Competitive Monte Carlo methods for the Pricing of Asian Options

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Abstract

We explain how a carefully chosen scheme can lead to competitive Monte Carlo algorithm for the computation of the price of Asian options. We give evidence of the efficiency of these algorithms with a mathematical study of the rate of convergence and a numerical comparison with some existing methods. KEY WORDS: Asian option, Monte Carlo methods, Numerical methods, Diffusion process.

1 Introduction

Monte Carlo methods are known to be useful when the state dimension is large. This is widely true but we will give here an example of a small dimension problem coming from finance where a Monte Carlo (helped by a variance reduction technique) can be more efficient than other known methods.

This example is based on the price of an Asian option (see subsection 2.1). This problem is known to be computationally hard and a lot of literature deals with this problem: using either analytic methods ([7], [6]), numerical methods based on the partial differential equation associated ([3], [5], [9], [12]) or Monte Carlo methods ([10]). The originality of this work is to propose (and to give precise results) new time schemes approximation for the integral of the Black & Scholes model: this point is really a source of concern when using Monte Carlo methods as noted by Madan, Fu and Wang ([6], page 14). We will show that, when we use a suitable scheme and variance reduction, a Monte Carlo method can be more competitive than other methods under some circumstances. In order to get precise and complete numerical results, we have undertaken extensive comparisons with most of the other known methods : Forward Shooting Grid ([2]), Hull and White ([8]), Finite Difference and already quoted Monte Carlo methods. For moderate precision 10^{-2}, tree methods (FSG and Hull and White) are the most

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efficient. But, surprisingly, really large precision (10^{-4}) can only be reached by Monte Carlo method with relatively large time step ($\sim$ 1 month).
This article is organized in the following way: we first introduce the mathematical context, then we present the Monte Carlo schemes in sections 1-2, section 3 contains the proof of their convergence in the L^p spaces and section 4 quickly describes the variance reduction method used in this article. The end of the paper is devoted to numerical tests and comparisons.

2 Mathematical Context

2.1 Financial Background

To describe the price of an asset at time t, we use the Black and Scholes model with a risky asset (a share of price S_t at time t) and a no-risk asset (whose price is S_t^0 at time t). The price S_t is given by the following stochastic differential equation

$$dS_t = S_t(\mu dt \ + \ \sigma dB_t),$$

where μ and σ are two positive constants and (B_t) is a standard Brownian motion. The price of the no-risk asset satisfies the ordinary differential equation

$$dS_t^0 = rS_t^0 dt.$$

As usual, we introduce the process $W_t = B_t + \frac{\mu - r}{\sigma} t$ which is a Brownian motion under an adapted probability, called the neutral risk probability and denoted by $\mathbb{P}$. Thus, the risky asset satisfies a new stochastic differential equation

$$dS_t = S_t(rdt \ + \ \sigma dW_t),$$

whose solution is

$$S_t = S_{T_0} \exp\left(\sigma W_t - \frac{\sigma^2}{2} t + rt\right).$$

S_{T_0} is the price of the asset at the beginning of the modeling.
Asian options (or options on average) is the general name for a class of options whose payoff depends of the mean of the price of the risky asset on a given period. Thus, the price of an Asian option with maturity T can be written:

$$V(t, S, A) = e^{-r(T-t)} \mathbb{E} f\left(S_t, A_S(T_0, t)\right).$$

where

$$A_S(T_0, t) = \frac{1}{t - T_0} \int_{T_0}^{t} S_u du.$$

From now on, we choose $T_0 = 0$, with no lost of generality. The function f depends of the type of the options

- For a call with fixed strike: $f(s, a) = (a - K)_+$
- For a put with fixed strike: $f(s, a) = (K - a)_+$
- For a call with floating strike: $f(s, a) = (s - a)_+$
- For a put with floating strike: $f(s, a) = (a - s)_+$

2.2 Practical Schemes

If we want to use Monte Carlo methods to compute a price, we have to simulate the average of S_t, therefore we need to approximate an integral. Here, it is not necessary to approximate S_t because it can be exactly simulated. With this aim in view, the interval $[0, T]$ will be divided into N steps. The step size will be noted $h = T/N$, and we define the times $t_k = kT/N = kh$.

The standard scheme We introduce three schemes to estimate $Y_T = \int_0^T S_u du$. Since we are able to simulate S_t at a given t, the integral can be approached by Riemann sums:

$$\mathbf{Y_T^{r,n} = h \sum_{k=0}^{n-1} S_{t_k}}. \tag{1}$$

For example, if M denote the number of drawing in the Monte Carlo method, an approximation of the price at maturity of a fixed strike Asian call is given by

$$\frac{e^{-rT}}{M} \sum_{j=1}^{M} \left(\frac{h}{T} \sum_{k=0}^{n-1} S_{t_k} - K \right)_+ .$$

Note that the time complexity of this algorithm is $O\left(\frac{1}{NM}\right)$ (this is true for every kind of Monte Carlo methods) and that it involves two kinds of errors: the Monte Carlo error and the time step error. The Monte Carlo error is of order $\frac{\sigma}{\sqrt{M}}$ and the time step error is harder to evaluate (see proposition 3.3).
The scheme (1) can be interpreted as the second variable in the Euler approximation for the following stochastic differential equation

$$dU_t = B(U_t)dt + \Sigma(U_t)dW_t \quad \text{with } U_t = \begin{bmatrix} S_t \\ Y_t \end{bmatrix},\ B(U_t) = \begin{bmatrix} rS_t \\ S_t \end{bmatrix} \text{ and } \Sigma(U_t) = \begin{bmatrix} \sigma(S_t) \\ 0 \end{bmatrix}$$

Higher accuracy schemes One way to obtain higher accuracy for the integral approximation is to remark that in L^2, the closest random variable to $\left(\frac{1}{T}\int_0^T S_s ds - K\right)_+$ when the $(S_{t_k},\ k = 0, \dots, N)$ are known is given by:

$$\mathbb{E}\left(\left(\frac{1}{T} \int_0^T S_u du - K \right)_+ \,\middle|\, \mathcal{B}_h \right), \tag{2}$$

with $\mathcal{B}_h$ is the σ-field generated by the $(S_{t_k},\ k = 0, \dots, N)$. Of course, it is theoretically impossible to compute explicitly this conditional expectation (it is certainly harder than to obtain an explicit formula for V). But, since the conditional law of W_u with respect to $\mathcal{B}_h$ for $u \in [t_k, t_{k+1}]$ is given by

$$\mathcal{L}(W_u \mid W_{t_k} = x,\ W_{t_{k+1}} = y) = \mathcal{N}\left(\frac{t_{k+1} - u}{h}x + \frac{u - t_k}{h}y, \frac{(t_{k+1} - u)(u - t_k)}{h}\right), \quad (3)$$

one can in principle compute

$$\left(\mathbb{E}\left(\frac{1}{T}\int_0^T S_u du \,\middle|\, \mathcal{B}_h\right) - K\right)_+ = \left(\frac{1}{T}\int_0^T \mathbb{E}\left(S_u \middle| \mathcal{B}_h\right) du \; - K\right)_+ \quad (4)$$

as a function of $(W_{t_k},\ k = 0, \dots, N)$. Jensen inequality proves that (4) is smaller than (2), but we will see that (2) is already a really good approximation of Y_T (see proposition 3.3). Using the law described by (3), we get

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du \,\middle|\, \mathcal{B}_h\right] \\
&\quad = \frac{1}{T}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} e^{(r-\frac{\sigma^2}{2})u} e^{\sigma\frac{t_{k+1}-u}{h}W_{t_k} + \sigma\frac{u-t_k}{h}W_{t_{k+1}} + \frac{\sigma^2}{2}\frac{(t_{k+1}-u)(u-t_k)}{h}} du \\
&\quad = \frac{1}{T}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} e^{\sigma\frac{u-t_k}{h}(W_{t_{k+1}} - W_{t_k}) - \frac{\sigma^2}{2}\frac{(u-t_k)^2}{h} + ru} e^{\sigma W_{t_k} - \frac{\sigma^2}{2}t_k} du
\end{aligned}$$

In Monte Carlo simulation, this approximation is used in a double loop (in times and in the number of simulation), so it is really necessary to simplify this formula. Hence, a formal Taylor expansion can be done with h small, which leads to a more practical scheme:

$$\mathbf{Y_T^{e,n}} = \frac{\mathbf{h}}{\mathbf{T}}\sum_{\mathbf{k=0}}^{\mathbf{n-1}} \mathbf{S_{t_k}}\left(\mathbf{1} + \frac{\mathbf{rh}}{\mathbf{2}} + \sigma\frac{\mathbf{W_{t_{k+1}}} - \mathbf{W_{t_k}}}{\mathbf{2}}\right). \quad (5)$$

Remark 2.1. Note that this scheme is equivalent to the well known trapezoidal method. We will prove that,

$$\mathbb{E}\left(Y_T^{e,n} - \frac{1}{T}\sum_{k=0}^{n-1} h\frac{S_{t_k} + S_{t_{k+1}}}{2}\right)^2 = O\left(\frac{1}{n^3}\right).$$

So, since the rate of convergence of (5) is in $1/n$ (see proposition 3.3), the error above is negligible.

Proof. This result can be obtained using a Taylor expansion:

$$\begin{aligned}
\frac{1}{T}\sum_{k=0}^{n-1} h\frac{S_{t_k} + S_{t_{k+1}}}{2} &= \frac{1}{T}\sum_{k=0}^{n-1}\frac{hS_{t_k}}{2}(e^{\sigma(W_{t_{k+1}} - W_{t_k}) - \frac{\sigma^2}{2}h + rh} + 1) \\
&= \frac{1}{T}\sum_{k=0}^{n-1}\frac{hS_{t_k}}{2}(2 + \sigma(W_{t_{k+1}} - W_{t_k}) + rh) + \dots
\end{aligned}$$

which is exactly the scheme (5). The term of order $\sigma^2 h$ and the quadratic variation of $\sigma(W_{t_{k+1}} - W_{t_k})$ cancel each other out, so we does not retain it. The mathematical proof of this remark is the same as proposition 3.3. □

Our last scheme is quite similar. As the Brownian motion is a Gaussian process, $\int_0^T W_u du$ has a normal density w.r.t. the Lebesgues measure on $\mathbb{R}$ and can easily be simulated. Hence,

$$\begin{aligned} Y_T &= \frac{1}{T}\int_0^T S_u du \\ &= \frac{1}{T}\sum_{k=0}^{n-1} S_{t_k}\int_{t_k}^{t_{k+1}} e^{\sigma(W_u - W_{t_k}) - \frac{\sigma^2}{2}(u-t_k) + r(u-t_k)} du. \end{aligned}$$

So, using a Taylor expansion again, we obtain the scheme:

$$\mathbf{Y_T^{p,n}} = \frac{\mathbf{1}}{\mathbf{T}}\sum_{\mathbf{k=0}}^{\mathbf{n-1}} \mathbf{S_{t_k}}\left(\mathbf{h} + \frac{\mathbf{rh^2}}{\mathbf{2}} + \sigma\int_{\mathbf{t_k}}^{\mathbf{t_{k+1}}} (\mathbf{W_u} - \mathbf{W_{t_k}})\mathbf{du}\right). \tag{6}$$

Remark 2.2. In practice, to simulate this scheme, we have, at each time step, to simulate $W_{t_{k+1}}$ knowing W_{t_k} and $(\int_{t_k}^{t_{k+1}} W_u du \mid W_{t_k}, W_{t_{k+1}})$. For the second random variable we use the law (2) and for the first one the fact that $(W_{t_{k+1}} - W_{t_k},\ k = 0, \ldots, N-1)$ are Gaussian i.i.d. variables.

Remark 2.3. This scheme can be generalized easily to a higher class of diffusion process. Let S_t be a diffusion with a drift $b(S_t)$ and a diffusion term $\sigma(S_t)$, and $Y_t = \int S_t dt$. One can use the Euler scheme S_t^n to simulate S_t and

$$Y_T^n = \sum_{k=0}^{n-1} S_{t_k}^n\left(h + \int_{t_k}^{t_{k+1}} (S_s^n - S_{t_k}^n) ds\right).$$

Note that when the diffusion S_t can be directly simulated, the following scheme is sufficient

$$Y_T^n = \sum_{k=0}^{n-1} S_{t_k}\left(h + \int_{t_k}^{t_{k+1}} (S_s^n - S_{t_k}) ds\right),$$

We can show that the weak convergence holds at the rate $1/n^{3/2}$.

We will now prove some results on the speed of convergence of these 3 schemes. The weak convergence can be obtained for the scheme (1) (at the rate $1/n$) and (6) (at the rate $1/n^{3/2}$) under a weak assumption on f using Malliavin calculus and techniques developed by Bally and Talay ([1]). We refer to [13] for a complete proof of these convergence results.

3 Convergence in the L^p spaces

In this section, we are interested by the strong convergence of the schemes. In L^p, one can only have some upper bound of the rate of convergence, but the precise expansion can only be obtained in the particular case $p = 2$.
Let us start with two well known but important results.

Proposition 3.1. *For a Black and Scholes diffusion,*

$$\mathbb{E}|S_t - S_s|^{2q} \leq C_q|t-s|^q$$

This proposition is available for any diffusion with Lipschitz coefficient (see [11]).
The following will also be useful (see [4] chapter 3 for a proof):

Lemma 3.1. *Let $Z_t = Z_0 + \int_0^t A_s dW_s + \int_0^t B_s ds$ where B_s is a vector in $\mathbb{R}^n$, A_s a matrix in $\mathbb{R}^{n\times d}$, and W_t a d dimensional Brownian motion. (Z_t is an Itô process so A and B are adapted, $\int |A_s|ds < +\infty$ and $\mathbb{E}\int B_s^2 ds < +\infty$).*
Then, Z_t satisfies

$$\mathbb{E}|Z_t|^p \leq \mathbb{E}|Z_0|^p + C\int_0^t \mathbb{E}(|Z_s|^p + |A_s|^p + |B_s|^p)ds$$

We can now obtain precise results for the rate of convergence of our schemes.

Proposition 3.2. *With the above notations, there exist three non decreasing maps $K_1(T)$, $K_2(T)$, $K_3(T)$ such that,*

$$\left(\mathbb{E}\left(\sup_{t\in[0,T]}|Y_t^{r,n} - Y_t|^{2q}\right)\right)^{\frac{1}{2q}} \leq \frac{K_1(T)}{n} \tag{7}$$

$$\left(\mathbb{E}\left(\sup_{t\in[0,T]}|Y_t^{e,n} - Y_t|^{2q}\right)\right)^{\frac{1}{2q}} \leq \frac{K_2(T)}{n} \tag{8}$$

$$\left(\mathbb{E}\left(\sup_{t\in[0,T]}|Y_t^{p,n} - Y_t|^{2q}\right)\right)^{\frac{1}{2q}} \leq \frac{K_3(T)}{n^{3/2}} \tag{9}$$

Proof. Let us begin with the inequality (7). If we note $\epsilon_t = Y_t^{r,n} - Y_t$, one has

$$\begin{aligned}\forall t \in [t_k, t_{k+1}] \quad \epsilon_t &= \epsilon_{t_k} + \int_{t_k}^t (S_s - S_{t_k})ds \\ &= \epsilon_{t_k} + \int_{t_k}^t (t-u) r S_u du + \int_{t_k}^t (t-u)\sigma S_u dW_u\end{aligned}$$

So, the lemma 3.1 implies that

$$\forall t \in [t_k, t_{k+1}] \quad \mathbb{E}|\epsilon_t|^{2q} \leq (\mathbb{E}|\epsilon_{t_K}|^{2q} + h^{2q+1}) + \int_{t_k}^t \mathbb{E}|\epsilon_s|^{2q} ds.$$

Applying Gronwall's lemma, we get that

$$\mathbb{E}|\epsilon_{t_{k+1}}|^{2q} \leq (\mathbb{E}|\epsilon_{t_K}|^{2q} + h^{2q+1})e^h$$

Since $x_{n+1} \leq ax_n + b$ implies $x_n \leq a^n x_0 + ne^{n(a-1)}b$ for $a \geq 1$, we have

$$\mathbb{E}|\epsilon_{t_k}|^{2q} \leq Ch^{2q}$$

We conclude with the inequality of Burkholder-Davis-Gundy. We use the same kind of arguments for the two other schemes writing them as:

$$\begin{aligned} Y_t - Y_t^{e,n} &= Y_{t_k} - Y_{t_k}^{e,n} + \int_{t_k}^t \int_{t_k}^s r(S_u - S_{t_k})duds + \int_{t_k}^t \int_{t_k}^s \sigma(S_u - \frac{h}{2}S_{t_k})dW_u ds \\ &= Y_{t_k} - Y_{t_k}^{e,n} + \int_{t_k}^t r(t-u)(S_u - S_{t_k})du + \int_{t_k}^t \sigma((t-u)S_u - \frac{h}{2}S_{t_k})dW_u \end{aligned}$$

and,

$$\begin{aligned} Y_t - Y_t^{p,n} &= Y_{t_k} - Y_{t_k}^{p,n} + \int_{t_k}^t \int_{t_k}^s r(S_u - S_{t_k})duds + \int_{t_k}^t \int_{t_k}^s \sigma(S_u - S_{t_k})dW_u ds \\ &= Y_{t_k} - Y_{t_k}^{e,n} + \int_{t_k}^t r(t-u)(S_u - S_{t_k})du + \int_{t_k}^t \sigma(t-u)(S_u - S_{t_k})dW_u. \end{aligned}$$

From this point, it is easy to get the inequalities (8) and (9).

□

These inequalities give an upper bound for the rate of convergence, but we can obtain more precise results. Moreover, if we study the L^2 convergence, we can obtain an expansion, and this proves the exact rate of convergence.

Proposition 3.3. *With the above notations and assumptions, it holds that*

$$\sqrt{\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du - \mathbb{E}\left(\frac{1}{T}\int_0^T S_u du \mid \mathcal{B}_h\right)\right]^2} = \frac{\sigma}{n}\sqrt{\frac{e^{(\sigma^2+2r)T}-1}{12(\sigma^2+2r)}} + O\left(\frac{1}{n\sqrt{n}}\right) \quad (10)$$

$$\sqrt{\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du - Y_T^{e,n}\right]^2} = \frac{\sigma}{n}\sqrt{\frac{e^{(\sigma^2+2r)T}-1}{12(\sigma^2+2r)}} + O\left(\frac{1}{n\sqrt{n}}\right) \quad (11)$$

$$\sqrt{\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du - Y_T^{r,n}\right]^2} = \frac{K_2}{n} + O\left(\frac{1}{n\sqrt{n}}\right) \text{ where } K_2 \geq \sigma\sqrt{\frac{e^{(\sigma^2+2r)T}-1}{12(\sigma^2+2r)}} \quad (12)$$

$$\sqrt{\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du - Y_T^{p,n}\right]^2} = \frac{1}{n\sqrt{n}}\sqrt{T\sigma^4\frac{e^{(\sigma^2+2r)T}-1}{12(\sigma^2+2r)}} + O\left(\frac{1}{n^2}\right). \quad (13)$$

Remark that the same rate of convergence occurs in equations (10) and (11), and futhermore the first term in the expansion is the same. It implies that we did not lose anything by approximating the conditional expectation with the scheme (5).
If we are interested in the strong convergence, it seems that the most efficient scheme is the last one. We also remark that the schemes (1) and (5) have the same rate of convergence and that the first terms in the expansions are quite similar.
The proof of these 4 equalities are similar and we will treat only the last one.

Proof of (13) Let us note $A_{u,t_k} = e^{\sigma(W_u - W_{t_k}) - \frac{\sigma^2}{2}(u-t_k) + r(u-t_k)} - 1 - r(u-t_k) - \sigma(W_u - W_{t_k})$. Then we get,

$$\mathbb{E}\left[\frac{1}{T}\int_0^T S_u du - Y_T^{p,n}\right]^2 = \frac{2}{T^2}\sum_{0\le j<i\le N-1}\int_{t_i}^{t_{i+1}} dv \int_{t_j}^{t_{j+1}} du\ \mathbb{E}(A_{v,t_i}A_{u,t_j}S_{t_i}S_{t_j}) + \frac{2}{T^2}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} dv \int_{t_k}^{u} du\ \mathbb{E}(A_{u,t_k}A_{v,t_k}S_{t_k}^2).$$

Since A_{v,t_i} is independent of $\mathcal{F}_{t_i}$, it follows that

$$\mathbb{E}(A_{v,t_i}A_{u,t_j}S_{t_i}S_{t_j}) = e^{(\sigma^2+r)t_j + rt_i}(e^{r(v-t_i)} - 1 - r(v-t_i))(e^{(r+\sigma^2)(u-t_i)} - 1 - (r+\sigma^2)(u-t_i)).$$

With a normalization of the variable in the integrand we get,

$$\frac{2}{T^2}\sum_{0\le j<i\le N-1}\int_{t_i}^{t_{i+1}} dv \int_{t_j}^{t_{j+1}} du\ \mathbb{E}(A_{v,t_i}A_{u,t_j}S_{t_i}S_{t_j}) = \frac{2}{T^2}\left(\sum_{0\le j<i\le N-1} e^{(\sigma^2+r)t_j + rt_i}\right) h^2 \int_0^1 dv \int_0^1 du\ (e^{rhv} - 1 - rhv)(e^{(r+\sigma^2)(hu)} - 1 - (r+\sigma^2)hu).$$

It is clear that $\left(\sum\limits_{0\le j<i\le N-1} e^{(\sigma^2+r)t_j+rt_i}\right) = O(\frac{1}{h^2})$, which leads to

$$\frac{2}{T^2}\sum_{0\le j<i\le N-1}\int_{t_i}^{t_{i+1}} dv \int_{t_j}^{t_{j+1}} du\ \mathbb{E}(A_{v,t_i}A_{u,t_j}S_{t_i}S_{t_j}) = O(h^4).$$

We use the same method for the second term,

$$\mathbb{E}(A_{u,t_k}A_{v,t_k}S_{t_k}^2) = e^{(\sigma^2+2r)t_k}\big(-(1+r(u-t_k))(e^{r(v-t_k)} - 1 - r(v-t_k)) + e^{r(u-t_k)}(e^{(\sigma^2+r)(v-t_k)} - 1 - (r+\sigma^2)(v-t_k)) + \sigma^2(v-t_k)(e^{r(v-t_k)} - 1)\big).$$

Consequently,

$$\begin{aligned}
&\frac{2}{T^2}\sum_{k=0}^{N-1}\int_{t_k}^{t_{k+1}} dv \int_{t_k}^{u} du\ \mathbb{E}(A_{u,t_k}A_{v,t_k}S_{t_k}^2) \\
&= \frac{2}{T^2}\left(\sum_{k=0}^{N-1} e^{(\sigma^2+2r)t_k}\right)\ h^2\int_0^1 du \int_0^u dv(\frac{\sigma^4 h^2 v^2}{2} + O(h^3)) \\
&= h^3 \frac{\sigma^4 e^{(\sigma^2+2r)T} - 1}{12(\sigma^2+2r)}.
\end{aligned}$$

□

4 Variance Reduction

To increase the efficiency of the Monte Carlo simulation a variance reduction method can be used. We follow the method developed by Kemna and Vorst [10]. It consists in approximating $\frac{1}{T}\int_0^T S_u du$ by $\exp\left(\frac{1}{T}\int_0^T \log(S_u)du\right)$. We can expect that these two random variables are similar since r and σ are not too large.
The random variable $Z' = \frac{1}{T}\int_0^T \log(S_u)du$ obviously has a normal law, and so we can compute explicitly

$$\mathbb{E}(e^{-rT}(\exp(Z') - K)_+).$$

As a consequence, we can choose the random variable Z defined by

$$Z = e^{-rT}(xe^{(r-\frac{\sigma^2}{2})\frac{T}{2}+\frac{\sigma}{T}\int_0^T W_u du} - K)_+,$$

as our control variable.
Note that the control variable has to be computed with the path of the Brownian motion already simulated. Consequently, each control variable has to be adapted to the schemes. So, we retain the same approximation for $\int_0^T W_u du$ as for $\int_0^T S_u du$:

$$\begin{array}{lrcl}
\text{For (1)} & Z_T^{r,n} & = & e^{-rT}(xe^{(r-\frac{\sigma^2}{2})\frac{T}{2}+\frac{\sigma}{T}\sum\limits_{k=0}^{n-1} hW_{t_k}} - K)_+ \\
\text{For (5)} & Z_T^{e,n} & = & e^{-rT}(xe^{(r-\frac{\sigma^2}{2})\frac{T}{2}+\frac{\sigma}{T}\sum\limits_{k=0}^{n-1} \frac{h}{2}(W_{t_k}+W_{t_{k+1}})} - K)_+ \\
\text{For (6)} & Z_T^{p,n} & = & e^{-rT}(xe^{(r-\frac{\sigma^2}{2})\frac{T}{2}+\frac{\sigma}{T}\sum\limits_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} W_u du} - K)_+
\end{array}$$

Note that, as it is explained in remark 2.2, when we use the scheme (6), we simulate at each step $W_{t_{k+1}}$ then $(\int_{t_k}^{t_{k+1}} W_u du \mid W_{t_k}, W_{t_{k+1}})$. By writing $\sum\limits_{k=0}^{n-1}\int_{t_k}^{t_{k+1}} W_u du$ in the last equation above, we intend the mean of the generated variables.

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