Fourier-Cosine Method for Pricing Commodity Options: Implementation in PREMIA

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Abstract

Applying the pricing method based on Fourier-Cosine series expansion proposed in [1] and [2], we implement the algorithms for pricing commodity options under the Ornstein-Uhlenbeck (OU, in short) model incorporate with the seasonality component.

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1 Commodity Model

The commodity price S_t including the seasonality effect is defined as

$$S_t = G(t)\mathrm{e}^{y_t} = \mathrm{e}^{g(t)+y_t}, \quad \text{with} \quad S_0 = G(0), \tag{1}$$

where $G(t)$ is a deterministic function to describe the seasonality effect and it is defined as

$$G(t) \equiv \mathrm{e}^{g(t)} = a_1 + a_2 \sin(a_3 t) \tag{2}$$

with constants a_1, a_2, a_3, and y_t is a stochastic zero-level-mean reverting process given by:

$$\mathrm{d}y_t = -\kappa y_t \mathrm{d}t + \sigma \mathrm{d}W_t^y, \quad \text{with} \quad y_0 = 0, \tag{3}$$

with W_t^y a Brownian motion, the constants κ and σ represent the speed of the mean reverting and the deterministic volatility respectively.

Apply Itô's formula to (1) and substitute (3), we obtain the dynamics of S_t as

$$\mathrm{d}S_t = \kappa(\theta(t) - \log S_t)S_t \mathrm{d}t + \sigma S_t \mathrm{d}W_t^y, \tag{4}$$

where $\theta(t) = g(t) + \left(\frac{1}{2}\sigma^2 + g'(t)\right)/\kappa$. By taking the log-transfor of the stock price $x_t = \log S_t$, we have

$$\mathrm{d}x_t = \kappa(\overline{\theta}(t) - x_t)\mathrm{d}t + \sigma \mathrm{d}W_t^y, \quad \text{with} \quad x_0 = \log S_0, \tag{5}$$

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and $\overline{\theta}(t) = \theta(t) - \sigma^2/2\kappa = g'(t)/\kappa + g(t)$.

From the above stochastic equation (5), we know that for $0 < s < t$ and given x_s, x_t is normally distributed, i.e. $x_t \sim \mathcal{N}(\mathbb{E}(x_t|x_s), \mathbb{V}ar(x_t|x_s))$ with

$$\begin{aligned} \mathbb{E}(x_t|\mathcal{F}_s) &= x_s \mathrm{e}^{-\kappa(t-s)} + g(t) - \mathrm{e}^{-\kappa(t-s)} g(s), \\ \mathbb{V}\mathrm{ar}(x_t|\mathcal{F}_s) &= \frac{\sigma^2}{2\kappa}(1 - \mathrm{e}^{-2\kappa(t-s)}). \end{aligned}$$

For the characteristic function of x_t condition on x_s denoted by $\phi(u; x, s, t) := \mathbb{E}(\mathrm{e}^{iuX_t}|x_s = x)$, by applying the Feymann-Kac formula, we have the PDE for the function $u(x, s) := \phi(u; x, s, t)$ as:

$$\frac{\partial u(x,s)}{\partial s} + \kappa[\overline{\theta}(s) - x]\frac{\partial u(x,s)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial u(x,s)}{\partial x^2} = 0. \tag{6}$$

Guess the function $\phi(u; x, s, t)$ of the form $\mathrm{e}^{xB(u,t-s)+A(u,t-s)}$ and subsititute it into (6), we derive the ODEs for $B(u, t-s)$ and $A(u, t-s)$:

$$\begin{cases} \frac{\partial B(u,t-s)}{\partial(t-s)} = -\kappa B(u, t-s) \\ \frac{\partial A(u,t-s)}{\partial(t-s)} = \kappa\overline{\theta}(s)B(u, t-s) + \frac{1}{2}\sigma^2 B^2(u, t-s), \end{cases} \tag{7}$$

with initial conditions $B(u, 0) = iu$, $A(u, 0) = 0$.

Solve the above ODEs, we have

$$\begin{cases} B(u, t-s) = iu\mathrm{e}^{-\kappa(t-s)}, \\ A(u, t-s) = iu\left[g(t) - \mathrm{e}^{-\kappa(t-s)}g(s)\right] - \frac{u^2\sigma^2}{4\kappa}\left[1 - \mathrm{e}^{-2\kappa(t-s)}\right]. \end{cases} \tag{8}$$

The conditional charactersitic function of the special form

$$\phi(u; x_s, s, t) = \mathrm{e}^{iux_s}\widetilde{\phi}_A(u, t-s),$$

where $\widetilde{\phi}_A(u, t-s)$ does not contain x_s, will be benefical for the pricing Bermudan option by the Fourier-Cosine method. From (8), the conditional characteristic function can be rewritten as

$$\phi(u; x_s, s, t) = \mathrm{e}^{iux_s}\mathrm{e}^{-iux_s(1-\mathrm{e}^{-\kappa(t-s)})+A(u,t-s)} = \mathrm{e}^{iux_s}\overline{\phi}(u; x_s, s, t),$$

where $\overline{\phi}(u; x_s, s, t) = \mathrm{e}^{A(u,t-s)}\mathrm{e}^{-iux_s(1-\mathrm{e}^{-\kappa(t-s)})}$. Note that $\overline{\phi}(u; x_s, s, t)$ contains x_s, to apply the efficient computation in the Fourier-Cosine algorithm, we use an approximation of $\overline{\phi}(u; x_s, s, t)$ which does not contain x_s:

$$\overline{\phi}(u; x_s, s, t) \approx \overline{\phi}(u; \mathbb{E}(x_s|\mathcal{F}_0), s, t) = \mathrm{e}^{A(u,t-s)}\mathrm{e}^{-iu\mathbb{E}(x_s|\mathcal{F}_0)(1-\mathrm{e}^{-\kappa(t-s)})}. \tag{9}$$

2 Fourier-Cosine Method

The Fourier-Cosine pricing method (COS, in short), is based on the risk-neutral option valuation formula (discounted expected payoff approach). For an European option,

$$v(x, s) = e^{-r\triangle t}\int_{-\infty}^{\infty} v(y, t)f(y|x)dy, \tag{10}$$

where $v(x, s)$ is the option value at time s, r the interest rate, $\triangle t = t - s$ and x, y can be any monotone functions of the underlying asset at initial time s and the expiration date t, respectively. Function $v(y, t)$, which equals the payoff of the European option, is known, but the transitional density function, $f(y|x)$, typically is not. For a chosen sufficient wide domain $[a, b]$, we approximate the integration in Equation (10)

$$v(x,s) \approx e^{-r\triangle t} \int_a^b v(y,t) f(y|x) dy. \tag{11}$$

Then the unknown conditional density function defined in $[a, b]$ can be recovered from its charateristic function by a truncated Fourier-Cosine expansion as:

$$f(y|x) \approx \frac{2}{b-a} {\sum_{k=0}^{N-1}}' \mathrm{Re}\left[\phi\left(\frac{k\pi}{b-a};x\right)\exp\left(-i\frac{ak\pi}{b-a}\right)\right]\cos\left(k\pi\frac{y-a}{b-a}\right), \tag{12}$$

where $\phi(u; x)$ the characteristic function of $f(y|x)$, Re means taking the real part of the argument, and the prime at the sum symbol indicates the the first term in the expansion is multiplied by one-half. The appropriate size of the integration interval can be determined with the help of the cumulants, we will give detail about it for the OU model and the Heston model.

Replacing $f(y|x)$ by its approximation (12) in Equation (11) and interchanging integration and summation gives the COS formula for computing the values of European options:

$$v(x,s) \approx {\sum_{k=0}^{N-1}}' \mathrm{Re}\left[\phi\left(\frac{k\pi}{b-a};x\right)\exp\left(-i\frac{ak\pi}{b-a}\right)\right] V_k, \tag{13}$$

where

$$V_k = \frac{2}{b-a}\int_a^b v(y,t)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \tag{14}$$

are the Fourier-Cosine coefficients of $v(y, t)$, available in closed form for several payoff functions.

Formula (13) also forms the basis for the pricing of Bermudan options. A Bermudan option can be exercised at pre-specified dates before maturity. The holder receives the exercise payoff by exercising the option. Let t_0 denote the initial time and $\{t_1, \cdots, t_M\}$ be the collection of all exercise dates with $\triangle t := t_m - t_{m-1}, t_0 < t_1 < \cdots < t_M = T$. The pricing formula for a Bermudan option with M dates reads, for $m = M, M-1, \cdots, 2$:

$$\begin{cases} c(x,t_{m-1}) & = \mathrm{e}^{-r\triangle t}\int_{\mathbb{R}} v(y,t_m) f(y|x)dy, \\ v(x,t_{m-1}) & = \max(g(x,t_{m-1}), c(x,t_{m-1})), \end{cases} \tag{15}$$

follows by

$$v(x,t_0) = \mathrm{e}^{-r\triangle t}\int_{\mathbb{R}} v(y,t_1) f(y|x)dy, \tag{16}$$

where x and y are state variables defined as

$$x := \ln(S(t_{m-1})), \quad \text{and} \quad y := \ln(S(t_m)),$$

$v(x,t), c(x,t)$ and $g(x,t)$ are the option value, the continuation value and the payoff at time t, respectively. For call and put option, the payoff at time t is

$$g(x,t) = \max[\alpha(\mathrm{e}^x - K), 0], \quad \alpha = \begin{cases} 1, & \text{for a call,} \\ -1, & \text{for a put.} \end{cases} \tag{17}$$

For the time t_M, the value of the option equals to the payoff, i.e.

$$v(x,t_M) \equiv g(x,t_M).$$

To apply COS expansion (13) for (16) to derive the Bermudan options price, we need to determine the Fourier-Cosine coefficients of the option value at time t_1, $V_k(t_1)$. We use iteration staring from the Fourier-Cosine coefficients of the option value $v(x,t_M)$ at time t_M, $V_k(t_M)$, which can be derived from (14) with explicit form of $v(x,t_M) = g(x,t_M)$ given in (17):

$$V_k(t_M) = \begin{cases} G_k(0,b), & \text{for a call} \\ G_k(a,0), & \text{for a put,} \end{cases} \tag{18}$$

where

$$G_k(l,u) := \frac{2}{b-a}\int_a^b g(y,t_m)\cos\left(k\pi\frac{y-a}{b-a}\right)dy. \tag{19}$$

For $g(y,t_m)$ given in (17), $G_k(t_M)$ has an explicit formula.

$$G_k(l,u) = \frac{2}{b-a}\alpha K[\chi_k(l^*,u^*) - \psi_k(l^*,u^*)], \quad \alpha = \begin{cases} 1, & \text{for a call} \\ -1, & \text{for a put,} \end{cases} \tag{20}$$

with

$$l^* = \begin{cases} \max(l,0), & \text{for a call} \\ \min(l,0), & \text{for a put,} \end{cases} \quad u^* = \begin{cases} \max(u,0), & \text{for a call} \\ \min(u,0), & \text{for a put.} \end{cases} \tag{21}$$

and

$$\begin{aligned} \chi_k(l^*,u^*) &:= \int_{l^*}^{u^*} \mathrm{e}^x \cos\left(n\pi\frac{x-a}{b-a}\right)dx \\ = \quad & \frac{1}{1+\left(\frac{n\pi}{b-a}\right)^2}\left[\cos\left(n\pi\frac{u-a}{b-a}\right)\mathrm{e}^u - \cos\left(n\pi\frac{l-a}{b-a}\right)\mathrm{e}^l\right. \\ & \left.\frac{n\pi}{b-a}\sin\left(n\pi\frac{u-a}{b-a}\right)\mathrm{e}^u - \frac{n\pi}{b-a}\sin\left(n\pi\frac{l-a}{b-a}\right)\mathrm{e}^l\right], \end{aligned}$$

$$\begin{aligned} \psi_k(l^*,u^*) &:= \int_{l^*}^{u^*} \cos\left(n\pi\frac{x-a}{b-a}\right)dx \\ = \quad & \begin{cases} \left[\sin\left(n\pi\frac{u-a}{b-a}\right) - \sin\left(n\pi\frac{l-a}{b-a}\right)\right]\frac{b-a}{n\pi}, & n \neq 0 \\ u-l, & n = 0. \end{cases} \end{aligned}$$

Then the continuation value $c(x,t_{M-1})$ can be calculated by COS method: for $m = M, \cdots, 2$,

$$c(x,t_{m-1}) := \mathrm{e}^{-r\triangle t}\sum_{k=1}^{N-1}\mathrm{Re}\left\{\phi\left(\frac{k\pi}{b-a};x\right)\mathrm{e}^{-i\kappa\pi\frac{a}{b-a}}\right\}V_k(t_m). \tag{22}$$

To derive $V_k(t_1)$ iteratelly from $V_k(t_M)$, we need to find out the *early-exercise point*, x_m^* at time t_m, which is the point where the continuation value equals to the payoff, i.e. $c(x_m^*, t_m) = g(x_m^*, t_m)$, and can be found by Newton's method. From (14) and split the integration interval by x_m^*, $V_k(t_m)$ can be computed from two parts: one on the interval $[a, x_m^*]$ and the other on $(x_m^*, b]$, i.e.

$$V_k(t_m) = \begin{cases} C_k(a, x_m^*, t_m) + G_k(x_m^*, b), & \text{for a call} \\ G_k(a, x_m^*) + C_k(x_m^*, b, t_m), & \text{for a put,} \end{cases} \tag{23}$$

where

$$C_k(x_1, x_2, t_m) := \frac{2}{b-a} \int_{x_1}^{x_2} c(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx, \tag{24}$$

$G_k(l, u)$ as defined by (19).

Substituting (13) into (15), we have the approximated continuation value at time t_m, $c(x, t_m)$, then inserting it into (24) and interchanging summation and integration, we derive the Fourier-Cosine coefficient for the continuation value $C_k(x_1, x_2, t_m)$:

$$C_k(x_1, x_2, t_m) := \mathrm{e}^{-r\triangle t} \sideset{}{'}\sum_{j=0}^{N-1} \mathrm{Re}\left[\widetilde{\phi}\left(\frac{j\pi}{b-a}, \triangle t\right) V_j(t_{m+1}) \cdot H_{k,j}(x_1, x_2)\right], \tag{25}$$

where $\widetilde{\phi}_A(u, t-s) := \overline{\phi}(u; \mathbb{E}(x_s|\mathcal{F}_0), s, t)$,

$$H_{k,j}(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} \mathrm{e}^{ij\pi \frac{x-a}{b-a}} \cos\left(k\pi \frac{x-a}{b-a}\right) dx. \tag{26}$$

$H_{k,j}(x_1, x_2)$ can be calculated by two parts

$$H_{k,j}(l, u) = -\frac{i}{\pi}\left(H_{k,j}^c(l, u) + H_{k,j}^s(l, u)\right),$$

where

$$H_{k,j}^c(l, u) := \begin{cases} \frac{(u-l)\pi i}{b-a}, & k = j = 0, \\ \frac{\exp\left(i(j+k)\frac{(u-a)\pi}{b-a}\right) - \exp\left(i(j+k)\frac{(l-a)\pi}{b-a}\right)}{j+k}, & \text{otherwise} \end{cases} \tag{27}$$

$$H_{k,j}^s(l, u) := \begin{cases} \frac{(u-l)\pi i}{b-a}, & k = j, \\ \frac{\exp\left(i(j-k)\frac{(u-a)\pi}{b-a}\right) - \exp\left(i(j-k)\frac{(l-a)\pi}{b-a}\right)}{j-k}, & k \neq j. \end{cases} \tag{28}$$

The matrices

$$H^c(l, u) := \{\mathcal{M}_{k,j}^c(l, u)\}_{k,j=0}^{N-1}, \quad H^s(l, u) := \{\mathcal{M}_{k,j}^s(l, u)\}_{k,j=0}^{N-1}$$

have special structure: $H^c(l, u)$ is a Hankel matrix and $H^s(l, u)$ is a Toeplitz matrix. Both of the Hankel matrix and the Toeplitz matrix yield the property that the products of a vector with Hankel matrix and the Toeplitz matrix respectively can be transformed into a circular convolution, then the Fast Fourier Transform can be applied directly for highly efficient matrix-vector multiplication. Thus we can efficiently recover $V_k(t_1)$ from $V_k(t_M)$ by iteration using (23) and (24). For more details on the COS method for Bermudan option, please refer to [1].

3 Program Manual

We implement the pricing of commodity options with early-exercise under OU processes with seasonality effects by Fourier Cosine expansion. The program HAS TO work with the pnl library.

Model Parameters:
kappa: the speed of the mean reversion, κ as given in (3).
sigma: the volatility of volatility, σ as given in (3).
a1: constants a_1 in seasonality effect function as given in (2).
a2: constants a_2 in seasonality effect function as given in (2).
a3: constants a_3 in seasonality effect function as given in (2).

Parameters of the product:
S0: the initial value of stock price.
strike: strike of the Bermudan option.
T: the maturity of the Bermudan option.
r: the discount interest rate.
M: number of early-exercise dates.

Parameters for Fourier-Cosine method:
N: number of Fourier-Cosine series, N in (12).

References

[1] Fang, F., Oosterlee, C. W., 2011, A Fourier-based valuation method for Bermudan and barrier options under Heston's model, *SIAM J. Fin. Math*, to appear.

[2] Zhang, B., Grzelak, L.A., Oosterlee, C.W., 2011, Efficient pricing of commodity options with early-exercise under the Ornstein-Uhlenbeck process, preprint, *http://ta.twi.tudelft.nl/mf/user/oosterle/oosterlee/BowenZhang.pdf*.

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