Upper bound for bermudan swaptions with Schoenmakers et al.(2007) algorithm

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1 LIBOR market model

Let us consider a set of dates $T_0, T_1, ..., T_N$ with $0 = T_0 < T_1 < ... < T_N$ and $T_{i+1} - T_i = \delta$.

We note $F_i(t)$, for a certain date $t \leq T_i$, the value at date t of the Libor rate settled at T_i and payed at T_{i+1}. We extend this definition to $t > T_i$ simply by $F_i(t) = F_i(T_i)$.

By absence of arbitrage, the Libor rates are related to Zero Coupon bond by :

$$F_i(t) = \frac{1}{\delta}\left(\frac{B(t, T_i)}{B(t, T_{i+1})} - 1\right)$$

In the Libor market model, we suppose the following dynamic for the forward Libor rates :

$$dF_i(t) = F_i(t)\gamma(t; T_i, T_{i+1})dW_t^{Q^{T_{i+1}}}$$

where $(W^{Q^{T_{i+1}}}; t \geq 0)$ is a D-dimensional Brownian motion under the forward probability $Q^{T_{i+1}}$ associated with the numeraire $B(t, T_{i+1})$ and $\gamma(t; T_i, T_{i+1})$ is a deterministic function, such that $\gamma(t; T_i, T_{i+1}) = 0$ for $t > T_i$.

In the next section we will be interested in the pricing of bermudan swaption with a *starting date* T_α and an *ending date* T_β such that $T_0 < T_\alpha < ... < T_\beta \leq T_N$. this contract gives the right of choosing, at each date T_i with $\alpha \leq i < \beta$, whether to enter or not into an european swaption over $[T_i, T_\beta]$

Exercising a payer $[T_i, T_\beta]$-european swaption with strike K means to be payed at time T_i the quantity :

$$H(T_i; T_i, T_\beta) = \left(\sum_{j=i+1}^{\beta} \delta B(T_i, T_j) \right) (S(T_i, T_i, T_\beta) - K)^+$$

where $S(T_i, T_i, T_\beta)$ is the swap rate at date T_i, settled at T_i and expiring at T_β. [1]

Thus the price of a bermudan swaption with starting date T_α and ending date T_β is given, under the measure Q^N associated with the numeraire $N(t)$, by :

$$U(t) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_t^N \left[\frac{N(t)}{N(\tau)} H(\tau; \tau, T_\beta) \right] \tag{1}$$

where $\mathcal{T}$ is the set of stopping time taking values in $\{T_{i_t}, ..., T_{\beta-1}\}$, with i_t is an integer verifying $T_{i_t - 1} \leq t < T_{i_t}$.

We define the N-discounted price $\widetilde{U}(t) = \dfrac{U(t)}{N(t)}$ and $\widetilde{H}(T_i) = \dfrac{H(T_i)}{N(T_i)}$. Then:

$$\widetilde{U}(t) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_t^N \left[\widetilde{H}(\tau; \tau, T_\beta) \right] \tag{2}$$

Standard theory of optimal stopping time ensures us that $\widetilde{U}(0)$ is the solution of the following dynamic programing problem:

$$\begin{cases} \widetilde{U}_{\beta-1} = \widetilde{H}_{\beta-1} \\ \\ \widetilde{U}_i = \max\left\{ \widetilde{H}_i,\ \mathbb{E}_i^N \left[\widetilde{U}_{i+1} \right] \right\}\ , \forall i = \alpha, ..., \beta - 2 \\ \\ \widetilde{U}_0 = \mathbb{E}_0^N \left[\widetilde{U}_\alpha \right] \end{cases} \tag{3}$$

with following notations : $\widetilde{U}_i = \widetilde{U}(T_i)$ and $\widetilde{H}_i = \widetilde{H}(T_i; T_i, T_\beta)$.

[1] We recall that the swap rate $S(T_i, T_i, T_\beta)$ and zero coupon bond $B(T_i, T_j)$ depend explicitly on the Libor rates.

2 Lower bound of the price $U(0)$

Using equation 2, one can compute a lower bound for the price of the bermudan swaption by choosing an exercise strategy τ_0 and compute the associated price $N(0)\mathbb{E}_0^N\left[\widetilde{H}(\tau_0)\right]$. For example, Andersen proposes in [Andersen 2000] a method that parameterizes the exercise policy and then optimizes these parameters over a set of simulated paths to determine an approximation to the optimal exercise strategy. An alternative way is to approximate the conditional expectation in the dynamical programing equation 3 by mean of least square regression method as proposed in [Longstaff and Schwartz 2001].

3 Primal-Dual methods

The duality approach was recently introduced by [Rogers, 2002] and [Haugh and Kogan, 1999]. It provides a method to compute an upper bound from the specification of some arbitrary martingale process.

In fact, if we consider a martingale M starting at $M_0 = 0$, then, using martingale property and Optional Sampling Theorem:

$$\begin{aligned}\widetilde{U}(0) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}_0^N\left[\widetilde{H}(\tau) - M(\tau) + M(\tau)\right] \\ &= \sup_{\tau\in\mathcal{T}} \mathbb{E}_0^N\left[\widetilde{H}(\tau) - M(\tau)\right] \\ &\leq \mathbb{E}_0^N\left[\max_{\alpha\leq i\leq\beta-1}\left(\widetilde{H}_i - M_i\right)\right]\end{aligned}$$

where $M_i = M(T_i)$.

We denote

$$\widetilde{U}^{up}(0) = \mathbb{E}_0^N\left[\max_{\alpha\leq i\leq\beta-1}\left(\widetilde{H}_i - M_i\right)\right] \tag{4}$$

It's an upper bound for the price of the bermudan swaption. Thus, choosing any martingale M, starting at 0, will give an upper bound for the price. Moreover, there exist a martingale such that the upper bound $\widetilde{U}^{up}(0)$ coincides with the price $\widetilde{U}(0)$.

Indeed, knowing that the process $\widetilde{U}(t)$ is a super-martingale, we can write it, by the Doob-Meyer decomposition:

$$\widetilde{U}(t) = \widetilde{U}(0) + M(t) - A(t)$$

where $M(t)$ is a martingale and $A(t)$ is an increasing process with $M(0) = A(0) = 0$. We can then prove that, with this martingale M, the "duality gap" $\widetilde{U}^{up}(0) - \widetilde{U}(0)$ is zero. This suggests to choose M as the martingale component of a good lower bound of the N-discounted price $\widetilde{U}(t)$. We can use for example [Longstaff and Schwartz 2001] algorithm or any parameterized exercise strategy, as in [Andersen 2000] to compute a lower bound $\widetilde{L}(t)$ then plug the martingale component of $\widetilde{L}(t)$ in equation 4 to get an upper bound of $\widetilde{U}(t)$.

[Andersen and Broadie 2004] proposed to compute the martingale component of $\widetilde{L}$ with a simulation within a simulation, using the following equation of M:

$$M_{j+1} = M_j + \widetilde{L}_{j+1} - \mathbb{E}_j^N\left[\widetilde{L}_{j+1}\right]$$

An inner simulation is used to estimate the conditional expectation is the above equation [2], then an outer simulation is used to estimate the expectation in equation 4.

4 Upper bound by Schoenmakers et al.(2007)

In the case we use Longstaff and Schwartz algorithm to compute $\widetilde{L}$, then the expectation $\mathbb{E}_j^N\left[\widetilde{L}_{j+1}\right]$ can be estimated using the regression coefficients computed along the algorithm, so we don't have to use inner simulation. But by doing so, the estimator of M may fail to verify the martingale property and thus ensure that the price calculated is biased high. To overcome this difficulty, [Schoenmakers et al. 2009] proposed to construct an estimator of M based on the martingale representation theorem [3].

Indeed, there exist a square integrable, vector valued, process $Z_t = (Z_t^1, ..., Z_t^D)$ satisfying:

$$M(T_j) = \int_0^{T_j} Z_t dW_t\,,\ j = 0, ..., \beta - 1 \tag{5}$$

where $W_t = (W_t^1, ..., W_t^D)$ is the Brownian motion, under N-measure, driving the dynamic of the Libor rates.

We have then: $M_j = M_\alpha + \int_{T_\alpha}^{T_j} Z_t dW_t\,,\ j = \alpha + 1, ..., \beta - 1$ and $M_\alpha = \widetilde{L}_\alpha - \widetilde{L}_0$.

Hence, to compute the values of the $(M_j)_j$, we first estimate the process Z_t on a time grid $\pi = \{t_0, ..., t_{\mathcal{J}}\} \subset [T_\alpha, T_{\beta-1}]$ such that $t_0 = T_\alpha$, $t_{\mathcal{J}} = T_{\beta-1}$, then approximate the continuous stochastic integral in equation 5 by a discrete integral. As noted in [Schoenmakers et al. 2009], for $d \in [1, D]$, $Z_{t_i}^d$ can be computed by:

[2] If at date T_j the swaption is not exercised, then $\mathbb{E}_j^N\left[\widetilde{L}_{j+1}\right] = \widetilde{L}_j$ ie: we don't need inner simulation to estimate $\mathbb{E}_j^N\left[\widetilde{L}_{j+1}\right]$.

[3] This means that the model should be in a Brownian motion setting.

$$Z^d_{t_i} = \frac{1}{t_{i+1} - t_i}\mathbb{E}^N_{t_i}\left[(W^d_{t_{i+1}} - W^d_{t_i})\widetilde{L}_{j+1}\right], \; T_j \le t_i < T_{j+1} \tag{6}$$

The corresponding discrete approximation of M_j, for $j = \alpha, ..., \beta - 2$:

$$M_{j+1} = M_j + \sum_{t_i \in \pi, T_j \le t_i < T_{j+1}} Z_{t_i}(W_{t_{i+1}} - W_{t_i}) \tag{7}$$

We recall that $M_\alpha = \widetilde{L}_\alpha - \widetilde{L}_0$ and $Z_{t_i}(W_{t_{i+1}} - W_{t_i})$ is a scalar product.

To estimate the conditional expectation in equation 6, we use the least squares regression as in Longstaff and Schwartz algorithm as explained in the next section.

5 Practical design

To evaluate the price of bermudan swaption, we need to simulate the Libor rates $F(t, T_i, T_{i+1})$. For this purpose we choose to use the simulation method proposed by [Glasserman and Zhao 2000], that preclude arbitrage among bonds and keep interest rates positive even after discretization. They transform Libor rates into two martingales they discretize and then recover the Libor rates from these discretized variables. We have at the end two methods to simulate Libor rates under two different measures: Terminal measure and Spot measure.

As a lower bound, we choose the price given by Longstaff and Schwartz method, so we have at hand at each exercise date T_j the regression coefficients noted ξ_j. We can then compute the price of swaption at each exercice date $\widetilde{L}_{T_j} = \max\left(H_{T_j}, \left\langle \xi_j, \psi(T_j, X_{T_j})\right\rangle\right)$ where X_{T_j} is an explanatory variable used in the projection basis ψ. To have a good stability in regression method, we can choose X_{T_j} to be the brownian motion simulated at date T_j in the SDE discretization scheme.

To compute the conditional expectation in equation 6, we follow the recommandation in [Schoenmakers et al. 2009] and use a least squares methods. We consider basis functions $\psi(t_i, .) = (\psi_k(t_i, .), \, k = 1, ..., K)$ and N_1 independants simulations of libor rates $(t_i, F^n_{t_i})\, n = 1, ..., N_1$ constructed with the brownian increments $\Delta W_{t_i} = W_{t_{i+1}} - W_{t_i}$, we define for $T_j \le t_i < T_{j+1}$,

$$\eta^d_{t_i} = \arg\min_{\eta \in \mathbb{R}^K} \left\{ \left| \sum_{n=1}^{N_1} \frac{\Delta_n W^d_{t_i}}{\Delta t_i} \widetilde{L}_{T_{j+1}} - \langle \eta, \, \psi(t_i, \, {}_nX_{t_i})\rangle \right|^2 \right\} \tag{8}$$

It's also recommanded to compute the coeffients $\eta^d_{t_i}$ only on the exercise dates $t_i = T_j$ to obtain $\eta^d_{T_j}$ then interpolate them locally constant: $\eta^d_t = \eta^d_{T_j}$ for $t \in [T_j, T_{j+1}[$.

This means that we will have to resolve, for $j = \alpha, ..., \beta - 1$:

$$\eta^d_{T_j} = \arg\min_{\eta \in \mathbb{R}^K} \left\{ \left| \sum_{n=1}^{N_1} \frac{{}_nW^d_{T_{j+1}} - {}_nW^d_{T_j}}{\delta} \widetilde{L}_{T_{j+1}} - \left\langle \eta,\, \psi(T_j,\, {}_nX_{T_j}) \right\rangle \right|^2 \right\}$$

where $\widetilde{L}_{T_{j+1}} = \max\left(H_{T_{j+1}},\, \left\langle \xi_{j+1},\, \psi(T_{j+1}, X_{T_{j+1}}) \right\rangle\right)$.

After having computed the coefficients $\eta^d_{T_j}$ for $d = 1, ..., D$ and $j = \alpha, ..., \beta - 2$, we simulate a new set of N_2 paths, independants of those used in the regression step, and we construct a discrete approximation of the continuous stochastic integral in equation 5 on a refined partition π by:

$$ {}_nM_{j+1} = {}_nM_j + \sum_{t_i \in \pi, T_j \leq t_i < T_{j+1}} \langle \eta_j,\, \psi(t_i,\, {}_nX_{t_i}) \rangle \left({}_nW_{t_{i+1}} - {}_nW_{t_i} \right) \tag{9}$$

Finaly we estimate the upper bound of the price by the mean:

$$\widetilde{U}^{up}(0) = \frac{1}{N_2} \sum_{n=1}^{N_2} \max_{\alpha \leq i \leq \beta - 1} \left({}_n\widetilde{H}_i - {}_nM_i \right) \tag{10}$$

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