

# EFFICIENT PRICING LOOKBACK OPTIONS UNDER LÉVY PROCESSES

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ABSTRACT. We use a general formula which was derived in Kudryavtsev and Levendorskiĭ (2011) for pricing options with barrier and/or lookback features in Lévy models. In the case of lookback options the pricing formula can be efficiently realized using the methodology developed in Kudryavtsev and Levendorskiĭ (Finance Stoch. 13: 531–562, 2009). We demonstrate advantages of our approach in terms of accuracy and convergence.

**AMS 2000 subject classifications:** Primary 60-08, 60J75; secondary 65R10, 47A68, 42A85, 91G60

**Keywords:** Lévy processes, KoBoL, CGMY, barrier options, lookback options, barrier-lookback options, Wiener-Hopf factorization, inverse Laplace transform, Fast Fourier transform

## Premia 18

### 1. INTRODUCTION

Lookback options are among the most popular path-dependent derivatives traded in exchanges worldwide. The payoffs of these options depend on the realized minimum (or maximum) asset price at expiration. A standard European lookback call (put) gives the option holder the right to buy (sell) an asset at its lowest (highest) price during the life of the option. Standard (also called floating strike) lookbacks were first studied in [32, 33], where closed-form pricing formulas were derived in the Black-Scholes framework. In addition to standard lookback options, paper [23] introduces fixed strike lookbacks. A fixed strike lookback call pays off the difference between the

realized maximum price and some prespecified strike or zero, whichever is greater. A lookback put with a fixed strike pays off the difference between the strike and the realized minimum price or zero, whichever is greater. Fixed strike lookback options can be priced also analytically in the Black-Scholes model [23]. In a discrete time setting the extremum of the asset price will be determined at discrete monitoring instants (see e.g. [2] for a detailed description). The continuity corrections for discrete lookback options in the Black-Scholes setting are given in [16]. For a description of lookback options with American type constraints and related pricing methods in diffusion models, we refer to [5, 31, 34, 47, 64] and the references cited therein.

Other types of lookback options include exotic lookbacks, percentage lookback options in which the extreme values are multiplied by a constant [23], and partial lookback options [35] (the monitoring interval for the extremum is a subinterval between the initial date and the expiry).

In recent years more and more attention has been given to stochastic models of financial markets which depart from the traditional Black-Scholes model. We concentrate on one-factor non-gaussian exponential Lévy models. These models provide a better fit to empirical asset price distributions that typically have fatter tails than Gaussian ones, and can reproduce volatility smile phenomena in option prices. For an introduction to applications of these models applied to finance, we refer to [14, 22].

Option valuation under Lévy processes has been dealt with by a host of researchers, therefore, an exhaustive list is virtually impossible. However, the pricing of path-dependent options in exponential Lévy models still remains a mathematical and computational challenge (see, e.g., [62, 41, 54, 10, 11, 9] for recent surveys of the state of the art of exotic option pricing in Lévy models).

The Wiener-Hopf factorization method is a standard tool for pricing path-dependent options. Nguyen-Ngoc and Yor [54] obtained formulas in terms of the Wiener-Hopf factors for the Laplace transform of continuously monitored barrier and lookback options in general Lévy models. The probabilistic approach used in the paper allows, in particular, to recover the results for barrier options derived in [14] using the analytical form of the Wiener-Hopf factorization method. The drawback of the

formulas in [14, 54] is the complexity of numerical calculations required, since, in general, numerical  $n$ -fold integrals (with  $n = 2, 3$ ) are needed.

In the case of jump diffusions with exponentially distributed Poisson jumps (a double-exponential jump diffusion process (DEJD) and its generalization: a hyper-exponential jump-diffusion model (HEJD)), the Laplace transform of the price w.r.t. time has a relatively simple explicit form. Formulas for DEJD model were obtained by Lipton [50] and Kou [41], and, for double-barrier options, by Sepp [61]; for HEJD case, see [20, 24, 36] and the bibliography in [9]. Note that papers [50, 40] consider continuously monitored barrier and lookback options, whereas the other papers cited above studied barrier options only. The Laplace transform of the price having being calculated, one uses a suitable numerical Laplace inversion algorithm to recover the option price. However, the problem of the inversion of the Laplace transform is non-trivial from the computational point of view. We refer the reader to [1] for a description of a general framework for related numerical methods.

Calculation of the Laplace transform of the price under a general Lévy process is non-trivial as well. To simplify calculations, one can approximate the initial process by a DEJD or, more generally, HEJD, and then use the Laplace transform method (see, e.g., [20, 24, 36]). However, this approximation introduces an additional error, which may be quite sizable near the barrier (see examples in [10, 11]).

Borovkov and Novikov [8] develop a method based on Spitzer's identity to price discrete lookback options in a general Lévy model; see also [55]. The methods of these papers are computationally expensive when the monitoring is frequent (e.g., daily monitoring). Using the Hilbert transform, Feng and Linetsky [29, 30] proposed a new computationally efficient method for pricing discrete barrier and lookback options and calculation of exponential moments of the discrete maximum of a Lévy process.

As the number of monitoring times goes to infinity, discrete (barrier) lookback options converge to continuous (barrier) lookbacks. However, the discrete options pricing methods described above converge to continuous prices rather slowly. The Richardson extrapolation is typically used to improve the rate of convergence of

various numerical schemes. Unfortunately, even for the standard lookbacks, there are no theoretical results, which can be used to find a generalization of the Richardson extrapolation procedure appropriate for estimation of continuous values from discrete ones (see the discussion in [30]). Motivated by the pricing of lookback options in exponential Lévy models, Dia and Lamberton [25] studied the difference between the continuous and discrete supremum of a general Lévy process. However, similar results for hybrid exotics are unavailable at present.

Kudryavtsev and Levendorskiĭ [42] developed a fast and accurate numerical method labelled Fast Wiener-Hopf factorization method (FWHF-method) for pricing continuously monitored barrier options under Lévy processes of a wide class. FWHF-method is based on an efficient approximation of the Wiener-Hopf factors in the exact formula for the solution and the Fast Fourier Transform (FFT) algorithm. In contrast to finite difference methods which require a detailed analysis of the underlying Lévy model, the FWHF-method deals with the characteristic exponent of the process.

In [44], Kudryavtsev and Levendorskiĭ derive a general formula which is applicable to barrier options, lookbacks, lookbarriers, barrier-lookbacks, and other similar types of options, using an operator form of the Wiener-Hopf factorization. An efficient numerical realization of the formula for lookback options in KoBoL(CGMY) and Kou models was implemented into Premia 14. As well as in [44] we use Gaver-Stehfest algorithm (see [1]) and the Fast Wiener-Hopf factorization method developed in [42]. The total computational cost of our algorithm is  $O(NM \log_2(M))$ , where  $N$  is the parameter of the Gaver-Stehfest algorithm (see Appendix A), and  $M$  is the number of discrete points used to compute the fast Fourier transform in FWHF-method (see Appendix B). In contrast to pricing methods based on approximations by options with discrete monitoring, our pricing method converges very fast to prices of options with continuous monitoring.

The rest of the paper is organized as follows. In Section 2, we give necessary definitions of the theory of Lévy processes, and we provide a general theorem on pricing of options with barrier and lookback features. In Section 3, we consider realizations of the general formula for several types of options with lookback and/or

barrier features, and describe an efficient numerical method for realization of the general formula. Numerical examples in Section 4 demonstrate advantages of our method in terms of accuracy and speed. Section 5 gives details on the method implemented into Premia 14.

## 2. LÉVY PROCESSES: BASIC FACTS

**2.1. General definitions.** A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see, e.g., [60]). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. A Lévy process  $X_t$  can be completely specified by its characteristic exponent,  $\psi$ , definable from the equality  $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$  (we confine ourselves to the one-dimensional case).

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty} (1 - e^{i\xi y} + i\xi y \mathbb{1}_{[-1;1]}(y))F(dy),$$

where  $\sigma^2 \geq 0$  is the variance of the Gaussian component,  $\mathbb{1}_A$  is the indicator function of the set  $A$ , and the Lévy measure  $F(dy)$  satisfies

$$\int_{\mathbb{R} \setminus \{0\}} \min\{1, y^2\} F(dy) < +\infty.$$

Assume that the riskless rate  $r$  is constant, and, under a risk-neutral measure chosen by the market, the underlying evolves as  $S_t = S_0 e^{X_t}$ , where  $X_t$  is a Lévy process. Then we must have  $E[e^{X_t}] < +\infty$ , and, therefore,  $\psi$  must admit the analytic continuation into the strip  $\text{Im } \xi \in (-1, 0)$  and continuous continuation into the closed strip  $\text{Im } \xi \in [-1, 0]$ .

Further, if  $d \geq 0$  is the constant dividend yield on the underlying asset, then the following condition (the EMM-requirement) must hold:  $E[e^{X_t}] = e^{(r-d)t}$ . Equivalently,

$$(2.1) \quad r - d + \psi(-i) = 0,$$

which can be used to express the drift  $\mu$  via the other parameters of the Lévy process:

$$(2.2) \quad \mu = r - d - \frac{\sigma^2}{2} + \int_{-\infty}^{+\infty} (1 - e^y + y \mathbb{1}_{[-1;1]}(y)) F(dy).$$

In empirical studies of financial markets, the following classes of Lévy processes are popular: the Merton model [53], double-exponential jump-diffusion model (DEJD) introduced to finance by Lipton [50] and Kou [38], generalization of DEJD model constructed by Levendorskii [48] and labelled later Hyper-exponential jump-diffusion model (HEJD), Variance Gamma Processes (VGP) introduced to finance by Madan with coauthors (see, e.g., [51]), Hyperbolic processes constructed in [26, 27], Normal Inverse Gaussian processes constructed by Barndorff-Nielsen [3] and generalized in [4], and extended Koponen's family introduced in [12, 13] and labelled KoBoL model in [14]. Koponen [37] introduced a symmetric version; Boyarchenko and Levendorskii [12, 13] gave a non-symmetric generalization; later, in [21], a subclass of this model appeared under the name CGMY-model.

*Example 2.1.* The characteristic exponent of a pure jump KoBoL (CGMY) process of order  $\nu \in (0, 2)$ ,  $\nu \neq 1$ , is given by

$$(2.3) \quad \psi(\xi) = -i\mu\xi + c\Gamma(-\nu)[\lambda_+^\nu - (\lambda_+ + i\xi)^\nu + (-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu],$$

where  $c > 0$ ,  $\mu \in \mathbb{R}$ , and  $\lambda_- < -1 < 0 < \lambda_+$ . Carr et al. (2002) [21] use different parameters labels  $C, G, M, Y$ :

$$(2.4) \quad \psi(\xi) = -i\mu\xi + C\Gamma(-Y)[G^Y - (G + i\xi)^Y + M^Y - (M - i\xi)^Y].$$

The relation between two parameterizations is quite easy to obtain:

$$(2.5) \quad c = C, \lambda_+ = G, \lambda_- = -M, \nu = Y.$$

*Example 2.2.* In DEJD model, the characteristic exponent is of the form

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where  $\sigma > 0$ ,  $\mu \in \mathbb{R}$ ,  $c_\pm > 0$  and  $\lambda_- < -1 < 0 < \lambda_+$ .

**2.2. The Wiener-Hopf factorization.** There are several forms of the Wiener-Hopf factorization. The Wiener-Hopf factorization formula used in probability reads:

$$(2.6) \quad E[e^{i\xi X_{T_q}}] = E[e^{i\xi \bar{X}_{T_q}}]E[e^{i\xi \underline{X}_{T_q}}], \quad \forall \xi \in \mathbb{R},$$

where  $T_q \sim \text{Exp } q$  is an exponentially distributed random variable independent of  $X$ , and  $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$  and  $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$  are the supremum and infimum processes. Introducing the notation

$$\begin{aligned}\phi_q^+(\xi) &= qE \left[ \int_0^\infty e^{-qt} e^{i\xi \bar{X}_t} dt \right] = E \left[ e^{i\xi \bar{X}_{T_q}} \right], \\ \phi_q^-(\xi) &= qE \left[ \int_0^\infty e^{-qt} e^{i\xi \underline{X}_t} dt \right] = E \left[ e^{i\xi \underline{X}_{T_q}} \right]\end{aligned}$$

we can write (2.6) as

$$(2.7) \quad \frac{q}{q + \psi(\xi)} = \phi_q^+(\xi) \phi_q^-(\xi).$$

Introduce the normalized resolvent of  $X$  or the expected present value operator (EPV-operator) under  $X$ . The name is due to the observation that, for a stream  $g(X_t)$ ,

$$(2.8) \quad \mathcal{E}_q g(x) = E \left[ \int_0^{+\infty} q e^{-qt} g(X_t) dt \mid X_0 = x \right].$$

Replacing in (2.8) process  $X$  with the supremum and infimum processes  $\bar{X}$  and  $\underline{X}$ , we obtain the EPV operators  $\mathcal{E}_q^\pm$  under supremum and infimum process. Equivalently,

$$(2.9) \quad \mathcal{E}_q u(x) = \mathbb{E}^x[u(X_{T_q})], \quad \mathcal{E}_q^+ u(x) = \mathbb{E}^x[u(\bar{X}_{T_q})], \quad \mathcal{E}_q^- u(x) = \mathbb{E}^x[u(\underline{X}_{T_q})].$$

Hence,  $\mathcal{E}_q$  and  $\mathcal{E}_q^\pm$  admit interpretation as expectation operators:

$$\mathcal{E}_q g(x) = \int_{-\infty}^{+\infty} g(x+y) P_q(dy), \quad \mathcal{E}_q^\pm g(x) = \int_{-\infty}^{+\infty} g(x+y) P_q^\pm(dy),$$

where  $P_q(dy)$ ,  $P_q^\pm(dy)$  are probability distributions with  $\text{supp } P_q^+ \subset [0, +\infty)$ ,  $\text{supp } P_q^- \subset (-\infty, 0]$ . The characteristic functions of the distributions  $P_q(dy)$  and  $P_q^\pm(dy)$  are  $q(q + \psi(\xi))^{-1}$  and  $\phi_q^\pm(\xi)$ , respectively.

The operator form of the Wiener-Hopf factorization is written as follows (see details in [59, p.81]):

$$(2.10) \quad \mathcal{E}_q = \mathcal{E}_q^+ \mathcal{E}_q^- = \mathcal{E}_q^- \mathcal{E}_q^+.$$

Note that (2.10) is understood as equalities for operators in appropriate function spaces, for instance, in the space of semi-bounded Borel functions. Under appropriate conditions on the characteristic exponent, the EPV operators are defined as operators in spaces of functions of exponential growth at infinity, and (2.10) holds in these spaces. See [14].

Finally, note that (2.7) is a special case of the Wiener-Hopf factorization of the symbol of a pseudo-differential operator (PDO). In applications to Lévy processes, the symbol is  $q/(q + \psi(\xi))$ , and the PDO is  $\mathcal{E}_q := q(q + \psi(D))^{-1}$ . Recall that a PDO  $A = a(D)$  acts as follows:

$$Au(x) = (2\pi)^{-1} \int_{-\infty}^{+\infty} e^{ix\xi} a(\xi) \hat{u}(\xi) d\xi,$$

where  $\hat{u}$  is the Fourier transform of a function  $u$ :

$$\hat{u}(\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} u(x) dx.$$

It is easily seen that

$$(2.11) \quad \mathcal{E}_q^\pm := \phi_q^\pm(D) = \mathcal{F}_{\xi \rightarrow x}^{-1} \phi_q^\pm(\xi) \mathcal{F}_{x \rightarrow \xi}.$$

where  $\mathcal{F}$  is the Fourier transform. We will construct appropriate approximations to each operator on the rightmost part of (2.11), and use (2.11) as the basis for a numerical realization of the main Theorem 2.1.

**2.3. Calculation of value functions using Laplace transform.** We consider options, whose payoff at maturity date  $T$  depends on  $(X_T, \underline{X}_T)$  but not on  $\bar{X}_T$ . This implies that the barrier  $H = e^h$  may play a role only when it is reached or crossed from above. The definitions and results below admit the straightforward reformulation for the case of options with payoffs depending on  $(X_T, \bar{X}_T)$  but not on  $\underline{X}_T$ . In this case, the barrier  $H = e^h$  may play a role only when it is reached or crossed from below.

Consider

$$(2.12) \quad V(T, x) = \mathbb{E}^x \left[ e^{-rT} g(X_T, \underline{X}_T) \mathbb{1}_{\{\tau_h^- > T\}} \right],$$

where time 0 is the beginning of a period under consideration (so that  $X_0 = \underline{X}_0 = x$ ),  $T$  is the final date,  $h$  is the absorbing barrier,  $\tau_h^-$  denotes the first entrance time into  $(-\infty, h]$ , and  $g(X_T, \underline{X}_T)$  is the payoff at time  $T$ .

Denote by  $\hat{V}(q, x)$  the Laplace transform of  $V(T, x)$  w.r.t.  $T$ . Applying Fubini's theorem, we obtain that  $\hat{V}(q, x)$  is the value function of the perpetual stream  $g(X_t, \underline{X}_t)$ , which is terminated the first moment  $X_t$  crosses  $h$ , the discounting factor



being  $q + r$ :

$$\begin{aligned}\hat{V}(q, x) &= \int_0^{+\infty} e^{-qt} \mathbb{E}^x \left[ e^{-rt} g(X_t, \underline{X}_t) \mathbb{1}_{\{\tau_h^- > t\}} \right] dt \\ &= \mathbb{E}^x \left[ \int_0^{\tau_h^-} e^{-(q+r)t} g(X_t, \underline{X}_t) dt \right].\end{aligned}$$

**Theorem 2.1.** *Let  $g$  be a measurable locally bounded function satisfying certain conditions on growth at  $\infty$ . Then*

$$(2.13) \quad \hat{V}(q, x) = (q + r)^{-1} \left( \mathcal{E}_{q+r}^- \mathbb{1}_{(h, +\infty)} w(q + r; \cdot) \right) (x),$$

where

$$(2.14) \quad w(q + r; x) = \mathcal{E}_{q+r}^+ g(x, y)|_{y=x},$$

and  $\mathcal{E}_{q+r}^+$  acts w.r.t. the first argument.

The proof of the Theorem 2.1 can be found in [44].

### 3. SPECIAL CASES AND NUMERICAL METHOD

In this section, we apply Theorem 2.1 to price different types of exotics on an underlying that pays dividends. More details can be found in [44].

We assume that, under an EMM chosen for pricing, the log-price of the stock  $X_t = \log S_t$  follows a Lévy process with the characteristic exponent  $\psi$ . The riskless rate  $r \geq 0$  and dividend rate  $d \geq 0$  are assumed constant.

**3.1. European floating strike lookback options.** In this subsection, we recover the formula for the European floating strike lookbacks from [54] using Theorem 2.1. Assume that the infimum process starts at  $x = X_0$ . Then the time-0 price of the lookback option with payoff  $g(X_T, \underline{X}_T)$  at maturity is given by

$$V(T, x) = \mathbb{E}^x \left[ e^{-rT} g(X_T, \underline{X}_T) \right].$$

This is a particular case of (2.12) with  $h = -\infty$ . Hence, Theorem 2.1 can be applied. Consider a floating strike lookback call. Then  $g(x, y) = e^x - e^y$ . Applying  $\mathcal{E}_{q+r}^+$  w.r.t.  $x$ , we obtain  $w(q + r; x) = \phi_{q+r}^+(-i)e^x - e^x = (\phi_{q+r}^+(-i) - 1)e^x$ , and then

$$\begin{aligned}\hat{V}(q, x) &= (q + r)^{-1} \left( \mathcal{E}_{q+r}^- (\phi_{q+r}^+(-i) - 1)e^\cdot \right) (x) \\ &= (q + r)^{-1} \phi_{q+r}^-(-i) (\phi_{q+r}^+(-i) - 1)e^x.\end{aligned}$$

Using the Wiener-Hopf factorization formula (2.7) with  $q + r$  in place of  $q$ , and the EMM condition (2.1), we arrive at

$$(3.1) \quad \hat{V}(q, x) = \left[ \frac{1}{q + d} - \frac{\phi_{q+r}^-(-i)}{q + r} \right] e^x.$$

The value  $\phi_{q+r}^-(-i)$  can be calculated using an integral representation for  $\ln \phi_{q+r}^-(\xi)$  (see e.g. [14]).

**3.2. European fixed strike lookback put.** In this case,  $g(x, y) = (K - e^y)_+$  is independent of  $x$ ,  $h = -\infty$ , and  $w(q + r; x) = (K - e^x)_+$ . Therefore, equation (2.13) simplifies:

$$\hat{V}(q, x) = (q + r)^{-1} \left( \mathcal{E}_{q+r}^-(K - e^{\cdot})_+ \right) (x).$$

Note that Eberlein and Papapantoleon [28] proved a symmetry relationship between floating-strike and fixed-strike lookback options for assets driven by general Lévy processes.

**3.3. Barrier-lookback options.** In the subsection, we suggest a new class of hybrid exotics: lookbacks with barriers. We will call these options *barrier-lookbacks*. The *barrier-lookback option* expires worthless if the extremum crosses some prefixed barrier during the option's life. Otherwise, the option's owner is entitled to the lookback's payoff  $g(X_T, \underline{X}_T)$  at maturity date  $T$ . We may consider down-and-out or up-and-out lookbacks with a floating (fixed) strike lookback's payoff.

For instance, we apply our approach to a down-and-out floating strike lookback call with the barrier  $H = e^h$ . Without loss of generality, we assume  $h = 0$ . The price  $V(T, x)$  of the option is given by (2.12) with  $g(X_T, \underline{X}_T) = e^{X_T} - e^{\underline{X}_T}$ . We apply Theorem 2.1 and obtain (2.13), where  $w(q; x)$  can be explicitly calculated:

$$(3.2) \quad w(q; x) = \mathcal{E}_{q+r}^+(e^x - e^y)|_{y=x} = \phi_{q+r}^+(-i)e^x - e^x.$$

Using the Wiener-Hopf factorization formula (2.7) with  $q + r$  in place of  $q$ , the EMM condition (2.1) and (3.2), we rewrite (2.13) as follows.

$$\begin{aligned}
 \hat{V}(q, x) &= (q + r)^{-1} \left( \mathcal{E}_{q+r}^- \mathbb{1}_{(0, +\infty)} (\phi_{q+r}^+(-i) - 1) e^\cdot \right) (x) \\
 &= (q + r)^{-1} \left( \mathcal{E}_{q+r}^- (\phi_{q+r}^+(-i) - 1) e^\cdot \right) (x) \\
 &\quad - (q + r)^{-1} \left( \mathcal{E}_{q+r}^- \mathbb{1}_{(-\infty; 0]} (\phi_{q+r}^+(-i) - 1) e^\cdot \right) (x) \\
 (3.3) \quad &= \frac{1}{q + d} e^x - \frac{\phi_{q+r}^-(-i)}{q + r} e^x - \left( \frac{1}{(q + d)\phi_{q+r}^-(-i)} - \frac{1}{q + r} \right) \left( \mathcal{E}_{q+r}^- \mathbb{1}_{(-\infty; 0]} e^\cdot \right) (x).
 \end{aligned}$$

*Remark 3.1.* In a numerical realization, we need to approximate operator  $\mathcal{E}_{q+r}^-$  only. Furthermore, since

$$\phi_{q+r}^-(-i) = (\mathcal{E}_{q+r}^- \mathbb{1}_{(-\infty; 0]} e^\cdot)(0),$$

it is unnecessary to write a separate subprogram for calculation of  $\phi_{q+r}^-(-i)$ .

Now, consider a down-and-out fixed strike lookback put with the barrier  $H = e^h$ . The price  $V(T, x)$  of the option under consideration at time zero is given by (2.12) with the payoff function  $g(X_T, \underline{X}_T) = (K - e^{\underline{X}_T})_+$ . Applying Theorem 2.1, we find the Laplace transform of  $V(T, x)$ :

$$(3.4) \quad \hat{V}(q, x) = (q + r)^{-1} \left( \mathcal{E}_{q+r}^- \mathbb{1}_{(h, +\infty)} (K - e^\cdot)_+ \right) (x).$$

**3.4. Partial barrier-lookback options.** In the current subsection, we combine the features of barrier-lookback options and of a class of partial barrier options introduced in [35]. Consider the option with two maturity dates  $T_1 < T_2$ . Provided that the underlying remains above the barrier level  $H = e^h$  during the early monitoring knock-out window  $[0, T_1]$ , at time  $T_1$ , the option becomes a seasoned lookback option  $g(X_{T_1}, \underline{X}_{T_1})$  with payoff  $g_0(X_{T_2}, \underline{X}_{T_2})$  at time  $T_2$ . We will call these options *partial barrier-lookbacks*.

The price  $V(T_1, T_2; x)$  of the partial knock-out barrier lookback at time zero is given by (2.12) with  $T = T_1$ , where

$$(3.5) \quad g(x, y) = \mathbb{E}_{T_1} \left[ e^{-r(T_2 - T_1)} g_0(X_{T_2}, \underline{X}_{T_2}) | X_{T_1} = x, \underline{X}_{T_1} = y \right].$$

Set  $T = T_2 - T_1$  and simplify the expression (3.5) for  $g(x, y)$  at  $x \geq y$ :

$$\begin{aligned} g(x, y) &= \mathbb{E}^x \left[ e^{-rT} g_0(X_T, \min\{\underline{X}_T, y\}) \right] \\ &= \mathbb{E}^x \left[ e^{-rT} g_0(X_T, \underline{X}_T) \mathbb{1}_{\{\tau_y^- \leq T\}} \right] + \mathbb{E}^x \left[ e^{-rT} g_0(X_T, y) \mathbb{1}_{\{\tau_y^- > T\}} \right] \\ &= \mathbb{E}^x \left[ e^{-rT} g_0(X_T, \underline{X}_T) \right] + \mathbb{E}^x \left[ e^{-rT} (g_0(X_T, y) - g_0(X_T, \underline{X}_T)) \mathbb{1}_{\{\tau_y^- > T\}} \right]. \end{aligned}$$

Hence,  $g(x, y)$  can be represented as follows.

$$(3.6) \quad g(x, y) = V_1(T, x) + V_2(T, x, y),$$

where  $V_1(T, x)$ ,  $V_2(T, x, y)$  are the prices of the lookback and barrier-lookback options with maturity  $T$  and payoffs  $g_0(X_T, \underline{X}_T)$ ,  $g_0(X_T, y) - g_0(X_T, \underline{X}_T)$ , respectively.

Now, we can apply the procedures from the previous subsections to obtain the prices of the partial barrier-lookbacks. An additional ingredient for an efficient numerical realization would be a piece-wise linear or polynomial approximation of  $g$  before the last step of calculation of the price of barrier-lookback options can be made.

*Remark 3.2.* Notice that (3.6) gives the time- $t$  price of a seasoned lookback with the time remaining to expiration  $T$ , conditional on  $X_t = x$  and the minimum of the underlying asset price observed prior to the current time  $t$  is  $e^y$ ,  $y < x$ .

**3.5. Numerical method.** In general Lévy models formulas (3.3) and (3.4) involve the double Fourier inversion (and one more integration needed to calculate one of the factors in the Wiener-Hopf factorization formula). Hence, it is difficult to implement these formulas in practice apart from the cases when explicit formulas for the factors are available. We suggest to use FWHF-method from [42] for calculation of  $\hat{V}(q, x)$  at points  $q$  chosen for the application of the Gaver-Stehfest algorithm. (Appendix A contains a description of the optimized version of the one-dimensional Gaver-Stehfest method.) For this reason, we refer to our algorithm as the “FWHF&GS-method”. The total computational cost of our method is  $O(NM \log_2(M))$ , where  $N$  is the parameter of the Gaver-Stehfest algorithm, and  $M$  is the number of discrete points used to compute the fast Fourier transform in FWHF-method. The short description of the FWHF-method can be found in Appendix B. In [61], [24], [43], it was found

numerically that the choice of 12-14 terms in the Gaver-Stehfest formula results in satisfactory accuracy (5–7 significant digits) for the case of Kou, HEJD and KoBoL models, respectively. Our numerical experiments in Section 4 confirm this statement. In this case, the standard double precision gives reasonable results.

*FWHF&GS algorithm for pricing options with barrier and/or lookback features*

- Using Theorem 2.1, find the Laplace transformed prices  $\hat{V}(q, x)$  at values  $q$  specified by the Gaver-Stehfest algorithm. Action of operators  $\mathcal{E}_{q+r}^{\pm}$  in (2.13) and (2.14) can be realized using FWHF-method.
- Use the Gaver-Stehfest inversion formula.

#### 4. NUMERICAL EXAMPLES

We check the performance of FWHF&GS-method against prices obtained by Monte Carlo simulation (MC-method). It is well known that the convergence of Monte Carlo estimators of quantities involving first passage is very slow. The recent exception is the Wiener-Hopf Monte-Carlo simulation technique developed in [46]. The method is numerically tractable for families of Levy processes whose Lévy measure can be written as a sum of a Lévy measure from the  $\beta$ -family [45] or hypergeometric family [46] and a measure of finite mass. However, class of KoBoL processes is not a subclass of any of these two families. Hence, we simulate trajectories of KoBoL using the code of J. Poirot and P. Tankov ([www.math.jussieu.fr/~tankov/](http://www.math.jussieu.fr/~tankov/)). The program uses the algorithm in [52], see also [58]. We used 3, 000, 000 paths with time step 0.00005. The option prices were calculated on a PC with characteristics Intel Core(TM)2 Due CPU, 1.8GHz, RAM 1024Mb, under Windows Vista.

We consider two types of lookback options under the KoBoL(CGMY) model (2.3), and use the same parameters of the KoBoL (CGMY) process as in [29]:  $c = 4$ ,  $\lambda_+ = 50.0$ ,  $\lambda_- = -60.0$ ,  $\nu = 0.7$  ( $C = 4$ ,  $G = 50.0$ ,  $M = 60.0$ ,  $Y = 0.7$  in CGMY parametrization); the remaining parameters are time to maturity  $T = 1$  and the dividend rate  $d = 0.02$  and interest rate  $r = 0.05$ . The drift parameter  $\mu$  is fixed by

(2.2), according to the EMM-requirement (2.1). The examples, which we analyze in detail below, are fairly representative.

For verification of the accuracy of our method, we calculate prices of European lookbacks without barrier. Table 1 reports prices for the floating strike lookback put calculated by using MC-Method, FWHF&GS-method and HT method [30] (HT-method). The HT-prices were obtained using the code implemented into the program platform Premia [56]. FWHF&GS-prices converge very fast and agree with MC-prices very well. However, discrete lookback option prices calculated by HT-method converge to the continuous lookback option prices rather slowly. If HT-method is expanded for the case of the barrier-lookbacks, we may expect the same behavior of the prices.

Next, we consider a barrier-lookback with a down-and-out barrier  $H$ , a floating strike lookback's payoff and time to expiry  $T$ . We will refer to this option as a down-and-out floating strike lookback call. In Table 2, the sample mean values are compared with the prices of FWHF&GS method.

#### APPENDIX A. A NUMERICAL LAPLACE TRANSFORM INVERSION: THE GAVER-STEHFEST ALGORITHM

Methods of numerical Laplace inversion that fit the framework of [1] have the following general feature: the approximate formula for  $f(\tau)$  can be written as

$$(A.1) \quad f(\tau) \approx \frac{1}{\tau} \sum_{k=1}^N \omega_k \cdot \tilde{f}\left(\frac{\alpha_k}{\tau}\right), \quad 0 < \tau < \infty,$$

where  $N$  is a positive integer and  $\alpha_k$ ,  $\omega_k$  are certain constants that are called the *nodes* and the *weights*, respectively. They depend on  $N$ , but not on  $f$  or on  $\tau$ . In particular, the inversion formula of the Gaver-Stehfest method can be written in the form (A.1) with  $\alpha_k = k \ln(2)$ ,

$$(A.2) \quad N = 2n;$$

$$(A.3) \quad \omega_k := \frac{(-1)^{n+k} \ln(2)}{n!} \sum_{j=[(k+1)/2]}^{\min\{k,n\}} j^{n+1} C_n^j C_{2j}^j C_j^{k-j},$$

TABLE 1. KoBoL model: floating strike lookback put prices

<b>A</b>				
	MC	FWHF&GS		
		$h = 0.01$	$h = 0.001$	$h = 0.0002$
Option prices	14.2693	14.4380	14.2880	14.2675
Errors w.r.t. MC	0.1%	1.2%	0.1%	0.0%
CPU-time (sec)	100,000	0.003	0.031	0.156

  

<b>B</b>				
	MC	HT		
		$N = 320$ $M = 4096$	$N = 640$ $M = 8192$	$N = 1280$ $M = 32768$
Option prices	14.2693	13.94	14.07	14.17
Errors w.r.t. MC	0.1%	-2.3%	-1.4%	-0.7%
CPU-time (sec)	100,000	28	129	1117

KoBoL parameters:  $c = 4$ ,  $\lambda_+ = 50.0$ ,  $\lambda_- = -60.0$ ,  $\nu = 0.7$ ,  $\mu = 0.207142$ .

Option parameters:  $S = 100$ ,  $T = 1$ ,  $r = 0.05$ ,  $d = 0.02$ .

Parameters of the FWHF&GS-method :  $h$  – space step,  $N = 14$ .

Parameters of the HM-method :  $N$  – the number of observations of the maximum,  $M$  – the number of discrete points used to compute the Hilbert transform.

Panel A: Option prices

Panel B: Relative errors w.r.t. MC; MC error indicates the ratio between the half-width of the 95% confidence interval and the sample mean.

where  $[x]$  is the greatest integer less than or equal to  $x$  and  $C_L^K = \frac{L!}{(L-K)!K!}$  are the binomial coefficients.

Because of the binomial coefficients in the weights, the Gaver-Stehfest algorithm tends to require high system precision in order to yield good accuracy in the calculations. If  $M$  significant digits are desired, then according to [63], the parameter  $n$  in (A.2) should be the least integer greater than or equal to  $1.1M$ ,  $N = 2n$ , and the required system precision is about  $1.1N$ . In particular, for  $M = 6$  and  $N = 14$  a standard double precision gives reasonable results. The precision requirement is driven by the coefficients  $\omega_k$  in (A.3).

Since constants  $\omega_k$  do not depend on  $\tau$ , they can be tabulated for the values of  $N$  that are commonly used in computational finance (e.g., 12 or 14).

TABLE 2. KoBoL model: down-and-out floating strike lookback call prices

**A**

	MC	FWHF&GS			
Spot price	Sample mean	$h = 0.01$	$h = 0.001$	$h = 0.0002$	$h = 0.0001$
$S = 81$	1.73270	1.40919	1.69624	1.72731	1.73151
$S = 90$	9.14148	8.99476	9.15197	9.16768	9.16946
$S = 100$	13.66470	13.6559	13.6864	13.6869	13.6864
$S = 110$	16.34670	16.4432	16.3807	16.3707	16.3688
CPU-time (sec)	100,000	0.003	0.031	0.156	0.31

**B**

	MC	FWHF&GS			
Spot price	MC error	$h = 0.01$	$h = 0.001$	$h = 0.0002$	$h = 0.0001$
$S = 81$	0.5%	-18.7%	-2.1%	-0.3%	-0.1%
$S = 90$	0.2%	-1.6%	0.1%	0.3%	0.3%
$S = 100$	0.1%	-0.1%	0.2%	0.2%	0.2%
$S = 110$	0.1%	0.6%	0.2%	0.1%	0.1%
CPU-time (sec)	100,000	0.003	0.031	0.156	0.31

KoBoL parameters:  $c = 4$ ,  $\lambda_+ = 50.0$ ,  $\lambda_- = -60.0$ ,  $\nu = 0.7$ ,  $\mu = 0.207142$ .

Option parameters:  $S$  – spot price,  $H = 80$ ,  $T = 1$ ,  $r = 0.05$ ,  $d = 0.02$ .

Algorithm parameters:  $h$  – space step,  $N = 14$ .

Panel A: Option prices

Panel B: Relative errors w.r.t. MC; MC errors indicate the ratio between the half-width of the 95% confidence interval and the sample mean.

## APPENDIX B. THE FAST WIENER-HOPF FACTORIZATION METHOD

We briefly review the framework proposed in [42]. The main contribution of the FWHF-method is an efficient numerical realization of EPV-operators  $\mathcal{E}$ ,  $\mathcal{E}^+$  and  $\mathcal{E}^-$ . As the initial step, we need an efficient procedure for calculation of the Wiener-Hopf factors  $\phi_q^\pm(\xi)$ .

It is well-known that the limit of a sequence of the Poisson type characteristic functions is an infinitely divisible characteristic function. The converse is also true. Every infinitely divisible characteristic function can be written as the limit of a sequence of finite products of Poisson type characteristic functions. Since  $\psi(\xi)$  is the characteristic exponent of Lévy process, then the function  $q/(q + \psi(\xi))$  is infinitely divisible characteristic function.



## 5. IMPLEMENTATION TO THE PREMIA 14

We implemented FWHF-method for four types of lookback options:

- European floating strike lookback put with a prefixed maximum;
- European fixed strike lookback put with a prefixed minimum;
- European floating strike lookback call with a prefixed minimum;
- European fixed strike lookback call with a prefixed maximum;

The method is implemented for two models:

- CGMY model (see Example 2.1);
- Kou model (see Example 2.2).

Note that in the program implemented into Premia 14 one can manage by two parameters of the algorithm: the space step  $d$  and the scale of logprice range  $L$ . Parameter  $L$  controls the size of the truncated region in  $x$ -space; it corresponds to the region  $(-L \ln(2)/d; L \ln(2)/d)$ . The typical values of the parameter are  $L = 1$ ,  $L = 2$  and  $L = 4$ . To improve the results one should decrease  $d$ , when  $L$  is fixed.

**A**

	MC	FWHF&GS			
Spot price	Sample mean	$h = 0.01$	$h = 0.001$	$h = 0.0002$	$h = 0.0001$
$S = 81$	1.05121	0.84865	1.02957	1.04854	1.05112
$S = 90$	6.16135	6.00966	6.15043	6.16031	6.16138
$S = 100$	9.27596	9.24288	9.28138	9.27738	9.27651
$S = 110$	10.9063	10.95320	10.91950	10.90330	10.90130
CPU-time (sec)	100,000	0.008	0.078	0.39	0.78

**B**

	MC	FWHF&GS			
Spot price	MC error	$h = 0.01$	$h = 0.001$	$h = 0.0002$	$h = 0.0001$
$S = 81$	0.4%	-19.3%	-2.1%	-0.3%	0.0%
$S = 90$	0.1%	-2.5%	-0.2%	0.0%	0.0%
$S = 100$	0.1%	-0.4%	0.1%	0.0%	0.0%
$S = 110$	0.1%	0.4%	0.1%	0.0%	0.0%
CPU-time (sec)	100,000	0.008	0.078	0.39	0.78

KoBoL parameters:  $c = 4$ ,  $\lambda_+ = 50.0$ ,  $\lambda_- = -60.0$ ,  $\nu = 0.7$ ,  $\mu = 0.207142$ .

Option parameters:  $S$  – spot price,  $H = 80$ ,  $T_1 = 0.5$ ,  $T_2 = 1$ ,  $r = 0.05$ ,  $d = 0.02$ .

Algorithm parameters:  $h$  – space step,  $N = 14$ .

Panel A: Option prices

Panel B: Relative errors w.r.t. MC; MC errors indicate the ratio between the half-width of the 95% confidence interval and the sample mean.

We approximate  $q/(q + \psi(\xi))$  by a periodic function  $\Phi$  with a large period  $2\pi/h$ , which is the length of the truncated region in the frequency domain, then approximate

the latter by a partial sum of the Fourier series, and, finally, use the factorization of the latter instead of the exact one.

Explicit formulas for approximations of  $\phi_q^\pm$  have the following form. For small positive  $h$  and large even  $M$ , set

$$\begin{aligned} b_k^h &= \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \ln \Phi(\xi) e^{-i\xi kh} d\xi, \quad k \neq 0, \\ b_{h,M}^+(\xi) &= \sum_{k=1}^{M/2} b_k^h (\exp(i\xi kh) - 1), \quad b_{h,M}^-(\xi) = \sum_{k=-M/2+1}^{-1} b_k^h (\exp(i\xi kh) - 1); \\ \phi_q^\pm(\xi) &\approx \exp(b_{h,M}^\pm(\xi)). \end{aligned}$$

We can also apply this realization after the reduction to symbols of order 0 has been made (see details in [42]). Numerical experiments show that there is no sufficient difference between both realizations.

Approximants for EPV-operators can be efficiently computed using the Fast Fourier Transform (FFT) for real-valued functions. Consider the algorithm of the discrete Fourier transform (DFT) defined by

$$G_l = DFT[g](l) = \sum_{k=0}^{M-1} g_k e^{2\pi i k l / M}, \quad l = 0, \dots, M-1.$$

The formula for the inverse DFT which recovers the set of  $g_k$ 's exactly from  $G_l$ 's is:

$$g_k = iDFT[G](k) = \frac{1}{M} \sum_{l=0}^{M-1} G_l e^{-2\pi i k l / M}, \quad k = 0, \dots, M-1.$$

In our case, the data consist of a real-valued array  $\{g_k\}_{k=0}^M$ . The resulting transform satisfies  $G_{M-l} = \bar{G}_l$ . Since this complex-valued array has real values  $G_0$  and  $G_{M/2}$ , and  $M/2 - 1$  other independent complex values  $G_1, \dots, G_{M/2-1}$ , then it has the same “degrees of freedom” as the original real data set. In this case, it is efficient to use FFT algorithm for real-valued functions (see [57] for technical details). To distinguish DFT of real functions we will use notation RDFT.

Fix the space step  $h > 0$  and number of the space points  $M = 2^m$ . Define the partitions of the normalized log-price domain  $[-\frac{Mh}{2}; \frac{Mh}{2})$  by points  $x_k = -\frac{Mh}{2} + kh$ ,  $k = 0, \dots, M-1$ , and the frequency domain  $[-\frac{\pi}{h}; \frac{\pi}{h}]$  by points  $\xi_l = \frac{2\pi l}{hM}$ ,  $l = -M/2, \dots, M/2$ . Then the Fourier transform of a function  $g$  on the real line can be

approximated as follows:

$$\hat{g}(\xi_l) \approx h e^{i\pi l} \overline{RDFT[g](l)}, \quad l = 0, \dots, M/2.$$

Here and below,  $\bar{z}$  denotes the complex conjugate of  $z$ , and  $.*$  is the element-wise multiplication of arrays that represent the functions. Using the notation  $p(\xi) = q(q + \psi(\xi))^{-1}$ , we can approximate  $\mathcal{E}_q$ :

$$(\mathcal{E}_q g)(x_k) \approx iRDFT[\bar{p}.*RDFT[g]](k), \quad k = 0, \dots, M-1.$$

Next, we define

$$\begin{aligned} b_k^h &= iRDFT[\ln \Phi](k), \quad k = 0, \dots, M-1, \\ p^\pm(\xi_l) &= \exp(b_{h,M}^\pm(\xi_l)), \quad l = -M/2, \dots, 0. \end{aligned}$$

The action of the EPV-operator  $\mathcal{E}_q^\pm$  is approximated as follows:

$$(\mathcal{E}_q^\pm g)(x_k) = iRDFT[\overline{p^\pm}.*RDFT[g]](k), \quad k = 0, \dots, M-1.$$

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