# Credit Derivatives

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# 1 The Student Copula

## 1.1 Preliminaries

The default indicator variable of the student copula and the conditional probabilities of the default time of this copula are given by :

$$X_i = \sqrt{W} Z_i,$$

Where $Z_i = \rho V + \sqrt{1-\rho^2} V_i$ and $\frac{\nu}{W} = \chi^2_\nu$, V , $V_1, ....V_M$ are $M+1$ independent standard gaussians and independent of $W$.

**Conditional probability with respect to** $(V, W)$ :

$$P(\tau_i < t/(V,W)) = \Phi(\frac{W^{\frac{-1}{2}} St_\nu^{-1}(Q(t)) - \rho V)}{\sqrt{1-\rho^2}})$$

**Default times**

In contrast with other copulas already implemented in *Premia* , the conditional probabilities depend on two variables $V$ and $W$.

## 1.2 Hull & White and Laurent & Gregory methods

Let $N_t := \sum_{i=1}^M \mathbf{1}_{\{\tau_i \leq t\}}$. We have that :

$$P(N_t = k) = \int_R \int_{R^+} (\Phi_k(v,w) g(w)) f(v) dv,$$

where $\Phi_k(v,w)$ are ceofficients given by Hull & White method's (resp. Laurent & Gregory meyhod's) and which are numerically defined in the pricer.(See file CDO_Pricer/src/laurent_gregory.c(or hull_white.c) ) Note that $f$ and $g$ are the functions densitiy of respectively $V$ and $W$. In order to compute efficiently this double integral, we employ the method of Gauss & Knorod

## 1.3 Gauss & Kronrod method's

The Gauss & Kronrod method's consists on choosing some specific points $x_i$ on a given grid and affects them weights $wg_i$ such that the integral of a given function $f$ is equal to a weighted sum of the $f(x_i)$. In our case we use a scheme of 21 points :

$$\int_a^b f(x)dx = \frac{b-a}{2}[f\frac{(a+b)}{2} wg[11] + \sum_{i=1}^{10} wg_i(f(x_i - \frac{a+b}{2}) + f(x_i - \frac{a+b}{2}))]$$

We adapt this method for computing the double integral given above. For this reason we will fix $v$ and use the Gauss & Kronrod method's a first time

$$C_k(v) = \int_{R^+} (\Phi_k(v,w) g(w))$$

$$D_k(v) = \int_{R^+} (\pi_k(v,w) g(w))$$

The points $w_i$ of $W$ will be initialized in the object points et and the weights $w_i$ will be initialized in the object weights of the structure student copula. In the other hand e have that :

$$P(N_t = k) = \int_R C_k(v) f(v) dv = \int_R D_k(v) f(v) dv$$

In the same way the points $v_i$ of $V$ will be initialized in the object points and the weights $w_i$ will be initialized in the object weights of the structure student copula. The same method will be used for the other algorithms on premia computing our conditional probabilities : FFT, Hull and White recursive approach, ...

**Initialisation of the Student copula**

$$\begin{array}{c} \text{copula } *cop; \\ \text{double } \rho = 0.02; \\ \text{double } \nu = 5; \\ cop = init\_student\_copula(\rho, \nu); \end{array}$$

# 2 Double t copula

In the framework of the double t copula we define :

$$X_i = \rho \sqrt{\frac{(t_1 - 2)}{t_1}} V + \sqrt{1 - \rho^2} \sqrt{\frac{(t_2 - 2)}{t_2}} V_i,$$

where $V$ is the student of parameter $t_1$ and $V_1, ..V_M$ are $M$ Students with $t_2$ parameter. Hence,

$$P(\tau_i < t/V) = P(X_i < F_{X_i}^{-1}(Q(t))/V) = st_{t_2}\{(\sqrt{\frac{t_2}{t_2 - 2}}) \frac{(F_{X_i}^{-1}(Q(t)) - \rho\sqrt{\frac{t_1 - 2}{t_1}} V)}{\sqrt{1 - \rho^2}}\}$$

The most difficult task in this algorithm is the implementation of the function $F_{X_i}$. In order to solve this problem we use the following method :

- Define a grid of points covring the domain of $X_i$ for all values of $\rho$,$t_1$,$t_2$.
- Compute $F_{X_i}$ for every point of the grid.
- Use interpolation method to have all the values of function $F$.

(See in file CDO_PRICER/src/double_t_copula.c :

$$\begin{array}{l} init\_points(copula * cop); \\ init\_cdf(copula * cop); \\ double\_t\_cdf(copula * cop, double\ x); ) \end{array}$$

**Initialisation of the double t copula**

$$\begin{array}{c} \text{copula } *cop; \\ \text{double } \rho = 0.02; \\ \text{double } t_1 = 5; \\ \text{double } t_2 = 7; \\ cop = init\_double\_t\_copula(\rho, t_1, t_2); \end{array}$$

# 3 Implicit approach : Base-Correlation & Implied Copula

In litterature, several methods permit to compute the spread of a given CDO. However, a crucial point, like usually in financial problems, is the calibration of the model parameters and the evaluation of exotic products or less tradable products on the market. In our framework, we will suppose that the CDO tranches are standard (the itraxx, CDX and IG tranches) and our aim is to evaluate the spread of a non standard tranche using informations on standard ones. The Base-Correlation and implied-copula are the most useful methods to solve this problem.

## 3.1 The Base Correlation method

The Base correlation method consists on computing the implicit correlations of the tranches not leading to arbitrage opportunities. Now, we introduce the "trancche correlation"

### 3.1.1 Tranche correlation

Given a spread on a tranche of a standard CDO, the tranche correlation is the obtained correlation by inverting the spread formula of a CDO tranche by the Laurent & Gregory method or by Hull & White one. In the following, we set $\rho_m$ for $m = 1, .., 5$ the computed tranche correlations and $\rho_{B^i}$ is the Base correlation of the $i^{th}$ tranche given by :

$$C(0, a_{H^i}, \rho_{B^i}) = \sum_{m=1}^{i} C(a_{L^m}, a_{H^m}, \rho_m) \qquad (1),$$

where $a_{L^m}$ and $a_{H^m}$ are respectively attachment points and posting ones associated to the $m$ tranche. By convention we set $a_{L^1} = 0$. The quantity $C(a_{L^m}, a_{H^m}, \rho_m)$ denotes the loss on the tranche $[a_{L^m}, a_{H^m}]$ by considering the correlation $\rho_m$. The equation (1) ensures the abscence of arbitrage opportunities : the loss on the tranche $[0, a_{H^i}]$ is equal to the sum of the losses on tranches $[0, a_{L^1}], [a_{L^1}, a_{L^2}], ....[a_{L^i}, a_{H^i}]$.

### 3.1.2 The spread computation of a non standard tranche using the Base correlation method

In order to compute the spread of a non standard tranche $[a_{L^m}, a_{H^m}]$ (e.g. [0.04,0.05]) using the "Base correlation" method, we will proceed as follows :

- First, we compute the base correlation of the standard tranches $[0, a^{'}_{L^m}]$ and $[0, a^{'}_{H^m}]$ ,$a^{'}_{L^m}$ and $a^{'}_{H^m}$ denote the posting points such that $a_{L^m} \in [0, a^{'}_{L^m}]$ and $a_{H^m} \in [0, a^{'}_{H^m}]$.
- Using an interpolation method we compute the base correlation of the tranches $[0, a_{L^m}]$ and $[0, a_{H^m}]$. Hence, we can compute the loss on these tranches and then deduce the loss on the tranche $[a_{L^m}, a_{H^m}]$.
- Given the loss on the tranche, we will compute the tranche correlation by inverting the loss formula.
- Finally, given the correlation we can compute the spread of a CDO tranche.

The pricer function computing the base correlation, the implicit correlation of a tranche and the spread tranche are in file CDO_PRICER/src/Base_Correlation.c

$$correl\_impl = Base\_Correl(produit * prod, double * s_1, double\ s_2)$$
$$correl\_tranche = rho\_Base(produit * prod, double * s_1, double\ s_2)$$
$$pl = pl\_Base(produit * prod, double * s_1, double\ s_2)$$
$$dl = dl\_Base(produit * prod, double * s_1, double\ s_2)$$

Here, prod is a structure containing the characteristics of a CDO (maturity ,number of firms , attachement points and posting points). The vector $s_1$ denotes the spread of the standard tranches and the vector $s_2$ denotes the spread the CDS Index.

## 3.2 Implied Copula

The Hull and White idea is to compute the intensities thanks to the market data. The distribution functions of these intensities will lead us to evaluate the non standard spread tranches.

### 3.2.1 How to choose the intensities

The intensities will be implicitly defined as a data given by the standard tranches and the CDS index. We have the data over six products : the spreads of the standard CDO tranches and the spread of the CDS index. The idea is to define a minmal intensity and a maximal one, then choose the intensities in this interval. More precisely, we set $\lambda_{min} = 0$ the minimal intensity and $\lambda_{max}$ the maximum one . The maximum intensity is chosen such that $P(\tau \geq T) = 1$, where T is the maturity. We denote :

$$V(s, \lambda) = PL(s, \lambda) - DL(s, \lambda)$$

the value of a contract product, where $PL(s, \lambda)$, $DL(s, \lambda)$ are respectively the premium leg and the default leg. We set :

$$V_{min} = \sum_{i=0}^{6} V_i(\lambda_{min}), \qquad V_{max} = \sum_{i=0}^{6} V_i(\lambda_{max}),$$

where $V_{min}$ denotes the sum of the contracts over our products by using the intensity $\lambda_{min}$. However, $V_{max}$ denotes the sum of the contracts by using the intensity $\lambda_{max}$.

The $n$ intensities $(\lambda_k)_{1 \leq k \leq n}$ that we will choose such that the sum of the contracts computed by using the intensities on the interval $[\lambda_{min}, \lambda_{max}]$ cover uniformly the interval $[V_{min}, V_{max}]$ :

$$\sum_{m=1}^{6} V_m(\lambda_k) = V_{min} + \frac{k-1}{n-1}(V_{max} - V_{min}),$$

for all $k = 1, ..., n$.

### 3.2.2 How to choose the probabilities

Given the implicit intensities, we have to define the probabilities :

$$\pi_k = P(\lambda = \lambda_k)$$

and the value of the contract product :

$$V_m = \sum_{k=1}^{n} \pi_k V_m(\lambda_k).$$

The probabilities $\pi_k$ associated to the intensities are defined such that the value of evry contract is equal to zero :

$V_m = \sum_{k=1}^{n} \pi_k V_m(\lambda_k) = 0 \qquad m = 1, ..., 6$

sc :$\pi_k \geq 0 \qquad \sum_{k=1}^{n} \pi_k = 1$. This will leads us to solve the following optimization problem :

$$min_\pi \{ \sum_{j=1}^{6} V_j^2(\pi)) + \frac{c}{2} \sum_{k=2}^{n-1} \frac{(\pi_{k+1} + \pi_{k-1} - 2\pi_k)^2}{\lambda_{k+1} - \lambda_{k-1}} \}$$

$$sc : \sum_{k=1}^{n} \pi_k = 1 \qquad \pi_k \geq 0$$

where $c$ is a penalization constant. The functions computing the PL and the DL of a given tranche using the implied copula method are in file CDO_PRICER/src/Implied_Copula.c.

$$pl = pl\_impl(produit * prod, double * s_1, double\ s_2)$$
$$dl = dl\_impl(produit * prod, double * s_1, double\ s_2)$$

# Références


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