Implementation : Gamma Expansion of Heston Stochastic Volatility Model

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Abstract

In this paper we show the implementation of the exact simulation methods presented in the paper of Galsserman and al. [2]. We derive two methods, the first one is based on the truncation series simulation, and the second one deals with inversion techniques of both the Laplace transform and the cumulative function.

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1 Introduction

We consider the following Heston model, given by the following stochastic differential equation

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu dt + \sqrt{V_t}(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2) \\
dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}dW_t^1,
\end{aligned}
$$

where (W_t^1, W_t^2) is a standard two dimensional Browian motion. The variable S_t describes the level of an underlying asset and Vt the variance of its instantaneous returns. The parameters κ, θ, σ (and typically also μ) are positive, and ρ takes values in $[-1, 1]$. We take the initial conditions S_0 and V_0 to be strictly positive.

It is well know [1], that the CIR process V_t is given by

$$
V_t \sim \frac{\sigma^2(1-e^{-\kappa t})}{4\kappa}\chi_\delta^2\left(\frac{4\kappa e^{-\kappa t}}{\sigma^2(1-e^{-\kappa t})}V_0\right),\ t > 0,\ \delta = \frac{4\kappa\theta}{\sigma^2},
$$

where $\chi_{delta}^2(\lambda)$ denotes non central chi-square variable with δ degrees freedom and non central parameter λ.

Broadie and Kaya rewrite the exact simulation of the couple (S_t, V_t) as follows

$$
\log(\frac{S_t}{S_0}) \sim \mathcal{N}\left((\mu - \frac{\rho\kappa\theta}{\sigma})t + (\frac{\kappa\rho}{\sigma} - \frac{1}{2})\int_0^t V_s ds + \frac{\rho}{\sigma}(V_t - V_0), (1-\rho^2)\int_0^t V_s ds\right),
$$

where $\mathcal{N}(m, var)$ is an independent Gaussian randon variable with mean m and variance var. It is then sufficient to know the joint distribution of $(V_t, \int_0^t V_s ds)$ to sample exactly the couple (S_t, V_t). The problem of the exact simulation can be reduced to sample exactly just

$$\left(\int_0^t V_s ds | V_0, V_t\right). \tag{1}$$

The main result of the paper is given by the following theorem

Theorem 1. — *The distribution of 1 admits the following representation*

$$\left(\int_0^t V_s ds | V_0, V_t\right) \sim X_1 + X_2 + X_3 \equiv X_1 + X_2 + \sim_{j=1}^{\eta} Z_j,$$

in which X_1, X_2, $\mu, Z_1, Z_2 \dots$ are mutually independent, the Z_j are independent copies of a random variable Z, and η is a Bessel random variable with parameter $\nu = \frac{\delta}{2} - 1$, and $z = \frac{2\kappa/\sigma^2}{sinh(\kappa t/2)} \sqrt{V_t V_0}$.

Moreover, X_1, X_2, and Z have the following representations:

$$X_1 \sim \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \sum_{j=1}^{N_n} Exp_j(1),\ X_2 \sim \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \Gamma(\delta/2, 1),\ Z \sim \sum_{n=1}^{\infty} \frac{1}{\gamma_n} \Gamma(2, 1), \tag{2}$$

where

$$\lambda_n = \frac{16\pi n^2}{\sigma^2 t(\kappa^2 t^2 + 4\pi^2 n^2)},\ \gamma_n = \frac{\kappa^2 t^2 + 4\pi^2 n^2}{2\sigma^2 t^2}. \tag{3}$$

The N_n are independent Poisson random variable with parameter $\lambda_n(V_0 + V_t)$, the $Exp_j(1)$ are independent, unit mean exponential random variable, and the $\Gamma(\alpha, \beta)$ denote the independent gamma random variable with shape of parameter α and scale one β.

2 First Method: Series truncation

This section concerns the method named mcGlassermanKim1. The idea is to go back to the thoerem and replace the infinite series by a finite one. The rest of each truncation is approximated by a non central chi-square random variable.

Technical details are given in [2].

Input: ordertr, θ, κ, σ, t, V_t and V_0
Output: The value of the random variable X_1
Truncation method
$X_1 \sim \sum_{n=1}^{\text{ordertr}} \frac{1}{\gamma_n} \sum_{j=1}^{N_n} \text{Exp}_j(1) +$ Chi-square random variable

Algorithm 1: The function X1Sample

Input: ordertr, θ, κ, σ, and t.
Output: The value of the random variable X_2
Truncation method $\sim \sum_{n=1}^{ordertr} \frac{1}{\gamma_n}\Gamma(\delta/2, 1) +$ Chi-square random variable

Algorithm 2: The function X2Sample

Input: ordertr, θ, κ, σ, and t.
Output: The value of the random variable Z
Truncation method $Z \sum_{n=1}^{ordertr} \frac{1}{\gamma_n}\Gamma(2, 1) +$ Chi-square random variable

Algorithm 3: The function X3Sample

Input: θ, κ, σ, t, V_t
Output: The value of the random variable $(V_t, \int_0^t V s ds)$
Using the representation of Theorem 1 taking a default value
$ordertr = 20$

Algorithm 4: The function SampleC

3 First Method: Series truncation

This last section concerns the method named mcGlassermanKim2. Since all variables X_1, X_2, Z given in Theorem 1 have an explicit Laplace transform, we can thus use the inverse of both the Laplace transform and the cumulative function of each variable to sample exactly. Technical details are given in [2]. However this method is very expensive in term of time computation. One has to notice that X_2 and X_3 do not depend on the initial value of the couple $(V_t, \int_0^t V s ds)$. We can then inverse the cumulative function of both variable, and be use on one shot for all available values of the couple $(V_t, \int_0^t V s ds)$. How ever, X_1 is given by its truncation value. The Cumulative function can be then computed using Abatt algorithm (used by Broadie and Kaya [3]).

Input: vector, mprecision, θ, κ, σ and t.

Output: The vector value of the inverse Cumulative function of the random variable X_2

We use the same technique as in [3], for a vector input we derive the inverse of the cumulative function with respect to the components value of the vector

Algorithm 5: The function `CumuX2M`

Input: vector, mprecision, θ, κ, σ and t.

Output: The vector value of the inverse Cumulative function of the random variable Z

We use the same technique as in [3], for a vector input we derive the inverse of the cumulative function with respect to the components value of the vector

Algorithm 6: The function `CumuX3M`

The method has to be used carefully, because one has to calculate the inverse of the cumulative function of X_2 and Z before doing Monte Carlo to compute the price.

References

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[3] M.Broadie O.Kaya. 2004 winter simulation conference (wsc'04)). *Exact Simulation of Option Greeks under Stochastic Volatility and Jump Diffusion Models*, 2:535–543, 2004. 4