Libor modelling using Affine processes

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February 18, 2016

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Remark: Most of what is presented here is taken from [Keller-Ressel et al. 2009].

1 Libor market model

Let us consider a set of dates $T_0, T_1, ..., T_N$ with $0 = T_0 < T_1 < ... < T_N$ and $T_{k+1} - T_k = \delta$.

We note $L_k(t)$, for a certain date $t \leq T_k$, the value at date t of the Libor rate settled at T_k and payed at T_{k+1}. We extend this definition to $t > T_k$ simply by $L_k(t) = L_k(T_k)$.

By absence of arbitrage, the Libor rates are related to Zero Coupon bond by :

$$L_k(t) = \frac{1}{\delta}\left(\frac{B(t,T_k)}{B(t,T_{k+1})} - 1\right)$$

The authors in this paper propose a Libor modeling framework that respects the basic requirement that Libor rates are non-negatives martingales under their own forward measure. Their approach, based on affine processes, ensure an analytically tractable dynamics for the Libor processes under any forward measure, which permits to have closed formulas for most liquid interest rate derivatives, i.e. caps and swaptions, so that the calibration to market data can be done in a reasonable time.

First let's recall the definition of the affine processes and some of their properties.

2 Affine Processes

Let $X = (X_t)_{0\leq t\leq T}$ be a stochastically continuous, time-homogeneous Markov process with state space $D = R^d_{\geq 0}$, starting from $x \in D$. The process X is called affine if the moment generating function satisfies:

1. $\mathbb{E}_x\left[e^{u\cdot X_t}\right] = \exp\left(\phi_t(u) + \psi_t(u)\cdot x\right)$
2. $0 \in I^0_T$

for some functions $\phi : [0,T] \times I_T \to R$ and $\psi : [0,T] \times I_T \to R^d$, and all $(t,u,x) \in [0,T] \times I_T \times D$, where $I_T := \{u \in R^d : \mathbb{E}_x\left[e^{u\cdot X_t}\right] < \infty, \forall t \in [0,T],\ \forall x \in D\}$.

These functions satisfy some useful properties:

1. $\phi_{t+s}(u) = \phi_t(u) + \phi_s(\psi_t(u))$
2. $\psi_{t+s}(u) = \psi_s(\psi_t(u))$
3. $\phi_0(u) = 0$ and $\psi_0(u) = u$
4. If $u \in I_T$, the process $(M^u_t)_{0\leq t\leq T}$ defined by

$$M^u_t = \mathbb{E}\left[e^{u\cdot X_T}|\mathcal{F}_t\right] = \exp\left(\phi_{T-t}(u) + \psi_{T-t}(u)\cdot X_t\right)$$

is a martingale. Moreover, if $u \in I_T \cap R^d_{\geq 0}$, then $M^u_t \geq 1$, $\forall t \in [0,T]$, $\forall X_0 \in R^d_{\geq 0}$.

For more details c.f. [Keller-Ressel et al. 2009] and the references therein.

3 Affine LIBOR model

Now, using the fact that discounted Zero Coupon Bond $\widetilde{B}(t,T_k) = \dfrac{B(t,T_k)}{B(t,T_N)}$ is a martingale under terminal measure $\mathbb{P}_{T_N}$ (corresponding to the numeraire $B(t,T_N)$), the libor rates are modeled as follows:

$$1 + \delta L_k(t) = \widetilde{B}(t, T_k) = M_t^{u_k},\ k \in [1, N]$$

for some affine process X, starting at x, and vectors u_k.

Under some reasonable conditions, the $(u_k)_{1\leq k\leq N}$ can be chosen in a way to match the initial yield curve. cf. Proposition 6.1 [Keller-Ressel et al. 2009]. In the same reference, the moment generating function of X is derived under other forward measures $\mathbb{P}_{T_k}$.

4 Pricing of caps and swaptions

In order to calibrate the model to market data, we need to have pricing methods for plain derivative products, like caps/floors and swaptions. Since the driving process X of the model is known by it's moment generating function components ϕ and ψ, it's reasonable to use Fourier method to calculate the price of these products. First we start with swaption.

4.1 Swaption

Recall that this contract gives the right of choosing at a date T_i, whether to enter or not into a swap over $[T_i, T_m]$ with $T_i,\ T_m \in \{T_1, ..., T_N\}$.

Exercising a payer $[T_i, T_m]$-european swaption with strike K means to be payed at time T_i the quantity :

$$\mathbb{S}_{T_i}(K, T_i, T_m) = \left(-\sum_{k=i}^{m} c_k B(T_i, T_k)\right)^+$$

, where $c_k = \begin{cases} -1, & k = i \\ \delta K, & k \in [i+1, m-1] \\ 1 + \delta K, & k = m \end{cases}$

The price of this product at time 0 is given by the $\mathbb{P}_{T_N}$-expectation:

$$\begin{aligned}
\mathbb{S}_0(K,T_i,T_m) &= B(0,T_N)\mathbb{E}_{\mathbb{P}_{T_N}}\left[\frac{\mathbb{S}_{T_i}(K,T_i,T_m)}{B(T_i,T_N)}\right] \\
&= B(0,T_N)\mathbb{E}_{\mathbb{P}_{T_N}}\left[\left(-\sum_{k=i}^{m} c_k \frac{B(T_i,T_k)}{B(T_i,T_N)}\right)^+\right] \\
&= B(0,T_N)\mathbb{E}_{\mathbb{P}_{T_N}}\left[\left(-\sum_{k=i}^{m} c_k M_{T_i}^{u_k}\right)^+\right] \\
&= B(0,T_N)\mathbb{E}_{\mathbb{P}_{T_N}}\left[f\left(X_{T_i}\right)\right]
\end{aligned}$$

where

$$f(x) = \left\{-\sum_{k=i}^{m} c_k e^{\phi_{T_N-T_i}(u_k)+\psi_{T_N-T_i}(u_k)\cdot x}\right\}^+$$

Hence, we apply Fourier methods by using theorem 2.2 in [Eberlein et al. 2009]. We get:

$$\mathbb{S}_0(K,T_i,T_m) = \frac{B(0,T_N)}{2\pi}\int_{\mathbb{R}} \Lambda_{X_{T_i}}(R-iv)\widehat{f}(v+iR)dv$$

where $R \in \mathbb{R}$ is the dampling factor and $\Lambda_{X_{T_i}}$ is the $\mathbb{P}_{T_N}$-moment generating function of the random variable X_{T_i}.

$$\begin{aligned}
\Lambda_{X_{T_i}}(z) &= \mathbb{E}_{\mathbb{P}_{T_N}}\left[e^{z\cdot X_{T_i}}\right] \\
&= \exp\left(\phi_{T_i}(z) + \psi_{T_i}(z)\cdot x\right)
\end{aligned}$$

and $\widehat{f}$ is Fourier transform of f.

To compute $\widehat{f}$, we use the same idea in [Keller-Ressel et al. 2009] to get:

$$\begin{aligned}
\widehat{f}(z) &= \int_{\mathbb{R}} e^{izx} f(x)dx \\
&= e^{izY}\sum_{k=i}^{m} c_k \frac{e^{\phi_{T_N-T_i}(u_k)+\psi_{T_N-T_i}(u_k)\cdot Y}}{\psi_{T_N-T_i}(u_k)+iz}
\end{aligned}$$

where Y is the unique zero of the function f.

Finaly:

$$\begin{aligned}\Lambda_{X_{T_i}}(R-iv) &= \exp\left(\phi_{T_i}(R-iv)+\psi_{T_i}(R-iv)\cdot x\right)\\ \widehat{f}(v+iR) &= e^{ivY}\sum_{k=i}^{m} c_k \frac{e^{\phi_{T_N-T_i}(u_k)+(\psi_{T_N-T_i}(u_k)-R)Y}}{\psi_{T_N-T_i}(u_k)-R+iv}\end{aligned}$$

Remark: The damping factor should satisfy some conditions in order to apply theorem 2.2 in [Eberlein et al. 2009]:

(C1) : $g:\ x \to e^{-Rx}f(x)$ is bounded continuous function in $L^1(\mathbb{R})$.

(C2) : $\Lambda_{X_{T_i}}(R)$ exists.

(C2) : $\widehat{g}$ Fourier transform of g is in $L^1(\mathbb{R})$.

These conditions imply : $R \in I_{T_i}$ and $R > \psi_{T_N-T_i}(u_i)$.

4.2 Caplet

Consider a caplet with fixing date T_i, payment date T_{i+1} and strike K. The price of this product is a special case of swaption, if we choose $T_m = T_{i+1}$.

In fact, the price of a caplet at time 0 is given by the $\mathbb{P}_{T_N}$-expectation:

$$\begin{aligned}\mathbb{C}_0(K,T_i,T_m) &= B(0,T_N)\mathbb{E}_{\mathbb{P}_{T_N}}\left[\left\{\frac{1}{B(T_i,T_N)} - \frac{B(T_i,T_{i+1})}{B(T_i,T_N)}\mathcal{K}\right\}^+\right]\\ &= B(0,T_N)\mathbb{E}_{\mathbb{P}_{T_N}}\left[\left(M_{T_i}^{u_i} - M_{T_i}^{u_{i+1}}\mathcal{K}\right)^+\right]\end{aligned}$$

where $\mathcal{K} = 1+\delta K$.

So in this case, we use the same pricing formula as for swaption with Y, the zero of the function f, being explicitly known: $Y = -\dfrac{\phi_{T_N-T_i}(u_{i+1}) - \phi_{T_N-T_i}(u_i) + \log(\mathcal{K})}{\psi_{T_N-T_i}(u_{i+1}) - \psi_{T_N-T_i}(u_i)}$.

5 Model specifications

In our implementation, we restrict our self to the one-dimensional affine processes. In this case, since I_{T_i} is a convex set, it's in fact an interval:

$$I_{T_i} \cap (0,\infty) = (0, u_{max}(T_i)) \text{ where } u_{max}(T_i) \geq 0$$

So to price a $[T_i, T_m]$-european swaption, we have to choose a damping factor:

$$\psi_{T_N - T_i}(u_i) < R < u_{max}$$

We implemented two specification of affine Libor model. The first has a CIR process as driving process, and a Γ-OU process for the second one.

- **CIR based model**

The first example is the Cox-Ingersoll-Ross (CIR) process, given by

$$dX_t = \lambda(\theta - X_t)dt + 2\eta\sqrt{X_t}dW_t,\ X_0 = x$$

where $x,\ \lambda,\ \theta,\ \eta \in \mathbb{R}_{\geq 0}$ are the process parameters.

The moment generating function is given by the two functions:

$$\phi_t(u) = -\frac{\lambda\theta}{2\eta^2}\log\left(1 - 2\eta^2 b(t)u\right) \text{ and } \psi_t(u) = \frac{a(t)u}{1 - 2\eta^2 b(t)u}$$

with $b(t) = \begin{cases} t, & \text{if } \lambda = 0 \\ \dfrac{1 - e^{\lambda t}}{\lambda}, & \text{if } \lambda \neq 0 \end{cases}$ and $a(t) = e^{\lambda t}$.

The functions ϕ_t and ψ_t are well defined for $u < u_{max}(t) = \dfrac{1}{2\eta^2 b(t)}$.

In the case of CIR driving process, it's also possible to derive a closed form pricing formula for caps and swaption, using the χ^2 distribution function. For more details and pricing formula, please refer to [Keller-Ressel et al. 2009].

- **Γ-OU based model**

The second example is a model driven by a process X_t such that:

$$dX_t = \lambda X_t dt + dH_t,\ X_0 = x$$

where H is a compound Poisson process with jump intensity $\lambda\beta$ and exponentially distributed jumps with parameters α. This process has 4 parameters $x, \lambda, \alpha, \beta \in R_{\geq 0}$.

X is an affine process and its moment generating function is given by the two functions:

$$\phi_t(u) = \beta\log\left(\frac{\alpha - e^{-\lambda t}u}{\alpha - u}\right) \text{ and } \psi_t(u) = e^{-\lambda t}u$$

The functions ϕ_t and ψ_t are well defined for $u < u_{max}(t) = \alpha$.

In addition to the parameters $u_1, ..., u_N$ used to match the initial yield curve[1], the two models have 4 parameters that can be used to match the market prices of standard interest rate products like caps and swaptions. The fact that we have closed-form valuation formulas, using Fourier transform, for both of these products is a great advantage so that the model can be calibrated to market data in reasonable time.

[1]The authors prove also that these two models can fit any initial yield curve.

References

[Keller-Ressel et al. 2009] Keller-Ressel, M., A. Papapantoleon, and J. Teichmann (2009) A new approach to LIBOR modeling. Preprint, arXiv/0904.0555. 1, 2, 3, 4, 6

[Eberlein et al. 2009] E. Eberlein, K. Glau, A. Papapantoleon (2009) Analysis of Fourier transform valuation formulas and applications. Applied Mathematical Finance (forthcoming) 4, 5

References