Backward Convolution Algorithm for Discretely Sampled Asian Options

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Abstract

This note give a summary of the backward price convolution algorithm used in [1] to price discretely sampled Asian options. For more details see [1].

1 Introduction

Let $\left(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ be a probability space. We consider an underlying price of a risky asset given by

$$S_j = S_0 \exp\left(\sum_{k=1}^{j} Z_k\right) \quad for\ j = \{1, \ldots, n\}, \tag{1.1}$$

and a process of partial sums defined by

$$I_j = \sum_{k=0}^{j} \lambda_k S_k \quad for\ j = \{1, \ldots, n\}, \tag{1.2}$$

where $(Z_k)_{1\leq k\leq n}$ is acollection of independent random vari- ables and the deterministic process λ depends of the type of the Asian option (see Table 1).

As proved by [1], the price of an Asian option amounts to calculating the quantity

$$\mathbb{E}\left(I_n^+\right). \tag{1.3}$$

Introduce a new filtration $\mathcal{G} := (G_i)_{1\leq i\leq n}$

$$\mathcal{G}_i = \sigma\left(Z_n, Z_{n-1}, \ldots, Z_{n-i+1}\right),$$

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Option type	λ_0	$\lambda_1, \ldots, \lambda_{n-1}$	λ_n
Call, fixed strike	$\frac{\gamma}{n+\gamma} - \frac{K}{S_0}$	$\frac{1}{n+\gamma}$	$\frac{1}{n+\gamma}$
Call, floating strike	$-\frac{\gamma\alpha}{n+\gamma}$	$-\frac{\alpha}{n+\gamma}$	$1 - \frac{\alpha}{n+\gamma}$
Put, fixed strike	$\frac{K}{S_0} - \frac{\gamma}{n+\gamma}$	$-\frac{1}{n+\gamma}$	$-\frac{1}{n+\gamma}$
Put, floating strike	$\frac{\gamma\alpha}{n+\gamma}$	$\frac{\alpha}{n+\gamma}$	$\frac{\alpha}{n+\gamma} - 1$

Table 1: Choice of λ corresponding to different types of Asian options. $\alpha > 0$ is the coefficient of partiality for floating strike options. Coefficient γ takes value 1 (0) when S_0 is (is not) included in the average.

and a process X defined by

$$\begin{aligned} X_k &:= \lambda_{n-k} + X_{k-1} \exp\left(Z_{n+1-k}\right) \\ X0 &:= \lambda_n : \end{aligned}$$

Proposition 2.1 of [1] shows that X is a $\mathcal{G}$-Markov process under $\mathbb{P}$.

2 Backward Price Convolution Algorithm

To price Asian options [1] have used a backward algorithm described in the following theorem (Theorem 3.1 in [1]). Note that we should have $\lambda_k > 0$ for any $k \in \{1, \ldots, n\}$. This is the case for the fixed strike Asian call.

Theorem 2.1. *Assume that for all k the CDF of Z_{n+1-k} has a probability density function f_k with respect to the Lebesgue measure on $\mathbb{R}$, satisfying*

$$\mu_k := \int_{\mathbb{R}} e^z f_k(z) dz < \infty.$$

Consider constants $\lambda_k > 0$, $0 < k \leq n$ and $\lambda_0 \in \mathbb{R}$. Define functions p_k: $\mathbb{R} \to \mathbb{R}$ for $0 < k \leq n$ and q_k, h_k: $\mathbb{R} \to \mathbb{R}$ for $0 \leq k < n$ as follows

$$\begin{aligned} &p_n(y) := \left(e^y + \lambda_0\right)^+; \\ &h_k(y) := \log\left(e^y + \lambda_{n-k}\right), \quad 0 < k < n, \\ &q_{k-1}(x) := \int_{\mathbb{R}} p_k(x+z) f_k(z) dz, \quad 0 < k \leq n, \\ &p_{k-1}(y) := q_{k-1}\left(h_{k-1}(y)\right), \quad 1 < k \leq n. \end{aligned}$$

The following statements hold:

1. *The forward price of an Asian call contract with parameters $(\lambda_j)_{0 \leq j \leq n}$ is given by*

$$\mathbb{E}\left(I_n^+\right) = S_0 \mathbb{E}\left(X_n^+\right) = S_0 q_0\left(\log\left(\lambda_n\right)\right).$$

2. *There are positive constants a_k, b_k such that for all $x, y \in \mathbb{R}$*

$$
\begin{aligned}
0 &\le p_k(y) \le a_k e^y + b_k, \\
0 &\le q_k(x) \le a_k e^x + b_k + 1.
\end{aligned}
$$

These constants are given recursively by

$$
\begin{aligned}
&a_n = 1, \;\; b_n = \lambda_0^+ \\
&a_{k-1} = a_k \mu_k \\
&b_{k-1} = b_k + a_{k-1}\lambda_{n-k+1}.
\end{aligned}
$$

The range of integration in the above theorem must be curtailed. So functions p_k and q_k are approximated by functions $\bar{p}_k$ and $\bar{q}_k$ defined by

$$
\begin{aligned}
\bar{q}_{k-1}(x) &:= \int_{\mathbb{R}} \bar{p}_k(x+z) f_k(z) dz \mathbb{1}_{[\bar{L}_{k-1}, \bar{U}_{k-1}]}(x), \quad 0 < k \le n \\
\bar{p}_{k-1}(y) &:= \bar{q}_{k-1}\left(h_{k-1}(y)\right) \mathbb{1}_{[L_{k-1}, U_{k-1}]}(y), \quad 1 < k \le n \\
\bar{p}_n y) &:= p_n(y) \mathbb{1}_{[L_n, U_n]}(y).
\end{aligned}
$$

The curtailed ranges $\left[\bar{L}_k, \bar{U}_k\right]$ for $k \in \{0, \ldots, n-1\}$ and $[L_k, U_k]$ for $k \in \{1, \ldots, n\}$ are defined in Theorem 3.2 in [1]. The pricing error caused by the curtailment is also estimated in the latter. The idea in [1] is to evaluate $\bar{p}_k$ and $\bar{q}_k$ in the Fourier space. Then, for $0 \le k < n$ and $x \in (\bar{L}_k, \bar{U}_k)$

$$
\bar{q}_{k-1}(x) = \mathcal{F}^{-1}\left(\mathcal{F}\left(\bar{p}_k\right) \bar{\phi}_k\right)(x),
$$

where $\mathcal{F}$ denotes the Fourier transform, ϕ_k is the characteristic function of Z_{n-k} and $\bar{\phi}_k$ denotes its complex conjugate. The Fourier transform is approximated by the general discrete Fourier transform (see [1], Definition 4.3). The backward algorithm for numerical implementations is summarized by Proposition 4.3 of [1].

References

[1] Cerny, A., Kyriakou I.. An Improved Convolution Algorithm for Discretely Sampled Asian Options. To appear in Quantitative Finance (2010). 1, 2, 3

References