PRICING OPTIONS UNDER STOCHASTIC VOLATILITY : A POWER SERIES APPROACH

MARIE AMORY

Premia 18

Contents

1. Model

1.1. **Framework.** Let [0,T] be a finite time interval and let $(\Omega, \mathcal{F}, Q)$ be a complete probability space on which two-dimensional standard Brownian motion (B, W) is defined. By $\{\mathcal{F}_t\}_{t\in[0,T]}$ we denote the filtration generated by B and W, augmented with the Q-null sets and made right-continuous. We study a market model with only two assets :
- a riskless one (bond). Its price is given by : $R_u = e^{ru}$ for some constant $r > 0$.
- a risky one (stock)
We assume that there are no arbitrage opportunities, no transaction costs, no restrictions on short-selling and that the assets are infinitely divisible.

By the fundamental theorem of asset pricing we know that there exists at least one probability $P \sim Q$, under which the discounted risky asset price is a martingale. From now on, we shall be able to work under a risk-neutral probability P, selected by some criterion. Hence, fixing $t \in [0, T]$ as our initial time, $x \in \mathbb{R}$, as the logarithm of the initial spot and $v > 0$, the dynamics of the price of the risky asset S for $t \leq s$

is given by :

$$S_s^{t,e^x,v} = e^x + \int_t^s r S_u^{t,e^x,v} du + \int_t^s f(v_u^{t,v}) S_u^{t,e^x,v} dZ_u, x \in \mathbb{R} \tag{1}$$

where $v_.$ represents the stochastic volatily and verifies :

$$v_s^{t,v} = v + \int_t^s \mu(v_u^{t,v}) du + \int_t^s \eta(v_u^{t,v}) dB_u \tag{2}$$

and Z and B are two ρ-correlated Brownian motions :

$$Z_t = \rho B_t + \sqrt{1-\rho^2} W_t$$

with B and W two independent Brownian motions and $-1 < \rho < 1$.

Besides, we make the hypothesis :

(H) *The functions $f, \mu, \eta : \mathbb{R} \to \mathbb{R}$ are well defined and in $C^\infty(\mathbb{R})$ with uniformly bounded derivatives of order greater than or equal 1. Moreover, we assume that (1) - (2) have a unique strong solution $(S_., v_.)$ such that for any $2 \le p$*

$$\mathbb{E}\left(\sup_{u \in [t,T]} (|S_u^{t,e^x,v}|^p + |v_u^{t,v}|^p) \right) < C$$

for some constant C depending on p, T, x, v.

1.2. **Expansion of the price in terms of the correlation ρ.** We want to price a European call with payoff
$\Psi(e^{X_T^{t,x,v}}) = (e^{X_T^{t,x,v}} - e^K)_+, K \in \mathbb{R}$. In the absence of arbitrage opportunities, the price has the representation

$$u(t, x, v; \rho) = \mathbb{E}(e^{-r(T-t)} \Psi(e^{X_T^{t,x,v}})). \tag{3}$$

We denote by $(\mathcal{F}_s^B)$ the filtration generated by the Brownian motion B on the interval $[t, s]$. The distribution of $X_s^{t,x,v}$ conditionally on $\mathcal{F}_s^B$ is normal.

$$\begin{aligned} X_s^{t,x,v}|_{\mathcal{F}_s^B} &\sim \mathcal{N}\Big(x + r(s-t) + \rho \int_t^s f(v_u^{t,v}) dB_u - \frac{1}{2} \int_t^s f^2(v_u^{t,v}) du; \\ &\quad (1-\rho^2) \int_t^s f^2(v_u^{t,v}) du\Big) \\ &\sim \mathcal{N}\Big(x + r(s-t) + \rho M_s^{t,v} - (\frac{1-\rho^2}{2} + \frac{\rho^2}{2}) \langle M^{t,v} \rangle_s; \\ &\quad (1-\rho^2) \langle M^{t,v} \rangle_s\Big) \end{aligned}$$

where we denote by $M_s^{t,v}$ the martingale :

$$M_s^{t,v} = \int_t^s f(v_u^{t,v}) dB_u \tag{5a}$$

$$\langle M^{t,v} \rangle_s = \int_t^s f^2(v_u^{t,v}) du \tag{5b}$$

Conditionally on $\mathcal{F}_s^B$, $(X_s^{t,x,v})_{t \le s \le T}$ follows the Black-Scholes dynamis with a time-dependent volatility. We obtain simply the explicit formula of the value of a call option thanks to the Black and Scholes formula changing the volatility into the root mean square of the volatility $f(v_s^{t,v})$ over the time intervalle $[t, T]$.
Let us denote :

$$c_{BS}(t, x, \eta) = e^x N(d_1(t, x, \eta)) - e^{K - r(T-t)} N(d_2(t, x, \eta))$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du \tag{7a}$$

$$d_1(t,x,\eta) = \frac{x - K + r(T-t) + \frac{1}{2}\eta^2}{\eta} \tag{7b}$$

$$d_2(t,x,\eta) = d_1(t,x,\eta) - \eta \tag{7c}$$

In our case, we can write :

$$\begin{aligned}
u(t,x,v;\rho) &= \mathbb{E}(e^{-r(T-t)}(e^{X_T^{t,x,v}} - e^K)_+) \\
&= \mathbb{E}\Big(\mathbb{E}(e^{-r(T-t)}(e^{X_T^{t,x,v}} - e^K)_+ | \mathcal{F}_T^B)\Big) \\
&= \mathbb{E}\Big(c_{BS}(t, x + \rho M_T^{t,v} - \frac{\rho^2}{2}\langle M^{t,v}\rangle_T, \sqrt{(1-\rho^2)\langle M^{t,v}\rangle_T})\Big) \\
&= \mathbb{E}\Big(e^{x+A(\rho)} N\Big(d_1(t, x + A(\rho), \sqrt{(1-\rho^2)\langle M^{t,v}\rangle_T})\Big) \\
&\quad - e^{K - r(T-t)} N\Big(d_2(t, x + A(\rho), \sqrt{(1-\rho^2)\langle M^{t,v}\rangle_T})\Big)\Big)
\end{aligned}$$

where $A(\rho) = \rho M_T^{t,v} - \frac{\rho^2}{2}\langle M^{t,v}\rangle_T$.

The main result of [1] is that it is possible to expand $u(t,x,v;\rho)$ into a series of powers of ρ.

Theorem *Let S, v satisfy 1 and 2 under hypothesis (H).*
If the condition :
(H1) *There exists a constant C_1, depending only on $T-t, v$ and the uniform bound on the derivatives of the coefficients, such that for any $1 \leq q$*

$$\mathbb{E}\left(\left(\frac{1}{\int_t^T f^2(v_r^{t,v})dr}\right)^q\right) \leq C_1^q$$

is also satisfied, then :
(i) The function $u(t,x,v;\rho)$ is C^∞ in a neighborhood of $\rho = 0$.
(ii) The series

$$u(t,x,v;\rho) = \sum_{0 \leq k} \frac{1}{k!} \frac{\partial^k u(t,x,v;\rho)}{\partial \rho^k}|_{\rho=0} \rho^k \tag{9}$$

converges for any $\rho \in (-R, R)$ for some appropriate constant $0 < R < 1$.

We refer to [1] for the proof.

1.3. **Identification of the coefficients.**

1.3.1. *Explicit computation.* With the previous theorem, we want to compute the different coefficients of the series. The first attempt will be to differenciate the solution u with respect to ρ. To compute the first order derivative

$$\frac{\partial u}{\partial \rho}(t,x,v;\rho)|_{\rho=0} = e^x \mathbb{E}\Big(M_T^{t,v} N\Big(d_1(t, x, \sqrt{\langle M^{t,v}\rangle_T})\Big)\Big)$$

the problem is the evaluation of the joint distribution of $\Big(M_T^{t,v}, \langle M^{t,v}\rangle_T\Big)$. That is why Antonelli and Scarlatti propose an alternative approach.
Due to the deterministic nature of the coefficients, the couple $(X_., v_.)$ is Markov. Hence taking those processes as state variables and applying Itô's lemma, we may associate the following PDE evaluation problem, u is a classical solution of which

for $x \in \mathbb{R}, v > 0, t \in [0,T]$:

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{f^2(v)}{2}\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\eta^2(v)}{2}\dfrac{\partial^2 u}{\partial v^2} + \rho\eta(v)f(v)\dfrac{\partial^2 u}{\partial x \partial v} \\ \qquad\qquad +(r - \dfrac{f^2(v)}{2})\dfrac{\partial u}{\partial x} + \mu(v)\dfrac{\partial u}{\partial v} = ru, \\ \qquad\qquad u(T,x,v) = \Psi(e^x). \end{cases}$$

We denote by $\mathcal{L}_p$ the operator in the first equality and we divide it into two parts :
$\mathcal{L}_p = \mathcal{L}_0 + \rho\mathcal{A}$:

$$\begin{cases} \mathcal{L}_0 = \frac{\partial u}{\partial t} + \frac{f^2(v)}{2}\frac{\partial^2 u}{\partial x^2} + \frac{\eta^2(v)}{2}\frac{\partial^2 u}{\partial v^2} + \big(r - \frac{f^2(v)}{2}\big)\frac{\partial u}{\partial x} + \mu(v)\frac{\partial u}{\partial v} - ru \\ \mathcal{A} = \eta(v)f(v)\frac{\partial^2 u}{\partial x \partial v} \end{cases}$$

We denote by $g_k(t,x,v)$ the series coefficients :

$$g_k(t,x,v) = \frac{1}{k!}\frac{\partial^k u}{\partial \rho^k}(t,x,v,\rho)|_{\rho=0} \tag{10}$$

If we differentiate with respect to ρ and specialize for $\rho = 0$ we find quickly that the series coefficients have to verify :

$$\begin{cases} \mathcal{L}_0 g_0(t,x,v) = 0 & x \in \mathbb{R}, v > 0 \\ g_0(T,x,v) = \Psi(e^x) & x \in \mathbb{R}, v > 0 \end{cases}$$

and for $1 \leq k$,

$$\begin{cases} \mathcal{L}_0 g_k(t,x,v) = -\mathcal{A}g_{k-1}(t,x,v) & x \in \mathbb{R}, v > 0 \\ g_k(T,x,v) = 0 & x \in \mathbb{R}, v > 0 \end{cases}$$

Then the series coefficients $g_k(t,x,v)$ are the solutions of inhomogeneous PDE, with zero terminal conditions. We can solve the PDE with classical methods. We can find a formula for each coefficient (in article [1]), which is

$$\begin{aligned} g_k(t,x,v) &= \int_t^T d\alpha_k \ldots \int_{\alpha_2}^T d\alpha_1 e^{K-r(T-t)}\mathbb{E}\Bigg((f\eta)(v^{t,v}_{\alpha_k}) \\ \text{x} &\quad \frac{\partial}{\partial v}\Big[(f\eta)(v^{t,v}_{\alpha_{k-1}})\frac{\partial}{\partial v}\Big[\ldots\frac{\partial}{\partial v}\Big[(f\eta)(v^{t,v}_{\alpha_1}) \\ \text{x} &\quad \frac{N^{(k+1)}(d_2(\alpha_1, X^{t,x,v}_{\alpha_1}, \sqrt{\langle M^{\alpha_1, v^{t,v}_{\alpha_1}}\rangle_T}))}{\Big(\langle M^{t,v}\rangle_{[\alpha_1,T]}\Big)^{\frac{k}{2}}} \\ \text{x} &\quad \frac{\partial}{\partial v}\big(\sqrt{\langle M^{\alpha_1, v^{t,v}_{\alpha_1}}\rangle_T}\big)\Big]\ldots\Big]\Big]\Bigg). \end{aligned}$$

With the flow property of v we have, for $\alpha_1 \in [t,T]$:

$$\langle M^{\alpha_1, v^{t,v}_{\alpha_1}}\rangle_T = \langle M^{t,v}\rangle_{[\alpha_1,T]} = \langle M^{t,v}\rangle_T - \langle M^{t,v}\rangle_{\alpha_1}$$

1.3.2. *The first coefficients.* The hope is that when approximating u by means of the truncated series, only a few coefficients g_k will be necessary to achieve a good degree of approximation.

The first coefficient is given by :

$$g_0(t,x,v) = \mathbb{E}\left(c_{BS}(t,x,\sqrt{\langle M^{t,v}\rangle_T})\right) \tag{13}$$

The expression of $g_1(t,x,v)$ is computed with the previous formula and:

$$g_1(t,x,v) = -e^{K-r(T-t)}\mathbb{E}\left(\frac{[d_2 N'(d_2)](t,x,\sqrt{\langle M^{t,v}\rangle_T})}{\langle M^{t,v}\rangle_T} c^{t,v}_{[t,T]}\right) \tag{14}$$

where :

$$c^{t,v}_{[t,T]} = \int_t^T (f\eta)(v^{t,v}_\alpha) \int_\alpha^T f(v^{t,v}_s) f'(v^{t,v}_s) \frac{\partial v^{t,v}_s}{\partial v} \left(\frac{\partial v^{t,v}_\alpha}{\partial v}\right)^{-1} ds d\alpha \tag{15}$$

and d_2 is given by (7)
See [1] for the proof

An explicit computation of the coefficients, and even $g_0(t,x,v)$ need the total knowledge of the law of $\langle M^{t,v}\rangle_T$ which is not possible for most of the models. So, we are going to focus a possible approximation procedure for the coeffictients.

1.4. **Approximation of the coefficients.** To approximate the coefficients, Antonelli and Scarlatti propose to perform a Taylor expansion (up to the zeroth order) of the functions appearing in the expectations with respect to the variable $\langle M^{t,v}\rangle_T$ around its mean. Pratically this corresponds to substituting in the formulas $\langle M^{t,v}\rangle_T$ with $\mathbb{E}(\langle M^{t,v}\rangle_T)$.
We can define the approximation of the first two coefficients :

$$\begin{aligned}
\overline{g_0}(t,x,v) &= c_{BS}(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)}) = e^x N(\overline{d_1}) - e^{K-r(T-t)} N(\overline{d_2}) \\
\overline{g_1}(t,x,v) &= -e^{K-r(T-t)} \frac{\overline{d_2} N'(\overline{d_2})}{\mathbb{E}(\langle M^{t,v}\rangle_T)} \mathbb{E}(c^{t,v}_{[t,T]})
\end{aligned}$$

Here $\overline{d_1}$ and $\overline{d_2}$ are approximation of d_1 and d_2 given by :

$$\overline{d_1} = \frac{x - K + r(T-t) + \frac{1}{2}\mathbb{E}(\langle M^{t,v}\rangle_T)}{\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)}} \tag{17a}$$

$$\overline{d_2} = \overline{d_1} - \sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)} \tag{17b}$$

If we denote by :

$$\overline{u}(t,x,v;\rho) = \sum_{0\le k} \overline{g_k}(t,x,v)\rho^k \tag{18}$$

Then, Antonelli and Scarlatti control the total error due to this approximation :

Theorem *Under the hypotheses (H) and (H1), for any $\rho \in [-R_1, R_1]$ for some sufficiently small $R_1 > 0$, it holds*

$$|u(t,x,v;\rho) - \overline{u}(t,x,v;\rho)| \le C Var(\langle M^{t,v}\rangle_T)^{\frac{1}{2}}$$

for some constant $C > 0$ depending only on $x, v, r, K, T-t, \rho$ and the constants appearing in hypotheses (H) and (H1)

See [1] for the proof

The idea, now, to refine our approximation is to push the Taylor expansion of the first term g_0 up to the second order. The aim is to obtain the order of $Var(\langle M^{t,v}\rangle_T)$ instead of $\mathbb{E}(\langle M^{t,v}\rangle_T)$. Using the second-order Taylor expansion of c_{BS} in $\langle M^{t,v}\rangle_T$ we obtain the following approximation for g_0

We denote by $\widetilde{g_0}(t,x,v)$:

$$\begin{aligned}\widetilde{g_0}(t,x,v) &= \overline{g_0}(t,x,v) + \frac{1}{2}\frac{\partial^2}{\partial y^2}c_{BS}(t,x,\sqrt{y})|_{y=\mathbb{E}(\langle M^{t,v}\rangle_T)}Var(\langle M^{t,v}\rangle_T)\\ &= \overline{g_0}(t,x,v) + \frac{e^{K-r(T-t)}}{8\mathbb{E}(\langle M^{t,v}\rangle_T)^{\frac{3}{2}}}(\overline{d_2 d_1}-1)N'(\overline{d_2})Var(\langle M^{t,v}\rangle_T).\end{aligned}$$

The total error has been reduced since according to [1],

$$\tilde{u}(t,x,v;\rho) = \widetilde{g_0}(t,x,v) + \sum_{1\le k}\overline{g_k}(t,x,v)\rho^k \tag{20}$$

is such

$$|u(t,x,v;\rho) - \tilde{u}(t,x,v;\rho)| \le L\Big(Var(\langle M^{t,v}\rangle_T) + |\rho|Var(\langle M^{t,v}\rangle_T)^{\frac{1}{2}}\Big)$$

where L is an appropriate constant depending on x, v, t, K.

1.5. **Pricing of a European put.** To compute the price of an European put we want to use the same method of power series approach. We want to make the same approximations as in the call options too. Instead of computing another time the function of price (from the Black and Scholes formula), we just use the call and put parity.

Denoting $p(t,x,v;\rho)$ the price of a European put, in the above stochastic volatility model, we have :

$$u(t,x,v;\rho) - p(t,x,v;\rho) = e^x - e^{K-r(T-t)} \tag{21}$$

We assume that for our computation :

$$\tilde{p}(t,x,v;\rho) = \tilde{u}(t,x,v;\rho) - e^x + e^{K-r(T-t)}$$

1.6. **Hedging.**

1.6.1. *Call option.* In this subsection we are going to compute an approximation of the hedging strategy. We are interested in strategies that incorporate a component with respect to the underlying (delta hedging). By our approach, we are going to write the option price in the series form and differentiate with respect to x. Then a possible approximation involving the first two terms might be :

$$h(t,x,v;\rho) = \frac{\partial g_0}{\partial x}(t,x,v) + \rho\frac{\partial g_1}{\partial x}(t,x,v)$$

And we simply have :

$$\begin{aligned}\frac{\partial g_0}{\partial x}(t,x,v) &= e^x\mathbb{E}\Big(N\Big(d_1(t,x,\sqrt{\langle M^{t,v}\rangle_T})\Big)\Big)\\ \frac{\partial g_1}{\partial x}(t,x,v) &= -e^{K-r(T-t)}\mathbb{E}\Big((1-d_2^2)N'(d_2)(t,x,\sqrt{\langle M^{t,v}\rangle_T})\frac{c^{t,v}_{[t,T]}}{(\langle M^{t,v}\rangle_T)^{\frac{3}{2}}}\Big)\end{aligned}$$

Then as in the others formula and in order to stay coherent with the rest we substitute $\langle M^{t,v}\rangle_T$ with $\mathbb{E}(\langle M^{t,v}\rangle_T)$ and we obtain :

$$\begin{aligned}
\tilde{h}(t,x,v;\rho) &:= h_0(t,x,v) + \rho h_1(t,x,v) \\
h_0(t,x,v) &:= e^x N\Big(d_1(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)})\Big) \\
h_1(t,x,v) &:= -e^{K-r(T-t)}(1-d_2^2)N'(d_2)(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)})\frac{\mathbb{E}(c^{t,v}_{[t,T]})}{(\mathbb{E}(\langle M^{t,v}\rangle_T))^{\frac{3}{2}}}
\end{aligned}$$

For the delta of the call option δ_c we use the following approximation :

$$\delta_c(t,x,v;\rho) = e^{-x}\big(h_0(t,x,v) + \rho h_1(t,x,v)\big)$$

1.6.2. *Put option.* In order to compute the hedging for a put option, we still use the call and put parity.

$$\begin{aligned}
p(t,x,v;\rho) &= u(t,x,v;\rho) - e^x + e^{K-r(T-t)} \\
\frac{\partial p}{\partial x}(t,x,v;\rho) &= \frac{\partial u}{\partial x}(t,x,v;\rho) - e^x
\end{aligned}$$

Then, we simply have (after first order approximation) for the delta of a put option δ_p

$$\delta_p(t,x,v;\rho) = \delta_c(t,x,v;\rho) - 1$$

2. Application to financial models

In these applications, we will compute :

$$\begin{aligned}
u_{exp}(0,x,v;\rho) &= \widetilde{g_0}(0,x,v) + \rho\overline{g_1}(0,x,v) && (26a)\\
p_{exp}(0,x,v;\rho) &= u_{exp}(0,x,v;\rho) - S_0 + e^{K-rT} && (26b)\\
\delta_{c_{exp}}(0,x,v;\rho) &= e^{-x}\big(h_0(0,x,v) + \rho h_1(0,x,v)\big) && (26c)\\
\delta_{p_{exp}}(0,x,v;\rho) &= \delta_{c_{exp}}(0,x,v;\rho) - 1 && (26d)\\
& && (26e)
\end{aligned}$$

2.1. **Heston model.** In the Heston model the stochastic volatility/asset price dynamics are given by :

$$\begin{aligned}
v^{t,v}_s &= v + \int_t^s b(a - v^{t,v}_u)dt + c\int_t^s \sqrt{v^{t,v}_u}dB_u && (27a)\\
S^{t,e^x,v}_s &= e^x + \int_t^s S^{t,e^x,v}_u(rdu + \sqrt{v^{t,v}_u})dZ_u && (27b)
\end{aligned}$$

with $v, a, b, c > 0$.
So :

$$\begin{aligned}
f(v) &= \sqrt{v}, & \mu(v) &= b(a-v), & \eta(v) &= c\sqrt{v} \\
f'(v) &= \frac{1}{2\sqrt{v}}, & \mu'(v) &= -b, & \eta'(v) &= \frac{c}{2\sqrt{v}}
\end{aligned}$$

Then we have :

$$\begin{aligned}
\langle M^{t,v}\rangle_s &= \int_t^s v_u^{t,v} du \\
\mathbb{E}(\langle M^{t,v}\rangle_T) &= a(T-t) + \frac{v-a}{b}(1-e^{-b(T-t)}) \\
c_{[\beta,T]}^{t,v} &= \frac{c}{2}\int_\beta^T v_\alpha^{t,v}\int_\alpha^T \frac{\partial v_u^{t,v}}{\partial v}(\frac{\partial v_\alpha^{t,v}}{\partial v})^{-1} ds d\alpha \\
\mathbb{E}(v_s^{t,v}) &= (v-a)e^{-b(s-t)} + a,
\end{aligned}$$

2.1.1. *Computation of* $\overline{g_1}(t,x,v)$. Explicity $\overline{g_1}(t,x,v)$ is :

$$\overline{g_1}(t,x,v) = -\frac{c}{2}e^{K-r(T-t)}\frac{\overline{d_2}N'(\overline{d_2})}{\mathbb{E}(\langle M^{t,v}\rangle_T)}\mathbb{E}(c_{[t,T]}^{t,v})$$

$\overline{g_1}(t,x,v)$ can be computed with :

$$\begin{aligned}
\frac{\partial v_s^{t,v}}{\partial v} &= e^{-b(s-t)} + c\int_t^s \frac{e^{-b(s-w)}}{2\sqrt{v_w^{t,v}}}\partial v_w^{t,v} dB_w \\
e^{bs}\frac{\partial v_s^{t,v}}{\partial v} &= e^{bt} + c\int_t^s \frac{e^{bw}}{2\sqrt{v_w^{t,v}}}\partial v_w^{t,v} dB_w \\
e^{bs}\frac{\partial v_s^{t,v}}{\partial v} &= e^{bt}exp(\frac{c}{2}\int_t^s \frac{1}{\sqrt{v_u^{t,v}}}dB_u - \frac{c^2}{2}\int_t^s \frac{1}{4v_u^{t,v}}du) \\
\frac{\partial v_s^{t,v}}{\partial v} &= e^{-b(s-t)}exp(\frac{c}{2}\int_t^s \frac{1}{\sqrt{v_u^{t,v}}}dB_u - \frac{c^2}{8}\int_t^s \frac{1}{v_u^{t,v}}du) \\
\frac{\partial v_s^{t,v}}{\partial v}(\frac{\partial v_\alpha^{t,v}}{\partial v})^{-1} &= e^{-b(s-\alpha)}e^{\frac{c}{2}\int_\alpha^s \frac{1}{\sqrt{v_u^{t,v}}}dB_u - \frac{c^2}{8}\int_\alpha^s \frac{1}{v_u^{t,v}}du}
\end{aligned}$$

Then we obtain :

$$\begin{aligned}
\mathbb{E}(c_{[\beta,T]}^{t,v}) &= \frac{c}{2}\int_\beta^T \mathbb{E}(v_\alpha^{t,v})\int_\alpha^T e^{-b(s-\alpha)} ds d\alpha \\
&= \frac{c}{2}\int_\beta^T ((v-a)e^{-b(\alpha-t)} + a)\frac{1}{b}(1-e^{-b(T-\alpha)})d\alpha \\
&= \frac{c}{2}(\frac{a}{b}(T-\beta) - \frac{v-a}{b}(T-\beta)e^{-b(T-t)} \\
&+ \frac{v-a}{b^2}(e^{-b(\beta-t)} - e^{-b(T-t)}) - \frac{a}{b^2}(1-e^{-b(T-\beta)}))
\end{aligned}$$

And $\overline{g_1}(t,x,v)$ has an explicit form :

$$\overline{g_1}(t,x,v) = -\frac{ce^{K-r(T-t)}\overline{d_2}N'(\overline{d_2})}{2b\mathbb{E}(\langle M^{t,v}\rangle_T)}(\frac{v-2a}{b}(1-e^{-b(T-t)}) + (T-t)(a-(v-a)e^{-b(T-t)}))$$

2.1.2. *Computation of* $\widetilde{g_0}(t,x,v)$. For $\widetilde{g_0}(t,x,v)$, it is given by :

$$\begin{aligned}\widetilde{g_0}(t,x,v) &= \overline{g_0}(t,x,v) + \frac{e^{K-r(T-t)}}{8\mathbb{E}(\langle M^{t,v}\rangle_T)^{\frac{3}{2}}}(\overline{d_2 d_1}-1)N'(\overline{d_2})Var(\langle M^{t,v}\rangle_T)\\ \overline{g_0}(t,x,v) &= e^x N(\overline{d_1}) - e^{K-r(T-t)}N(\overline{d_2})\end{aligned}$$

Then we need to know the value of the variance of $\langle M^{t,v}\rangle_T$, so $\mathbb{E}(\langle M^{t,v}\rangle_T^2)$

$$\begin{aligned}\mathbb{E}(\langle M^{t,v}\rangle_T^2) &= \mathbb{E}((\int_t^T v_u^{t,v}du)^2)\\ &= \mathbb{E}(\int_t^T\int_t^T v_u^{t,v}v_w^{t,v}dudw)\\ &= \int_t^T\int_t^T \mathbb{E}(v_u^{t,v}v_w^{t,v})dudw\end{aligned}$$

for $u \leq w$

$$\begin{aligned}v_w^{t,v} &= v_u^{t,v} + \int_u^w b(a-v_s^{t,v})ds + c\int_u^w \sqrt{v_s^{t,v}}dB_s\\ &= v_u^{t,v}e^{-b(w-u)} + ab\int_u^w e^{-b(w-s)}ds + c\int_u^w e^{-b(w-s)}\sqrt{v_s^{t,v}}dB_s\end{aligned}$$

Hence :

$$\mathbb{E}(v_u^{t,v}v_w^{t,v}) = e^{-b(w-u)}\mathbb{E}((v_u^{t,v})^2) + a(1-e^{-b(w-u)})\mathbb{E}(v_u^{t,v})$$

The calculation of $\mathbb{E}((v_u^{t,v})^2)$ is necessary . We obtain with $v_u^{t,v} = ve^{-b(u-t)} + a(1-e^{-b(u-t)}) + c\int_t^u e^{-b(u-s)}\sqrt{v_s^{t,v}}dB_s$

$$\begin{aligned}\mathbb{E}((v_u^{t,v})^2) &= (ve^{-b(u-t)} + a(1-e^{-b(u-t)}))^2 + c^2\int_t^u e^{-2b(u-s)}\mathbb{E}(v_s^{t,v})ds\\ &= (ve^{-b(u-t)} + a(1-e^{-b(u-t)}))^2 + c^2\Bigg((v-a)\frac{e^{-b(u-t)}-e^{-2b(u-t)}}{b}\\ &+ \frac{a}{2b}(1-e^{-2b(u-t)})\Bigg)\end{aligned}$$

Then, we have :

$$\begin{aligned}\mathbb{E}(v_u^{t,v}v_w^{t,v}) &= e^{-b(w-u)}\mathbb{E}((v_u^{t,v})^2) + a(1-e^{-b(w-u)})\mathbb{E}(v_u^{t,v})\\ &= \frac{1}{2b}(2\,e^{b(-w-u+2\,t)}bv^2 + 2\,bve^{b(t-w)}a - 4\,e^{b(-w-u+2\,t)}bva\\ &- 2\,ba^2e^{b(t-w)} + 2\,e^{b(-w-u+2\,t)}ba^2 - 2\,e^{b(-w-u+2\,t)}c^v\\ &+ e^{b(-w-u+2\,t)}c^2a + 2\,c^2e^{b(t-w)}v + e^{b(-w+u)}c^2a\\ &- 2\,c^2e^{b(t-w)}a + 2\,bve^{b(t-u)}a - 2\,ba^2e^{b(t-u)} + 2\,ba^2)\end{aligned}$$

We can compute $\mathbb{E}(\langle M^{t,v}\rangle_T^2)$ dividing the formula according to $u \leq w$ and $w \leq u$.

$$\begin{aligned}
\mathbb{E}(\langle M^{t,v}\rangle_T^2) &= \int_t^T \int_t^T \mathbb{E}(v_u^{t,v} v_w^{t,v}) du dw \\
&= \int_t^T \int_t^w \mathbb{E}(v_u^{t,v} v_w^{t,v}) du dw + \int_t^T \int_w^T \mathbb{E}(v_u^{t,v} v_w^{t,v}) du dw \\
&= \frac{1}{2b^3}(2\,c^2 v + 2\,bv^2 + c^2 a e^{-2\,b(T-t)} - 2\,c^2 v e^{-2\,b(T-t)} \\
&- 4\,v^2 e^{-b(T-t)} b - 4\,a^2 e^{-b(T-t)} b + 2\,v^2 e^{-2\,b(T-t)} b + 2\,b^3 a^2 T^2 \\
&- 4\,b^2 a^2 T - 4\,b^3 a^2 t T + 4\,b^2 v a T + 8\,v a e^{-b(T-t)} b + 2\,c^2 a T b + 4\,b^2 a^2 t \\
&+ 4\,v t c^2 e^{-b(T-t)} b - 4\,a t c^2 e^{-b(T-t)} b + 4\,b^2 v a t e^{-b(T-t)} \\
&- 4\,b^2 a^2 t e^{-b(T-t)} - 4\,v a e^{-2\,b(T-t)} b - 4\,b v a - 4\,b^2 v a t + 2\,a^2 e^{-2\,b(T-t)} b \\
&- 2\,b a t c^2 + 2\,b a^2 - 5\,c^2 a + 4\,c^2 a e^{-b(T-t)} + 2\,b^3 a^2 t^2 - 4\,c^2 v e^{-b(T-t)} b T \\
&+ 4\,c^2 a e^{-b(T-t)} b T - 4\,v a e^{-b(T-t)} b^2 T + 4\,a^2 e^{-b(T-t)} b^2 T)
\end{aligned}$$

Then for the variance of $\langle M^{t,v}\rangle_T$:

$$\begin{aligned}
Var(\langle M^{t,v}\rangle_T) &= \mathbb{E}(\langle M^{t,v}\rangle_T^2) - (\mathbb{E}(\langle M^{t,v}\rangle_T))^2 \\
&= \frac{c^2}{2b^3}(2\,v + 4\,v t e^{-b(T-t)} b - 2\,v e^{-2\,b(T-t)} + a e^{-2\,b(T-t)} + 4\,a e^{-b(T-t)} \\
&+ 2\,a b T + 4\,a e^{-b(T-t)} b T - 4\,a t e^{-b(T-t)} b - 4\,v e^{-b(T-t)} b T - 2\,a b t - 5\,a)
\end{aligned}$$

2.1.3. *Computation of Put and Hedging.* We have for the put in Heston model : $p_{exp}(0,x,v;\rho) = u_{exp}(0,x,v;\rho) - S_0 + e^{K-rT}$ which gives us immediatly the result.

For the hedging :

$$\begin{aligned}
h_0(t,x,v) &:= e^x N\Big(d_1(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)})\Big) \\
h_1(t,x,v) &:= -e^{K-r(T-t)}(1-d_2^2)N'(d_2)(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)})\frac{\mathbb{E}(c_{[t,T]}^{t,v})}{(\mathbb{E}(\langle M^{t,v}\rangle_T))^{\frac{3}{2}}}
\end{aligned}$$

$h_0(t,x,v)$ is explicit and for $h_1(t,x,v)$ we just have to compute $\mathbb{E}(c_{[t,T]}^{t,v})$ which has been already made for $\overline{g_1}(t,x,v)$:

$$\mathbb{E}(c_{[t,T]}^{t,v}) = \frac{c}{2}(\frac{a}{b}(T-t) - \frac{v-a}{b}(T-t)e^{-b(T-t)} + \frac{v-2a}{b^2}(1-e^{-b(T-t)}))$$

Then the calculation of $\delta_{c_{exp}}$ and $\delta_{p_{exp}}$ are explicit.

3. Stein and Stein model

In the generalized Stein and Stein model, the stochastic volatility is described by means of Ornstein-Uhlenbeck process, hence :

$$v_s^{t,v} = v + \int_t^s b(a - v_u^{t,v})du + \int_t^s c dB_u \tag{42a}$$

$$S_s^{t,e^x,v} = e^x + \int_t^s S_u^{t,e^x,v}(rdu + v_u^{t,v})dZ_u \tag{42b}$$

So :

$$\begin{array}{rclccc} f(v) & = & v, & \mu(v) = b(a-v), & \eta(v) = c \\ f'(v) & = & 1, & \mu'(v) = -b, & \eta'(v) = 0 \end{array}$$

Then we obtain :

$$\langle M^{t,v} \rangle_s = \int_t^s (v_u^{t,v})^2 du$$

The stochastic volatility follows a Vasicek's model, so we have :

$$\begin{aligned} v_s^{t,v} &= ve^{-b(s-t)} + a(1 - e^{-b(s-t)}) + ce^{-bs} \int_t^s e^{bu} dB_u \\ \mathbb{E}(v_s^{t,v}) &= ve^{-b(s-t)} + a(1 - e^{-b(s-t)}) \\ \mathbb{E}(\langle M^{t,v} \rangle_T) &= \left(a^2 + \frac{c^2}{2b}\right)(T-t) + 2a(v-a)\frac{1 - e^{-b(T-t)}}{b} \\ &+ \left((v-a)^2 - \frac{c^2}{2b}\right)\frac{1 - e^{-2b(T-t)}}{2b} \end{aligned}$$

And for $\alpha \leq s$

$$\begin{aligned} \mathbb{E}(v_s^{t,v} v_\alpha^{t,v}) &= (a + (v-a)e^{-b(\alpha-t)})(a + (v-a)e^{-b(s-t)}) + \frac{c^2}{2b}(e^{-b(s-\alpha)} - e^{-b[(\alpha-t)+(s-t)]}) \\ \frac{\partial v_s^{t,v}}{\partial v} &= e^{-b(s-t)} \\ c_{[\beta,T]}^{t,v} &= c \int_\beta^T v_\alpha^{t,v} \int_\alpha^T v_s^{t,v} e^{-b(s-\alpha)} ds d\alpha \end{aligned}$$

3.1. **Computation of $\overline{g_1}(t,x,v)$.** We can compute $\overline{g_1}(t,x,v)$ which is given by :

$$\overline{g_1}(t,x,v,\rho) = -ce^{K-r(T-t)} \frac{d_2 N'(d_2)}{\mathbb{E}(\langle M^{t,v} \rangle_T)} \int_t^T \int_\alpha^T e^{-b(s-\alpha)} \mathbb{E}(v_\alpha^{t,v} v_s^{t,v}) ds d\alpha$$

The computation of $\overline{g_1}(t,x,v,\rho)$ is a double integral which is elementary and we can obtain an explicit and analytical expression.

$$\begin{array}{rcl}\overline{g_1}(t,x,v,\rho) & = & -ce^{K-r(T-t)}\dfrac{d_2N'(d_2)}{\mathbb{E}(\langle M^{t,v}\rangle_T)}\dfrac{1}{4b^3}\Big[-4ab^2v(T-t)(e^{-b(T-t)}-e^{-2b(T-t)}) \\ & + & bc^2(T-t)(1-e^{-2b(T-t)})-2b^2(T-t)e^{-2b(T-t)}(v^2+a^2)+4abv(1-e^{-b(T-t)})^2 \\ & + & 4a^2b^2(T-t)(e^{-b(T-t)}+1)+bv^2(1-e^{-b(T-t)}) \\ & - & c^2(1-e^{-b(T-t)})-3a^2b(e^{-2b(T-t)}+4e^{-b(T-t)}-3)\Big]\end{array}$$

3.2. **Computation of $\widetilde{g_0}(t,x,v)$.** To compute $\widetilde{g_0}(t,x,v)$ we just need to have $Var(\langle M^{t,v}\rangle_T)$, so
$\mathbb{E}((\langle M^{t,v}\rangle_T)^2)$:

$$\begin{array}{rcl}\mathbb{E}(\langle M^{t,v}\rangle_s^2) & = & (\displaystyle\int_t^s (v_u^{t,v})^2du)^2 \\ & = & \displaystyle\int_t^s\int_t^s \mathbb{E}((v_u^{t,v}v_w^{t,v})^2)dudw\end{array}$$

To compute $\mathbb{E}((v_u^{t,v}v_w^{t,v})^2)$ we use the fact that $(v_u^{t,v})_{t\le u\le T}$ is a Gaussian process.
For $w\le u$

$$v_u^{t,v}\sim\mathcal{N}(ve^{-b(u-t)}+a(1-e^{-b(u-t)}),\frac{c^2(1-e^{-2b(u-t)})}{2b})$$

$$\begin{pmatrix}v_u^{t,v}\\ v_w^{t,v}\end{pmatrix}\sim\mathcal{N}_2\begin{pmatrix}ve^{-b(u-t)}+a(1-e^{-b(u-t)})\\ ve^{-b(w-t)}+a(1-e^{-b(w-t)})\end{pmatrix},\Sigma\Bigg)$$

where :

$$\Sigma=\begin{pmatrix}\frac{c^2}{2b}(1-e^{-2b(u-t)}) & \frac{c^2}{2b}e^{-b(u+w)}(e^{2bw}-e^{2bt})\\ \frac{c^2}{2b}e^{-b(u+w)}(e^{2bw}-e^{2bt}) & \frac{c^2}{2b}(1-e^{-2b(w-t)})\end{pmatrix}$$

If we have :

$$\begin{pmatrix}X\\ Y\end{pmatrix}\sim\mathcal{N}_2\left(\begin{matrix}0\\0\end{matrix},\begin{pmatrix}a & b\\ b & c\end{pmatrix}\right)$$

Then $cov(Y-\frac{bX}{a},X)=0$
Our problem is to compute $\mathbb{E}((x+X)^2(y+Y)^2)$
And we have :

$$\begin{array}{rcl}\mathbb{E}((x+X)^2(y+Y)^2) & = & \mathbb{E}((x^2+2xX+X^2)(y^2+2yY+Y^2)) \\ & = & x^2y^2+x^2c+4xyb+y^2a+ac+2b^2\end{array}$$

Then we can compute without any problem $\mathbb{E}((v_u^{t,v}v_w^{t,v})^2)$ for $u \leq w$ and $w \leq u$ replacing x, y, X, Y with the corresponding terms. Just with the separation of an integral we obtain $\mathbb{E}((\langle M^{t,v}\rangle_T)^2)$ and then $Var(\langle M^{t,v}\rangle_T)$.

$$\begin{aligned}
Var(\langle M^{t,v}\rangle_T) &= \frac{-c^2}{8b^4}\Big[32\,b^2vate^{-2\,b(T-t)} - c^2e^{-4\,b(T-t)} \\
&+ 16\,vae^{-3\,b(T-t)}b + 4\,v^2e^{-4\,b(T-t)}b + 16\,v^2e^{-2\,b(T-t)}b^2T - 16\,b^2a^2te^{-2\,b(T-t)} \\
&- 112\,a^2be^{-b(T-t)} + 48\,ba^2e^{-2\,b(T-t)} - 16\,b^2v^2te^{-2\,b(T-t)} \\
&- 16\,a^2e^{-3\,b(T-t)}b + 16\,a^2e^{-2\,b(T-t)}b^2T + 4\,a^2e^{-4\,b(T-t)}b - 8\,vae^{-4\,b(T-t)}b \\
&- 32\,b^2vate^{-b(T-t)} + 8\,c^2te^{-2\,b(T-t)}b - 8\,c^2e^{-2\,b(T-t)}bT + 5\,c^2 - 4\,c^2e^{-2\,b(T-t)} \\
&+ 76\,ba^2 + 32\,b^2a^2t + 4\,c^2bt - 4\,v^2b + 48\,abve^{-b(T-t)} - 32\,avbe^{-2\,b(T-t)} - 32\,b^2a^2T \\
&- 4\,c^2bT - 24\,avb + 32\,vae^{-b(T-t)}b^2T + 32\,b^2a^2te^{-b(T-t)} \\
&- 32\,a^2e^{-b(T-t)}b^2T - 32\,vae^{-2\,b(T-t)}b^2T\Big]
\end{aligned}$$

3.3. **Computation of Put and Hedging.** We have for the put in Stein and Stein model :
$p_{exp}(0,x,v;\rho) = u_{exp}(0,x,v;\rho) - S_0 + e^{K-rT}$ which gives us immediatly the result.

For the hedging :

$$\begin{aligned}
h_0(t,x,v) &:= e^xN\Big(d_1(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)})\Big) \\
h_1(t,x,v) &:= -e^{K-r(T-t)}(1-d_2^2)N'(d_2)(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)})\frac{\mathbb{E}(c_{[t,T]}^{t,v})}{(\mathbb{E}(\langle M^{t,v}\rangle_T))^{\frac{3}{2}}}
\end{aligned}$$

$h_0(t,x,v)$ is explicit and for $h_1(t,x,v)$ we just have to compute $\mathbb{E}(c_{[t,T]}^{t,v})$ which is easily computable as in $\overline{g_1}(t,x,v)$:

$$\begin{aligned}
\mathbb{E}(c_{[t,T]}^{t,v}) &= c\int_t^T\int_\alpha^T \mathbb{E}(v_\alpha^{t,v}v_s^{t,v})e^{-b(s-\alpha)}dsd\alpha \\
&= \frac{c}{4b^3}\Big[-4ab^2v(T-t)(e^{-b(T-t)} - e^{-2b(T-t)}) \\
&+ bc^2(T-t)(1-e^{-2b(T-t)}) - 2b^2(T-t)e^{-2b(T-t)}(v^2+a^2) + 4abv(1-e^{-b(T-t)})^2 \\
&+ 4a^2b^2(T-t)(e^{-b(T-t)}+1) + bv^2(1-e^{-b(T-t)}) \\
&- c^2(1-e^{-b(T-t)}) - 3a^2b(e^{-2b(T-t)} + 4e^{-b(T-t)} - 3)\Big]
\end{aligned}$$

Then the calculation of $\delta_{c_{exp}}$ and $\delta_{p_{exp}}$ are explicit.

4. Hull and White model

In the Hull and White model, the stochastic volatility is described by :

$$v_s^{t,v} = v + \int_t^s \mu v_u^{t,v} du + \int_t^s c v_u^{t,v} dB_u \tag{55a}$$

$$S_s^{t,e^x,v} = e^x + \int_t^s S_u^{t,e^x,v}(rdu + v_u^{t,v})dZ_u \tag{55b}$$

So,

$$\begin{array}{rclccc} f(v) & = & v, & \mu(v) = \mu v, & \eta(v) = cv \\ f'(v) & = & 1, & \mu'(v) = \mu, & \eta'(v) = c \end{array}$$

$v_\cdot^{t,v}$ follows the Black and Scholes's model, then :

$$\begin{aligned} v_u^{t,v} &= ve^{(\mu - \frac{c^2}{2})(u-t)+cB_u)} \\ (v_u^{t,v})^2 &= v^2 e^{(2\mu+c^2)(u-t)+(2c)B_u - \frac{(2c)^2}{2}(u-t)} \\ \mathbb{E}(\langle M^{t,v}\rangle_s) &= \mathbb{E}(\int_t^s (v_u^{t,v})^2 du) \\ &= \int_t^s v^2 e^{(2\mu+c^2)(u-t)} du \\ &= \frac{v^2}{2\mu+c^2}(e^{(2\mu+c^2)(s-t)} - 1) \end{aligned}$$

using the fact that : $e^{\sigma B_u - \frac{\sigma^2}{2}u}$ is a martingale for B_u a Brownian motion.

4.1. **Computation of $\overline{g_1}(t,x,v)$.** We can compute easily $\overline{g_1}(t,x,v)$ since we have the expression of $v_u^{t,v}$.

$$\overline{g_1}(t,x,v) = -e^{K-r(T-t)} \frac{\overline{d_2} N'(\overline{d}_2)}{\mathbb{E}(\langle M^{t,v}\rangle_T)} \mathbb{E}(c_{[t,T]}^{t,v})$$

And $c_{[t,T]}^{t,v}$ is given by :

$$c_{[\beta,T]}^{t,v} = c\int_\beta^T v_\alpha^{t,v} \int_\alpha^T (v_s^{t,v})^2 ds d\alpha$$

because

$$\frac{\partial v_s^{t,v}}{\partial v} = \frac{v_s^{t,v}}{v}.$$

And for $\mathbb{E}(v_\alpha^{t,v}(v_s^{t,v})^2)$ we have, when $\alpha \le s$:

$$\begin{aligned} v_s^{t,v} &= v_\alpha^{t,v} exp(\mu(s-\alpha) - \frac{c^2}{2}(s-\alpha) + cB_s) \\ (v_s^{t,v})^2 v_\alpha^{t,v} &= v^3 exp\Bigg((3\mu + 3c^3)(\alpha - t) - \frac{9c^2}{2}(\alpha - t) + 3c(B_\alpha - B_t \\ &+ (2\mu + c^2)(s-\alpha) + 2c(B_s - B_\alpha) - 2c^2(s-\alpha)\Bigg) \\ \mathbb{E}(v_\alpha^{t,v}(v_s^{t,v})^2) &= v^3 exp((3\mu + 3c^2)(\alpha - t) + (2\mu + c^2)(s-\alpha)) \end{aligned}$$

So, we have :

$$\begin{aligned}\overline{g_1}(t,x,v) &= -ce^{K-r(T-t)}\frac{\overline{d_2}N'(\overline{d_2})}{\mathbb{E}(\langle M^{t,v}\rangle_T)}\left(\frac{e^{(3\mu+3c^2)(T-t)}(2\mu+c^2)}{(\mu+2c^2)(3\mu+3c^2)}\right.\\ &- \left.\frac{e^{(2\mu+c^2)(T-t)}}{\mu+2c^2}+\frac{1}{3\mu+3c^2}\right)\frac{v^3}{2\mu+c^2}\end{aligned}$$

4.2. **Computation of $\widetilde{g_0}(t,x,v)$.** The problem is the same as in the two previous models : the variance of $\langle M^{t,v}\rangle_T$

$$\begin{aligned}\mathbb{E}((\langle M^{t,v}\rangle_T)^2) &= \mathbb{E}((\int_t^T(v_s^{t,v})^2)^2)\\ &= \int_t^T\int_t^T \mathbb{E}((v_s^{t,v}v_\alpha^{t,v})^2)dsd\alpha\\ &= \int_t^T\int_t^\alpha \underbrace{\mathbb{E}((v_s^{t,v}v_\alpha^{t,v})^2)}_{s\le\alpha}dsd\alpha+\int_t^T\int_\alpha^T\underbrace{\mathbb{E}((v_s^{t,v}v_\alpha^{t,v})^2)}_{\alpha\le s}dsd\alpha\end{aligned}$$

Calculation of $\mathbb{E}\big((v_s^{t,v}v_\alpha^{t,v})^2\big)$ for $\alpha\le s$:

$$\begin{aligned}v_s^{t,v} &= v_\alpha^{t,v}exp(\mu(s-\alpha)-\frac{c^2}{2}(s-\alpha)+cB_s)\\ (v_s^{t,v}v_\alpha^{t,v})^2 &= v^4exp((4\mu+6c^2)(\alpha-t)-8c^2(\alpha-t)+4c(B_t-B_\alpha)+(2\mu+c^2)(s-\alpha)-2c^2(s-\alpha)+2c\\ \mathbb{E}((v_\alpha^{t,v}v_s^{t,v})^2) &= v^4exp((4\mu+6c^2)(\alpha-t)+(2\mu+c^2)(s-\alpha))\\ \mathbb{E}((\langle M^{t,v}\rangle_T)^2) &= \frac{v^4(2\mu+5c^2-(4\mu+6c^2)e^{(2\mu+c^2)(T-t)}+(2\mu+c^2)e^{(4\mu+6c^2)(T-t)})}{(2\mu+5c^2)(2\mu+c^2)(2\mu+3c^2)}\end{aligned}$$

Then, we have :

$$Var(\langle M^{t,v}\rangle_T)=\mathbb{E}((\langle M^{t,v}\rangle_T)^2)-\mathbb{E}(\langle M^{t,v}\rangle_T)^2$$

which can be computed explicity. And the result is :

$$\begin{aligned}\widetilde{g_0}(t,x,v) &= e^xN(d_1)-e^{K-r(T-t)}N(d_2)\\ &+ \frac{e^{K-r(T-t)}}{8\mathbb{E}(\langle M^{t,v}\rangle_T)^{\frac{3}{2}}}(d_2d_1-1)N'(d_2)Var(\langle M^{t,v}\rangle_T)\end{aligned}$$

4.3. **Computation of Put and Hedging.** We have for the put in Hull and White model :
$p_{exp}(0,x,v;\rho)=u_{exp}(0,x,v;\rho)-S_0+e^{K-rT}$ which gives us immediatly the result.

For the hedging :

$$\begin{aligned}h_0(t,x,v) &:= e^xN\Big(d_1(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)})\Big)\\ h_1(t,x,v) &:= -e^{K-r(T-t)}(1-d_2^2)N'(d_2)(t,x,\sqrt{\mathbb{E}(\langle M^{t,v}\rangle_T)})\frac{\mathbb{E}(c_{[t,T]}^{t,v})}{(\mathbb{E}(\langle M^{t,v}\rangle_T))^{\frac{3}{2}}}\end{aligned}$$

$h_0(t,x,v)$ is explicit and for $h_1(t,x,v)$ we just have to compute $\mathbb{E}(c^{t,v}_{[t,T]})$ which is easily computable as in $\overline{g_1}(t,x,v)$:

$$\mathbb{E}(c^{t,v}_{[t,T]}) = \frac{cv^3}{2\mu+c^2}\left(\frac{e^{(3\mu+3c^2)(T-t)}(2\mu+c^2)}{(\mu+2c^2)(3\mu+3c^2)} - \frac{e^{(2\mu+c^2)(T-t)}}{\mu+2c^2} + \frac{1}{3\mu+3c^2}\right)$$

Then the calculation of $\delta_{c_{exp}}$ and $\delta_{p_{exp}}$ are explicit.

References

[1] Antonelli, F. Scarlatti, S. Pricing options under stochastic volatility : a power series approach, Finance Stoch **13**, 269-303 (2009)
3, 4, 5, 6

[3] Lamberton, D. Lapeyre, B. Introduction au calcul stochastique appliqué à la finance, Mathématiques et Applications, Paris (1992)