Fourier-Cosine Method for Pricing Butterfly Options in Uncertain Volatility Model: Implementation in PREMIA

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Abstract

Applying the pricing method based on Fourier-Cosine series expansion (COS) proposed in [3], we implement the algorithms for pricing butterfly options under the uncertain volatility model. The idea is to formulate the pricing problem into a stochastic control problem and then apply the Fourier-Cosine series expansion method to solve the stochastic control problem.

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1 Butterfly Option under Uncertain Volatility Model

We assume that the dynamics of the asset price follows the uncertian volatility model:

$$\mathrm{d}S_s = (r-d)S_s\mathrm{d}s + \alpha_s S_s \mathrm{d}W_s, \quad S_0 \quad \text{given.} \tag{1}$$

$(\alpha_s)_{0\le s\le T}$ is an uncertain volatility process, which is valued in the interval $[a^-, a^+]$ and r is the risk-neutral interest rate, d is the dividend. The payoff function of the butterfly options with strike K_1 and K_2 at maturity T is

$$g(S_T) = (S_T - K_1)^+ - 2\left(S_T - \frac{K_1+K_2}{2}\right)^+ + (S_T-K_2)^+. \tag{2}$$

The price of butterfly option under the uncertain volatility model is the worst case of expected payoff for an investor with a long position of this option, that is

$$v(t,S) = \inf_{\alpha\in[\alpha^-,\alpha^+]} J(t,S,\alpha) = \inf_{\alpha\in[\alpha^-,\alpha^+]} \mathbb{E}\left[\mathrm{e}^{-r(T-t)}g(S_T)\right]. \tag{3}$$

Thus the problem of pricing butterfly options under uncertain volatility model converts to the stochastic control problem where the value function is $v(t,S)$ and taking the infimum over the gain function $J(t,S,\alpha) = \mathbb{E}\left[\mathrm{e}^{-r(T-t)}g(S_T)\right]$.

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2 Dynamics Programming Principle

The optimal control problem can be solved backward recursively by the Bellman's optimality principle, which is also called the dynamic programming priciple. This priciple shows that if one has taken an optimal control path until some arbitrary observation time θ, given this information, it remains optimal ot use it after that observation time, which is stated as follows:

Remark 1. *Let $(t, x) \in [0, T] \times \mathbb{R}^n$. Then we have*

$$v(t, x) = \sup_{\alpha \in [\alpha^-, \alpha^+]} \mathbb{E}\left[e^{-\rho(\theta - t)} v(\theta, S_\theta^{t,x})\right], \tag{4}$$

for any stopping time $\theta \in [t, T]$ and $S_\theta^{t,x}$ stands for the asset price at stopping time θ given $S_t = x$.

Remark 2. *By the dynamic programming principle, on can derive the well-known Hamilton-Jacobi-Bellman (HJB) equation corresponding to the problem (3).*

$$-\frac{\partial v}{\partial t}(t, S) + rv(t, S) - \min_{\alpha \in [\alpha^-, \alpha^+]} \left[rS \frac{\partial v}{\partial S}(t, S) + \frac{1}{2}\alpha^2 S^2 \frac{\partial^2 v}{\partial S^2}(t, S) \right] = 0, \tag{5}$$

$\forall (t, S) \in [0, T) \times \mathbb{R}_+$, *which yields*

$$\begin{cases} \textit{if} \quad \frac{\partial^2 v}{\partial S^2} \leq 0 \quad \Rightarrow \quad \textit{take} \quad \alpha = \alpha^+ \\ \textit{if} \quad \frac{\partial^2 v}{\partial S^2} > 0 \quad \Rightarrow \quad \textit{take} \quad \alpha = \alpha^-. \end{cases} \tag{6}$$

This means that the infimum achieves either when $\alpha = \alpha^-$ or when $\alpha = \alpha^+$, so that it allows us to restirct the set of possible control values to $A = \{a^-, a^+\}$.

3 COS method for stochastic control problem

To solve the stochastic control problem (3) by dynamic programming priciple, the time interval $[t, T]$ is divided into M grids: $t = t_0 < t_1 < \cdots < t_M = T$, with $\triangle t := t_m - t_{m-1}$. In each time grid $[t_m, t_{m+1})$, a constant volatility $a_m \in \{a^-, a^+\}$ is taken such that the gain function is minimized. The dynamic programming priciple is used to determine the value function backward recursively by the following equation

$$\begin{aligned} v(t_{m-1}, x) &= \min_{\alpha_{m-1} \in \{\alpha^-, \alpha^+\}} \mathrm{e}^{-\rho \triangle t} \mathbb{E}[v(t_m, X_{t_m}) | X_{t_{m-1}} = x, \alpha_{m-1}] \\ &= \min[c(t_{m-1}, x, \alpha^-), c(t_{m-1}, x, \alpha^+)], \end{aligned} \tag{7}$$

where $X_s := \log(S_s), s \in [t, T]$ and the continuation value

$$\begin{aligned} c(t_{m-1}, x, \alpha) &= \mathrm{e}^{-\rho \triangle t} \mathbb{E}[v(t_m, X_{t_m}) | X_{t_{m-1}} = x, \alpha_{m-1} = \alpha] \\ &= \mathrm{e}^{-\rho \triangle t} \int_{\mathbb{R}} v(t_m, y) f(y \mid x, \alpha) \mathrm{d}y. \end{aligned} \tag{8}$$

The numerical method is based on the cosine series expansions of the value function at the next time level and the density function. The resulting equation is called the COS formula, due to the use of Fourier-cosine series expansions. Fourier series expansions and their convergence properties have been discussed in [1]. The conditional density function $f(y \mid x, \alpha)$ decays to zero rapidly as $y \to \pm\infty$, so that we can, for given x, truncate the infinite integration range of the expectation to some interval $[a, b] \subset \mathbb{R}$ without loosing significant accuracy. This gives the approximation

$$\begin{aligned} c_1(t_{m-1}, x, \alpha) &= \mathrm{e}^{-\rho\triangle t} \int_a^b v(t_m, y) f(y \mid x, \alpha) \mathrm{d}y \\ &= \mathrm{e}^{-\rho\triangle t} \int_a^b v(t_m, y) \sum_{k=0}^{+\infty}{}' G_k(x, \alpha) \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y \end{aligned} \tag{9}$$

In the second equation in (9), the conditional density is replaced by its Fourier cosine expansion in y on $[a, b]$, with series coefficients $\{G_k(x, \alpha)\}_{k=0}^{\pm\infty}$ defined by

$$\begin{aligned} G_k(x, \alpha) &:= \frac{2}{b-a} \int_a^b f(y \mid x, \alpha) \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y \\ &\approx \frac{2}{b-a} \mathrm{Re}\left(\psi\left(\frac{k\pi}{b-a} \mid x, \alpha\right) \mathrm{e}^{-ik\pi \frac{a}{b-a}}\right). \end{aligned} \tag{10}$$

$\sum{}'$ in (9) indicates that the first term in the summation is weighted by one-half and $\psi(u|x, \alpha)$ in (10) is the conditional characteristic function of the logarithm of the asset price X_{t_m} given $X_{t_{m-1}} = x$. We interchange summation and integration and define:

$$V_k(t_m) := \frac{2}{b-a} \int_a^b v(t_m, y) \cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y \tag{11}$$

which are the Fourier cosine series coefficients of $v(t_m, y)$ on $[a, b]$. Then the continuation value can be approximated as

$$\begin{aligned} c(t_{m-1}, x, \alpha) &\approx \mathrm{e}^{-\rho\triangle t} \sum_{k=0}^{N-1}{}' \mathrm{Re}\left(\psi_{levy}\left(\frac{k\pi}{b-a} | \alpha\right) \mathrm{e}^{ik\pi \frac{x-a}{b-a}}\right) V_k(t_m) \\ &:= \hat{c}(t_{m-1}, x, \alpha). \end{aligned} \tag{12}$$

The characteristic function $\psi_{levy}(u|\alpha)$ is the stationary increment of X_s in time interval $\triangle t$, which is $\psi_{levy}(u|\alpha) = \exp(iu(r - d - \frac{1}{2}\alpha^2))\triangle t - \frac{1}{2}u^2\alpha^2\triangle t)$. Note that it depends on the volatility α but independent of the initial value of x.

3.1 Recursion formula for coefficients V_k

The algorithm for solving the stochastic control problems is based on the recursive recovery of the coefficients V_k, starting with the coefficients at the terminal time:

$$
\begin{aligned}
& V_k(t_M) \\
= \; & \frac{2}{b-a}\int_a^b v(T,y)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y = \frac{2}{b-a}\int_a^b g(\mathrm{e}^y)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \\
= \; & \frac{2}{b-a}\Bigg[\chi_k(\log K_1,b,a,b) - 2\chi_k\left(\frac{K_1+K_2}{2},b,a,b\right) + \chi_k(\log K_2,b,a,b) \\
& -K_1\varphi_k(\log K_1,b,a,b) + (K_1+K_2)\varphi_k\left(\frac{K_1+K_2}{2},b,a,b\right) - K_2\varphi_k(\log K_2,b,a,b)\Bigg],
\end{aligned}
$$

where $\chi_k(l,u,a,b)$ and $\varphi_k(l,u,a,b)$ are given as follows:

$$
\begin{aligned}
& \chi_k(l,u,a,b) := \int_l^u \mathrm{e}^x\cos\left(k\pi\frac{x-a}{b-a}\right)\mathrm{d}x \\
= \; & \frac{1}{1+\left(\frac{k\pi}{b-a}\right)^2}\Bigg[\cos\left(k\pi\frac{u-a}{b-a}\right)\mathrm{e}^u - \cos\left(k\pi\frac{l-a}{b-a}\right)\mathrm{e}^l \\
& \frac{k\pi}{b-a}\sin\left(k\pi\frac{u-a}{b-a}\right)\mathrm{e}^u - \frac{k\pi}{b-a}\sin\left(k\pi\frac{l-a}{b-a}\right)\mathrm{e}^l\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
& \varphi_k(l,u,a,b) := \int_l^u \cos\left(k\pi\frac{x-a}{b-a}\right)\mathrm{d}x \\
= \; & \begin{cases} \left[\sin\left(k\pi\frac{u-a}{b-a}\right) - \sin\left(k\pi\frac{l-a}{b-a}\right)\right]\frac{b-a}{k\pi}, & k \neq 0 \\ u-l, & k = 0. \end{cases}
\end{aligned}
$$

The coefficients $V_k(t_M), k = 0,\cdots,N$ are used for the approximation of the continuation value at time t_{M-1}. The other time level of the Fourier coefficients $\hat{V}_k(t_m)$ are approximated by

$$
\hat{V}_k(t_m) = \frac{2}{b-a}\int_a^b \min[\hat{c}(t_m,y,\alpha^-),\hat{c}(t_m,y,\alpha^+)]\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y, \tag{13}
$$

where $m = 1,\cdots,M-1$. To calculate $\hat{V}_k(t_m)$, we divide the integration interval $[a,b]$ into sub-domains $\mathcal{D}_m^-$ and $\mathcal{D}_m^+$, for which the optimal control values at control time t_m are α_m^- and α_m^+, respectively:

$$
\begin{aligned}
\hat{V}_k(t_m) \;=\; & \frac{2}{b-a}\int_{\mathcal{D}_m^-}\hat{c}(t_m,y,\alpha^-)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \\
& +\frac{2}{b-a}\int_{\mathcal{D}_m^+}\hat{c}(t_m,y,\alpha^+)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y \\
:= \; & \hat{C}_k(t_m,\mathcal{D}_m^-,\alpha^-) + \hat{C}_k(t_m,\mathcal{D}_m^+,\alpha^+)
\end{aligned} \tag{14}
$$

On the integrands of terms $\hat{C}_k$ we can apply again the Fourier cosine series expansion by inserting equation (12):

$$\begin{aligned}
&\hat{C}_k(t_m, z_1, z_2, \alpha) \qquad (15)\\
= &\frac{2}{b-a}\int_{z_1}^{z_2} \hat{c}(t_m, y, \alpha)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y\\
= &\mathrm{e}^{-\rho\triangle t}\frac{2}{b-a}\int_{z_1}^{z_2}\left(\sum_{j=0}^{N-1}{}'\mathrm{Re}\left(\psi_{levy}\left(\frac{j\pi}{b-a}\mid\alpha\right)\mathrm{e}^{ij\pi\frac{y-a}{b-a}}\right)V_j(t_{m+1})\right)\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y\\
= &\mathrm{e}^{-\rho\triangle t}\mathrm{Re}\left(\sum_{j=0}^{N-1}{}'\psi_{levy}\left(\frac{j\pi}{b-a}\mid\alpha\right)V_j(t_{m+1})M_{k,j}(z_1, z_2)\right)
\end{aligned}$$

where z_1, z_2 are the bounds of integration domain $\mathcal{D}^-$ or $\mathcal{D}^+$ and the elements of matrix $M(z_1, z_2)$ are given by:

$$M_{k,j}(z_1, z_2) := \frac{2}{b-a}\int_{z_1}^{z_2} e^{ij\pi\frac{y-a}{b-a}}\cos\left(k\pi\frac{y-a}{b-a}\right)\mathrm{d}y. \qquad (16)$$

Finally, we end up with the vector form

$$\hat{V}(t_m) = \mathrm{e}^{-\rho\triangle t}\mathrm{Re}(M(\mathcal{D}_m^-)w^-) + \mathrm{e}^{-\rho\triangle t}\mathrm{Re}(M(\mathcal{D}_m^+)w^+) \qquad (17)$$

where $w^q = \{w_j^q\}_{j=0}^{N-1}, q = \{-,+\}$ with

$$w_j^q = \psi\left(\frac{j\pi}{b-a}\mid\alpha_m^q\right)\hat{V}_j(t_{m+1}), \quad w_0^q = \frac{1}{2}\psi(0\mid\alpha_m^q)\hat{V}_0(t_{m+1}).$$

The parameters of the matrices M are the boundary values of their respective integration ranges.

Remark 3. *(Efficient Computation of $\hat{C}(t_m, z_1, z_2, \alpha)$)*
The matrix-vector product $M(z_1, z_2)w$ can be computed in $O(N\log_2 N)$ operations, with the help of the Fast Fourier Transform (FFT) algorithm.

The key insight of this efficient computation is the equality

$$M_{k,j}(z_1, z_2) = -\frac{i}{\pi}(M_{k,j}^c(z_1, z_2) + M_{k,j}^s(z_1, z_2)),$$

where matrix M^c is a Hankel matrix and M^s is a Toeplitx matrix. The special matrix structure enables us to use the FFT algorithm for the matrix-verctor products. More details on this matrix-vector product is included in [2].

Applying FFT to recover $\hat{V}_1$ from $\hat{V}_M$ backward recursively, then substituting $\hat{V}_1$ into (12), the continuation value $c(t_0, x, \alpha)$ is derived, hence the option price $v(t_0, x)$ is obtained by (7).

3.2 Algotithm

We can recover the terms $\hat{V}_k(t_m)$ recursively, starting with $\hat{V}_k(t_M)$, to $\hat{V}_k(t_1)$. The algorithm to solve the discrete-time stochastic control problem (3) backwards in time then reads:

Algorithm 1. *(COS method for stochastic control problems)*
Initialization:
Calculate coefficients $V_k(t_M)$ for $k = 0, 1, \cdots, N-1$.
Main loop to recover $\{\hat{V}_k(t_m)\}_{k=0,\cdots,N-1}$: For $m = M-1$ to 1:

- Determine the sub-domains $\mathcal{D}_m^q$ for which the optimal control value is α_m^q
- Compute $\hat{V}_k(t_m)$ by equations (14) and (15), with the help of the FFT algorithm.

Final step:
Compute $v(t_0, x)$ by inserting $\hat{V}_k(t_1)$ into equation (8) and with (7).

The computational complexity of the algorithm is $O(MKN \log_2 N)$, as we need to computer M time steps, and K sub-domains.

4 Program Manual

We implement the Butterfly options pricing by Fourier Cosine expansion. The program HAS TO work with the pnl library.

Model Parameters:
sigma_min: the minimum volatility of uncertain volatility model, α^- in (3).
sigma_max: the maximum volatility of uncertain volatility model, α^+ in (3).

Parameters of the product:
S0: the initial value of stock price.
k1: strike K_1 of the Butterfly option.
k2: strike K_2 of the Butterfly option.
T: the maturity of the Butterfly option.
r: the discount interest rate.
divident: the payout dividend.

Parameters for Fourier-Cosine method:
N: number of Fourier-Cosine series, N in (9).
M: numbers of grids in time interval $[t, T]$.

References

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