The Singular Points Binomial Method for pricing American path-dependent options

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Abstract

We introduce a new numerical approach, called "Singular Points Method", for pricing American path-dependent options. This method, based on a continuous representation of the price at each node of the binomial tree, allows us to obtain very precise upper and lower bounds of the discrete binomial price. Moreover, the method provides a-priori estimates of the difference between upper and lower bounds. The algorithm is convergent and provides efficient estimates of the continuous price value. We apply the method to the case of Asian and lookback American options.

Keywords: option pricing, American options, Asian options, lookback options, tree methods.

Introduction

A path-dependent option is an option whose payoff depends not only on the value of the stock price at maturity but also on the past history of the underlying asset price. In this paper we are mainly interested in the case of Asian and lookback options.

The pay-off of an Asian option is based on several forms of averaging of the underlying asset price over the life of the option. The most common cases are those for which the average is arithmetic or geometric. Lookback options are options whose payoff depend on the maximum or on the minimum of the underlying asset price reached during the life of the option.

American lookback and American Asian options cannot be valued by closed-form formulae, even in the Black-Scholes model, and their valuation requires the use of numerical methods. Here we consider tree methods for pricing these type of options.

The difficulty of applying Cox-Ross-Rubinstein (CRR) method to Asian options with arithmetic average is well known. This is because the number of possible averages increases exponentially with the number of tree steps. For this reason Hull and White ([7]) and similarly Barraquand-Pudet ([2]), proposed more feasible approaches. The main idea behind these procedures is to restrict the range of all the possible arithmetic averages to a set of representative values. These values are selected in order to span all the possible values of the averages achievable at each node of the tree. The price is then computed by a backward induction procedure in which the prices associated to the averages not included in the set of representative values, are obtained by interpolation.
In comparison with the CRR binomial method, these two techniques significantly reduce the number of computations. In fact the computational complexity of both methods is $O(n^3)$ (where n is the number of tree steps). However, these techniques have some drawbacks related both to the precision of the approximations and to the convergence to the continuous value, as observed by Forsyth et al in [11]. Forsyth et al proved that a procedure of order $O(n^{\frac{7}{2}})$ is necessary in order to assure the convergence of these algorithms.

Later Chalasani et al ([3], [4]) proposed a totally different approach which allowed them to obtain thin upper and lower bounds of the exact CRR binomial price for American Asian options. Their method requires a forward procedure and a backward induction. This algorithm significantly increases the precision of the estimates but it requires a very large amount of memory and has computational complexity $O(n^4)$.

More recently, very efficient PDE-based methods have been introduced by Vecer[10] and D'Halluin et al [6]. Vecer proposed a one-dimensional PDE method that runs in $O(n^2)$. This approach cannot be applied to American fixed strike Asian options, which, on the other hand, can be treated using the semi-lagrangian approach of D'Halluin et al.

As regards lookback options, the complexity of the exact CRR binomial algorithm is of order $O(n^3)$ and the methods proposed in [7] and [2] do not improve the efficiency. Babbs ([1]) gave an efficient and accurate solution to the problem of American floating strike lookback options through a procedure of complexity of order $O(n^2)$. He used a change of "numeraire" approach, which cannot be applied in the fixed strike case.

In this paper we will introduce a general binomial framework for pricing European/American path-dependent options in a efficient way. In particular, we apply it for the pricing of both American Asian and American lookback options.

The method provides very precise upper and lower bounds for the exact binomial discrete value and it significantly reduces the time of computation with respect to the previous tree techniques.

The main idea is to give a continuous representation, at each node of the tree, of the option prices as a piecewise linear convex function of the path-dependent variable (average or maximum/minimum). These functions are characterized just by a set of points, which we name "singular points". All such functions can be evaluated by backward induction in a straightforward way. Consequently the method provides an alternative and more efficient approach to evaluate the exact binomial price associated with the path-dependent options. Moreover, the convexity property of the piecewise linear function representing the price, allows us to obtain, simply and naturally upper and lower bounds of the discrete binomial price. A further appeal of the procedure is that it is possible to fix an a-priori level of precision for the distance between the estimates and the exact binomial value. This can be done very efficiently keeping the amount of time and memory space at low level. Moreover, the error control process permits to automatically obtain the convergence of the approximations to the continuous value.

The choice of providing an a-priori control of the option price error in the discrete model gives rise to problems in determining the theoretical complexity of the procedure. Nevertheless, the numerical experiments show that the method is very competitive in practice. Moreover, the observed complexity is $O(n^3)$.

The paper is organized as follows: in Section 1 we will describe the standard binomial techniques for American Asian and lookback options. Section 2 is devoted to the singular points method in the Asian case, including a description of the implementation of the algorithm. In Section 3 we will propose an algorithm for American lookback options. Finally, in Section 4, we will compare our technique with the best lattice based methods known in the literature. Furthermore, we will study the convergence of our method to the continuous price value by comparing it with the PDE-based methods.

1 The exact binomial algorithm

In this paper, we consider a market model where the evolution of a risky asset is governed by the Black-Scholes stochastic differential equation

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dB_t, \quad S_0 = s_0, \tag{1}$$

where $(B_t)_{0\leq t\leq T}$ is a standard Brownian motion, under the risk neutral measure Q. The nonnegative constant r is the force of interest rate, q is the continuous dividend yields and σ is the volatility of the risky asset. Then S_T is the value of the underlying asset at maturity T:

$$S_T = s_0 e^{(r-q-\frac{\sigma^2}{2})T+\sigma B_T}.$$

We will consider two examples of path-dependent options written on the underlying S_t: arithmetic Asian options and lookback options.

1.1 American Asian options

The price of an American Asian option of initial time 0 and maturity T is:

$$P(0,S_0,A_0) = \sup_{\tau\in\mathcal{T}_{0,T}} E\left[e^{-r\tau}\psi(S_\tau,A_\tau)|S_0=s_0,A_0=s_0\right],$$

where: $\mathcal{T}_{0,T}$ is the set of all stopping times with values in $[0,T]$, ψ denotes the payoff function and A_τ is the arithmetic average of the price of the underlying asset over the period $[0,\tau]$, i.e. $A_\tau = \frac{1}{\tau}\int_0^\tau S_t dt$.

Let K be the strike price. Some examples of payoff functions useful for Asian options pricing are:

- Fixed Asian Call: the payoff is $(A_T - K)_+$

- Fixed Asian Put: the payoff is $(K - A_T)_+$
- Floating Asian Call: the payoff is $(S_T - A_T)_+$
- Floating Asian Put: the payoff is $(A_T - S_T)_+$.

Consider now the binomial approach. Let n be the number of steps of the binomial tree and $\Delta T = \frac{T}{n}$ the corresponding time-step. The lognormal diffusion process $(S_{i\Delta T})_{0\leq i\leq n}$ is approximated by the Cox-Ross-Rubinstein binomial process

$$S_i = (s_0 \prod_{j=1}^{i} Y_j)_{0\leq i\leq n}$$

where the random variables $Y_1, \ldots, Y_n$ are independent and identically distributed with values in $\{d, u\}$. Let us denote by $\pi = \mathbb{P}(Y_n = u)$. The Cox-Ross-Rubinstein tree corresponds to the choice $u = \frac{1}{d} = e^{\sigma\sqrt{\Delta T}}$ and

$$\pi = \frac{e^{r\Delta T} - e^{-\sigma\sqrt{\Delta T}}}{e^{\sigma\sqrt{\Delta T}} - e^{-\sigma\sqrt{\Delta T}}}$$

In a discrete-time setting, the payoff function at maturity n of an Asian option is given by $f(S_n, A_n)$ where

$$A_n = \frac{1}{n+1}\sum_{i=0}^{n} S_i$$

and the average process $(A_i)_{0\leq i\leq n}$ is recursively computed by

$$A_{i+1} = \frac{(i+1)A_i + S_{i+1}}{i+2}, A_0 = s_0.$$

In the Cox-Ross-Rubinstein model, the price at time 0 of the American (resp. European) Asian option with payoff function ψ is given by $v(0, s_0, s_0)$ where the functions $v(i, x, y)$ can be computed by the following backward dynamic programming equations

$$\begin{cases} v(n,x,y) = \psi(x,y) \\ v(i,x,y) = \max\left(\psi'(x,y), e^{-r\Delta T}\left[\pi v(i+1, xu, \frac{(i+1)y + xu}{i+2}) + (1-\pi)v(i+1, xd, \frac{(i+1)y+xd}{i+2})\right]\right), \end{cases} \tag{2}$$

where $\psi' = \psi$ in the American case and $\psi' \equiv 0$ in the European case.

The obtained tree is not recombining so that, from a practical point of view, the valuation of $v(0, s_0, s_0)$ is unfeasible just for very small number of steps.

1.2 Lookback options

The price of an American lookback option is:

$$P(0, S_0, S_0^*) = \sup_{\tau\in\mathcal{T}_{0,T}} E\left[e^{-r\tau}\psi(S_\tau, S_\tau^*)|S_0 = s, S_0^* = s\right].$$

where ψ denotes the payoff function of the option and

$$S_\tau^* = M_\tau = \max_{u\in[0,\tau]} S_u \quad or \quad S_\tau^* = m_\tau = \min_{u\in[0,\tau]} S_u$$

Let K be the strike. Some examples of payoff function useful in lookback option pricing are:

- Fixed lookback Call: the payoff is $(M_T - K)_+$.
- Fixed lookback Put: the payoff is $(K - m_T)_+$.
- Floating lookback Call: the payoff is $(S_T - m_T)_+$.
- Floating lookback Put: the payoff is $(M_T - S_T)_+$.

In a discrete-time setting, the payoff at maturity n of an European lookback option, written on the maximum, is given by $\psi(S_n, M_n)$ where

$$M_n = \max(S_0, ..., S_n).$$

The maximum process $(M_i)_{0\leq i\leq n}$ can be computed recursively by

$$M_{i+1} = \max(M_i, S_{i+1}), M_0 = s_0.$$

In the Cox-Ross-Rubinstein model, the price at time 0 of the corresponding American lookback option is given by $v(0, s_0, s_0)$ where the functions $v(i, x, y)$ can be computed by the following backward dynamic programming equations

$$\begin{cases} v(n,x,y) = \psi(x,y) \\ v(i,x,y) = \max\left(\psi(x,y), e^{-r\Delta T}\Big[\pi v(i+1, xu, \max(xu,y)) + (1-\pi)v(i+1, xd, y)\Big]\right), \end{cases} \tag{3}$$

where $\psi(x, y)$ is the payoff function. The valuation of $v(0, s_0, s_0)$ requires a number of computations of order $O(n^3)$.

2 The Singular Points Method

In this section we will introduce a new backward procedure. The main idea is to give a continuous representation of the option price as a piecewise linear function at each node of the tree, which describes the path-dependent nature of the option. Such a representation only depends on a finite number of points (i.e. the points were the slope of the function changes) called "singular points".

2.1 Piecewise linear convex functions and singular points

Henceforth we will use the following notations:

Definition 1. *Given a set of points: $(x_1, y_1), ..., (x_n, y_n)$, such that $a = x_1 < x_2 < ... < x_n = b$ and*

$$\frac{y_i - y_{i-1}}{x_i - x_{i-1}} < \frac{y_{i+1} - y_i}{x_{i+1} - x_i}, \qquad i = 2, ..., n-1, \tag{4}$$

let us consider the function $f(x)$, $x \in [a, b]$, obtained by interpolating the given points linearly. The points $(x_1, y_1), ..., (x_n, y_n)$ (which characterize the piecewise linear function f), will be called the singular points of f, while $x_1, ..., x_n$ will be called the singular values of f.

Remark 1. *In the previous definition we considered only piecewise linear functions with strictly increasing slopes, this implies that the resulting function f is convex.*

From here on we shall consider only piecewise linear functions that are continuous and convex on the interval $[a, b]$. For each of these functions we can find a set of singular points characterizing them and satisfying equation (4).

The following results, which have a very simple geometrical interpretation (see Fig.1 and Fig.2), allow us to construct the upper and the lower bounds of the discrete option price.

Lemma 1. *Let f be a piecewise linear and convex function defined on $[a, b]$, and let $C = \{(x_1, y_1), ..., (x_n, y_n)\}$ be the set of its singular points.*

Removing a point (x_i, y_i), $2 \leq i \leq n-1$, from the set C, the resulting piecewise linear function $\widetilde{f}$, whose set of singular points is $C \setminus \{(x_i, y_i)\}$, is again convex in $[a, b]$ and we have:

$$f(x) \leq \widetilde{f}(x), \quad \forall x \in [a, b].$$

Proof. The previous inequality and the convexity of $\widetilde{f}$ follow from the fact that $\widetilde{f}$ is the maximum between f and the function given by the straight line joining the points (x_{i-1}, y_{i-1}), (x_{i+1}, y_{i+1}). ♢

Remark 2. *From the previous Lemma it follows that every piecewise linear function $\widetilde{f}$ whose singular points are a subset of C (containing the first and the last singular point) is still convex and satisfies $\widetilde{f} \geq f$.*

Lemma 2. *Let f be a piecewise linear and convex function defined on $[a, b]$, and let $C = \{(x_1, y_1), ..., (x_n, y_n)\}$ be the set of its singular points. Let us denote by $(\overline{x}, \overline{y})$ the intersection between the straight line joining (x_{i-1}, y_{i-1}), (x_i, y_i) and the one joining (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}), $2 \leq i \leq n-2$.*

If we consider the new set of $n-1$ singular points

$$\{(x_1, y_1), ..., (x_{i-1}, y_{i-1}), (\overline{x}, \overline{y}), (x_{i+2}, y_{i+2}), ..., (x_n, y_n)\},$$

the associated piecewise linear function $\widetilde{f}$ is again convex on $[a, b]$ and we have:

$$f(x) \geq \widetilde{f}(x), \quad \forall x \in [a, b].$$

Proof. The singular points of f satisfy the property of increasing slopes (4). The set of slopes associated to the singular points of $\widetilde{f}$ are obtained removing the slope of the line joining (x_i, y_i), (x_{i+1}, y_{i+1}), hence (4) is again satisfied and $\widetilde{f}$ is convex. The inequality $f \geq \widetilde{f}$ is trivial. ♢

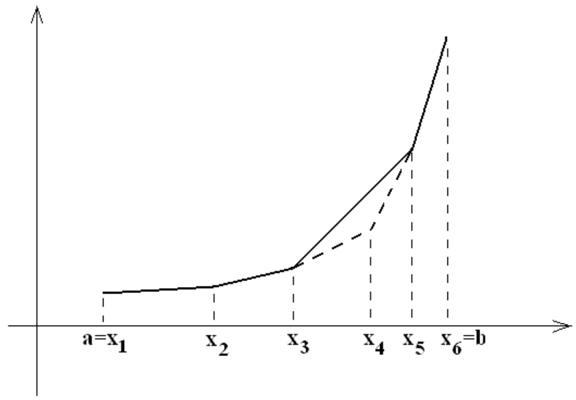

Figure 1: *Upper estimate: x_4 has been removed.*

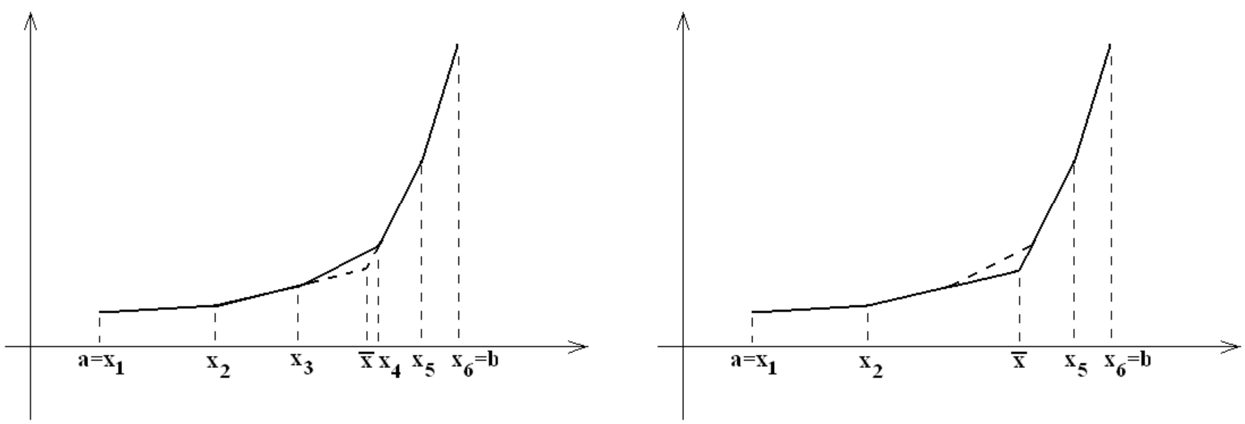

Figure 2: *Lower estimate: x_3 and x_4 have been removed, $\overline{x}$ has been inserted.*

2.2 Fixed strike European Asian options

First at all we will describe the proposed algorithm in the framework of a fixed strike European Asian call option.

In this case the method consists in valuating the price of the option, at each node of the tree, for each possible choice of the average at that point. So we consider not only averages which are effectively achievable, but all the possible averages between the minimum and maximum realized at that point. In this way, we will show that it is possible to give a continuous representation of the price function as a piecewise linear convex function of the average. This function is characterized just by its singular points.

We will now introduce some further notations.
Let us denote by $N_{i,j}$ the node of the tree whose underlying asset is $S_{i,j} = s_0 u^{2j-i}$, $i = 0, ..., n$, $j = 0, ..., i$.
To each node $N_{i,j}$ we will associate a set of singular points, whose number is $L_{i,j}$. The singular

points will be denoted by
$$(A^l_{i,j}, P^l_{i,j}), \quad l = 1, ..., L_{i,j}.$$
As regards Asian options, the singular values $A^l_{i,j}$ are called singular averages and $P^l_{i,j}$ are called singular prices.

Let us consider first the nodes $N_{n,j}$, $j = 0, ..., n$, of the tree at maturity. At each node the average values vary between a minimum average $A^{min}_{n,j}$ (corresponding to the path with $n - j$ down movements followed by j up movements) and a maximum average $A^{max}_{n,j}$ (corresponding to the path with j up movements followed by $n - j$ down movements). These minimum and maximum are easily valuable:

$$A^{min}_{n,j} = \frac{s_0}{n+1}(\frac{1-d^{n-j+1}}{1-d} + d^{n-j}(\frac{1-u^{j+1}}{1-u} - 1)),$$

$$A^{max}_{n,j} = \frac{s_0}{n+1}(\frac{1-u^{j+1}}{1-u} + u^{j}(\frac{1-d^{n-j+1}}{1-d} - 1)).$$

For each $A \in [A^{min}_{n,j}, A^{max}_{n,j}]$ the price of the option can be continuously defined by $v_{n,j}(A) = (A - K)_+$
(remark that $v_{n,j}(A) \equiv v(n, S_{n,j}, A)$ where $v(n, x, y)$, is the price function introduced in Section 1.1).

Note that the function $v_{n,j}(A)$ is a piecewise linear function satisfying Definition 1, whose singular points are valuable in a straightforward way. In fact:

- if $K \in (A^{min}_{n,j}, A^{max}_{n,j})$ then the price value function $v_{n,j}(A)$ is characterized by the 3 singular points $(A^l_{n,j}, P^l_{n,j})$, $l = 1, 2, 3$ (hence $L_{n,j} = 3$), where
$$\begin{array}{ll} A^1_{n,j} = A^{min}_{n,j}, & P^1_{n,j} = 0; \\ A^2_{n,j} = K, & P^2_{n,j} = 0; \\ A^3_{n,j} = A^{max}_{n,j}, & P^3_{n,j} = A^{max}_{n,j} - K. \end{array} \tag{5}$$
.
- if $K \notin (A^{min}_{n,j}, A^{max}_{n,j})$ then the price value function $v_{n,j}(A)$ is characterized by the 2 singular points $(A^l_{n,j}, P^l_{n,j})$, $l = 1, 2$, $(L_{n,j} = 2)$, where
$$\begin{array}{ll} A^1_{n,j} = A^{min}_{n,j}, & P^1_{n,j} = (A^{min}_{n,j} - K)_+ \ ; \\ A^2_{n,j} = A^{max}_{n,j}, & P^2_{n,j} = (A^{max}_{n,j} - K)_+ \ . \end{array} \tag{6}$$
- In the case $j = 0$ and $j = n$ the minimum and maximum of the averages coincide and $L_{n,j} = 1$.

Therefore we can conclude

Lemma 3. *At each node at maturity the function $v_{n,j}(A)$ that provides the price of the option, is a piecewise linear function on the interval $[A^{min}_{n,j}, A^{max}_{n,j}]$. Moreover, such a function is convex on its domain.*

Now consider the step i, $0 \le i \le n-1$. At the node $N_{i,j}$ we can evaluate recursively the minimum and the maximum of the averages, respectively

$$A^{min}_{i,j} = \frac{(i+2)A^{min}_{i+1,j+1} - S_{i+1,j+1}}{i+1}, \qquad A^{max}_{i,j} = \frac{(i+2)A^{max}_{i+1,j} - S_{i+1,j}}{i+1}.$$

Lemma 4. *At each node $N_{i,j}$, $i = 0, ..., n$, $j = 0, ..., i$, the function $v_{i,j}(A)$, which provides the price of the option as function of the average A, is piecewise linear and convex in the interval $[A^{min}_{i,j}, A^{max}_{i,j}]$.*

Proof. The claim is true at step $i = n$ (at maturity) by Lemma 3. At step $i = n-1$, the price function $v_{i,j}(A)$, with $A \in [A^{min}_{i,j}, A^{max}_{i,j}]$, is obtained by considering the discounted expectation value:

$$v_{i,j}(A) = e^{-r\Delta T}[\pi v_{i+1,j+1}(A') + (1-\pi)v_{i+1,j}(A'')], \tag{7}$$

where

$$A' = \frac{(i+1)A + s_0 u^{2j-i+1}}{i+2}, \qquad A'' = \frac{(i+1)A + s_0 u^{2j-i-1}}{i+2}. \tag{8}$$

As $v_{n,j}(A)$ is piecewise linear and convex on its domain and $h_1(A) = v_{i+1,j+1}(\frac{(i+1)A+s_0u^{2j-i+1}}{i+2})$ is a function composed by a linear function of A and a piecewise linear convex function, then $h_1(A)$ is piecewise linear and convex as a function of A. The same holds true for $h_2(A) = v_{i+1,j}(\frac{(i+1)A+s_0u^{2j-i-1}}{i+2})$. We can conclude that $v_{i,j}(A)$ is piecewise linear and convex on its domain.

The claim of the Lemma now follows by backward induction. ♢

Figure 3: *The price function $v_{n-1,j}(A)$ is obtained from $v_{n,j}(A)$ and $v_{n,j+1}(A)$. It is piecewise linear and convex and its internal singular points arise from the singular points of $v_{n,j}(A)$ and $v_{n,j+1}(A)$.*

Consider again the step $i = n-1$ and the node $N_{i,j}$. By Lemma 4, $v_{i,j}(A)$ is piecewise linear and convex, hence it is characterized by its singular points (see Fig.3).

The valuation of the singular points can be carried out recursively by a backward algorithm, which will be described in the sequel.

Each average $A^l_{i+1,j}$, $l = 1, ..., L_{i+1,j}$, associated to a singular point of the node $N_{i+1,j}$ is projected in a new average value B^l at the node $N_{i,j}$ by the relation

$$B^l = \frac{(i+2)A^l_{i+1,j} - s_0 u^{2j-i-1}}{i+1}. \tag{9}$$

Note that B^l is the average evaluated at the node $N_{i,j}$ which becomes $A^l_{i+1,j}$ after a down movement of the underlying.

Observe that B^l is increasing with respect to l, $B^{L_{i+1,j}} = A^{max}_{i,j}$ for all j, and $B^1 \notin [A^{min}_{i,j}, A^{max}_{i,j}]$ if $0 < j < i$. Each B^l belonging to the interval $[A^{min}_{i,j}, A^{max}_{i,j}]$ becomes the first coordinate of a singular point associated to the node $N_{i,j}$.

In order to evaluate the price $v_{i,j}(B^l)$ associated to the singular average $B^l \in [A_{i,j}^{min}, A_{i,j}^{max}]$, we remark that after a down movement of the underlying, B^l transforms into $A_{i+1,j}^l$ and the corresponding price is $P_{i+1,j}^l$. Consider now an up movement of the underlying. In this case B^l transforms into the average: $B_{up}^l = \frac{(i+1)B_l + s_0 u^{2j-i+1}}{i+2}$. This average clearly could not belong to the set of singular averages associated to the node $N_{i+1,j+1}$. Therefore we need to evaluate the index s such that $B_{up}^l \in [A_{i+1,j+1}^s, A_{i+1,j+1}^{s+1}]$. Since in this interval the price function is linear, we have

$$v_{i+1,j+1}(B_{up}^l) = \frac{P_{i+1,j+1}^{s+1} - P_{i+1,j+1}^s}{A_{i+1,j+1}^{s+1} - A_{i+1,j+1}^s}(B_{up}^l - A_{i+1,j+1}^s) + P_{i+1,j+1}^s.$$

We can evaluate the price associated to the singular average B^l evaluating the discounted expectation value:

$$v_{i,j}(B^l) = e^{-r\Delta T}[\pi v_{i+1,j+1}(B_{up}^l) + (1-\pi)v_{i+1,j}(A_{i+1,j}^l)]. \tag{10}$$

In a similar way each singular average $A_{i+1,j+1}^l$, $l = 1, ..., L_{i+1,j+1}$ associated to the node $N_{i+1,j+1}$ is projected in a new average C^l at the node $N_{i,j}$ by the relation

$$C^l = \frac{(i+2)A_{i+1,j+1}^l - s_0 u^{2j-i+1}}{i+1}. \tag{11}$$

Now $C^1 = A_{i,j}^{min}$ for all j, and $C^{L_{i+1,j+1}} \notin [A_{i,j}^{min}, A_{i,j}^{max}]$ if $0 < j < i$. For each $C^l \in [A_{i,j}^{min}, A_{i,j}^{max}]$ we can evaluate the corresponding price $v_{i,j}(C^l)$ similarly as before.

Finally we proceed by sorting the averages B^l and C^l belonging to $[A_{i,j}^{min}, A_{i,j}^{max}]$, obtaining an ordered set $\{(A_{i,j}^1, P_{i,j}^1), ..., (A_{i,j}^{L_{i,j}}, P_{i,j}^{L_{i,j}})\}$ of singular points at the node $N_{i,j}$. By the previous construction these are exactly all the singular points associated to this node. Remark that $L_{i,j} \leq L_{i+1,j} + L_{i+1,j+1} - 2$.

The previous argument can be applied at every step $i = n-1, ..., 0$ and it holds for all $j = 1, ..., i-1$. At the nodes $N_{i,i}$, $N_{i,0}$, there is only a singular point whose price is given by

$$P_{i,0}^1 = e^{-r\Delta T}[\pi P_{i+1,0}^1 + (1-\pi)P_{i+1,1}^1], \qquad P_{i,i}^1 = e^{-r\Delta T}[\pi P_{i+1,i+1}^1 + (1-\pi)P_{i+1,i}^{L_{i+1,i}}]; \tag{12}$$

so that we get a complete description of the price function $v_{i,j}(A)$ at each node of the tree.

The value $P_{0,0}^1$ is exactly the binomial price relative to the tree with n steps of fixed strike European Asian call option. In fact, the method provides the price corresponding to every possible average at each node, in particular to the averages which are effectively realized on the binomial tree.

2.3 Fixed strike American Asian options

Consider now the American case. At maturity we have the same situation as in the European case. The price function is $v_{n,j}(A) = (A-K)_+$ for $A \in [A_{n,j}^{min}, A_{n,j}^{max}]$, and it is characterized by the same singular points.

Consider the step $i = n-1$. At the node $N_{i,j}$ we first compute, by using the procedure described in the previous section, the singular points associated to this node, obtaining in this way the continuation value function $v_{i,j}^c(A)$.

Taking into account of the American feature, the price function $v_{i,j}(A)$ is obtained by comparing the continuation value with the early exercise:

$$v_{i,j}(A) = \max\{v^c_{i,j}(A), A - K\}.$$

Let us remark that $v_{i,j}(A)$, $A \in [A^{min}_{i,j}, A^{max}_{i,j}]$, is still a piecewise linear convex function. For this reason we can characterize it again by its singular points.

In order to compute the singular points associated to the American case we first remark that the slopes characterizing the piecewise linear convex function $v^c_{i,j}(A)$ are all smaller than 1. This follows by virtue of equations (7), (8) and by differentiating $v^c_{i,j}(A)$ in the open intervals $(A^l_{i,j}, A^{l+1}_{i,j})$, $l = 1, ..., L_{i,j}$. Therefore there are two possible cases:

1. $A^{max}_{i,j} - K \leq v^c_{i,j}(A^{max}_{i,j})$ then $v_{i,j} \equiv v^c_{i,j}$, so the singular points do not change;
2. $A^{max}_{i,j} - K > v^c_{i,j}(A^{max}_{i,j})$. Here we have two subcases:
   - $A^{min}_{i,j} - K \geq v^c_{i,j}(A^{min}_{i,j})$ then $v_{i,j}(A) = A - K$ for all $A \in [A^{min}_{i,j}, A^{max}_{i,j}]$, so the set of singular points consists only on two points
   $$(A^{min}_{i,j}, A^{min}_{i,j} - K),\ (A^{max}_{i,j}, A^{max}_{i,j} - K);$$
   - $A^{min}_{i,j} - K < v^c_{i,j}(A^{min}_{i,j})$ then there is an unique average $\overline{A}$ where the continuation value is equal to the early exercise. Let j_0 be the largest index such that $A^{j_0}_{i,j} < \overline{A}$. The new set of singular points becomes (see also Fig.4):
   $$\{(A^1_{i,j}, P^1_{i,j}), ..., (A^{j_0}_{i,j}, P(A^{j_0}_{i,j})), (\overline{A}, \overline{A} - K), (A^{max}_{i,j}, A^{max}_{i,j} - K)\}.$$

The same argument can be applied at every step $i = n - 2, ..., 0$. This allows us to compute $P^1_{0,0}$ which provides the exact American binomial price relative to the tree with n steps.

Remark 3. *The number of singular points associated to a node could decrease in the American case, so the American procedure could be faster than the European one.*

Remark 4. *In the case of Asian put option the procedure is similar.*

Remark 5. *In the floating strike case the procedure is modified as follows: at maturity the singular points depend not more on the strike K but on the underlying value at each node $S_{i,j}$. Therefore the new singular points are obtained by replacing K by $S_{i,j}$. The backward procedure is the same as before, just taking into account properly the new intrinsic values.*

Figure 4: *American case: the point $\overline{A}$ has been inserted, A_4 and A_5 have been removed.*

2.4 Upper and lower bound

In the previous subsections we have introduced a new method in order to evaluate the exact binomial price in a discrete setting of an European or American Asian option.

As $L_{i,j} \leq L_{i+1,j} + L_{i+1,j+1} - 2$, the resulting algorithm can be of exponential complexity as the standard binomial technique.

The main advantage of our technique is that it allows us to obtain easily both an upper and a lower bound of the binomial price, drastically reducing the amount of computational time and memory requirements. Moreover a further appeal is given by the possibility to obtain an a-priori control of the distance of the estimates from the exact binomial price. Actually all these results are simple consequences of the previous Lemma 1 and Lemma 2.

More precisely, in order to get an upper bound of the exact binomial price, we just remove some singular points at each node. Lemma 1 ensures that the value obtained in such a way is an upper estimate of the exact binomial price.

There are several possible criteria to remove the singular points. Here we propose the following:
Consider the set of singular points $C = \{(A^1_{i,j}, P^1_{i,j}), ..., (A^L_{i,j}, P^L_{i,j})\}$ $(L = L_{i,j})$, associated to the node $N_{i,j}$ and the corresponding price value function $v_{i,j}(A)$. Let $v'_{i,j}(A)$ be the price value function obtained by removing a point $(A^l_{i,j}, P^l_{i,j})$ from C. We have

$$|v_{i,j}(A) - v'_{i,j}(A)| \leq \epsilon_l, \qquad \forall A \in [A^{min}_{i,j}, A^{max}_{i,j}] \tag{13}$$

where

$$\epsilon_l = v'_{i,j}(A^l_{i,j}) - v_{i,j}(A^l_{i,j}) = \frac{P^{l+1}_{i,j} - P^{l-1}_{i,j}}{A^{l+1}_{i,j} - A^{l-1}_{i,j}}(A^l_{i,j} - A^{l-1}_{i,j}) + P^{l-1}_{i,j} - P^l_{i,j}. \tag{14}$$

Therefore, given a real number $h > 0$ we choose to remove the point $(A^l_{i,j}, P^l_{i,j})$ if $\epsilon_l < h$. Repeating this procedure sequentially at each node of the tree, avoiding the elimination of two consecutive singular points, we can conclude that the obtained upper estimate differs from the exact binomial value at most for nh.

The algorithm for the computation of the lower bound is similar and follows by Lemma 2. Removing the points $(A^{l-1}_{i,j}, P^{l-1}_{i,j})$, $(A^l_{i,j}, P^l_{i,j})$, $l = 2, ..., L-2$, and adding the point $(\overline{x}, \overline{y})$ (see Lemma 2) the difference between the values of the associated piecewise linear functions is less or equal to δ_l, where

$$\delta_l = \frac{P^l_{i,j} - P^{l-1}_{i,j}}{A^l_{i,j} - A^{l-1}_{i,j}}(\overline{x} - A^{l-1}_{i,j}) + P^{l-1}_{i,j} - \overline{y}. \tag{15}$$

This replacement will take place only if $\delta_l < h$. Inductively and using the scheme proposed in the next remark, we get that the obtained lower estimate differs again from the exact binomial value at most for nh.

Remark 6. *In the case of the lower estimate, in order to obtain an error smaller than h at every step we propose the following algorithm:*
we start considering the points $(A^1_{i,j}, P^1_{i,j})$, $(A^2_{i,j}, P^2_{i,j})$, $(A^3_{i,j}, P^3_{i,j})$, $(A^4_{i,j}, P^4_{i,j})$. If $\delta_3 < h$ then we add the point $(\overline{x}, \overline{y})$ and delete $(A^2_{i,j}, P^2_{i,j})$, $(A^3_{i,j}, P^3_{i,j})$. Moreover the procedure will continue considering the new four points $(\overline{x}, \overline{y})$, $(A^4_{i,j}, P^4_{i,j})$, $(A^5_{i,j}, P^5_{i,j})$, $(A^6_{i,j}, P^6_{i,j})$. On the other hand if

$\delta_3 \geq h$ then we don't remove points and the procedure will continue considering the new four points $(A^2_{i,j}, P^2_{i,j})$, $(A^3_{i,j}, P^3_{i,j})$, $(A^4_{i,j}, P^5_{i,j})$, $(A^5_{i,j}, P^5_{i,j})$. We repeat this procedure completing the sequence of singular points of the node $N_{i,j}$.

Remark 7. *Jiang and Dai [8] proved the convergence of the exact binomial algorithm for European/ American path-dependent options. In particular they proved that the rate of convergence of the exact binomial algorithm to the continuous value is $O(\Delta T)$.*

The possibility of obtaining estimates of the exact binomial price with an error control allows us to prove easily the convergence of our method to the continuous value. Choosing h depending on n and so that $nh(n) \to 0$ we have that the corresponding sequences of upper and lower estimates converge to the continuous price value. Moreover, choosing $h(n) = O(\frac{1}{n^2})$, we are able to guarantee that the order of convergence is $O(\Delta T)$.

Remark 8. *The key issue in assessing the complexity of our algorithm is in the upper and lower bound computation. A theoretical complexity analysis combined with the above upper and lower bounds is out of reach. In fact, the control of the error with respect of the exact binomial algorithm does not permit us to control of the number of singular points. Nevertheless, the numerical results in section 4.2 indicate that the present method is very competitive in practice.*

2.5 Sketch of the algorithm in the American Asian case

Let us finally summarize the algorithm in order to obtain an upper and a lower bound of the exact binomial price for a fixed strike American Asian call option with an error smaller that nh $(h > 0)$.

- **STEP n**

  - Compute the singular points at maturity by using (5) and (6).

- **STEP i**, for $i = n-1, ..., 0$

  - Evaluate $P^1_{i,0}$, $P^1_{i,i}$ by comparing the continuation values given in (12) with the early exercise.

  - For each node $N_{i,j}$, $j = 1, ..., i-1$, compute the set of the singular points by the following steps:

  1. for each average $A^l_{i+1,j}$, $l = 1, ..., L_{i+1,j}$ compute B_l by (9),
  2. for $B_l \in [A^{min}_{i,j}, A^{max}_{i,j}]$ compute $v^c_{i,j}(B_l)$ by (10),
  3. for each average $A^l_{i+1,j+1}$, $l = 1, ..., L_{i+1,j}$ compute C_l by (11),
  4. for $C_l \in [A^{min}_{i,j}, A^{max}_{i,j}]$ compute $v^c_{i,j}(C_l)$,
  5. sort the set of the singular averages B_l and $C_l \in [A^{min}_{i,j}, A^{max}_{i,j}]$ obtaining the set of $L_{i,j}$ singular points associated to the node $N_{i,j}$,
  6. compute the American price according to Case 1 or 2 of Section 2.3 getting a new set of singular points with a new cardinality denoted, for simplicity, again by $L_{i,j}$,
  7. compute upper and lower bounds with error smaller than h:

upper bound: remove sequentially all the singular points $(A^l_{i,j}, P^l_{i,j})$, $l = 2, ..., L_{i,j} - 1$, for which $\epsilon_l < h$ (see (14)) avoiding the elimination of two consecutive singular points, obtaining a new set with a new cardinality denoted again by $L_{i,j}$,

lower bound: for each l, $l = 2, ..., L_{i,j} - 2$, for which $\delta_l < h$ (see (15)), remove the points $(A^{l-1}_{i,j}, P^{l-1}_{i,j})$, $(A^l_{i,j}, P^l_{i,j})$ and add the point $(\overline{x}, \overline{y})$ given by the intersection between the two straight lines joining $(A^{l-2}_{i,j}, P^{l-2}_{i,j})$, $(A^{l-1}_{i,j}, P^{l-1}_{i,j})$ and $(A^l_{i,j}, P^l_{i,j})$, $(A^{l+1}_{i,j}, P^{l+1}_{i,j})$, respectively (following the scheme described in Remark 6). We obtain again a new set of singular points with a new cardinality $L_{i,j}$.

$P^1_{0,0}$ is the upper [lower] estimate of the exact binomial price with error smaller that nh.

3 Lookback American options

We can apply the same procedure described in the Asian case, to the lookback options. Actually, in this case the algorithm admits several simplifications.

Consider a fixed strike American lookback call option. At the nodes $N_{n,j}$, $j = 0, ..., n$ (at maturity) the maximum of the underlying varies between a minimum value $M^{min}_{n,j}$ and a maximum value $M^{max}_{n,j}$ given by

$$M^{min}_{n,j} = \max\{S_{n,j}, s_0\}, \qquad M^{max}_{n,j} = s_0 u^j.$$

For all $M \in [M^{min}_{n,j}, M^{max}_{n,j}]$ the price of the option can be continuously defined by $v_{n,j}(M) = (M - K)_+$.

As in the Asian case, the function $v_{n,j}(M)$ is piecewise linear and its singular points are valuable by relations (5), (6) where M replaces A.

Consider now the step i, $0 \le i \le n-1$. At the node $N_{i,j}$ we can evaluate recursively the minimum and the maximum value of the maximum of the underlying by the relations

$$M^{min}_{i,j} = \max\{s_0, M^{min}_{i+1,j+1}/u\}, \qquad M^{max}_{i,j} = M^{max}_{i+1,j}. \tag{16}$$

Lemma 5. *At each node $N_{i,j}$, $i = 0, ..., n$, $j = 0, ..., i$, $v_{i,j}(M)$ is a piecewise linear and convex function on the interval $[M^{min}_{i,j}, M^{max}_{i,j}]$.*

Proof. The claim is true at step $i = n$. Consider the step $i = n-1$. We extend the function $v_{i+1,j+1}$ to the interval $[M^{min}_{i+1,j+1}/u, M^{max}_{i+1,j+1}]$ setting $v_{i+1,j+1}(M) = v_{i+1,j+1}(M^{min}_{i+1,j+1})$ for $M \in [M^{min}_{i+1,j+1}/u, M^{min}_{i+1,j+1})$. With such an extension the continuation value price function $v^c_{i,j}(M)$, becomes

$$v^c_{i,j}(M) = e^{-r\Delta T}[\pi v_{i+1,j+1}(M) + (1-\pi) v_{i+1,j}(M)]. \tag{17}$$

As $v_{i+1,j+1}(M)$ and $v_{i+1,j}(M)$ are piecewise linear and convex we can conclude that the same holds true for $v^c_{i,j}(M)$. Moreover $v_{i,j}(M) = \max\{v^c_{i,j}(M), M - K\}$, therefore $v_{i,j}(M)$ is still piecewise linear and convex. Inductively we have the claim. ♢

By the previous lemma, the price of an American lookback option can be obtained by computing only the singular points of the price function at each node. For this purpose we could use an algorithm similar to the one described in the Asian case. So the procedure for American lookback options consists in evaluating first the singular points $(M^1_{i,j}, P^1_{i,j}), ..., (M^L_{i,j}, P^L_{i,j})$ of $v^c_{i,j}(M)$. Then we can get the singular points of $v_{i,j}(M)$ in an easy way:

- if $M_{i,j}^{max} - K \leq v_{i,j}^c(M_{i,j}^{max})$ then the sets of singular points of $v_{i,j}(M)$ and $v_{i,j}^c(M)$ coincide;
- if $M_{i,j}^{min} - K \geq v_{i,j}^c(M_{i,j}^{min})$ then the set of singular points is composed only by two points: $(M_{i,j}^{min}, M_{i,j}^{min} - K)$, $(M_{i,j}^{max}, M_{i,j}^{max} - K)$;
- if $M_{i,j}^{min} - K < v_{i,j}^c(M_{i,j}^{min})$ and $M_{i,j}^{max} - K > v_{i,j}^c(M_{i,j}^{max})$ then there is an unique critical value $\overline{M}_{i,j} \in (M_{i,j}^{min}, M_{i,j}^{max})$ where the continuation value coincides with the early exercise value. Then the set of singular points of $v_{i,j}$ is composed by all the singular points of $v_{i,j}^c$ whose singular value belongs to $[M_{i,j}^{min}, \overline{M}_{i,j})$, with the addition of the points: $(\overline{M}_{i,j}, \overline{M}_{i,j} - K)$, $(M_{i,j}^{max}, M_{i,j}^{max} - K)$.

It is important to note that the particular structure of the tree in the lookback case allows us to obtain a simpler and more efficient procedure for the valuation of the singular points of $v_{i,j}$. This procedure, described in the next Proposition 1, is based on the possibility of computing the singular points in a direct way avoiding the sorting procedure. For this purpose we first need some properties which are strictly related to the lookback case:

Lemma 6. *The price value function $v_{i,j}(M)$, $M \in [M_{i,j}^{min}, M_{i,j}^{max}]$ has the following properties:*
a) if $K \in (M_{i,j}^{min}, M_{i,j}^{max})$ then $v_{i,j}(M)$ is constant in $[M_{i,j}^{min}, K]$,
b) if $M \in [M_{i,j}^{min}, M_{i,j-1}^{max}]$ and $v_{i,j}(M) = M - K$ then $v_{i,j-1}(M) = M - K$,
c) if $M \in [M_{i+1,j+1}^{min}, M_{i,j}^{max}]$ and $v_{i+1,j+1}(M) = M - K$ then $v_{i,j}(M) = M - K$,
d) assume that $x_1 = s_0u^l$, $x_2 \in (s_0u^l, s_0u^{l+1})$, $x_3 = s_0u^{l+1}$ are singular values of $v_{i,j}$. If we delete the singular point $(x_2, v_{i,j}(x_2))$ then $v_{0,0}(s_0)$ does not change.

Proof. The first two properties follow easily by induction on the tree. Property (c) follows by (b).

The claim of (d) follows by the fact that the value of the option at the nodes $N_{i,0}, N_{i,i}$, $i = 0, ..., n-1$, depends only on the values that $v_{i+1,j}$ assumes at the nodal stock values of the tree. $\diamondsuit$

By Lemma 6(d) it follows that every singular value which lies between two consecutive nodal stock values and which are singular values as well, can be removed. This implies that we may delete the critical value $\overline{M}_{i,j}$, during the backward procedure, if it lies between two consecutive nodal singular values.

In the next proposition we shall see that the set of internal singular values of $v_{i,j}$ at each node can be reduced to a sequence of consecutive nodal singular values which are singular values of $v_{i+1,j+1}$ as well, with the eventual addition of K. $\overline{M}_{i,j}$ lies always between two consecutive nodal singular values, so that it is not necessary to compute it in the backward procedure.

Proposition 1. *Consider the price value function $v_{i,j}$ and denote by l_0 the smallest index l such that $s_0u^l > \max\{K, M_{i,j}^{min}\}$. The set of singular values of $v_{i,j}$ can be reduced to:*
$M_{i,j}^{min}, M_{i,j}^{max}$, K if $K \in (M_{i,j}^{min}, M_{i,j}^{max})$ and a set (eventually empty) of consecutive nodal stock values $\{s_0u^{l_0}, s_0u^{l_0+1}, ..., s_0u^{l_0+k}\}$ which are singular values of $v_{i+1,j+1}$ as well.
Moreover if $M = s_0u^{l_0+k} < \frac{M_{i,j}^{max}}{u}$, then $v_{i,j}(M) = M - K$.

Proof. See Appendix.

Remark 9. *As in the case of Asian options, our procedure allows us to obtain an upper and a lower bound of the price in a simple way. However in this case the singular points are very few and their distance is much more relevant than in the Asian case.*

For this reason is not useful to compute upper and lower bounds unless we need to consider an extremely large number of time steps.

3.1 Sketch of the algorithm in the American lookback case

Let us summarize the algorithm in order to obtain the exact binomial price for a fixed strike American lookback call option.

- **STEP n**
  - Compute the singular points at maturity by using (5) and (6) where M replaces A.

- **STEP i**, for $i = n-1, ..., 0$

  - compute $P^1_{i,0}$, $P^1_{i,i}$ by comparing the continuation values given in (12) with the early exercise,

  - for each node $N_{i,j}$, $j = 1, ..., i-1$, compute the set of the singular points by the following steps:

  1. evaluate $v^c_{i,j}(M^{min}_{i,j})$, $v^c_{i,j}(M^{max}_{i,j})$,
     if $v^c_{i,j}(M^{min}_{i,j}) \leq M^{mim}_{i,j} - K$ then there are only two singular points: $(M^{min}_{i,j}, M^{min}_{i,j} - K)$, $(M^{max}_{i,j}, M^{max}_{i,j} - K)$ and the computation is concluded; otherwise insert $(M^{min}_{i,j}, v_{i,j}(M^{min}_{i,j}))$, $(M^{max}_{i,j}, v_{i,j}(M^{max}_{i,j}))$,
  2. if $K \in (M^{min}_{i,j}, M^{max}_{i,j})$ then insert $(K, v_{i,j}(K))$,
  3. for each singular value M of the node $N_{i+1,j+1}$ belonging to $(K, M^{max}_{i,j})$ add $(M, v^c_{i,j}(M))$. If $v^c_{i,j}(M^{max}_{i,j}) \geq M^{max}_{i,j} - K$ then $v^c_{i,j}$ and $v_{i,j}$ coincide so the computation is concluded. Otherwise (in this case $\overline{M}_{i,j}$ exists) remove all the singular points with singular value internal to $[M^{min}_{i,j}, M^{max}_{i,j}]$ and singular price given by the early exercise, except from the one which has the smallest singular value.

4 Numerical Comparisons

In this section we will illustrate numerically the efficiency of the singular points method, previously introduced, for pricing fixed Asian and lookback options in the American case.

We will first compare our algorithm with the most efficient tree methods for the fixed strike American Asian call options. Then we will study the behavior of convergence to the continuous price. Our comparison will include the PDE-based methods. Finally we will consider lookback options.

All the computations were performed in double precision on a PC equipped with a processor Centrino at 1.6 Ghz and 512 Mb of RAM.

4.1 Fixed strike American Asian call options: comparison with the tree methods

In order to check the behavior of the singular points algorithm, we will compare:

1. the exact CRR binomial method;
2. the Hull-White method (HW) with $h = 0.005$ (see [7]);
3. the linear interpolation forward shooting grid method (FSG) of Barraquand-Pudet choosing $\rho = 0.1$ (see [2],[11]);
4. the Chalasani et al. method (CJEV) providing an upper and a lower bound (see [3]);
5. the singular points method providing an upper and a lower bound with a level of error smaller than nh, with two different choices of h: $h = 10^{-4}$ (SP_1), $h = 10^{-5}$ (SP_2).

We will assume that the initial value of the stock price is $s_0 = 100$, the maturity $T = 1$, the force of interest rate $r = 0.1$, the continuous dividend yield $q = 0.03$. We will consider two choices for volatility $\sigma = 0.2$, $\sigma = 0.4$ and two choices for the strike $K = 90$ and $K = 110$.

We will consider various time steps $n = 25, 50, 100, 200, 400, 800$ and we will report the price estimates and the corresponding time of computation (in brackets). The exact binomial method is available only for $n = 25$, while CJEV is available only for $n = 25, 50, 100$ because of insufficient memory capacity. In the case of CJEV the global computational time in order to obtain the upper and lower estimates (by means of a single procedure) has been reported. As regards the SP methods, the two estimates are obtained separately.

	n	HW	FSG	CJEV		SP_1		SP_2		Exact BIN
				down	up	down	up	down	up	
$\sigma = 0.2$	25	14.60279	14.25496	14.24607	14.24723	14.24610	14.24628	14.24615	14.24617	14.24616
		(0.028)	(0.032)		(0.012)	(0.008)	(0.008)	(0.011)	(0.009)	
	50	14.49228	14.48210	14.47778	14.47887	14.47767	14.47830	14.47788	14.47793	-
		(0.15)	(0.20)		(0.20)	(0.03)	(0.03)	(0.07)	(0.05)	
	100	14.64524	14.62562	14.62141	14.62220	14.62095	14.62258	14.62150	14.62165	-
		(0.78)	(1.64)		(3.02)	(0.11)	(0.09)	(0.28)	(0.20)	
	200	14.74383	14.71127	-	-	14.70601	14.70954	14.70727	14.70762	-
		(4.70)	(13.20)			(0.48)	(0.34)	(1.33)	(0.95)	
	400	14.80389	14.76035	-	-	14.75368	14.76072	14.75641	14.75715	-
		(37.94)	(105.82)			(2.08)	(1.45)	(6.19)	(4.31)	
	800	14.83690	14.78716	-	-	14.77751	14.79049	14.78318	14.78464	-
		(201.86)	(836,80)			(9.94)	(6.73)	(30.19)	(20.66)	
$\sigma = 0.4$	25	18.32871	18.32871	17.84535	17.85148	17.84666	17.84687	17.84672	17.84674	17.84672
		(0.046)	(0.031)		(0.013)	(0.009)	(0.008)	(0.013)	(0.011)	
	50	18.21421	18.21659	18.20337	18.20847	18.20428	18.20493	18.20448	18.20454	-
		(0.28)	(0.20)		(0.20)	(0.04)	(0.03)	(0.09)	(0.06)	
	100	18.46687	18.46077	18.44834	18.45253	18.44918	18.45091	18.44975	18.44992	-
		(1.78)	(1.64)		(3.01)	(0.16)	(0.12)	(0.41)	(0.30)	
	200	18.62676	18.60731	-	-	18.59546	18.59922	18.59676	18.59714	-
		(9.51)	(13.23)			(0.80)	(0.55)	(2.28)	(1.56)	
	400	18.72696	18.69170	-	-	18.67888	18.68645	18.68168	18.68247	-
		(50.70)	(104.60)			(4.33)	(3.00)	(13.28)	(9.09)	
	800	18.78597	18.73771	-	-	18.72242	18.73669	18.72822	18.72979	-
		(302.72)	(825,66)			(31.34)	(21.11)	(96.95)	(64.91)	

Table 1: *Fixed strike American Asian call options with* $T = 1$, $s_0 = 100$, $r = 0.1$, $q = 0.03$ *and* $K = 90$

	n	HW	FSG	CJEV		SP_1		SP_2		Exact BIN
				down	up	down	up	down	up	
$\sigma = 0.2$	25	2.21580	2.21376	2.20952	2.21154	2.20973	2.21000	2.20982	2.20984	2.20983
		(0.031)	(0.031)		(0.013)	(0.008)	(0.006)	(0.011)	(0.009)	
	50	2.25529	2.24769	2.24348	2.24475	2.24353	2.24451	2.24384	2.24393	-
		(0.14)	(0.20)		(0.20)	(0.03)	(0.02)	(0.06)	(0.05)	
	100	2.28191	2.26623	2.26213	2.26290	2.26169	2.26416	2.26245	2.26269	-
		(0.78)	(1.64)		(3.03)	(0.11)	(0.08)	(0.25)	(0.17)	
	200	2.29734	2.27597	-	-	2.27054	2.27562	2.27220	2.27275	-
		(4.70)	(13.07)			(0.44)	(0.31)	(1.22)	(0.84)	
	400	2.30536	2.28080	-	-	2.27359	2.28331	2.27705	2.27815	-
		(37.92)	(104.59)			(1.95)	(1.34)	(5.73)	(3.98)	
	800	2.30944	2.28292	-	-	2.27221	2.28977	2.27919	2.28120	-
		(201.59)	(828,03)			(9.50)	(6.47)	(28.95)	(19.59)	
$\sigma = 0.4$	25	6,78517	6.79136	6.77940	6.78672	6.78106	6.78131	6.78115	6.78117	6.78116
		(0.047)	(0.031)		(0.013)	(0.009)	(0.008)	(0.011)	(0.009)	
	50	6.88872	6.89089	6.87917	6.88433	6.88086	6.88184	6.88118	6.88127	-
		(0.28)	(0.20)		(0.20)	(0.04)	(0.03)	(0.08)	(0.06)	
	100	6.95445	6.94958	6.93817	6.94173	6.93943	6.94190	6.94024	6.94048	-
		(1.78)	(1.63)		(3.03)	(0.16)	(0.11)	(0.39)	(0.26)	
	200	6.99555	6.98150	-	-	6.97075	6.97591	6.97249	6.97302	-
		(9.47)	(13.12)			(0.78)	(0.53)	(2.22)	(1.52)	
	400	7.01909	6.99776	-	-	6.98572	6.99571	6.98935	6.99047	-
		(50.53)	(104.75)			(4.27)	(2.95)	(13.23)	(8.89)	
	800	7.03155	7.00538	-	-	6.99025	7.00849	6.99772	6.99980	-
		(301.80)	(827,48)			(31.23)	(20.98)	(95.44)	(64.20)	

Table 2: *Fixed strike American Asian call options with* $T = 1$, $s_0 = 100$, $r = 0.1$, $q = 0.03$ *and* $K = 110$

	$\sigma = 0.2$, $K = 90$			$\sigma = 0.4$, $K = 90$			$\sigma = 0.2$, $K = 110$			$\sigma = 0.4$, $K = 110$		
n	CJEV	SP_1	SP_2	CJEV	SP_1	SP_2	CJEV	SP_1	SP_2	CJEV	SP_1	SP_2
25	.00016	.00018	.00002	.00613	.00021	.00002	.00202	.00027	.00002	.00732	.00025	.00002
50	.00109	.00063	.00005	.00510	.00065	.00006	.00127	.00098	.00009	.00516	.00098	.00009
100	.00079	.00163	.00015	.00419	.00173	.00017	.00077	.00247	.00024	.00356	.00247	.00024
200	-	.00353	.00035	-	.00376	.00038	-	.00508	.00055	-	.00516	.00053
400	-	.00704	.00074	-	.00757	.00079	-	.00972	.00110	-	.01001	.00112
800	-	.01298	.00146	-	.01427	.00157	-	.01756	.00201	-	.01824	.00208

Table 3: *Difference between the upper and lower estimates for CJEV, SP methods*

The numerical results obtained in Table 1 and 2 confirm the reliability of the singular points method. In comparison with the Chalasani et al. method (see also Table 3) we obtained an actual improvement in precision for the the upper and lower bounds in a lower CPU times and without problems of memory. With respect to Hull-White and the forward shooting grid methods the improvements seem to be significant.

4.2 Analysis of convergence of the approximations to the continuous value

In this section we will address more thoroughly the complexity and the convergence to the continuous price value of our algorithm both in the European and the American cases. We will compare our algorithm with the modified linear interpolation forward shooting grid method (M-FSG), which guarantees the convergence to the continuous price value (see [11]), and with two PDE-based methods (see [6] and [10]). We used the Richardson extrapolation in order to speed-up the convergence of the tree methods. In the European case we used the two-points extrapolation $2P_n - P_{\frac{n}{2}}$, whereas in the American case the three points extrapolation $\frac{8}{3}P_n - 2P_{\frac{n}{2}} + \frac{1}{3}P_{\frac{n}{4}}$ was adopted.

As regards to the convergence analysis, we will compare the following algorithms:

1. the PDE-based method of d'Halluin et al. (DFL) available for both the European and the American Asian options(see [6]);

2. the PDE-based method of Vecer available in the European Asian option case (see [10]);

3. the modified linear interpolation forward shooting grid method (M-FSG) of Barraquand-Pudet (see [2],[11]). We chose $\rho = 0.1$ and $n\sqrt{n}$ grid points in the Asian direction in order to guarantee the convergence (see the Premia implementation [9]);

4. the modified FSG algorithm with the Richardson extrapolation (M-FSG-Rich);

5. the singular points method (SP) providing an upper bound with a level of error smaller than nh with $h = \frac{0.1}{n^2}$ (see Remark 7);

6. the previous singular points upper algorithm combined with the Richardson extrapolation (SP-Rich).

For the PDE-based method we will use the numerical results provided in [6]. In order to compare the convergence behavior we consider the convergence ratio R proposed in [6],

$$R = \frac{P_{\frac{n}{2}} - P_{\frac{n}{4}}}{P_n - P_{\frac{n}{2}}}$$

In Tables 4 and 5 the European Asian call case is considered using low and high volatility. Table 6 refers to the American Asian put case. The PDE-based algorithms of Vecer and d'Halluin et al. are almost second order in time (see [6]). The singular points and the modified FSG algorithms exhibit, as expected, first-order convergence. The use of the Richardson extrapolation speeds up the convergence both in the case of our method and the modified FSG method.

As concern the computational analysis we have to take into account the computational time (see Remark 8). We will compare our algorithm (SP) with: M-SFG of complexity $O(n^{\frac{7}{2}})$, FSG of complexity $O(n^3)$ and the Vecer method of complexity $O(n^2)$. Fig.5 offers a number of steps/time of computation graph using data of Table 4. The comparison indicates that the present method can effectively be competitive in practice and it seems to be of complexity $O(n^3)$. More estensive numerical experiments have confirmed this order of complexity.

n	DFL		Vecer		M-FSG		M-FSG-Rich		SP		SP-Rich	
	Price	R	Price	R	Price	R	Price	R	Price	R	Price	R
25	1.857193	n.a.	1.839863	n.a.	1.845628	n.a.	n.a.	n.a.	1.845841	n.a.	n.a.	n.a.
50	1.853254	n.a.	1.848642	n.a.	1.848535	n.a.	1.851442	n.a.	1.848745	n.a.	1.851655	n.a.
100	1.852120	3.475	1.839863	3.974	1.850044	1.927	1.851553	n.a.	1.850204	1.996	1.851661	n.a.
200	1.851781	3.338	1.851407	3.979	1.850814	1.960	1.851583	3.616	1.850902	2.087	1.851600	-0.102
400	1.851660	2.815	1.851546	3.987	1.851202	1.983	1.851590	4.603	1.851248	2.016	1.851594	10.992

Table 4: *Fixed strike European Asian call options with* $T = 0.25$, $s_0 = 100$, $K = 100$, $r = 0.1$, $q = 0$, $\sigma = 0.1$

n	DFL		Vecer		M-FSG		M-FSG-Rich		SP		SP-Rich	
	Price	R	Price	R	Price	R	Price	R	Price	R	Price	R
25	6.010203	n.a.	6.009821	n.a.	5.995543	n.a.	n.a.	n.a.	5.995682	n.a.	n.a.	n.a.
50	6.015092	n.a.	6.014848	n.a.	6.005734	n.a.	6.015924	n.a.	6.005903	n.a.	6.016124	n.a.
100	6.016344	3.905	6.016251	3.582	6.011129	1.889	6.016525	n.a.	6.011255	1.910	6.016607	n.a.
200	6.016651	4.085	6.016619	3.816	6.013911	1.939	6.016693	3.559	6.013982	1.962	6.016710	4.673
400	6.016723	4.219	6.016713	3.915	6.015321	1.974	6.016730	4.546	6.015361	1.979	6.016740	3.485

Table 5: *Fixed strike European Asian call options with* $T = 0.25$, $s_0 = 100$, $K = 100$, $r = 0.05$, $q = 0$, $\sigma = 0.5$

n	DFL		M-FSG		M-FSG-Rich		SP		SP-Rich	
	Price	R	Price	R	Price	R	Price	R	Price	R
25	2.220443	n.a.	2.047501	n.a.	n.a.	n.a.	2.079846	n.a.	n.a.	n.a.
50	2.195726	n.a.	2.091589	n.a.	n.a	n.a.	2.124386	n.a.	n.a.	n.a.
100	2.188555	3.447	2.118518	1.637	2.148704	n.a.	2.151597	1.637	2.182102	n.a.
200	2.186717	3.903	2.134301	1.706	2.151631	1.948	2.167529	1.708	2.185012	2.054
400	2.186243	3.847	2.143060	1.802	2.152397	3.818	2.176364	1.803	2.185777	3.802

Table 6: *Fixed strike American Asian put options with* $T = 0.25$, $s_0 = 100$, $K = 100$, $r = 0.05$, $q = 0$, $\sigma = 0.15$

Figure 5: *Number of steps / time of computation table and graphic in log-log scale*

4.3 American fixed strike lookback call options

In the lookback case we will simply compare our technique with an optimized version of the exact binomial method. As already observed there are very few singular points involved in the computation, so that the valuation of an upper and a lower bound is not significant. Therefore, the price obtained through the use of the singular points method coincides with the exact binomial, but with an improvement in the computational time (see Table 7 and 8 where the parameters of Section 4.1 are used).

Clearly, when compared with the previous literature, improvements are less significant in the lookback case than in the Asian case. Nevertheless, the data confirm the power of the method and its relevance in pricing American path-dependent options.

σ	n	Bin	SP
0.2	100	27.73002	27.73002
		(0.004)	(0.003)
	200	28.02747	28.02747
		(0.025)	(0.016)
	400	28.24333	28.24333
		(0.184)	(0.077)
	800	28.39866	28.39866
		(1.28)	(0.30)
	1600	28.51033	28.51033
		(10.75)	(1.55)

σ	n	Bin	SP
0.4	100	44.31762	44.31762
		(0.004)	(0.003)
	200	45.00766	45.00766
		(0.026)	(0.017)
	400	45.50900	45.50900
		(0.183)	(0.080)
	800	45.87045	45.87045
		(1.47)	(0.42)
	1600	46.12961	46.12961
		(12.02)	(2.11)

Table 7: *Fixed strike American lookback call options with* $K = 90$ *for binomial method and SP method*

σ	n	Bin	SP
0.2	100	11.06517	11.06517
		(0.004)	(0.003)
	200	11.27996	11.27996
		(0.025)	(0.016)
	400	11.43759	11.43759
		(0.184)	(0.077)
	800	11.55096	11.55096
		(1.45)	(0.38)
	1600	11.63192	11.63192
		(12.02)	(2.00)

σ	n	Bin	SP
0.4	100	27.28512	27.28512
		(0.004)	(0.003)
	200	27.85271	14.3245
		(0.024)	(0.017)
	400	28.26777	28.26777
		(0.183)	(0.081)
	800	28.57206	28.57206
		(1.46)	(0.43)
	1600	28.79142	28.79142
		(11.98)	(2.16)

Table 8: *Fixed strike American lookback call options with $K = 110$ for binomial method and SP method*

5 Conclusion

We have introduced a new general binomial framework, called 'singular points method', for pricing path-dependent options of European/American type. We have applied it in the case of Asian and lookback options. The procedure provides upper and lower bounds of the exact binomial price with a prescribed level of error. The control of the error allows us to immediately prove the convergence of order $O(\Delta T)$ to the continuous value. The method is competitive in practice and the observed computational complexity is $O(n^3)$. The numerical results showed that the singular points method is an improvement on the previous tree methods.

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2, 18, 19
2, 17
13

6 Appendix: Proof of Proposition 1

Proof. Consider the case $i = n-1$. Take first $j \geq int[\frac{i+1}{2}]$ and $j < n-1$ (the case $j = n-1$ is trivial). 19

At the node $N_{i,j}$ the singular values of $v^c_{i,j}$ are $M^{min}_{i,j}$, $M^{max}_{i,j}$, K if $K \in (M^{min}_{i,j}, M^{max}_{i,j})$ and eventually $uM^{min}_{i,j} = M^{min}_{i+1,j+1}$. By Lemma 6(a) $uM^{min}_{i,j}$ is a singular value of $v^c_{i,j}$ if and only if $uM^{min}_{i,j} \geq K$. 2, 18, 19

Take now the value function $v_{i,j}$. The possible singular points of $v_{i,j}$ are the same of $v^c_{i,j}$ with the possible addiction of $\overline{M}_{i,j}$ (when it exists). If $\overline{M}_{i,j}$ exists then $\overline{M}_{i,j} \geq K$. If $\overline{M}_{i,j} = K$ then the claim follows, otherwise necessarily $uM^{min}_{i,j} = M^{min}_{i+1,j+1}$ is a singular value and $K < uM^{min}_{i,j}$. Hence $v_{i+1,j+1}(uM^{min}_{i,j}) = uM^{min}_{i,j} - K$ and, by Lemma 6(c), $v_{i,j}(uM^{min}_{i,j}) = uM^{min}_{i,j} - K$. We can conclude that $\overline{M}_{i,j} \in [M^{min}_{i,j}, uM^{min}_{i,j}]$ and by Lemma 6(d) it can be removed. Hence the claim holds. 2, 17, 18, 19

In the case $j < int[\frac{i+1}{2}]$ there are no singular values in $(K, M^{max}_{i,j})$ so the claim is trivial.

Consider now the case $i < n-1$ and take $0 < j < n-i$ (the cases $j = 0$, $j = n-i$ are trivial).

All the singular values of $v^c_{i,j}$ are either singular values of $v_{i+1,j+1}$ or singular values of $v_{i+1,j}$. We claim that every singular value of $v_{i+1,j}$ belonging to $(M^{min}_{i,j}, M^{max}_{i,j})$ is a singular values of $v_{i+1,j+1}$ as well. In fact if $M > \min\{K, M^{min}_{i,j}\}$ is a singular value of $v_{i+1,j}$ then, by induction, it is a singular value of $v_{i+2,j+1}$ as well, therefore it is a singular value of $v^c_{i+1,j+1}$. By Lemma 6(b) we can conclude that it is a singular value of $v_{i+1,j+1}$ as well. Therefore the set of all the singular values of $v^c_{i,j}$ is composed by $M^{min}_{i,j}$, $M^{max}_{i,j}$, eventually K and a sequence of consecutive nodal values $\{s_0u^{l_0}, s_0u^{l_0+1}, ..., s_0u^{l_0+k}\}$ which are singular values of $v_{i+1,j+1}$ as well.

Take now $v_{i,j}$. If $v^{i,j}(M^{max}_{i,j}) \geq M^{max}_{i,j} - K$ then $v_{i,j} \equiv v^c_{i,j}$ and their singular points coincide. If $s_0u^{l_0+k} < \frac{M^{max}_{i,j}}{u}$ then $s_0u^{l_0+k+1}$ is not a singular value of $v_{i+1,j+1}$. By induction $v_{i+1,j+1}(s_0u^{l_0+k}) = s_0u^{l_0+k} - K$. By Lemma 6(c) $v_{i,j}(s_0u^{l_0+k}) = s_0u^{l_0+k} - K$.

Assume $v^{i,j}(M_{i,j}^{max}) < M_{i,j}^{max} - K$. If $v_{i,j}(M_{i,j}^{min}) \leq M_{i,j}^{min} - K$ then there are no singular points in $(M_{i,j}^{min}, M_{i,j}^{max})$ and the claim holds. If $v_{i,j}(M_{i,j}^{min}) > M_{i,j}^{min} - K$ then $\overline{M}_{i,j}$ exists and $\overline{M}_{i,j} \geq K$. Let l_1 be the largest index l such that $s_0 u^l$ is a singular point of $v_{i,j}^c$ and $s_0 u^l \in (K, M_{i,j}^{max})$. If $s_0 u^{l_1} = M_{i,j}^{max}/u$ then the sequence of singular values of $v_{i,j}^c$ includes all the nodal stock values from $s_0 u^{l_0}$ to $M_{i,j}^{max}$. Denoting by $\bar{l}$ the smallest index such that $\overline{M}_{i,j} \leq s_0 u^{\bar{l}}$, we have that the singular values $s_0 u^{\bar{l}+1}, ..., s_0 u^{l_1}$ can be removed, hence the claim holds. If $s_0 u^{l_1} < M_{i,j}^{max}/u$ by the induction hypothesis $v_{i+1,j+1}(s_0 u^{l_1}) = s_0 u^{l_1} - K$. By Lemma 6(c) $\overline{M}_{i,j} \leq s_0 u^{l_1}$. Again $s_0 u^{\bar{l}+1}, ..., s_0 u^{l_1}$ can be removed and $v_{i,j}(s_0 u^{\bar{l}}) = s_0 u^{\bar{l}} - K$, proving the claim. $\diamondsuit$