Comparison of Finite Difference Methods for Pricing American options on two stocks.

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February 18, 2016

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Keywords:*American Options, Obstacle problem, Alternating Direction Implicit Method.*

AMS CLASSIFICATION: 49L25, 60G40, 90A09

Abstract

In this paper we establish numerical comparison of finite difference approximations for pricing two-colours Rainbow American options in the Black-Scholes model. This paper has two aims. First, we present new methods based on the alternating direction implicit algorithm due to Peaceman and Rachford. Second, we test these methods empirically comparing them with dynamic programming preconditioned generalized minimal residual (DP-GMRES) and projected succesive overrelaxation algorithms (PSOR).

Introduction

The valuation of American options on two stocks, also called two-colours Rainbow options by practitioners, is an important problem in financial economics since a wide variety of contracts that are traded in the O.T.C. market involve such options (Exchange options, Best-of options). Unlike European options, American options cannot be valued by closed-form formulae, even in the Black-Scholes model, and require the use of numerical methods. The optimality of early exercise may lead to strong difficulties from a computational viewpoint. Moreover, Broadie and Detemple [3] (see also [12]) proved that the exercise regions may exhibit interesting properties. Early works focused on the case of American options on a single asset. Cox, Ross and Rubinstein [6] introduced the binomial method whereas Brennan and Schwartz [2] introduced a very accurate finite difference method whose convergence has been proved in [8] for the American Put option. Extension of the binomial approach for pricing American options on two stocks has been made by Boyle, Evnine and Gibbs [4]. In this paper, we are interested in the extension of finite difference Brennan Schwartz approximation to two space dimensions. It has been known for more than forty years that the alternating direction implicit (ADI) algorithm of Peaceman-Rachford [9] is efficient for solving a large scale system of linear equations arising from the finite difference discretization of elliptic or parabolic equations. Our idea is to adapt the ADI algorithm to solve the linear complementarity problem (LCP) arising from the discretization of the parabolic variational inequalities related on the pricing of American options. The accuracy of such a method comes from the reduction of the dimension since we solve one-dimensional implicit step as part of an ADI method. Stability and convergence of these schemes have been proved in [13].

The paper is organized as follows. In order to fix ideas, we choose to present the ADI method for pricing two-colours rainbow options in the European case in the first section. Section 2 is devoted to the presentation of two new methods for pricing two-colours Rainbow American options. The first one is related to the dynamic programming principle and leads to the resolution at each time step of a parabolic equation by ADI method similarly to section 1. The second one is based on the resolution at each time step of two one-dimensional LCP. We present the computation of numerical results issued from ADI methods in section 3 and compare them with some well-known numerical methods, in European case and in the American case.

1 European Options on Two Stocks

We consider European options written on two dividend-paying stocks. Let $S_t^i (i = 1, 2)$ be the stock-price of the asset i at time t which satisfies the following stochastic differential equation:

$$\frac{dS_t^i}{S_t^i} = (r - \delta_i)dt + \sigma_i dW_t^i, \quad S_0^i = s_i, \quad i = 1, 2$$

where (W_t^1, W_t^2) are correlated Brownian motion with the correlation ρ.
The price at time 0 of a European option on two stocks with maturity T

and payoff ψ is given by

$$P_E(0,s_1,s_2) = E\left[e^{-rT}\psi(S_T^{1,s_1}, S_T^{2,s_2})\right].$$

Let $\alpha_i = r - \delta_i - \frac{1}{2}\sigma_i^2$ and $x_i = \log s_i$, $i = 1,2$.
After a standard logarithmic transformation $(X_t^1, X_t^2) = (\log(S_t^1), \log(S_t^2))$, the price at time 0 of the option can be formulated in terms of the solution $u(t,x_1,x_2)$ to the following partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma_1^2}{2}\frac{\partial^2 u}{\partial x_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 u}{\partial x_2^2} + \alpha_1\frac{\partial u}{\partial x_1} + \alpha_2\frac{\partial u}{\partial x_2} + \rho\sigma_1\sigma_2\frac{\partial^2 u}{\partial x_1 \partial x_2} - ru = 0 & \text{in } [0,T[\times I\!R^2 \\ u(T,x_1,x_2) = \psi(e^{x_1}, e^{x_2}) \end{cases} \tag{1}$$

by $P_E(t,s_1,s_2) = u(t, \ln s_1, \ln s_2)$.
Let us now recall the usual numerical approximation of (1).

1.1 Finite Difference Methods in two space dimensions

We start by limiting the integration domain in space. The problem will be solved in a finite interval $\Omega_l =]-l,l[^2$:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\sigma_1^2}{2}\frac{\partial^2 u}{\partial x_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 u}{\partial x_2^2} + \alpha_1\frac{\partial u}{\partial x_1} + \alpha_2\frac{\partial u}{\partial x_2} + \rho\sigma_1\sigma_2\frac{\partial^2 u}{\partial x_1 \partial x_2} - ru = 0 & \text{in } [0,T[\times \Omega_l \\ u(T,x_1,x_2) = \psi(e^{x_1}, e^{x_2}) \end{cases} \tag{2}$$

with Dirichlet boundary condition $u = \psi$ on $[0,T[\times\partial\Omega_l$.
For the numerical resolution of (2) by finite difference method, we introduce a grid of mesh points (nk, ih, jh) where h,k are mesh parameters which are thought of as tending to zero. Denote by $N = \left[\frac{T}{k}\right]$ and M the great integer such that $M + \frac{1}{2} \leq l$. For each point $x_{ij} = (ih, jh)$, we associate a square

$$C_{ij}^{(h)} = \left](i-\frac{1}{2})h, (i+\frac{1}{2})h\right[\times \left](j-\frac{1}{2})h, (j+\frac{1}{2})h\right[$$

we denote by

$$\begin{aligned} \Omega_h &= \left\{x_{ij}\,;\, C_{ij}^{(h)} \subset \Omega_l\right\} \\ &= \{x_{ij}\,;\, -M \leq i,j \leq M\} \end{aligned}$$

and V_h the space generated by $\chi_{ij}^{(h)}$ where $\chi_{ij}^{(h)}$ is the indicator function of $C_{ij}^{(h)}$, $-M \leq i,j \leq M$.
If $u_h \in V_h$, we write

$$u_h(x) = \sum_{i,j=-M}^{M} u_{ij}\chi_{ij}^{(h)}(x).$$

Note that $u_{ij} = u(ih, jh)$.
We denote by $\phi_{h,k}$ the approximation of the payoff function ϕ in the grid defined by

$$\begin{aligned}\phi_{h,k}(t,x) &= \sum_{n=0}^{N} \phi_h(x)\mathbf{1}_{[nk,(n+1)k[}(t) \\ &= \sum_{n=0}^{N} \left(\sum_{i,j=-M}^{M} \phi(x_{ij})\chi_{ij}^{(h)}(x) \right) \mathbf{1}_{[nk,(n+1)k[}(t)\end{aligned}$$

where $\mathbf{1}_{\mathrm{I}}$ is the indicator function of interval I. As usual, we construct recursively the approximate solution

$$u_{h,k} = \sum_{n=0}^{N} u_h^n(x)\mathbf{1}_{[nk,(n+1)k[}(t)$$

starting from $u_h^N = \phi$ with $u_h^n \in V_h$ for $0 \leq n \leq N$. One approximates the differential operator

$$A\phi := \frac{\sigma_1^2}{2}\frac{\partial^2\phi}{\partial x_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2\phi}{\partial x_2^2} + \alpha_1\frac{\partial\phi}{\partial x_1} + \alpha_2\frac{\partial\phi}{\partial x_2} + \rho\sigma_1\sigma_2\frac{\partial^2\phi}{\partial x_1\partial x_2} - r\phi$$

by a discrete operator A_h acting on functions u_h^n defined on V_h by:

$$A_h u_h^n(x) = \sum_{i,j=-M}^{M} (A_h u_h)_{i,j}^n \chi_{ij}^{(h)}(x).$$

where

$$(A_h u)_{i,j}^n = \frac{1}{2}\sigma_1^2\frac{\delta^2 u_{i,j}^n}{\delta x_1^2} + \frac{1}{2}\sigma_1^2\frac{\delta^2 u_{i,j}^n}{\delta x_2^2} + \alpha_1\frac{\delta u_{i,j}^n}{\delta x_1} + \alpha_2\frac{\delta u_{i,j}^n}{\delta x_2} + \rho\sigma_1\sigma_2\frac{\delta^2 u_{i,j}^n}{\delta x_1\delta x_2} - r u_{i,j}^n$$

with centered space derivatives

$$\frac{\delta^2 u_{i,j}^n}{\delta x_1^2} = \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2}$$

$$\frac{\delta^2 u_{i,j}^n}{\delta x_2^2} = \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2}$$

$$\frac{\delta u_{i,j}^n}{\delta x_1} = \frac{u_{i+1,j}^n - u_{i-1,j}^n}{2h}$$

$$\frac{\delta u_{i,j}^n}{\delta x_2} = \frac{u_{i,j+1}^n - u_{i,j-1}^n}{2h}$$

The mixed partial derivate $\frac{\partial^2 u^n_{i,j}}{\partial x_1 \partial x_2}$ is approximated by

$$\frac{\delta^2 u^n_{i,j}}{\delta x_1 \delta x_2} = \frac{-u^n_{i-1,j+1} - u^n_{i+1,j-1} + u^n_{i+1,j+1} + u^n_{i-1,j-1}}{4h^2}.$$

If we consider an explicit scheme in time, we have to solve directly at each time step

$$\begin{cases} \frac{u^{n+1}-u^n}{k} + A_h(u^{n+1}+v) = 0 \\ u^N = \psi \end{cases}$$

with

$$v_{-M,j} = \psi_{-(M+1),j}\,,\; v_{M,j} = \psi_{M+1,j} v_{i,-M} = \psi_{i,-(M+1)}\,,\; v_{i,M} = \psi_{i,M+1}$$

takes into account the Dirichlet boundary conditions. It is well-known that this explicit scheme is conditionally stable and thus is convergent if $\frac{k}{h^2}$ tends to zero.
Therefore we would rather work with a fully implicit scheme in time, where we have to solve at each time step

$$\begin{cases} \frac{u^{n+1}-u^n}{k} + A_h(u^n+v) = 0 \\ u^N = \psi \end{cases} \tag{3}$$

Let us introduce the vector U^n in $I\!R^{(2M+1)^2}$:

$$U^n = \left[\; u^n_{-M,M}, \cdots u^n_{-M,M}, u^n_{-M+1,-M} \cdots u^n_{-M+1,M} \cdots\cdots\cdots u^n_{M,-M} \cdots u^n_{M,M} \;\right]^t$$

We then obtain from (3) the linear system

$$U^{n+1} = AU^n + V \tag{4}$$

where the $(2M+1)^2 \times (2M+1)^2$ matrix A is block tridiagonal, with each of the blocks is a $(2M+1) \times (2M+1)$ square matrix. More precisely,

$$A = \begin{pmatrix} B & C & 0 & \cdots & 0 & 0 \\ D & B & C & 0 & \cdots & 0 \\ 0 & D & B & C & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & D & B & C \\ 0 & 0 & 0 & \cdots & D & B \end{pmatrix}. \tag{5}$$

where,

$$B = \begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ c & a & b & 0 & \cdots & 0 \\ 0 & c & a & b & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & c & a & b \\ 0 & 0 & 0 & \cdots & c & a \end{pmatrix}.$$

$$D = \begin{pmatrix} d & f & 0 & \cdots & 0 & 0 \\ j & d & f & 0 & \cdots & 0 \\ 0 & j & d & f & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & j & d & f \\ 0 & 0 & 0 & \cdots & j & d \end{pmatrix}.$$

and

$$C = \begin{pmatrix} e & i & 0 & \cdots & 0 & 0 \\ g & e & i & 0 & \cdots & 0 \\ 0 & g & e & i & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & g & e & i \\ 0 & 0 & 0 & \cdots & g & e \end{pmatrix}.$$

with respectively

$$a = 1 + k(r + \frac{\sigma_1^2}{h^2} + \frac{\sigma_2^2}{h^2}), \quad b = -k(\frac{\sigma_1^2}{2h^2} + \frac{\alpha_1}{2h}), \quad c = -k(\frac{\sigma_2^2}{2h^2} - \frac{\alpha_1}{2h})$$

$$e = -k(\frac{\sigma_2^2}{2h^2} - \frac{\alpha_2}{2h}), \quad i = -k\rho\sigma_1\sigma_2, \quad g = k\rho\sigma_1\sigma_2$$

$$d = -k(\frac{\sigma_2^2}{2h^2} + \frac{\alpha_2}{2h}), \quad f = k\rho\sigma_1\sigma_2, \quad j = -k\rho\sigma_1\sigma_2$$

and $V = kA_h v$.
In our comparaison tests we solve the linear system (4) with two different methods: the stationary iterative successive over-relaxation (SOR), and the non stationary iterative preconditioned generalized minimal residual (GMRES) with diagonal preconditioner, an algorithm applicable to nonsymmetric matrices ([10]).

1.2 Alterning Direction Implicit Methods

The purpose of this subsection is to describe a faster and accurate algorithm for pricing European options in the bidimensional Black-Scholes setting based upon ADI methods of Peaceman-Rachford ([9]).
In order to adapt the ADI algorithm to the discretization of the parabolic equation related on the pricing of European options, we'd rather work with the underlying bidimensional Brownian motion.
In the Black-Scholes model, the stock-price of asset i at time t may also be modeled by the following stochastic differential equation:

$$\frac{dS_t^i}{S_t^i} = (r - \delta_i)dt + \sum_{j=1}^{2} \sigma_{ij} dW_t^j, \quad S_0^i = s_i, \quad i = 1, 2$$

where (W^1, W^2) is a standard bidimensionnel Brownian motion. Let us assume that the covariance matrix

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}.$$

is invertible and let us recall the notations of section 1,

- $\alpha = (r - \delta_1 - \frac{1}{2}\sigma_{11}^2 - \frac{1}{2}\sigma_{12}^2, r - \delta_2 - \frac{1}{2}\sigma_{21}^2 - \frac{1}{2}\sigma_{22}^2)$
- $x = (x_1, x_2) = (\log s_1, \log s_2), \exp x = (e^{x_1}, e^{x_2})$.

Denote by $\phi_x(t, W_t) = \psi(\exp(x + \alpha t + \sigma W_t))$ the payoff function.
The price of a European option on two stocks is given by

$$u(t, W_t^1, W_t^2) = E\left(e^{-rT}\phi_x(T, W_T^1, W_T^2)|\mathcal{F}_t\right)$$

and can be approximated by the solution to the two dimensional partial differential equation

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial w_1^2} + \frac{1}{2}\frac{\partial^2 u}{\partial w_2^2} - ru = 0 \quad \text{in } [0,T[\times\Omega_l \\ u(T, w_1, w_2) = \phi_x(0, w_1, w_2) \end{cases}$$

with Dirichlet boundary condition $u(t, w_1, w_2) = \phi_x(T-t, w_1, w_2)$ for $(t, w_1, w_2) \in [0,T[\times\partial\Omega_l$.
For the numerical solution of the problem by finite difference method, we proceed as in section 1.1 and we construct recursively the approximate solution.

$$u_{h,k} = \sum_{n=0}^{N} u_h^n(x)\mathbf{1}_{[nk,(n+1)k[}(t)$$

starting from $u_h^N = \phi$ and computing u_h^n for $0 \le n \le N$ in two steps by Alternate Direction Implicit method (A.D.I.). A.D.I. methods were proposed by Peachman Rachford ([9]) and they may be described as follows: At each time step, one can integrate "in each direction" by using the usual finite difference method for unidimensional problems in two steps:

$$\begin{cases} \frac{u^{n+1}-u^{n+\frac{1}{2}}}{k/2} + \Delta_h^2(u^{n+\frac{1}{2}} + b) + \Delta_h^1(u^{n+1} + c) - \frac{1}{2}ru^{n+\frac{1}{2}} - \frac{1}{2}ru^{n+1} = 0 \\ \frac{u^{n+\frac{1}{2}}-u^n}{k/2} + \Delta_h^1(u^{n+\frac{1}{2}} + b) + \Delta_h^2(u^n + c) - \frac{1}{2}ru^{n+\frac{1}{2}} - \frac{1}{2}ru^n = 0 \end{cases} \tag{6}$$

where Δ_h^1 and Δ_h^2 are operators acting on V_h, with

$$\begin{aligned} \Delta_h^1 u_{i,j}^n &= \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} \\ \Delta_h^2 u_{i,j}^n &= \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \end{aligned}$$

Moreover, the Dirichlet condition impose

$$b_{Mj}=\frac{1}{2}\phi_{M+1,j}\quad b_{-Mj}=\frac{1}{2}\phi_{-M-1,j}\quad b_{ij}=0 \text{ for } |i|\le M-1$$

$$c_{iM}=\frac{1}{2}\phi_{i,M+1}\quad c_{i,-M}=\frac{1}{2}\phi_{i,-M-1}\quad c_{ij}=0 \text{ for } |j|\le M-1$$

Because each time step is an implicit unidimensional problem, it requires the solution of a linear system with a tridiagonal matrix, so that one can use the LU factorization for tridiagonal matrix.

2 American Options on Two Stocks

The price at time 0 of an American option on two stocks in the Black-Scholes setting is given by

$$P_A(0,s_1,s_2)=\sup_{\tau\in\mathcal{T}_{0,T}} E\left[e^{-r\tau}\psi(S^1_\tau,S^2_\tau)\right].$$

Proceeding analogously as in the European case, this price can be formulated, after a logarithm change of variable, in terms of the solution u to the following variational inequality (see e.g. [8]),

$$\begin{cases} \max\left(\psi-u,\frac{\partial u}{\partial t}+Au\right)=0, & (t,x_1,x_2) \text{ in } [0,T[\times R^2 \\ u(T,x_1,x_2)=\psi(e^{x_1},e^{x_2}) \end{cases} \tag{7}$$

by $P_A(t,s_1,s_2)=u(t,\ln s_1,\ln s_2)$.
Let us now describe the usual numerical approximation of (7).

2.1 Linear complementarity problem

Consider the following approximating obstacle problem on $Q_l=[0,T]\times\Omega_l$

$$\begin{cases} \max(\frac{\partial u}{\partial t}+Au,\psi-u)=0 \\ u(T,x_1,x_2)=\psi(e^{x_1},e^{x_2}) \end{cases} \tag{8}$$

with Dirichlet boundary condition $u=\psi$ on $[0,T[\times\partial\Omega_l$.
In order to make the numerical analysis of the obstacle problem (8), we introduce a finite difference grid in space similar to the European case and construct an approximation u^n satisfying

$$\begin{cases} u^N=\psi \\ u^n\ge\psi \\ \frac{u^{n+1}-u^n}{k}+A_hu^n\le 0 \\ (\frac{u^{n+1}-u^n}{k}+A_hu^n,u^n-\psi)=0 \end{cases} \tag{9}$$

Hereafter, we denote by Φ the vector

$$\left[\ \psi^n_{-M,M}, \cdots \psi^n_{-M,M}, \psi^n_{-M+1,-M} \cdots \psi^n_{-M+1,M} \cdots\cdots\cdots \psi^n_{M,-M} \cdots \psi^n_{M,M}\ \right]^t.$$

For a better understanding, we refer to [7] for a detailed presentation of the numerical analysis of variational inequalities. Moreover, it is proved in this book that the variational inequality in finite dimension (9) can be expressed as a linear complementarity problem. More precisely, we have to solve at each time step n,

$$\begin{cases} AU^n \geq U^{n+1} \\ X \geq \Phi \\ (AU^n - U^{n+1}, X - \Phi) = 0 \end{cases} \tag{10}$$

with $A = (I - kA_h)$. The $(2M+1)^2 \times (2M+1)^2$ matrix A is block tridiagonal (cf section 1.1). In our comparaison tests we solve the LCP (10) with projected successive over-relaxation (PSOR) methods of Cryer [5].

2.2 Dynamic Programming

We will give an alternate method to solve variational inequalities in finite dimension (9) which is not related to linear complementarity problems but to splitting methods.
The splitting methods can be viewed as an analytic version of dynamic programming. The idea contained in such a scheme is to split the American problem in two steps: we construct recursively the approximate solution u^n starting from $u^N = \psi$ and computing u^n for $n = 0, .., N-1$ as follows:

- Step 1 We solve the following Cauchy problem on $[nk, (n+1)k[\times\Omega_l$ with Dirichlet boundary conditions

$$\begin{cases} \frac{\partial v}{\partial t} + Av = 0, & \text{in } [nk, (n+1)k[\times\Omega_l \\ v((n+1)k, .) = u((n+1)k, .) \end{cases}$$

  Denote by $S_k[u((n+1)k, .)]$ the solution.

- Step 2

$$u(nk, .) = \max\left(\psi(.), S_k[u((n+1)k, .)]\right)$$

Barles-Daher-Romano ([1]) prove the convergence of this scheme. As described for the European case, we solve the first step using fully implicit schemes. In our comparaison tests we solve the linear system related to the first step with GMRES method (DP-GMRES).

2.3 DP-ADI Method

We start by remarking that the price of an American option on two stocks may also be expressed as a function of the underlying bidimensionnel Brownian motion,

$$u_A(0,0,0) = \sup_{\tau \in \mathcal{T}_{0,T-t}} E\left[e^{-r\tau}\phi_x(\tau, W_\tau^1, W_\tau^2)\right].$$

and therefore can be formulated in terms of the solution to the following obstacle problem:

$$\begin{cases} \max\left(\phi - u, \frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial^2 u}{\partial w_1^2} + \frac{1}{2}\frac{\partial^2 u}{\partial w_2^2} - ru\right) = 0, & \text{in } [0,T[\times \Omega_l \\ u(T, w_1, w_2) = \phi_x(0, w_1, w_2) \end{cases}$$

with a Dirichlet boundary condition $u = \phi_x$ on $[0,T[\times \partial\Omega_l$.
The idea of the DP-ADI method is to split as above in two steps but we achieve the resolution of step 1 using ADI method. More precisely, we compute the approximate solution u^n for $n = 0, .., N-1$ starting from $u^N = \phi_x^N$ as follows:

- Step 1 We solve the following Cauchy problem on $[nk, (n+1)k[\times \Omega_l$ with Dirichlet boundary conditions by ADI methods:

$$\begin{cases} \frac{u^{n+1} - u^{n+\frac{1}{2}}}{k/2} + \Delta_h^2(u^{n+\frac{1}{2}} + b) + \Delta_h^1(u^{n+1} + c) - \frac{1}{2}ru^{n+\frac{1}{2}} - \frac{1}{2}ru^{n+1} = 0 \\ \frac{u^{n+\frac{1}{2}} - u^n}{k/2} + \Delta_h^1(u^{n+\frac{1}{2}} + b) + \Delta_h^2(u^n + c) - \frac{1}{2}ru^{n+\frac{1}{2}} - \frac{1}{2}ru^n = 0 \end{cases} \tag{11}$$

  Denote by $S_k[u((n+1)k, .)]$ the solution.

- Step 2

$$u(nk, .) = \max\left(\phi(nk, .), S_k[u((n+1)k, .)]\right)$$

At any time step, we have to compute the solution of a bidimensional linear system issued from (11). The convergence of this scheme is rigorously proved in [13] under stability conditions. But, unlike explicit schemes, the practical computation seems to indicate that the convergence is unconditionally stable which makes this method very accurate.
Nevertheless, we can always solve the first step of the splitting method directly with an explicit scheme in time (DP-EXPLICIT method).

2.4 LCP-ADI Method

We propose the following approximation based on linear complementarity problem formulation.
The idea contained in such a scheme is to split the linear complementarity problem (9) in two unidimensional LCP. We construct recursively the approximate solution u^n starting from $u^N = \phi_x^N$ and computing u^n for for $n = 0, .., N-1$ in two steps as follows:

- Step 1 We solve

$$\begin{cases} u^{n+\frac{1}{2}} \geq \phi_x^{n+\frac{1}{2}} \\ \frac{u^{n+1}-u^{n+\frac{1}{2}}}{k/2} + \Delta_h^2 u^{n+\frac{1}{2}} + \Delta_h^1 u^{n+1} - \frac{1}{2} r u^{n+\frac{1}{2}} - \frac{1}{2} r u^{n+1} \leq 0 \\ (\frac{u^{n+1}-u^{n+\frac{1}{2}}}{k/2} + \Delta_h^2 u^{n+\frac{1}{2}} + \Delta_h^1 u^{n+1} - \frac{1}{2} r u^{n+\frac{1}{2}} - \frac{1}{2} r u^{n+1}, u^{n+\frac{1}{2}} - \phi_x^{n+\frac{1}{2}}) = 0 \end{cases} \tag{12}$$

  Let $u^{n+\frac{1}{2}}$ the solution of first step:

- Step 2 We then solve,

$$\begin{cases} u^{n} \geq \phi_x^{n} \\ \frac{u^{n+\frac{1}{2}}-u^{n}}{k/2} + \Delta_h^1 u^{n+\frac{1}{2}} + \Delta_h^2 u^{n}_{i,j} - \frac{1}{2} r u^{n+\frac{1}{2}} - \frac{1}{2} r u^{n} \leq 0 \\ (\frac{u^{n+\frac{1}{2}}-u^{n}}{k/2} + \Delta_h^1 u^{n+\frac{1}{2}} + \Delta_h^2 u^{n} - \frac{1}{2} r u^{n+\frac{1}{2}} - \frac{1}{2} r u^{n}, u^{n} - \phi_x^{n}) = 0 \end{cases} \tag{13}$$

We impose the usual Dirichlet boundary condition. Hence, the pricing of American option is now reduced to the computation of linear complementarity problem (12) and (13) involving tridiagonales Minkowski matrices.
For numerical purpose we use the pivoting method of Brennan-Schwartz (DP-BS method), a very fast algorithm beacause it uses Gauss eliminations. We are not able to provide a rigourous justification of the convergence of this method in this case.
Moreover, we proved in [13] that the approximate solutions obtained by dynamic programming ADI method are bounded above by those obtained by linear complementarity ADI method.

3 Numerical Results

This section reports comparative solution times for the pricing of two-colours rainbow options in the European case and in the American case. We choose to evaluate the American Put option on the minimum of two underlying assets with payoff $\psi = (K - \min(S_1, S_2))_+$.
We assume that the initial values of the stock prices are $s_1 = 40$, $s_2 = 40$, the volatility $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, the interest rate $r = log(1.05)$, the continuous dividend rates $\delta_1 = log(1.02)$, $\delta_2 = log(1.02)$, while the values of the exercise price and of the correlation vary $K = 36, 40, 44$ and $\rho = -0.5, 0.0, 0.5$.
In the European case we compare the analytical formula given in [11] with the finite difference algorithms of section 1 :

1. the ADI algorithm
2. the SOR-IMPLICIT algorithm

3. the Preconditioned GMRES-IMPLICIT algorithm

4. the EXPLICIT algorithm

In American case, we take as the “true” reference price, the one issued of the multinomial BEG tree-method ([4]) with 1000 step and compare it with the finite difference algorithms of section 2:

1. the PSOR-algorithm

2. the Brennan Schwartz BSADI algorithm

3. the DPADI algorithm

4. the explicit DPEXP algorithm

5. the Preconditioned GMRES-IMPLICIT algorithm (DPGS).

All computation was performed in double precision on a PC Pentium III 700 MH computer with 128 Mb of RAM.
SOR and PSOR algorithm was used with a relaxation parameter $\omega = 1.5$. For all iterative methods algorithms convergence tolerance was set to 10^{-6} and initial value equal to the previous time step's solution.
The next two tables display comparative solutions pricing in the European and American case with equals variyng spatial and time discretization steps $M, N = 50, 100, 200$. For the explicit scheme, the spatial discretization step varies whereas the time discretisation is obtained via the stability condition.

ρ	K	$N \times M$	ADI	SOR	$GMRES$	EXP	$TRUE$
-0.5	36	50×50	3.3507	3.3355	3.3423	3.3460	3.3568
		100×100	3.3522	3.3488	3.3533	3.3570	
		200×200	3.3534	3.3505	3.3562	3.3528	
	40	50×50	6.1508	6.1248	6.1316	6.1572	6.1598
		100×100	6.1549	6.1464	6.1509	6.1572	
		200×200	6.1555	6.1515	6.1577	6.1550	
	44	50×50	9.5782	9.5592	9.5661	9.5762	9.5876
		100×100	9.5816	9.5766	9.5811	9.5842	
		200×200	9.5828	9.5805	9.5865	9.5823	
0.0	36	50×50	3.1298	3.1171	3.1258	3.1279	3.1377
		100×100	3.1338	3.1299	3.1358	3.1352	
		200×200	3.1349	3.1315	3.1383	3.1343	
	40	50×50	5.6428	5.6175	5.6266	5.6409	5.6503
		100×100	5.6461	5.6377	5.6438	5.6480	
		200×200	5.6466	5.6425	5.6498	5.6468	
	44	50×50	8.7756	8.7581	8.7675	8.7768	8.7837
		100×100	8.7789	8.7739	8.7847	8.7801	
		200×200	8.7795	8.7740	8.7847	8.7791	
0.5	36	50×50	2.8070	2.7955	2.8043	2.8044	2.8126
		100×100	2.8087	2.8059	2.8119	2.8102	
		200×200	2.8098	2.8070	2.8138	2.8092	
	40	50×50	5.0184	4.9987	5.0083	5.0168	5.0263
		100×100	5.0221	5.0155	5.0219	5.0239	
		200×200	5.0228	5.0195	5.0268	5.0223	
	44	50×50	7.8477	7.8369	7.8471	7.8475	7.8562
		100×100	7.8516	7.8486	7.8555	7.8531	
		200×200	7.8524	7.8510	7.8585	7.8521	

ρ	K	$N \times M$	$BSADI$	$DPADI$	$PSOR$	$DPEXP$	$DPGS$	$TRUE$
-0.5	36	50×50	3.4382	3.4374	3.4165	3.4375	3.4153	3.4463
		100×100	3.4420	3.4415	3.4333	3.4455	3.4325	
		200×200	3.4441	3.4438	3.4358	3.4442	3.4383	
	40	50×50	6.3123	6.3110	6.2775	6.3209	6.2727	6.3236
		100×100	6.3200	6.3193	6.3046	6.3255	6.3016	
		200×200	6.3222	6.3218	6.3114	6.3228	6.3127	
	44	50×50	9.8102	9.8085	9.7796	9.7796	9.7718	9.8236
		100×100	9.8184	9.8175	9.8043	9.8248	9.7995	
		200×200	9.8211	9.8206	9.8109	9.8222	9.8110	
0.0	36	50×50	3.2029	3.2024	3.1852	3.2046	3.1878	3.2129
		100×100	3.2090	3.2087	3.2009	3.2122	3.2028	
		200×200	3.2110	3.2108	3.2029	3.2110	3.2078	
	40	50×50	5.7796	5.7788	5.7485	5.7835	5.7488	5.7898
		100×100	5.7867	5.7862	5.7730	5.7915	5.7734	
		200×200	5.7887	5.7884	5.7790	5.7893	5.7830	
	44	50×50	8.9871	8.9860	8.9617	8.9959	8.9589	8.9999
		100×100	8.9952	8.9946	8.9836	9.0008	8.9822	
		200×200	8.9978	8.9975	8.9877	8.9989	8.9920	
0.5	36	50×50	2.8687	2.8682	2.8542	2.8690	2.8580	2.8758
		100×100	2.8723	2.8720	2.8662	2.8753	2.8689	
		200×200	2.8741	2.8739	2.8671	2.8742	2.8727	
	40	50×50	5.1364	5.1358	5.1150	5.1396	5.1171	5.1462
		100×100	5.1435	5.1431	5.1333	5.1477	5.1352	
		200×200	5.1455	5.1453	5.1364	5.1460	5.1424	
	44	50×50	8.0404	8.0397	8.0285	8.0466	8.0286	8.0534
		100×100	8.0491	8.0487	8.0422	8.0539	8.0431	
		200×200	8.0518	8.0564	8.0426	8.0528	8.0497	

The next table displays comparative solutions times (in seconds) in the American case with variyng time-spatial grid.

NXM	*DPADI*	*BSADI*	*PSOR*	*DPEXP*	*DP-GS*
50X50	0.07	0.07	0.26	0.05	1.93
100X100	0.60	0.63	1.94	1.02	28.31
200X200	10.73	10.41	26.95	27.13	338.3

It appears that the numerical ADI direct methods proposed in [13] are finally faster than iterative algorithms used in litterature and those methods seem to be efficient for solving variational inequalities in two space dimensions. Moreover, the results confirm that the approximate solution obtained by dynamic programming ADI methods are bounded above by those obtained by linear complementarity ADI method as proved in [13].

4 Conclusion

The computation of American option values is considerably difficult and especially in higher spatial dimensions. We proposed two new algorithms and compare them to existing American option finite difference approximations based on speed and accuracy. The multinomial methods has been choosen as the reference solution for its simplicity and adaptibility to a large class of options. It appears that the ADI methods are significant improvements over existing iterative methods both for its simplicity (we solve one-dimensional problem) and for its accuracy (the results show that the ADI methods converge unconditionally and smoothly). However, one drawback of the ADI methods is the robustness to spatial dimensions greater than three.

References

[1] Barles G, Daher Ch and Romano M. Convergence of numerical Schemes for problems arising in Finance theory, *Mathematical models and methods in applied Sciences*, 5 , no 1, 125-143 (1995). 9

[2] Brennan M.J and Schwartz E. The valuation of the American put option, *J. of Finance*, 32, 449-462 (1977). 2

[3] Broadie M, Detemple J. The Valuation of American Options on Multiple Assets, *Mathematical Finance*, Vol 7, **3**, 241-286 (1996). 2

[4] Boyle P., Evnine J. and Gibbs S. Numerical Evaluation of Multivariate Contingent Claims, *Review of Financial Studies*, 2, 241-250 (1989). 2, 12

[5] Cryer C. W. The solution of a quadratic programming problem using systematic overrelaxation, *SIAM J. Control*, 9 ,385-392 (1971). 9

[6] Cox, J.C., S.A. Ross and M. Rubinstein. Option Pricing: A simplified Approach, *Journal of Financial Economics*, **7**, 229-263, (1979). 2

[7] Glowinski R, Lions J.L and R. Trémolières. Analyse numérique des inéquations variationnelles. Dunod, Paris (1976). 9

[8] Jaillet P, Lamberton D and B. Lapeyre. Variational inequalities and the pricing of American options, *Acta Appl.Math.* **21**, 263-289 (1990). 2, 8

[9] Peaceman P.W and Rachford H.H. The numerical solution of Parabolic and Elliptic differential equation, *Jour. Soc. Ind. Appl. Math.* vol III, 28-42 (1955). 2, 6, 7

[10] Saad, Y. and M. Schultz, G.M.R.E.S.. a generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Static. Comput.*, **7**, 856-869 (1986). 6

[11] Stulz R. Options on the minimum or the maximum of two risky assets. *Journal of Financial Economics*, **10**, 161-185, (1992). 11

[12] Villeneuve S. Exercise Region of American Options on several assets, *Finance and Stochastics*, **3**, 295-322 (1999) 2

[13] Villeneuve, S. and A. Zanette. Parabolic A.D.I. methods for pricing American options on two stocks. *Technical report 25-98 Université de Marne la Vallée, submitted to Math. Oper. Research.* 2, 10, 11, 15