Chang and Palmer: the center binomial model

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Let us assume that in the risk-neutral world the underlying asset price is the solution over $[0, T]$ of the SDE

$$dS_t = rS_t dt + \sigma S_t dB_t, \quad S_0 = s_0,$$

where r is the interest rate, σ is the volatility parameter and B is a Brownian motion in $\mathbb{R}$. The *center binomial model* is a new binomial model developed in Chang and Palmer [1] in which the binomial approximation error for European call and digital call options, i.e. the difference between the binomial and the Black-Scholes prices, is $O(\frac{1}{n})$, where $n \in \mathbb{N}$ denotes the number of time steps of the tree.
It is known that the rate of convergence of the classical Cox, Ross and Rubinstein (CRR) model provided in [2] is of order $1/n$ for European call options (Diener and Diener [3], Chang and Palmer [1]) and, more generally, for European options on continuous payoff functions (Walsh [4]). But when the payoff is assumed to be a generic discontinuous function, the CRR binomial approximation error is of the type $1/\sqrt{n}$ (Walsh [4]). In particular, a precise formula for the coefficient of $1/\sqrt{n}$ in the error term for European digital call options is provided in Chang and Palmer [1].
In [1] the authors consider a general class of binomial schemes, including the CRR model, and they get the following result:

Theorem 1. *In the n-period binomial model, with*

$$u = e^{\sigma\sqrt{h}+\lambda\sigma^2 h}, d = e^{-\sigma\sqrt{h}+\lambda\sigma^2 h}, p = \frac{e^{rh} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}, \tag{1}$$

where $h = T/n$, $\lambda = \lambda(n)$ is an arbitrary bounded function of n, s_0 is the initial stock price, K is the strike price and T is the maturity, the binomial approximation error $Err_{call\text{-}dig}(n)$ related to European digital call options is

$$Err_{call\text{-}dig}(n) = \frac{e^{-rT}e^{-\frac{d_{12}^2}{2}}}{\sqrt{2\pi}}\left[\frac{\Delta_n^K}{\sqrt{n}} - \frac{d_{12}(\Delta_n^K)^2}{2n} + \frac{B_n}{n}\right] + O\left(\frac{1}{n^{3/2}}\right); \tag{2}$$

the binomial approximation error $Err_{call}(n)$ related to European call options is

$$Err_{call}(n) = \frac{s_0 e^{-\frac{d_{11}^2}{2}}}{24\sigma\sqrt{2\pi T}} \frac{A_n - 12\sigma^2 T((\Delta_n^K)^2 - 1)}{n} + O\left(\frac{1}{n^{3/2}}\right), \tag{3}$$

where

$$\begin{aligned}
\Delta_n^K &= 1 - 2frac\left(\frac{\log(s_0/K) - n\log d}{\log(u/d)}\right),\\
d_{11} &= \frac{\log(s_0/K) + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \quad d_{12} = d_{11} - \sigma\sqrt{T},\\
B_n &= \frac{d_{11}^3 + d_{11}d_{12}^2 + 2d_{12} - 4d_{11}}{24} + \frac{(2 - d_{12}d_{22} - d_{12}^2)\sqrt{T}}{6\sigma}(r - \lambda\sigma^2)\\
&+ \frac{Td_{11}}{2\sigma^2}(r - \lambda\sigma^2)^2,\\
A_n &= -\sigma^2 T(6 + d_{11}^2 + d_{12}^2) + 4T(d_{11}^2 - d_{12}^2)(r - \lambda\sigma^2) - 12T^2(r - \lambda\sigma^2)^2.
\end{aligned}$$

We recall that for every $x \in \mathbb{R}$, $frac(x)$ is the fractional part of x, i.e. $frac(x) = x - \lfloor x \rfloor$ with $\lfloor x \rfloor$ denoting the largest integer not greater than x.

We notice that $\lambda = 0$ gives the CRR tree. The reason why Chang and Palmer consider this mode general model with u and d defined in (1) is that it allows them to achieve smooth convergence by a suitable choice of λ.
In fact the quantity Δ_n^K that appears in the coefficient of order $1/\sqrt{n}$ in (2) measures the position of the strike K in the log scale in relation to two adjacent terminal stock prices. If we call $(S_{i,j})_{i,j} = s_0 u^j d^{n-j}$ for $i = 0, ..., n$, $j = 0, ..., i$ the values of the stock price at the nodes of the n-step binomial tree, then there exists an integer j_K such that

$$S_{n,j_K-1} = s_0 u^{j_K-1} d^{n-j_K+1} < K \le S_{n,j_K} = s_0 u^{j_K} d^{n-j_K}$$

and so the "effective" strike price in the binomial tree is S_{n,j_K}. It is also possible to write in the log-scale the strike K as a geometric average of S_{n,j_K-1} and S_{n,j_K}, i.e.

$$\log K = \alpha \log S_{n,j_K} + (1-\alpha)\log S_{n,j_K-1}, \quad \text{where} \quad \alpha = \frac{1 + \Delta_n^K}{2}. \tag{4}$$

We remark that $-1 \le \Delta_n^K \le 1$ and from (4) we deduce that

- $\Delta_n^K = 0 \Rightarrow \log K = \frac{1}{2}\log S_{n,j_K} + \frac{1}{2}\log S_{n,j_K-1}$;
- $\Delta_n^K = -1 \Rightarrow \log K = \log S_{n,j_K-1}$;
- $\Delta_n^K = 1 \Rightarrow \log K = \log S_{n,j_K}$.

From the explicit expression of the binomial approximation error in (2), Chang and Palmer observe that if the parameter λ is chosen such that in the tree structure the strike price is situated exactly halfway between two stock prices at maturity (i.e. $\Delta_n^K = 0$), then the convergence both for European call and European digital call options is of order $1/n$. Then they define

$$\lambda = \frac{\log \frac{K}{s_0} - (2j_0 - 1 - n)\sigma\sqrt{h}}{n\sigma^2 h} \tag{5}$$

where $j_0 = \lceil \tilde{\gamma} \rceil = \min\{m \in \mathbb{N} : m \geq \tilde{\gamma}\}$ and $\tilde{\gamma} = \frac{\log \frac{K}{s_0} + n\sigma\sqrt{h}}{2\sigma\sqrt{h}}$. This choice of λ and the choice of u, d and p as in (1) give the so-called *center binomial model*. As a direct consequence of Theorem 1, one gets that for the n-period center binomial model the rate of convergence for call and digital call options is of order $1/n$.

References

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