A CLOSED-FORM EXACT SOLUTION FOR PRICING VARIANCE SWAPS WITH STOCHASTIC VOLATILITY

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Most of what is presented here is taken from [1].

The authors use the Heston (1993) stochastic volatility model to describe the dynamics of the underlying asset. To evaluate the discretely sampled realized variance swaps, we employ the dimension reduction technique proposed by Little and Pant (2001) to analytically solve the associated PDE

The forward price process S_t under the Heston stochastic volatility model is defined by the following stochastic differential equations

$$\begin{cases} \frac{dS_t}{S_t} = r dt + \sqrt{V_t} dW_t^1, \\ dV_t = (a - bV_t)\, dt + \sigma \sqrt{V_t} dW^2, \;\; d\langle W^1, W^2 \rangle_t = \rho dt, \end{cases} \tag{0.1}$$

Variance swaps are forward contracts on the future realized variance of the returns of the specified underlying asset. The long position of a variance swap pays a fixed delivery price at expiry and receives the floating amounts of annualized realized variance, whereas the short position is just the opposite. Thus it can be easily used for investors to gain exposure to volatility risk.

Usually, the value of a variance swap at expiry can be written as $(\sigma_R^2 - K_{var}) \times L$, where $\sigma_R^2$is the annualized realized variance over the contract life $[0, T]$, K_{var} is the annualized delivery price for the variance swap, and L is the notional amount of the swap in dollars per annualized volatility point squared. T is the life time of the contract.

A typical formula for the measure of realized variance

$$\sigma_R^2 = \frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \times 100^2 \tag{0.2}$$

where S_{t_i} is the closing price of the underlying asset at the ith observation time t_i , and there are altogether N observations. AF is the annualized factor converting this expression to an annualized variance. If the sampling frequency is every trading day, then $AF = 252$, assuming that there are 252 trading days in 1 year, if every week then $AF = 52$, if every month then $AF = 12$, and so on. We assume equally spaced discrete observations in this paper so that the annualized factor is of a simple expression $AF = \frac{1}{\Delta t} = \frac{N}{T}$.

In the risk-neutral world, the value of a variance swap at time t is the expected present value of the future payoff, $V_t = \mathbb{E}_t^Q \left[e^{-r(T-t)} (\sigma_R^2 - K_{var}) L \right]$. The fair variance delivery price can be easily defined as $K_{var} = \mathbb{E}_0^Q \sigma_R^2$

The expected value of realized variance in the risk neutral world is defined as

$$\mathbb{E}_0^Q \sigma_R^2 = \frac{100^2}{N\Delta t} \sum_{i=1}^{N} \mathbb{E}_0^Q \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2$$

So the problem of pricing variance swap is reduced to calculating the N expectations in the form of

$$\mathbb{E}_0^Q \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 \tag{0.3}$$

According to SONG-PING ZHU AND GUANG-HUA LIAN, (cf [1]), we have

$$\mathbb{E}_0^Q \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} \right)^2 = e^{r\Delta t} f_i(V_0) \tag{0.4}$$

where

$$f_i(V_0) = e^{\tilde{C}(\Delta t) + \frac{c_i e^{-\kappa t_{i-1}}}{c_i - \tilde{D}(\Delta t)} \tilde{D}(\Delta t) V_0} \left(\frac{c_i}{c_i - \tilde{D}(\Delta t)} \right)^{\frac{2\kappa\theta}{\sigma^2}} + e^{-r\Delta t} - 2$$

where $c_i = \frac{2\kappa}{\sigma^2(1-e^{-\kappa t_{i_1}})}$ and

$$\begin{cases} \tilde{C}(\tau) = r\tau + \frac{\kappa\theta}{\sigma^2}\left[(\tilde{a}+\tilde{b})\tau - 2\ln\left(\frac{1-\tilde{g}e^{\tilde{b}\tau}}{1-\tilde{g}}\right)\right], \\ \tilde{D}(\tau) = \frac{\tilde{a}+\tilde{b}}{\sigma^2}\ \frac{1-\tilde{g}}{1-\tilde{g}e^{\tilde{b}\tau}}, \\ \tilde{a} = \kappa - 2\rho\sigma, \quad \tilde{b} = \sqrt{\tilde{a}^2 - 2\sigma^2}, \\ \tilde{g} = (\frac{\tilde{a}}{\sigma})^2 - 1 + \frac{\tilde{a}}{\sigma}\ \sqrt{(\frac{\tilde{a}}{\sigma})^2 - 2} \end{cases} \tag{0.5}$$

And

$$\mathbb{E}_0^Q\left(\frac{S_{t_i} - S_0}{S_0}\right)^2 = e^{r\Delta t} f(V_0) \tag{0.6}$$

where

$$f(v) = e^{\tilde{C}(\Delta t)+\tilde{D}(\Delta t)v} + e^{-r\Delta t} - 2$$

References

[1] A Closed-Form Exact Solution for Pricing Variance Swaps with Stochastic Volatility. S.Zhu, G. Lian. *Mathematical Finance, Volume 21, Issue 2, April 2011*

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