

Title : Double Heston Fourier transform

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We consider the following model :

$$\begin{cases} \frac{dS_t}{S_t} &= (r - \delta)dt + \sum_{j=1}^2 \sqrt{V_t^j} \left(\rho_j dW_t^j + \sqrt{1 - \rho_1^2} dB_t^1 \right) \\ dV_t^j &= b_j(\theta_j - V_t^j)dt + \sigma_j \sqrt{V_t^j} dW_t^j \end{cases} \quad j = 1, 2 \quad (1)$$

1 Characteristic function

Let $X_t = \ln(e^{-(r-\delta)t} S_t)$, then we want to compute the following expression (when finite) for any $k \in \mathbb{C}$:

$$\psi(k, t, x, V_0^1, V_0^2) = \mathbb{E}_{(x, v_0, v_1)} [e^{ikX_t}] . \quad (2)$$

On one hand, thanks to [DFS03], we know that the model is affine, and that the characteristic function has the following form :

$$\psi(k, t, x, V_0^1, V_0^2) = e^{ik_1 x + A(k; t) + B_1(k; t) V_0^1 + B_2(k; t) V_0^2} \quad (3)$$

where $A(k; 0) = B_1(k; 0) = B_2(k; 0) = 0$.

On the other hand, we know by Ito's formula that $\partial_t \psi = L\psi$ where L is the operator associated to the three-dimensionnal affine process (X, V^1, V^2) and that $\psi(k, 0, x, V_0^1, V_0^2) = e^{ikx}$.

Proceeding by identification with respect to x, V_0^1, V_0^2 , we obtain Ricatti equations on B_1, B_2 and a trivial ODE on A . Fortunately, these equations are solvable. By introducing for $j = 1, 2$:

$$\begin{aligned} d_j &= \sqrt{(b_j - i\rho_j \sigma_j k)^2 + \sigma_j^2 k(k + i)} \\ G_j &= \frac{b_j - i\rho_j \sigma_j k - d_j}{b_j - i\rho_j \sigma_j k + d_j}, \end{aligned} \quad (4)$$

we get :

$$\begin{aligned} A(k; t) &= \sum_{j=1}^2 \frac{b_j \theta_j}{\sigma_j^2} \left[(b_j - i\rho_j \sigma_j k - d_j) t - 2 \ln \left(\frac{1 - G_j e^{-d_j t}}{1 - G_j} \right) \right] \\ B_j(k; t) &= \frac{b_j - i\rho_j \sigma_j k - d_j}{\sigma_j^2} \frac{1 - e^{-d_j t}}{1 - G_j e^{-d_j t}} \quad \text{for } j = 1, 2 \end{aligned} \quad (5)$$

where all the complex functions are to be understood as their principal branch. I.e if for some $a, b \in \mathbb{R}$, $z = e^{a+ib}$ then $\sqrt{z} = e^{\frac{a+ib}{2}}$ and if moreover $b \in]-\pi, \pi]$, then $\ln(z) = a + ib$.

2 Fourier pricing

Following [GP10] and [CM99] and [R. 04], for any $\tilde{K} \in \mathbb{R}_+^*$ we compute the Fourier transform of the following both integrable and square integrable function

$x \in \mathbb{R} \mapsto ((e^x - \tilde{K})^+ - e^x)e^{-\frac{x}{2}} \in \mathbb{R}$. We obtain :

$$((e^x - \tilde{K})^+ - e^x)e^{-\frac{x}{2}} = \int_{\mathbb{R}} e^{i2\pi\xi x} \frac{e^{(\frac{1}{2} + i2\pi\xi) \ln \tilde{K}}}{-\frac{1}{4} - (2\pi\xi)^2} d\xi. \quad (6)$$

Since we have :

$$\mathbb{E} [e^{-rT} (S_T - K)^+] = e^{-\delta T} \mathbb{E}_{\ln S_0, V_0^1, V_0^2} \left[\left(e^{X_T} - K e^{-(r-\delta)T} \right)^+ \right]. \quad (7)$$

and

$$\mathbb{E} [e^{-rT} S_T] = S_0 e^{-\delta T} = e^{-\delta T} \mathbb{E}_{\ln S_0, V_0^1, V_0^2} [e^{X_T}]. \quad (8)$$

Using Fubini's theorem, we get the following pricing formula :

$$\begin{aligned} C &= \mathbb{E} [e^{-rT} (S_T - K)^+] \\ &= S_0 e^{-\delta T} + e^{-\delta T} \int_{\mathbb{R}} \psi \left(2\pi\xi - \frac{i}{2}, T, \ln S_0, V_0^1, V_0^2 \right) \frac{e^{(-(r-\delta)T + \ln K)(\frac{1}{2} + i2\pi\xi)}}{-\frac{1}{4} - (2\pi\xi)^2} d\xi. \end{aligned}$$

Endly, we use the Gauss Kronrod procedure implemented in the Premia Numerical Library to compute the integral involving in this formula.

References

- [CM99] P. Carr and D. B. Madan. Option valuation using the fast Fourier transform. *J. Comput. Finance*, 2(4):61–73, 1999. [1](#)
- [DFS03] D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *Ann. Appl. Probab.*, 13:984–1053, 2003. [1](#)
- [GP10] P Gauthier and D Possamai. Efficient simulation of the double heston model. Technical report, Daiwa Capital Markets and Ecole Polytechnique, January 2010. [1](#)
- [R. 04] R. W. Lee. Option pricing by transform methods: extensions, unification and error control. *Journal of Computational Finance*, 7, 2004. [1](#)