and delta depend on K and $M_{0,t}$.

- IF $K > M_{0,t}$ THEN

$$\begin{aligned} \text{PRICE} \quad C(t, x) &= xe^{-\delta\theta}N(d_1) - Ke^{-r\theta}N(d_2) \\ &\quad + xe^{-r\theta}\frac{\sigma^2}{2b}\left[-\left(\frac{x}{K}\right)^{-\frac{2b}{\sigma^2}}N\left(d_1 - \frac{2b}{\sigma}\sqrt{\theta}\right) + e^{b\theta}N(d_1)\right] \\ \text{DELTA} \quad \frac{\partial C(t, x)}{\partial x} &= e^{-\delta\theta}N(d_1)\left(1 + \frac{\sigma^2}{2b}\right) + e^{-\delta\theta}\frac{n(d_1)}{\sigma\sqrt{\theta}} - e^{-r\theta}\frac{K}{x}\frac{n(d_2)}{\sigma\sqrt{\theta}} \\ &\quad + e^{-r\theta}\left(\frac{x}{K}\right)^{-\frac{2b}{\sigma^2}}N\left(d_1 - \frac{2b}{\sigma}\sqrt{t}\right)\left(1 - \frac{\sigma^2}{2b}\right) \end{aligned}$$

- IF $K \leq M_{0,t}$ THEN

PRICE $C(t, x) = e^{-r\theta}(M_{0,t} - K) + xe^{-\delta\theta}N(e_1) - M_{0,t}e^{-r\theta}N(e_2)$

$$+ xe^{-r\theta} \frac{\sigma^2}{2b} \left[- \left(\frac{x}{M_{0,t}} \right)^{-\frac{2b}{\sigma^2}} N \left(e_1 - \frac{2b}{\sigma} \sqrt{\theta} \right) + e^{b\theta} N(e_1) \right]$$

DELTA $\frac{\partial C(t, x)}{\partial x} = e^{-\delta\theta}N(e_1) \left(1 + \frac{\sigma^2}{2b} \right) + e^{-\delta\theta} \frac{n(e_1)}{\sigma\sqrt{\theta}} - e^{-r\theta} \frac{M_{0,t}}{x} \frac{n(d_2)}{\sigma\sqrt{\theta}}$

$$+ e^{-r\theta} \left(\frac{x}{M_{0,t}} \right)^{-\frac{2b}{\sigma^2}} N \left(e_1 - \frac{2b}{\sigma} \sqrt{t} \right) \left(1 - \frac{\sigma^2}{2b} \right)$$

Routine `cf_fixed_calllookback.c`

5.2 Fixed Lookback Put Options

PAYOFF $P_T = (K - m_{t,T})_+$

Both price and delta depend on K and $m_{0,t}$.

- IF $K < m_{0,t}$ THEN

PRICE $P(t, x) = Ke^{-r\theta}N(-d_2) - xe^{-\delta\theta}N(-d_1)$

$$+ xe^{-r\theta} \frac{\sigma^2}{2b} \left[\left(\frac{x}{K} \right)^{-\frac{2b}{\sigma^2}} N \left(-d_1 + \frac{2b}{\sigma} \sqrt{\theta} \right) - e^{b\theta} N(-d_1) \right]$$

DELTA $\frac{\partial P(t, x)}{\partial x} = e^{-\delta\theta}N(d_1) \left(1 + \frac{\sigma^2}{2b} \right) + e^{-\delta\theta} \frac{n(d_1)}{\sigma\sqrt{\theta}} - e^{-r\theta} \frac{K}{x} \frac{n(d_2)}{\sigma\sqrt{\theta}}$

$$+ e^{-r\theta} \left(\frac{x}{K} \right)^{-\frac{2b}{\sigma^2}} N \left(-d_1 + \frac{2b}{\sigma} \sqrt{t} \right) \left(\frac{\sigma^2}{2b} - 1 \right) - e^{-\delta\theta} \left(\frac{\sigma^2}{2b} - 1 \right)$$

- IF $K \geq m_{0,t}$ THEN

PRICE $P(t, x) = e^{-r\theta}(K - m_{0,t}) - xe^{-\delta\theta}N(-f_1) + m_{0,t}e^{-r\theta}N(-f_2)$

$$+ xe^{-r\theta} \frac{\sigma^2}{2b} \left[\left(\frac{x}{m_{0,t}} \right)^{-\frac{2b}{\sigma^2}} N \left(-f_1 + \frac{2b}{\sigma} \sqrt{\theta} \right) - e^{b\theta} N(-f_1) \right]$$

DELTA $\frac{\partial P(t, x)}{\partial x} = e^{-\delta\theta} \left(1 + \frac{\sigma^2}{2b} \right) (N(f_1) - 1) + e^{-\delta\theta} \frac{n(f_1)}{\sigma\sqrt{\theta}} - e^{-r\theta} \frac{M_{0,t}}{x} \frac{n(d_2)}{\sigma\sqrt{\theta}}$

$$+ e^{-r\theta} \left(\frac{x}{M_{0,t}} \right)^{-\frac{2b}{\sigma^2}} N \left(-f_1 + \frac{2b}{\sigma} \sqrt{t} \right) \left(\frac{\sigma^2}{2b} - 1 \right)$$

[Routine cf_fixed_putlookback.c](#)

5.3 Floating Lookback Call Options

PAYOFF $C_T = S_T - m_{t,T}$

PRICE
$$C(t, x) = xe^{-\delta\theta}N(f_1) - m_{0,t}e^{-r\theta}N(f_2) + xe^{-r\theta}\frac{\sigma^2}{2b} \left[\left(\frac{x}{m_{0,t}} \right)^{-\frac{2b}{\sigma^2}} N\left(-f_1 + \frac{2b}{\sigma}\sqrt{\theta}\right) - e^{b\theta}N(-f_1) \right]$$

DELTA
$$\frac{\partial C(t, x)}{\partial x} = e^{-\delta\theta}N(f_1) \left(1 + \frac{\sigma^2}{2b} \right) + e^{-\delta\theta} \frac{n(a_1)}{\sigma\sqrt{\theta}} - e^{-r\theta} \frac{\sigma^2}{2b} + e^{-r\theta} \left(\frac{x}{m_{0,t}} \right)^{-\frac{2b}{\sigma^2}} N\left(-f_1 + \frac{2b}{\sigma}\sqrt{t}\right) \left(\frac{\sigma^2}{2b} - 1 \right)$$

[Routine cf_floating_calllookback.c](#)

5.4 Floating Lookback Put Options

PAYOFF $P_T = M_{t,T} - S_T$

PRICE
$$P(t, x) = M_{0,t}e^{-r\theta}N(-e_2) - xe^{-\delta\theta}N(-e_1) + xe^{-r\theta}\frac{\sigma^2}{2b} \left[- \left(\frac{x}{M_{0,t}} \right)^{-\frac{2b}{\sigma^2}} N\left(e_1 - \frac{2b}{\sigma}\sqrt{\theta}\right) + e^{b\theta}N(b_1) \right]$$

DELTA
$$\frac{\partial P(t, x)}{\partial x} = e^{-\delta\theta}N(e_1) \left(1 + \frac{\sigma^2}{2b} \right) + e^{-r\theta} \left(\frac{x}{M_{0,t}} \right)^{-\frac{2b}{\sigma^2}} N\left(e_1 - \frac{2b}{\sigma}\sqrt{t}\right) \left(1 - \frac{\sigma^2}{2b} \right) - e^{-r\theta} \left(\frac{x}{M_{0,t}} \right) \left(\frac{n(b_1)}{\sigma\sqrt{\theta}} - 1 \right) + e^{-\delta\theta} \left(\frac{n(b_1)}{\sigma\sqrt{\theta}} - 1 \right)$$

[Routine cf_floating_putlookback.c](#)

6 Standard 2D European Options

Consider the pair of processes $S_t = (S_t^1, S_t^2)$ solution to

$$\begin{cases} dS_t^1 = S_t^1((r - \delta_1)dt + \sigma_1 dW_t^1), & S_0^1 = x^1 \\ dS_t^2 = S_t^2((r - \delta_2)dt + \sigma_2 dW_t^2), & S_0^2 = x^2. \end{cases}$$

where $(W_t^1, t \geq 0)$ and $(W_t^2, t \geq 0)$ denote two real-valued Brownian motions with instantaneous correlation ρ . The price of option with payoff f is:

$$F(t, x^1, x^2) = E \left[e^{-rT} f \left(S_T^1(x^1), S_T^2(x^2) \right) \right].$$

Here, closed formulas due to Johnson and Stulz are presented [1],[6].

We set

- $d = \frac{\log \frac{x_1}{x_2} + \left(\delta_2 - \delta_1 + \frac{\sigma^2}{2} \right) \theta}{\sigma \sqrt{\theta}}$
- $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}$
- $d_i = \frac{\log \left(\frac{x_i}{K} \right) + \left(r - \delta_i + \frac{\sigma_i^2}{2} \right) \theta}{\sigma_i \sqrt{\theta}}, \quad i = 1, 2$
- $\rho_1 = \frac{\sigma_1 - \rho\sigma_2}{\sigma}$
- $\rho_2 = \frac{\sigma_2 - \rho\sigma_1}{\sigma}$

and M as the cumulative bivariate normal distribution function:

$$M(a, b; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^a \int_{-\infty}^b e^{-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}} dx dy.$$

6.1 Call On the Maximum

PAYOFF	$C_T = (\max(S_T^1, S_T^2) - K)_+$
PRICE	$C(t, x_1, x_2) = x_1 e^{-\delta_1 \theta} M(d_1, d; \rho_1) + x_2 e^{-\delta_2 \theta} M(d_2, -d + \sigma \sqrt{\theta}; \rho_2) - K e^{-r\theta} (1 - M(-d_1 + \sigma_1 \sqrt{\theta}, -d_2 + \sigma_2 \sqrt{\theta}; \rho))$
DELTA	$\frac{\partial C(t, x_1, x_2)}{\partial x_1} = e^{-\delta_1 \theta} M(d_1, d; \rho_1)$ $\frac{\partial C(t, x_1, x_2)}{\partial x_2} = e^{-\delta_2 \theta} M(d_2, -d + \sigma \sqrt{\theta}; \rho_2)$

Routine `cf_callmax.c`

6.2 Put On the Minimum

PAYOFF $P_T = (K - \min(S_T^1, S_T^2))_+$

PRICE $P(t, x_1, x_2) = Ke^{-r\theta} - c_0 + c_1$

DELTA
$$\frac{\partial P(t, x_1, x_2)}{\partial x_1} = e^{-\delta_1\theta}(1 - N(d)) + e^{-\delta_1\theta}M(d_1, -d; -\rho_1)$$

$$\frac{\partial P(t, x_1, x_2)}{\partial x_2} = e^{-\delta_2\theta}N(d - \sigma) + e^{-\delta_2\theta}M(d_2, d - \sigma\sqrt{\theta}; -\rho_2)$$

where

- $c_0 = x_1e^{-\delta_1\theta}(1 - N(d)) + x_2e^{-\delta_2\theta}N(d - \sigma\sqrt{\theta})$
- $c_1 = x_1e^{-\delta_1\theta}M(d_1, -d, -\rho_1) + x_2e^{-\delta_2\theta}M(d_2, d - \sigma\sqrt{\theta}; -\rho_2) - Ke^{-r\theta}M(d_1 - \sigma_1\sqrt{\theta}, d_2 - \sigma_2\sqrt{\theta}; \rho)$

Routine `cf_putmin.c`

6.3 Exchange Options

PAYOFF $E_T = (S_T^1 - \lambda S_T^2)_+$

PRICE $E(t, x_1, x_2) = x_1e^{-\delta_1\theta}N(\hat{d}_1) - \lambda x_2e^{-\delta_2\theta}N(\hat{d}_2)$

where

$$\hat{d}_1 = \frac{\log\left(\frac{x_1}{\lambda x_2}\right) + \left(\delta_2 - \delta_1 + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}, \quad \hat{d}_2 = \hat{d}_1 - \sigma\sqrt{\theta}$$

Routine `cf_exchange.c`

6.4 Best Of Option

The payoff is $B_T = (\max(S_T^1 - K_1, S_T^2 - K_2))_+$.

References

- [1] H.JOHNSON. Options on the maximum of the minimum of several assets. *J.Of Finance and Quantitative Analysis*, 22:227–283, 1987. 15
- [2] B.M.GOLDMAN H.B.SOSIN M.A.GATTO. Path dependent options: buy at low, sell at high. *J. of Finance*, 34:111–127, 1979. 12
- [3] E.REINER M.RUBINSTEIN. Breaking down the barriers. *Risk*, 4:28–35, 191. 4
- [4] F.BLACK M.SCHOLES. The pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81:635–654, 1973. 2
- [5] N.KUNIMOTO N.IKEDA. Pricing options with curved boundaries. *Mathematical finance*, 2:275–298, 1992. 9
- [6] R.STULZ. Options on the minimum or the maximum of two risky assets. *J. of Finance*, 10:161–185, 1992. 15
- [7] A.CONZE R.VISWANATHAN. Path dependent options: the case of lookback options. *J. of Finance*, 46:1893–1907, 1992. 12