Model free volatility index and volatility swap

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1. Quadratic variation and volatility derivatives

Let S_t denote the value of a stock or stock index at time t. We assume that the volatility product starts at time zero and ends at time T. Set Assume that S_t is modeled by some diffusion process of the type:

$$\frac{dS_t}{S_t} = \mu(t, S_t, ...)dt + \sigma(t, S_t, ...)dW_t$$

where the drift μ and the volatility σ are either deterministic or stochastic and W_t is a Wiener process.

The annualized expected quadratic variation of log-returns over the time interval $[0; T]$ is determined by

$$Q_T = \frac{1}{T}\mathbf{E}\Big[\int_0^T \sigma^2(t, S_t, ...)dt\Big] = \frac{1}{T}\mathbf{E}\Big[[\log S, \log S]_T\Big],$$

where $[\log S, \log S]$ denotes the quadratic variation of $\log S$.

Now consider swaps written on the volatility $Q_T^{1/2}$. A volatility swap is an instrument which allows investors to trade future realized (or historical) volatility against current implied volatility. The quantity

$$IV = Q_T^{1/2} \cdot 100$$

is called "fair strike of a volatility swap in annual volatility points". If the underlying asset is a stock index, IV can be used as a volatility index of the local financial market.

Volatility swaps on volatility are derivatives written on $Q_T^{1/2}$:

Volatility swap with fixed strike K pays the holder

$$VolS(K, T) = Q_T^{1/2} - K.$$

In the most well-known model-free approaches such as the CBOE method, the price of variance derivative may be approximated by some portfolio (the so called replicating portfolio) of some amount of underlying and derivatives on it. One may find the construction of replicating portfolio e.g. in [2, 3].

We will concentrate on the alternative approach based on the implied volatility integration. Both methods begin with the following auxiliary formula

$$\frac{1}{T}\mathbf{E}\Big[[\log S, \log S]_T\Big] = -2\mathbf{E}\left[\ln\left(\frac{S_T}{\mathbf{E}[S_T]}\right)\right], \tag{1}$$

where $\mathbf{E}[]$ is the risk-neutral expectation. Notice that (1) is correct if S is a continuous semimartingale.

Denote by F the forward price of S, K the strike price and $P(K)$ the market put price at strike level K. Let us change the variable $k = \ln(K/F)$. The main result of the alternative free model volatility index formula in terms of market European put prices is summarized in the theorem 2.1 [4].

Theorem 1.1. *Define* $Put_{BS} : \mathbf{R} \times (0; +\infty) \to (0; +\infty)$ *by*

$$Put_{BS}(k, \sigma) := Fe^k N(-d_2(k, \sigma)) - FN(-d_1(k, \sigma)),$$

where $N()$ *is the standard normal distribution function,*

$$d_2(k, \sigma) := -\frac{k}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2},$$

$$d_1(k, \sigma) = d_2(k, \sigma) + \sigma\sqrt{T}.$$

Then the Black-Scholes implied volatility function $\sigma(k) : R \to [0; 1)$ *is well-defined by*

$$\sigma(k) := P_{BS}(k, \cdot)^{-1}(P(Fe^k))$$

or, equivalently, $P_{BS}(k, \sigma(k) = P(Fe^k)$.

Moreover, the mapping $d_2 : k \to d_2(k, \sigma(k))$ *is a decreasing function and it holds that*

$$Q_T = \int \sigma^2(g(z))\phi(z)dz,$$

where g *is the inverse function of* d_2 *and* $\phi(\cdot)$ *is the standard normal density.*

The same result can be reformulated in terms of market European call prices.

The algorithm implemented into the program platform Premia consist of the following steps (for details see [4]).

- input a set of European put and European call prices of the same maturity at different strike levels and a strike volatility swap;
- selecting a set of valid out-of-the-money (OTM) calls and puts to be used;
- approximating the function $z \to \sigma^2(g(z))$ in terms of a set of cubic polynomials;
- integrating the obtained function with respect to the normal density $\phi(\cdot)$;
- output "fair strike of a volatility swap " IV and "volatility swap price" $VolS$ in annual volatility points.

References

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[4] Fukasawa, M., Ishida, I., Maghrebi, N., Oya, K., Ubukata, M. and Yamazaki, K., "MODEL-FREE IMPLIED VOLATILITY: FROM SURFACE TO INDEX", International Journal of Theoretical and Applied Finance (IJTAF), 14, issue 04, 2011, p. 433-463. 2