Saddlepoint approximation method for pricing CDOs

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1 Default and portfolio loss distributions

We introduce the general framework for the default of the reference credits underlying a CDO, and derive basic formulas for the distribution of a portfolio loss up to any time $t \in [0, T]$. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space that contains all of the random elements. We will interpret $\mathbb{P}$ as the risk-neutral or pricing probability measure. The basic entities are:

- M reference credits with notional amounts of $N_j, j = 1, 2, \ldots, M$;
- the default time τ_j of the jth credit, an $\mathcal{F}_t$ stopping time;
- the fractional recovery R_j after default of the jth credit;
- the loss $l_j = (1 - R_j)N_j/N$ caused by default of the jth credit as a fraction of the total notional $N = \sum_j N_j$;
- the cumulative portfolio loss $L(t) = \sum_j l_j \mathbf{1}_{\{\tau_j \leq t\}}$ up to time t as a fraction of the total notional.

Assumptions:

1. The discount factor is $v(t) = e^{-rt}$ for a constant interest rate $r \geq 0$.
2. The fractional recovery values R_j and hence l_j are deterministic constants;
3. there is a sub σ-algebra $\mathcal{H} \subset \mathcal{F}$ generated by a d-dimensional random variable Y, the condition such that the default times τ_j are mutually conditionally independent under $\mathcal{H}$. The marginal distribution of Y is denoted by $\mathbb{P}_y$ and has probability density function $\rho(y),\ y \in \mathbb{R}^d$.

The most important consequence of these assumptions is that conditioned on $\mathcal{H}$, the fractional loss $L(t)$ is a sum of independent (but not identical) Bernoulli random variables. In the following, we fix a value for the time t and conditioning random variable Y, and denote $\hat{L} := L(t)|_Y$. Then $\hat{L} \sim \sum_j l_j X_j$ where $X_j \sim$ Bern(p_j), $p_j = \mathbb{P}(\tau_j \leq t | Y = y)$. We introduce the following functions associated to the random variable $\hat{L}$:

1. The probability distribution function (**PDF**) $\rho(x)$ (it is a sum of delta functions supported on the interval $[0,1]$);
2. the cumulative distribution function (**CDF**) $F^{(0)}(x) = \mathbb{E}(\mathbf{1}_{\{\hat{L}\le x\}})$;
3. the conditional moment one function or tranche function $F^{(1)}(x) = \mathbb{E}[(x-\hat{L})_+]$;
4. the cumulant generating function of $\hat{L}$ (**CGF**) $\Psi(u) = \log(\mathbb{E}[e^{-u\hat{L}}])$.

In the following, we will make explicit the dependence on t, Y by writing

$$F^{(1)}(x|t,y) = \mathbb{E}[(x-\hat{L})_+|Y=y],$$

$$F^{(1)}(x|t) = \mathbb{E}[(x-\hat{L})_+] = \int_{\mathbb{R}^d} F^{(1)}(x|t,y)\rho(y)\,dy.$$

The explicit form of the (**CGF**) of $\hat{L}$ is $\Psi(u) = \sum_j \log[1-p_j+p_je^{-ul_j}]$. the Fourrier integral

$$\rho(x) = \frac{1}{2\pi}\int_{-\infty+i\alpha}^{+\infty+i\alpha} e^{ikx}e^{\Psi(ik)}\,dk$$

for the **PDF** existis as a distribution for any $\alpha\in\mathbb{R}$. The value of this integral is independent of $\alpha\in\mathbb{R}$ since the moment generating function $e^{\Psi(u)}$ is entire analytic in u. After noting

$$\frac{F^{(1)}(x)}{dx} = F^{(0)}(x),$$

one can prove that the tranche function is given by related Fourier transform

$$F^{(1)}(x) = \frac{1}{2\pi}\int_{-\infty+i\alpha}^{+\infty+i\alpha} (ik)^{-m-1}e^{ikx}e^{\Psi(ik)}\,dk.$$

Now one needs to account for the presence of the single pole $k=0$. The last quantity is consistent with the boundary condition $F^{(1)}(0)=0$ for any countour with $\alpha<0$. We also define

$$G^{(1)}(x) = \frac{1}{2\pi}\int_{-\infty+i\alpha}^{+\infty+i\alpha} (ik)^{-m-1}e^{ikx}e^{\Psi(ik)}\,dk.$$

for any countour with $\alpha>0$. An easy application of the Cauchy integral formula shows that

$$F^{(1)}(x) = G^{(1)}(x) - \mathbb{E}(\hat{L}) + x.$$

2 The structure of CDO

Under the assumptions and definitions of the previous section we have

1. The fair price of the basic default leg with attachment level a is

$$W(a) = a - e^{-rT}F^{(1)}(a|T) - \int_0^T re^{-rt}F^{(1)}(a|t)\,dt.$$

2. The fair price of the basic premium leg with attachment level a and continous payments is

$$V(a) = \int_0^T e^{-rt} F^{(1)}(a|t)\, dt.$$

3. The $[a,b]$-tranche spread is that multiplier X of the premium tranche leg which solves the balance equation

$$X(V(b) - V(a)) = W(b) - W(a).$$

3 The saddlepoint method

In order to compute the integral

$$F^{(1)}(x) = \frac{1}{2\pi} \int_{-\infty+i\alpha}^{+\infty+i\alpha} e^{ikx+\Psi(ik)-2\log(ik)}\, dk$$

we use the saddlepoint method. We denote by u^* the solution of

$$x + \Psi'(u) - 2/u = 0.$$

The Taylor expansion

$$ikx+\Psi(ik)-2\log(ik) = u^*x+\Psi(u^*)-2\log(u^*)+\sum_{n=2}^{\infty} \frac{1}{n!}\left(\Psi-2\log\right)^{(n)}(u^*)(ik-u^*)^n$$

is plugged into exponent, and the contour is chosen with $\alpha = -u^*$ leading to

$$F^{(1)}(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp[u^*x+\Psi(u^*)-2\log(u^*)+\sum_{n=2}^{\infty} \frac{1}{n!}\left(\Psi-2\log\right)^{(n)}(u^*)(ik)^n]\, dk.$$

Using the resulting gaussian integrals, noting that the odd order terms vanish we obtain the asymptotic expansion

$$F^{(1)}(x) \sim \frac{e^{u^*x+\Psi(u^*)-2\log(u^*)}}{\sqrt{2\pi\Psi^{(2)}(u^*)+2/u^{*2}}}\left[1 + \frac{\Psi^{(4)}(u^*)+8/u^{*4}}{8(\Psi^{(2)}(u^*)+2/u^{*2})^2} + \dots\right].$$

In order to compute the optimal u^* we use a modified Newton-Raphson method for solving

$$x + \Psi'(u) - 2/u = 0.$$

with an initial $u^{(0)} > 0$ if the tranche level x satisfies $0 < x < \mathbb{E}[\hat{L}]$ or with an initial $u^{(0)} < 0$ if the tranche level x satisfies $\mathbb{E}(\hat{L}) < x < \sum_j l_j$.

The function computing u^* is called `double ***Uoptimal` . The function computing $F^{(1)}(x|t,y)$ is called `double **saddlepoint` and the function computing the default leg and the premium leg are respectively `double *default_leg_sadd` and `double *payment_leg_sadd`. All these functions are in file `cdo.c`.

References

[1] Jingping Yang, T. R. Hurd, Xuping Zhang "Saddlepoint approximation method for pricing CDOs", Journal of Computational Finance 10, 1-20, 2006.