Risk sensitive portfolio optimization with transaction costs

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Abstract

We develop methods of risk sensitive impulsive control theory in order to solve an optimal asset allocation problem with transaction costs and a stochastic interest rate. The optimal trading strategy and the risk-sensitized expected exponential growth rate of the investor's portfolio are characterized in terms of a nonlinear quasi-variational inequality. This problem can then be interpreted as the ergodic Isaac-Hamilton-Jacobi equation associated with a min-max problem. We use a numerical method based on an extended two-stage policy iteration algorithm for min-max problems and provide numerical results for the case of two assets and one factor that is a Vasicek interest rate.

1 Introduction

The mathematical problem of optimally managing a portfolio of securities when there are transaction costs has received considerable research attention

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in recent years. See Cadenillas [9] for a recent survey of this literature. Most such research has been concerned with the classical economic objectives of maximizing expected utility of terminal wealth and/or utility of consumption over a planning horizon that can be finite or infinite. Researchers such as Akian, Menaldi, and Sulem [1], Akian, Sulem, and Taksar [2], Cvitanic and Karatzas [14], and Shreve and Soner [26] were concerned with cases where the cost of a transaction is proportional to the amount of money that is shifted between securities. Typically the optimal strategy is characterized by a no-trade region, with "local-time" trading on the boundary used to keep a certain process in the region. Other researchers such as Korn [18], Morton and Pliska [22], Øksendal and Sulem [25] and Chancelier, Øksendal and Sulem [12] assumed the transaction cost has a fixed component, thereby precluding the optimality of local-time trading because it is, essentially, continuous. In this case the optimal strategy is again typically characterized by a no-trade region, only when a certain process such as the vector of portfolio proportions hits the boundary, a transaction is made causing the process to jump to a point in the interior of the region, from which the process resumes as before.

In a somewhat different direction, researchers such as Buckley and Korn [8], Connor and Leland [13], and Leland [20] have looked at optimal tracking problems, where the tracking error is some measure of the differences between specified fixed target proportions and the actual proportions for each asset with respect to the total value of the portfolio. Since there are transaction costs, there is an obvious trade-off between large tracking errors and large transaction costs. The optimal strategies that emerge from these studies resemble those mentioned in the preceding paragraph, with no-trade regions and so forth.

For all of these studies the underlying models of the securities are complete. Indeed, in most cases the asset appreciation rates and volatilities are constants, that is, the so-called "investment opportunity sets" are constants. Meanwhile, in a seminal paper, Merton [21] proposed a portfolio optimization model, the so-called intertemporal capital asset pricing model (ICAPM), where the asset appreciation rates and volatilities depend on one or more exogenous, stochastic factors. The added realism of his model comes with a cost, however: perhaps because the model is incomplete, explicit results are known for only a very few, special cases.

The objective for Merton's ICAPM was the classical, economic one of maximizing expected utility of terminal wealth and/or consumption. With the aim of obtaining explicit results for a wider variety of cases, in recent years researchers such as Bielecki and Pliska [3], [5], Bielecki, Pliska, and Sherris [7], Fleming and Sheu [16], and Kuroda and Nagai [19] have studied the very same model, only replacing the original economic objective with a so-called "risk sensitive" one: maximizing the portfolio's risk adjusted growth rate. Indeed, this approach has produced explicit results for cases with many assets, many Gaussian factors, and, very recently (see Nagai [23]), even non-Gaussian factors. And these explicit results come with only a small cost for abandoning classical expected utility criteria, because it has been shown that the risk sensitive criterion is simply an approximation of a fundamental objective in financial practice: the trade-off between a portfolio's average return and its average volatility. See Bielecki and Pliska [6] for a recent study of the economic properties of the risk sensitive criterion for portfolio management.

For most of these ICAPM models, whether they involve classical or risk sensitive objectives, there are no transaction costs. But there are a few exceptions. Weiner [28] studied a problem where there is a trade-off involving stochastic volatility, which is the factor. Another exception is a study by Bielecki and Pliska [4] that combined the risk sensitive ICAPM model with transaction costs having a fixed component. They used impulse control methods to show that solutions can be obtained via risk sensitive quasi-variational inequalities (RSQVIs). They demonstrated their methods by obtaining explicit, numerical results for a simple zero-factor case involving just two assets (in particular, for a modest generalization of the two-asset model by Morton and Pliska [22]). Bielecki and Pliska did not solve more complicated risk sensitive quasi-variational inequalities due to computational challenges.

Indeed, while theoretical results for transaction cost problems are available for rather general models featuring many assets, many stochastic factors, and various kinds of transaction costs, it remains true that numerical results, let alone explicit results, have been obtained only for a very few, relatively simple cases. The theory seems to be far ahead of the practical matter of actually solving problems. The research challenge is to be able to solve meaningful transaction cost problems in order to obtain economic insight if not optimal trading strategies for realistic applications.

The main goal of this paper is to advance this "computational barrier" by providing a computational algorithm and numerical results for a version of the Bielecki and Pliska [4] model. Our analysis is for a case where there are two assets (a risky asset and a bank account) and a single factor that one can interpret as the (Gaussian) short interest rate for the bank account. The stochastic dynamics of the risky asset explicitly depend on this interest rate, and the transaction cost is a fixed fraction of the portfolio's value when a transaction occurs. While this is only a modest generalization of the Bielecki-Pliska [4] numerical example, now with one factor instead of none, our solution approach is not simply a straightforward application of established computational procedures such as dynamic programming iterative methods, discrete time Markov chain approximations, or finite difference methods. First we make a transformation that reduces the number of underlying state variables from five to two, namely, the interest rate factor and the fraction of the funds that are invested in the risky asset. This is possible due to the form of the transaction cost. Then, due to non-linearities in the differential equation that is part of the risk sensitive QVI, we make a further transformation that results in what can be interpreted as the ergodic Isaac-Hamilton-Jacobi equation associated with a max-min problem. Finally, we use a numerical method based on an extended two-stage policy iteration algorithm to solve this min-max problem.

¿From the economic standpoint our numerical results for the optimal trading strategy are as one would intuitively anticipate. For a fixed interest rate the no-trade region is an interval, and, with lower values of the interest rate factor being bullish for the risky asset, the end points of the interval are decreasing, in a continuous fashion, with respect to the factor level. When the two-dimensional process hits the boundary, one rebalances by restarting the process from a specified point in the interior of the interval corresponding to the current factor value. And as one changes the risk sensitivity parameter so as to make the investor more risk averse, the no-trade region shifts in a direction toward smaller proportions in the risky asset.

The plan for this paper is as follows. In the next section we formulate the problem and present the RSQVI which must be solved for the optimal trading strategy. In Section 3 we explain how to transform this RSQVI into an equivalent Isaac-Hamilton-Jacobi equation. Our computational approach is presented in Section 4, and then in Section 5 we validate our approach by using it to reconstruct the results obtained with different methods by Bielecki and Pliska [4] for their zero-factor example. Then in Section 6 we illustrate our computational approach by numerically solving an example which includes the stochastic interest rate factor. These results not only demonstrate the efficiency of our algorithm, but by including comparative statics analyses they also provide some economic understanding of the underlying portfolio optimization problem. We conclude with some remarks in Section 7 on the vanishing transaction cost case and explicit solutions.

2 Model and Problem Formulation

We start with a two-asset, one-factor market model consisting of a bank account S_0

$$\frac{dS_0(t)}{S_0(t)} = (a_0 + A_0 X(t))dt, \quad S_0(0) = s_0,$$

a risky security S_1 such as a stock or stock index

$$\frac{dS_1(t)}{S_1(t)} = (a_1 + A_1 X(t))dt + \sigma_1 dW_1(t), \quad S_1(0) = s_1,$$

and one exogenous economic factor $X(t)$

$$dX(t) = (b + BX(t))dt + \lambda_1 dW_1(t) + \lambda_2 dW_2(t), \quad X(0) = x.$$

Here $W(t) = (W_1(t), W_2(t))$ is a two-dimensional Brownian motion while a_0, A_0, a_1, A_1, $\sigma_1, b, B, \lambda_1$, and λ_2 are various scalar parameters. Presumably, one could take $B < 0$ so that the factor process will have the "mean reverting" property. Notice that this factor can explicitly affect the appreciation rate of the risky asset. Moreover, if one takes $a_0 = 0$ and $A_0 = 1$, then the factor coincides with the bank's interest rate, and so one in this case one should interpret the factor as the short interest rate, as in the so-called Vasicek model.

In this market we consider an investor who is dynamically trading the two securities: S_0 and S_1. The information available to this investor is modeled by the filtration $\mathcal{G}_t := \sigma((S_1(s), X(s)), 0 \leq s \leq t)$. However, due to the presence of fixed transaction costs, the investor does not trade continuously in time. Rather, the investor is restricted to the use of *impulsive investment strategies* of the form $u = ((\tau_k, N_k), k = 0, 1, 2, \ldots)$ where

- $\tau_0 \equiv 0 < \tau_1 < \ldots < \tau_k < \tau_{k+1} < \ldots$ are $\mathcal{G}_t$-stopping times (portfolio rebalancing times) with $\tau_k \to \infty$ a.s. when $k \to \infty$, and
- $N_k := [N_{k,0}, N_{k,1}]^T$ is $\mathcal{G}_{\tau_k}$-measurable, where $N_{k,i}$ is the number of shares of security i to which the investor rebalances his portfolio at time τ_k, and $N_{k,i} \geq 0$ (no borrowing or short selling is allowed).

As we mentioned above, the trading is subject to *transaction costs*. Let $C(s, N, N')$ denote the cost of the transaction when the security prices are $s = (s_0, s_1)^T$ and the portfolio share-holding positions change from $N := (n_0, n_1)$ to $N' := (n'_0, n'_1)$. We envision two main examples:

- Proportional to the transaction volume:

$$C(s, N, N') := c + c_1 |s \cdot (N - N')|, \quad c, c_1 > 0$$

- Proportional to the investor's wealth level (as in Morton-Pliska [22])

$$C(s, N, N') := \alpha s \cdot N, \quad \alpha \in (0, 1).$$

We consider the following set of *admissible strategies*:

$$\mathcal{U} := \{u = ((\tau_k, N_k), k = 0, 1, \ldots), N_k \in \mathcal{A}(S(\tau_k), N_{k-1}), S(\tau_k) \cdot N_k > 0\}$$

where

$$\mathcal{A}(s, N) := \{N'[0, +\infty)^2 : \ \ s \cdot N - C(s, N, N') \geq s \cdot N'\}.$$

Thus for a trading strategy to be admissible it must be *self-financing*, that is, the portfolio value immediately after a transaction cannot be greater than the portfolio value immediately before a transaction less the cost of the transaction. Moreover, it is required that the investor retains a positive amount of money in the portfolio after any transaction.

Let $N(t) = N^u(t) := N_k \in \mathbb{R}^2_+, \ t \in [\tau_k, \tau_{k+1}), \ \ k = 0, 1, 2, \ldots,$ denote the *share holding process*. The investor's objective is to trade optimally according to the *risk-sensitive performance criterion*, that is, for $\theta > 0$, maximize the *risk sensitive expected exponential growth rate* of the investor's portfolio:

$$J^u_\theta := \liminf_{t \to \infty} \Big(-\frac{2}{\theta} \Big) t^{-1} \ln \mathbb{E}[S(t) \cdot N^u(t))^{-\frac{\theta}{2}}]. \tag{2.1}$$

As discussed above and explained more fully in various references cited above, this criterion can be interpreted as providing a trade-off between a portfolio's exponential growth rate and its asymptotic variance, that is, its average volatility. Moreover, the bigger the value of the parameter θ, the more risk averse the investor.

This model is a special case of the one in Bielecki and Pliska [4], so to solve this optimization problem we shall initially follow their approach. Consider the following *state process* (a piecewise Itô process):

$$Y_t^u = (S(t), X(t), N^u(t)) \in \mathcal{O} := (0,\infty)^2 \times \mathbb{R} \times [0,\infty)^2.$$

We denote the current state by $y = (s_0, s_1, x, n_0, n_1)$.

The wealth process $Z_t^u := S(t) \cdot N^u(t)$ evolves according to:

$$dZ_t^u = Z_t^u(f(Y_t^u)dt + \bar{\gamma}(Y_t^u)dW_t) + \sum_{k=0}^{\infty} \Delta Z_{\tau_k}^u \mathbf{1}_{\{\tau_k \le t\}}$$

where

$$f(y) = (a_0 + A_0 x)\frac{s_0 N_0}{s_0 N_0 + s_1 N_1} + (a_1 + A_1 x)\frac{s_1 N_1}{s_0 N_0 + s_1 N_1}$$

$$\bar{\gamma}(y) = (0, \gamma(y))^T$$

$$\gamma(y) = \left(\sigma_1 \frac{s_1 N_1}{s_0 N_0 + s_1 N_1}, 0\right)$$

Note for future use that the functions $\bar{\gamma}$ and f depend on $y = (s_0, s_1, x, N_0, N_1)$ only through x and the fraction of wealth held in the risky security: $\frac{s_i N_i}{s_0 N_0 + s_1 N_1}$.

Using Itô's formula, we obtain:

$$\begin{aligned} d\ln(Z_t^u) &= (f(Y_t^u) - 1/2\|\gamma(Y_t^u)\|^2)dt \\ &+ \bar{\gamma}(Y_t^u)dW_t + \sum_{k=0}^{\infty} \Delta \ln(Z_{\tau_k}^u)\mathbf{1}_{\{\tau_k \le t\}}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}[(Z_t^u)^{-\theta/2}] = \mathbb{E}[\exp &\left(-\frac{\theta}{2}(\int f_\theta(Y_r^u)dr + \sum_{k=0}^{\infty} \Delta \ln Z_{\tau_k}^u \mathbf{1}_{\{\tau_k \le t\}})\right) \\ &\exp\left(-1/2\int_0^t \|\gamma_\theta(Y_r^u)\|^2 dr + \int_0^t \gamma_\theta(Y_r^u)dW_r\right)] \end{aligned}$$

where

$$f_\theta(y) = f(y) - 1/2(\frac{\theta}{2} + 1)\|\gamma(y)\|^2$$

and $\gamma_\theta = -\frac{\theta}{2}\gamma$.

For each $u \in \mathcal{U}$ and $\theta > 0$, define an equivalent measure $P^{u,\theta}$ by

$$\frac{dP^{u,\theta}}{dP}\Big|_{\mathcal{F}_t} = \exp\left(-1/2\int_0^t \|\gamma_\theta(Y_r^u)\|^2 dr + \int_0^t \gamma_\theta(Y_r^u) dW_r\right),$$

where $\mathbf{F} := (\mathcal{F}_t,\ t \geq 0)$ is the Wiener filtration associated with the Brownian motions W_1 and W_2. Then the performance can be rewritten:

$$J_\theta^u(s,x) = \liminf_{t\to\infty} -\frac{2}{\theta} t^{-1} \ln \mathbb{E}_{P^{u,\theta}}^{s,x}\left(\exp(-\frac{\theta}{2}(\int_0^t f_\theta(Y_r^u)dr + \sum_{k=0}^{\infty} \Delta \ln Z_{\tau_k}^u \mathbf{1}_{\{\tau_k \leq t\}}))\right).$$

We know by Bielecki and Pliska [4] that the solution to the investor's problem can be characterized in terms of the following *Risk-Sensitive Quasi-Variational Inequality (RSQVI)*:

Find a scalar λ and a suitable, real-valued function Φ solving

$$\begin{cases} \max\{L^\theta\Phi(y) - \frac{\theta}{4}\|\Phi_y(y)\cdot\beta(y)\|^2 - \lambda + f_\theta(y), M\Phi - \Phi\} = 0 & \text{in } \mathrm{int}(\mathcal{O}) \\ M\Phi - \Phi \leq 0 & \text{on } \partial\mathcal{O} \end{cases}$$

where

$$L^\theta\Phi(y) := (a_0 + A_0 x)s_0\Phi_{s_0} + [(a_1 + A_1 x)s_1 - \tfrac{\theta}{2}\tfrac{\sigma^2 s_1^2 N_1}{s_0N_0 + s_1N_1}]\Phi_{s_1}$$

$$+[(b + Bx) - \frac{\theta}{2}\frac{\lambda_1\sigma_1 s_1 N_1}{s_0N_0 + s_1N_1}]\Phi_x + \frac{1}{2}s_1^2\sigma_1^2\Phi_{s_1s_1} + \lambda_1\sigma_1 s_1\Phi_{xs_1} + \frac{1}{2}(\lambda_1^2 + \lambda_2^2)\Phi_{xx},$$

$$M\Phi(y) := \sup_{N' \in \mathcal{A}(s,N)} \{\ln(s\cdot N') - \ln(s\cdot N) + \Phi(s,x,N')\},$$

$$\beta(y) := \begin{pmatrix} 0 & 0 \\ s\sigma_1 & 0 \\ \lambda_1 & \lambda_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$f_\theta(y) := f(y) - 1/2(\frac{\theta}{2} + 1)\|\gamma(y)\|^2,$$

$$f(y) := (a_0 + A_0 x)\frac{s_0 n_0}{s_0 n_0 + s_1 n_1} + (a_1 + A_1 x)\frac{s_1 n_1}{s_0 n_0 + s_1 n_1},$$

and

$$\gamma(y) := (\sigma_1 \frac{s_1 n_1}{s_0 n_0 + s_1 n_1}, 0).$$

The optimal trading strategy can be constructed from the solution (λ, Φ) in a very straightforward fashion. The *Verification Theorem* associated with this characterization was provided (for a more general version of this model) by Bielecki and Pliska [4], but that was as far as they took it. It remains to find the scalar λ and the smooth, real-valued function Φ satisfying this RSQVI.

3 Preliminary Transformations

We concentrate on the case where the transaction cost is

$$C(s, N, N') = \alpha s \cdot N, \ \ \alpha \in (0, 1).$$

The following change of variables reduces the number of state variables from five to two, thereby simplifying the analysis of our RSQVI. Consider the new state variable

$$z := \frac{n_1 s_1}{n_0 s_0 + n_1 s_1},$$

representing the current fraction of wealth in the risky asset (we call this the *risky fraction*), and define a new function by taking

$$\Psi(z, x) := \Phi(s_0, s_1, x, n_0, n_1)$$

on $\mathcal{D} := (0, 1) \times \mathbb{R}$. Now writing various partial derivatives of Φ in terms of Ψ, substituting these in the above RSQVI, and doing some algebra, it becomes apparent that our original RSQVI is equivalent to the following one:

$$\begin{cases} \max\{\mathcal{L}\Psi(z,x) - \frac{\theta}{4}(\sigma_1 z(1-z)\Psi_z(z,x) + \lambda_1 \Psi_x(z,x))^2 - \frac{\theta}{4}\lambda_2^2 \Psi_x(z,x)^2 & \\ -\lambda + \bar{f}(z,x), \mathcal{M}\Psi(z,x) - \Psi(z,x)\} = 0 & \text{in } \mathcal{D} \\ \mathcal{M}\Psi - \Psi(z,x) \le 0 & \text{on } \partial\mathcal{D} \end{cases} \tag{3.2}$$

where

$$\mathcal{L}\Psi(z,x) := b_1(z,x)\Psi_z + b_2(z,x)\Psi_x + d_{11}(z)\Psi_{zz} + d_{22}\Psi_{xx} + d_{12}(z)\Psi_{zx} \tag{3.3}$$

$$\mathcal{M}\Psi(z,x) := \ln(1-\alpha) + \sup_{0 \le z' \le 1} \Psi(z', x) \tag{3.4}$$

with

$$
\begin{array}{ll}
b_1(z,x) & := ((a_1-a_0)+(A_1-A_0)x-(\theta/2+1)\sigma_1^2 z)\, z(1-z), \\
b_2(z,x) & := b+Bx-\frac{\theta}{2}\lambda_1\sigma_1 z, \\
d_{11}(z) & := \frac{1}{2}\sigma_1^2 z^2(1-z)^2, \;\; d_{22}:=\frac{1}{2}(\lambda_1^2+\lambda_2^2), \;\; d_{12}(z):=\lambda_1\sigma_1 z(1-z), \\
\bar{f}(z,x) & := (a_0+A_0x)(1-z)+(a_1+A_1x)z-\frac{\sigma_1^2}{2}(\frac{\theta}{2}+1)z^2.
\end{array}
\tag{3.5}
$$

The economic intuition associated with this transformation is fairly clear: the two resulting state variables, namely, the risky fraction process and the factor process, together constitute a sufficient statistic for the purposes of the optimization problem.

Unfortunately, the terms in (3.2) that are nonlinear with respect to partial derivatives of Ψ present difficulties when one attempts to solve this RSQVI with a standard computational approach. Our approach for circumventing this difficulty is to linearise the RSQVI by using an auxiliary control variable.

The quadratic part in RSQVI (3.2), namely

$$
-\frac{\theta}{4}(\sigma_1 z(1-z)\Psi_z+\lambda_1\Psi_x)^2-\frac{\theta}{4}\lambda_2^2\Psi_x^2,
$$

can be expressed as

$$
\frac{\theta}{2}\min_{c_1\in\mathbb{R}}\{-(\sigma_1 z(1-z)\Psi_z+\lambda_1\Psi_x)c_1+\frac{1}{2}c_1^2\}+\frac{\theta}{2}\min_{c_2\in\mathbb{R}}\{-\lambda_2\Psi_x c_2+\frac{1}{2}c_2^2\}.
$$

The RSQVI can then be rewritten as

$$
\begin{cases}
\max\{\min_{c\in\mathbb{R}^2}[\mathcal{L}_c\Psi-\lambda+g_c]\mathcal{M}\Psi-\Psi\}=0 & \text{in } \mathcal{D} \\
\mathcal{M}\Psi-\Psi\le 0 & \text{on } \partial\mathcal{D}
\end{cases}
\tag{3.6}
$$

where

$$
\mathcal{L}_c\Psi := (b_1-\frac{\theta}{2}c_1\sigma_1 z(1-z))\Psi_z+(b_2-\frac{\theta}{2}\lambda_1c_1-\frac{\theta}{2}\lambda_2c_2)\Psi_x+d_{11}\Psi_{zz}+d_{22}\Psi_{xx}+d_{12}\Psi_{zx},
\tag{3.7}
$$

$$
g_c(z,x) := \bar{f}(z,x)+\frac{\theta}{4}c_1^2+\frac{\theta}{4}c_2^2,
$$

and $\mathcal{M}$ is given in (3.4). Thus we have introduced a game theoretic aspect to our problem, thereby making it more complicated in some respects. However, now all the partial derivatives of Ψ enter the RSQVI in a linear fashion, and so it is ready to be solved for (λ,Ψ), as will be explained in the following section.

Remark 3.1 *It is essential to emphasize here that our RSQVI (3.6) should not be mistaken for an eigenfunction/eigenvalue type problem (or a Sturm-Liouville type problem). In fact, we do not know of any way of characterizing the optimal investment problem considered in this paper in terms of an eigenfunction/eigenvalue type problem. One of the reasons that such a characterization does not appear to be possible is that the optimization problem we deal with is essentially a free-boundary problem, where the free boundary depends on the bias function. Moreover, our problem is an ergodic type problem, which – in distinction from discounted type problems – does not lend itself to the eigenfunction/eigenvalue characterization. Thus a special computational approach had to be devised for dealing with our problem.*

4 Computational Approach

The purpose of this section is to present an algorithm for computing a solution to the max-min problem (3.6) and thus the original RSQVI.

4.1 Localisation

With regard to boundary conditions, the original, transformed problem sits on a strip in $\mathbb{R}^2$. The risky fraction process lives between 0 and 1 due to the nature of the problem, which stipulates that after any rebalancing the fraction in the risky asset must start strictly between 0 and 1. For a continuous time diffusion the boundaries 0 and 1 are probably totally inaccessible (for example, see Morton and Pliska [22]), and so the boundary condition for the risky fraction component should not matter; we make it reflecting. The factor process lives between plus and minus infinity, but for computational purposes this must be changed to a compact interval. Since the factor process is stable (i.e., mean reverting), its boundaries can be set at levels which are rarely reached. If they are set in this way, then the specific kind of boundary behavior is not important; we shall make these reflecting. Consequently, the discretized problem ends up being an approximation of a two dimensional process that lives on a compact rectangle $\Omega = [0, 1] \times [L_1, L_2]$ with reflection on all four boundaries.

4.2 Finite-difference approximation

The operator $\mathcal{L}_c$ defined in (3.7) can be rewritten as:

$$\mathcal{L}_c\Psi := \bar{b}_1\Psi_z + \Psi_x + d_{11}\Psi_{zz} + d_{22}\Psi_{xx} + d_{12}\Psi_{zx},$$

where

$$\begin{aligned}\bar{b}_1 &:= b_1 - \theta/2c_1\sigma_1 z(1-z)\\ \bar{b}_2 &:= b_2 - \theta/2(\lambda_1 c_1 + \lambda_2 c_2)\end{aligned}$$

and b_1 and b_2 defined in (3.5). We consider a finite difference approximation, based upon a two-dimensional grid that is denoted $\mathcal{D}_\delta = E_Z \times E_X$ and has space discretization steps denoted δ_1 and δ_2 such that $1/\delta_1 \in \mathbb{N}$ and $\frac{L_2-L_1}{\delta_2} \in \mathbb{N}$. Here $E_Z = \{i\delta_1, i = 0....1/\delta_1\}$ and $E_X = \{L_1 + j\delta_2, j = 0....\frac{L_2-L_1}{\delta_2}\}$ denote respectively the z-grid and the x-grid.

Let $(i\delta_1, j\delta_2)$ be a point of the grid $\mathcal{D}_\delta$. We approximate $\mathcal{L}_c\Psi(i\delta_1, j\delta_2)$ by $\mathcal{L}_c^\delta\Psi(i\delta_1, j\delta_2)$ defined as:

$$\mathcal{L}_c^\delta\Psi = \bar{b}_1\partial_z^{sgn\bar{b}_1}\Psi + \bar{b}_2\partial_x^{sgn\bar{b}_2}\Psi + d_{11}\partial_{zz}\Psi + d_{22}\partial_{xx} + d_{12}\partial_{zx}\Psi,$$

where

$$\begin{aligned}
\partial_z^+(i\delta_1, j\delta_2) &= \frac{\Psi((i+1)\delta_1, j\delta_2) - \Psi(i\delta_1, j\delta_2)}{\delta_1},\\
\partial_z^-(i\delta_1, j\delta_2) &= \frac{\Psi(i\delta_1, j\delta_2) - \Psi((i-1)\delta_1, j\delta_2)}{\delta_1},\\
\partial_{zz}(i\delta_1, j\delta_2) &= \frac{\Psi((i+1)\delta_1, j\delta_2) - 2\Psi(i\delta_1, j\delta_2) + \Psi((i-1)\delta_1, j\delta_2)}{\delta_1^2},\\
\partial_{zx}(i\delta_1, j\delta_2) &= \frac{1}{\delta_1}\delta_2\Big(2\Psi(i\delta_1, j\delta_2) + \Psi((i+1)\delta_1, (j+1)\delta_2) + \Psi((i-1)\delta_1, (j-1)\delta_2),\\
&-\Psi((i+1)\delta_1, j\delta_2)\Psi((i-1)\delta_1, j\delta_2)\Psi(i\delta_1, (j+1)\delta_2)\Psi(i\delta_1, (j-1)\delta_2)\Big),
\end{aligned}$$

and $\partial_x^\pm$ and ∂_{xx} are similarly defined. The discrete operator $\mathcal{L}_c^\delta$ has a square matrix representation denoted by L_c^δ. The approximation is stable in the sense of the L_∞-norm when the matrix L_c^δ is invertible and if there exists a uniform upper bound of $\|(L_c^\delta)^{-1}\|_\infty$. This is achieved when L_c^δ is diagonally dominant. This condition is met when the spatial steps δ_1 and δ_2 satisfy

$$\frac{d_{11}}{\delta_1} - \frac{|d_{12}|}{\delta_2} \geq 0 \quad \text{and} \quad \frac{d_{22}}{\delta_2} - \frac{|d_{12}|}{\delta_1} \geq 0 \tag{4.8}$$

(see [15, p.108]), which is equivalent to

$$\frac{|\lambda_1|\sigma_1 z(1-z)}{\lambda_1^2+\lambda_2^2} \leq \frac{\delta_1}{\delta_2} \leq \frac{\sigma_1 z(1-z)}{|\lambda_1|} \text{ for all } z \in .$$

This is true if

$$\max_{z\in[0,1]} \frac{|\lambda_1|\sigma_1 z(1-z)}{\lambda_1^2+\lambda_2^2} \leq \frac{\delta_1}{\delta_2} \leq \min_{z\in[0,1]} \frac{\sigma_1 z(1-z)}{|\lambda_1|}.$$

Note that this cannot be satisfied for $z=0$ and 1. In order to be satisfied for all $z \in E_Z$ we add a viscosity term $\varepsilon\Delta$ to the $\mathcal{L}_c$ operator [1]. This leads to the following sufficient stability condition:

$$\frac{|\lambda_1|\sigma_1}{4(\lambda_1^2+\lambda_2^2)} \leq \frac{\delta_1}{\delta_2} \leq \frac{\sigma_1\varepsilon}{|\lambda_1|}. \tag{4.9}$$

We thus obtain the discrete approximation of Equation (3.6):

$$\max\left(\min_{c\in} \mathcal{B}_c^\delta(\lambda, V), \mathcal{M}^\delta V - V\right) = 0 \quad \text{in} \quad \mathcal{D}_\delta, \tag{4.10}$$

where

$$\begin{array}{rcl} \mathcal{B}_c^\delta(\lambda, V) & = & -\lambda + \mathcal{L}_c^{\delta,\epsilon} V + g_c \\ \mathcal{M}^\delta V(z,x) & = & \max\limits_{z'\in E_Z} V(z',x) + \ln(1-\alpha) \end{array}$$

and $\mathcal{L}_c^{\delta,\epsilon}$ is the discrete operator obtained from the finite difference of $\mathcal{L}_c + \varepsilon\Delta$.

Note that equation (4.10) is valid for the points of the grid situated on the boundary $\mathcal{D}_\delta \cap \partial\Omega$ since we have set reflecting boundary conditions (homogeneous Neuman limit conditions): the value of the function V at fictitious points situated outside the grid is equal to the value of the function V at the inner points, symmetric with respect to the boundary. Moreover $\mathcal{M}V - V \leq 0$ holds on the boundary since it is implied by the equation.

Denote by $L_c^{\delta,\epsilon}$ the matrix representation of $\mathcal{L}_c^{\delta,\epsilon}$. Take $k \leq |(L_c^{\delta,\epsilon})_{ii}|^{-1}$ for all diagonal entries $(L_c^{\delta,\epsilon})_{ii}$. Condition (4.9) implies that $P_c \equiv I + kL_c^{\delta,\epsilon}$ is a Markov transition matrix, and equation (4.10) can be rewritten as:

$$\max\left(\min_{c\in\mathbb{R}^2} \left\{-\lambda k + (P_c - I)V + kg_c\right\}, \mathcal{M}^\delta V - V\right) = 0 \quad \text{in} \quad \mathcal{D}_\delta. \tag{4.11}$$

[1]As suggested by Nagai [24], we could also perform the change of variable $y = \ln z - \ln(1-z)$

4.3 A policy interation algorithm for discrete-time max-min ergodic problems

To solve the discrete time max-min ergodic problem (4.10) we use an extended double stage policy iteration algorithm. Policy iteration algorithms for ergodic games are studied in [10], [11], and [17]. Convergence proofs are based on structure properties of the set of fixed points of the operators involved.

We discribe now the algorithm but we will not address here the issue of the proof of convergence. The algorithm involves a series of major iterations which we index by k. And each such major iteration involves a series of minor iterations that we index by p. We denote by ϵ an *a priori* prescribed precision, and for any function $\xi : (E_Z, E_X) \to E_Z$ and any function $V(z,x)$ we denote by $\mathcal{N}_\xi$ the function defined as

$$\mathcal{N}_\xi V(z,x) = V(\xi(z,x),x) + \ln(1-\alpha).$$

[S_1] **Initialization:** We fix an initial partition $\mathcal{D}^1_{\delta,1} \cup \mathcal{D}^1_{\delta,2}$ of $\mathcal{D}_\delta$ (take e.g. $\mathcal{D}^1_{\delta,1} = \mathcal{D}_\delta$ and $\mathcal{D}^1_{\delta,2} = \emptyset$) and an initial function $\xi_1 : (E_Z, E_X) \to E_Z$.

For $k \geq 1$ we have a major iteration comprised of the following steps:

[S_2] **Step** $2k-1$**:** Find (λ^k, V^k), a solution of

$$\begin{cases} \min_{c\in\mathbb{R}^2} \mathcal{B}^\delta_c(\lambda, V) = 0 & \text{in} \quad \mathcal{D}^k_{\delta,1} \\ V = \mathcal{N}_{\xi_k} V & \text{in} \quad \mathcal{D}^k_{\delta,2}, \end{cases} \tag{4.12}$$

and compute the optimal strategy c^k as given by

$$c^k \in \operatorname{argmin}_{c\in\mathbb{R}^2} \mathcal{B}^\delta_c(\lambda^k, V^k).$$

Equation (4.12) is solved by using a policy iteration algorithm, a separate procedure described below.

[S_3] **Step** $2k$**:** Compute $\xi^{k+1}(,x) \in \operatorname{argmax}_{z'\in E_Z} V^k(z',x)$ and define a new partition $\mathcal{D}^{k+1}_{\delta,1} \cup \mathcal{D}^{k+1}_{\delta,2}$ as

$$\begin{aligned} \mathcal{D}^{k+1}_{\delta,1} &= \left\{ (z,x) \in \mathcal{D}_\delta \min_{c\in\mathbb{R}^2} \mathcal{B}^\delta_c(\lambda^k, V^k) \geq \mathcal{N}_{\xi^{k+1}} V^k \right\} \\ \mathcal{D}^{k+1}_{\delta,2} &= \mathcal{D}_\delta \backslash \mathcal{D}^{k+1}_{\delta,1}. \end{aligned} \tag{4.13}$$

[S_4] **Stop:** If $|\lambda^{k+1} - \lambda^k| \leq \epsilon$ we stop; else we go back to [S_2] to perform step **Step** $2k+1$.

Resolution of (4.12). Given a partition $\mathcal{D}_{\delta,1} \cup \mathcal{D}_{\delta,2}$ and a function $\xi : (E_Z, E_X) \to E_Z$, the policy iteration algorithm used in step $2k-1$ is described now. This algorithm is a series of iterations that we call the minor iterations. Each such iteration, indexed by $p \geq 1$, consists of the following steps.

$[s_1]$ **Initialisation:** Let's c^0 be a given initial strategy: $\mathcal{D}_{\delta,1} \mapsto \mathbb{R}^2$

$[s_2]$ **Step** $2p-1$**:** given a strategy c^p :

$$\begin{aligned} c^p: \quad & \mathcal{D}_{\delta,1} \mapsto \mathbb{R}^2 \\ & (z,x) \mapsto (c_1, c_2) \end{aligned}$$

we solve the linear system in (λ, V):

$$\begin{cases} \mathcal{B}^{\delta}_{c^p}(\lambda, V) = 0 & \text{in} \quad \mathcal{D}_{\delta,1} \\ V = N_{\xi} V & \text{in} \quad \mathcal{D}_{\delta,2} \end{cases} \tag{4.14}$$

with the additional constraint $V(z,x) = 1$ because V is only defined within an additive constant by (4.12). We denote by (λ^p, V^p) the solution of the system (4.14).

$[s_3]$ **Step** $2p$**:** compute the strategy $c^{p+1} : \mathcal{D}_{\delta,1} \mapsto \mathbb{R}^2$ defined as

$$c^{p+1} = \text{argmin}_{\{c \in \mathbb{R}^2\}} \mathcal{B}^{\delta}_{c}(\lambda^p, V^p).$$

$[s_4]$ If $|\lambda^{p+1} - \lambda^p| \epsilon$ stop; else go back to $[s_2]$ to perform **Step** $2p+1$.

Remark : Note that problem (4.12) is not always well defined : the equation $V = \mathcal{N}_{\xi_k} V$ (in $\mathcal{D}^k_{\delta,2}$) might have no solutions, for example if there is a cycle in $\mathcal{D}^k_{\delta,2}$. However, this situation did not happen in the numerical simulations.

5 Comparison with the Risk Sensitive Extension of the Morton-Pliska Problem

As mentioned in the introduction, Morton and Pliska [22] studied a two-asset model with the same dynamics as here, only there were no factors, so $A_0 = A_1 = 0$. Their transaction cost was also the same as here, namely, proportional to the investor's wealth at the time of the transaction. Their optimality criterion consisted in maximizing the investor's long run, exponential growth rate; this is equivalent to our risk sensitive criterion with risk aversion parameter $\theta = 0$. Subsequently, Bielecki and Pliska [4] extended the Morton-Pliska results to allow for any positive value of the risk aversion parameter θ. In particular, they were able to find the explicit solution of the differential equation corresponding to the continuation-region portion of the RSQVI in their case, and so by using suitable computer software they were able to numerically compute the full solution of their RSQVI and thus their optimal strategy. Since their numerical example is a special case of the one-factor model studied in this paper, it provides a benchmark for the purpose of validating our double stage policy iteration algorithm approach.

Using the notation of this paper, the Morton-Pliska problem solved in Bielecki-Pliska [4] corresponds to the following choice of parameters (specifications of our space mesh are also given):

- $a_0 = 0.07, A_0 = 0, a_1 = 0.15, A_1 = 0, \sigma_1 = 0.4, \alpha = 0.001$.
- $[L_1, L_2] = [0, L_x]$ with $L_x = 0.12$
- $N_x = 30$ number of grid points on the x-axis process $X(t)$
- $N_z = 50$ number of grid points on the z-axis process $z(t)$

We computed the optimal trading strategy for $\theta \in \{0.1, 2, 4\}$. When $\theta = 4$, we obtained $\lambda = 0.07568$ (see Figure 1). The numerical results that we get are very close to the results published by Bielecki and Pliska in [4].

In Figure 1 and in subsequent figures, the horizontal axis is the risky fraction ($z \in [0, 1]$) and the vertical axis is the interest rate factor ($x \in [0, L_x]$). The gray region represents the rebalancing portfolio region, the white one is the no transaction region, and the black line is the set of levels reached after a transaction.

The optimal policy is characterized here by the triple $(\underline{z}, z^\star, \overline{z})$ such that $\{(x, z), z \leq \underline{z}\}$ is the buying region, $\{(x, z), z \geq \overline{z}\}$ is the selling region, and $z^\star$ is th optimal rebalancing portfolio. Figure 2 displays the sensitivity of $(\underline{z}, z^\star, \overline{z})$ to the size of the z-grid.

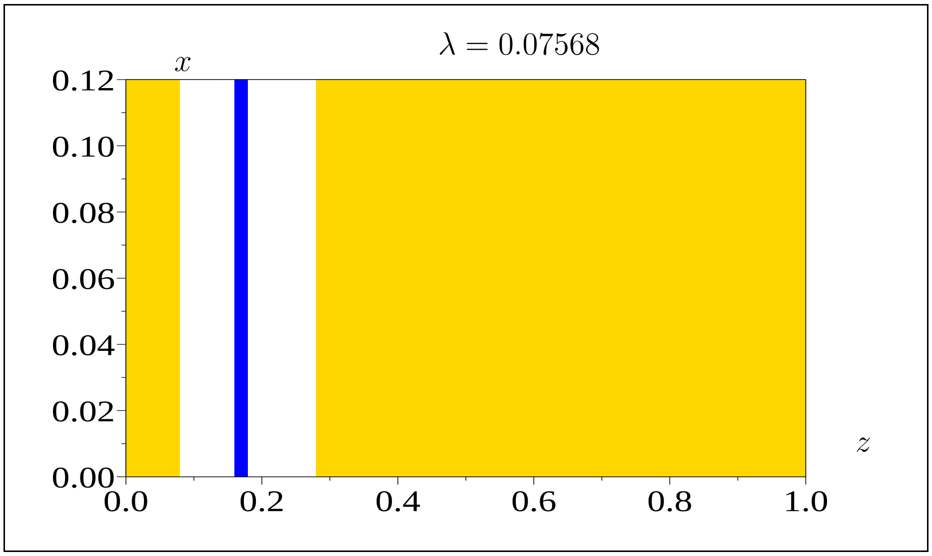

Figure 1: Optimal investment strategy for the risk sensitive Morton Pliska problem with $\alpha = 0.001$, $\theta = 4.00$, $N_x = 30$, $N_z = 50$

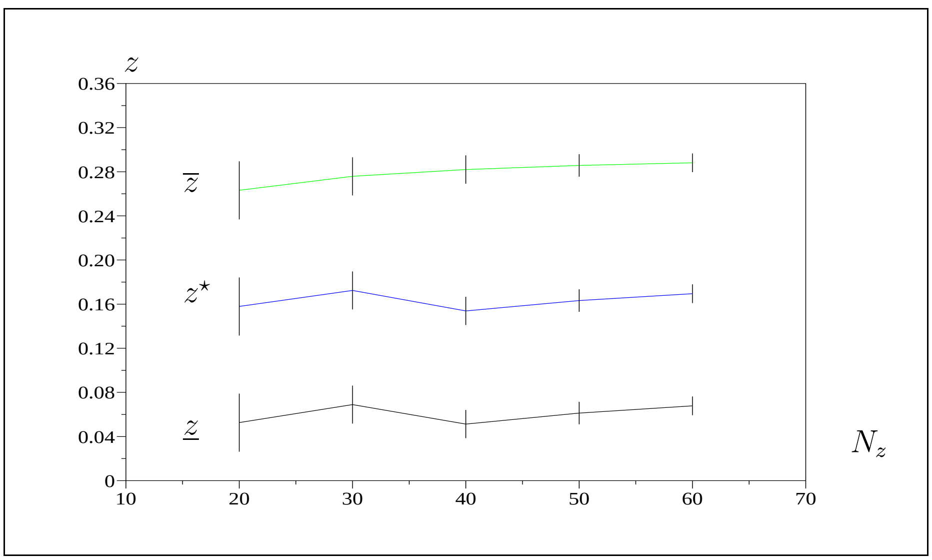

Figure 2: $\underline{z}$, $z^\star$ and $\overline{z}$ for the risk sensitive Morton Pliska problem with $\alpha = 0.001$, $\theta = 4.00$, $N_x = 30$

6 Numerical results for the risk sensitive problem with the interest rate factor

In this section we provide numerical results for the risk sensitive problem with a factor, obtained by implementing our algorithm.[2] Not only does this validate the efficiency of our algorithm, but this provides some economic understanding of the underlying problem. In fact, we do a comparative statics analysis, investigating the sensitivity of the optimal strategy to some of the data parameters.

Figure 3 represents a typical optimal strategy such as we obtained for figures 4 to 10. While the locations of the various boundaries will of course vary, the qualitative nature of the strategies corresponding to Figures 4 to 10 will all resemble the hypothetical strategy illustrated in Figure 3.

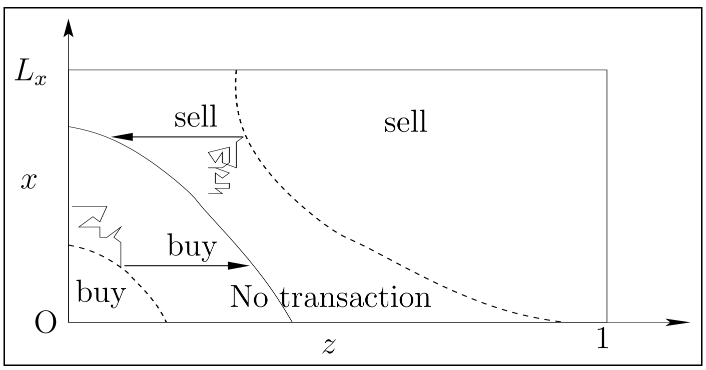

Figure 3: Typical optimal strategy

For Figure 4 the following "baseline" parameter values are used:

- $a_0 = 0, A_0 = 1, a_1 = 0.18, A_1 = -1, \sigma_1 = 0.4, b = 0.06, B = -1, \lambda_1 = -0.001, \lambda_2 = 0.005, \theta = 2, \alpha = 0.01$.
- $[L_1, L_2] = [0, L_x]$ with $L_x = 0.12$
- $N_x = 70$: number of grid points on the x-axis, $N_z = 70$: number of grid points on the z-axis.

[2]The `C` and `Scilab` [27] programs which were used for numerical computations can be found at url : `http://cermics.enpc.fr/~jpc/bcp-19-nov-2003/`

For these values of parameters the risk sensitive performance λ is 0.05 (it varies from 0.063 to 0.05 when the number of grid points increases from 30 points to 70 points in both directions).

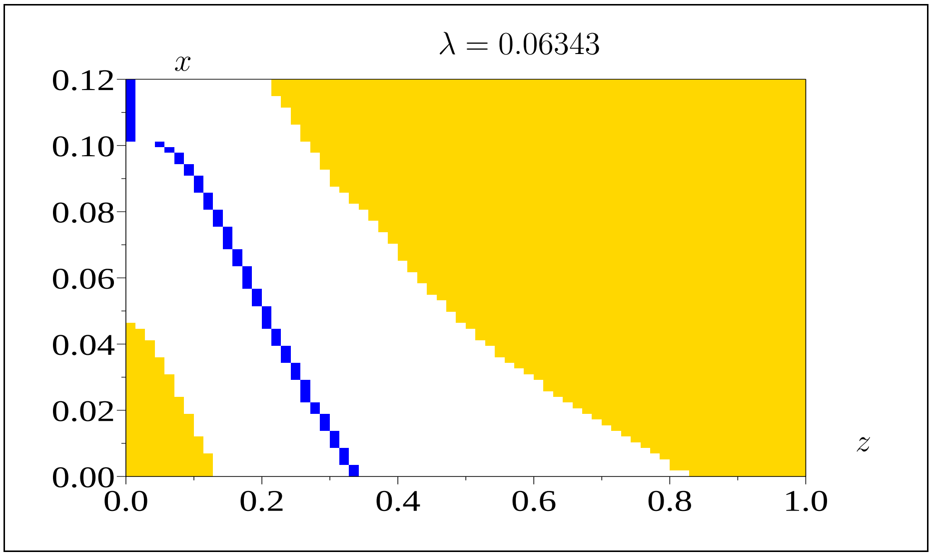

Figure 4: Optimal investment strategy for the risk sensitive problem $\alpha = 0.010$, $\theta = 2.00$, $N_x = 70$, $N_z = 70$

Given various values of x, Table 1 provides the values of the optimal thresholds $(\underline{z}(x), z^\star(x), \overline{z}(x))$ such that buying is optimal at (x, z) when $z \leq \underline{z}(x)$, selling is optimal at (x, z) when $z \geq \overline{z}(x)$, and the optimal rebalancing fraction is $z^\star(x)$. This is done for the parameters used for Figure 4 and also for a coarser grid $N_x = N_z = 50$.

Continuation region versus risk sensitive parameter. We investigate the sensitivity of the numerical results to changes in the investor's risk aversion parameter. As expected, as θ increases, the investor keeps more money in the less risky asset; see Figures 5, 6, and 7 and Tables 2 and 3 (here for $N_x = N_z = 50$).

Continuation region versus transaction cost. We also investigate the sensitivity of results to changes in the transaction cost (here for $N_x = N_z =$

	$N_z = N_x = 50$			$N_z = N_x = 70$		
x	$\underline{z}$	$z^\star$	$\overline{z}$	$\underline{z}$	$z^\star$	$\overline{z}$
0.000	0.122	0.347	0.837	0.116	0.333	0.841
0.030	0.061	0.245	0.612	0.058	0.246	0.609
0.060	–	0.184	0.449	–	0.174	0.435
0.090	–	0.102	0.327	–	0.101	0.304
0.120	–	0.000	0.224	–	0.000	0.217

Table 1: $\underline{z}$,$z^\star$ and $\overline{z}$ as in Figure 4

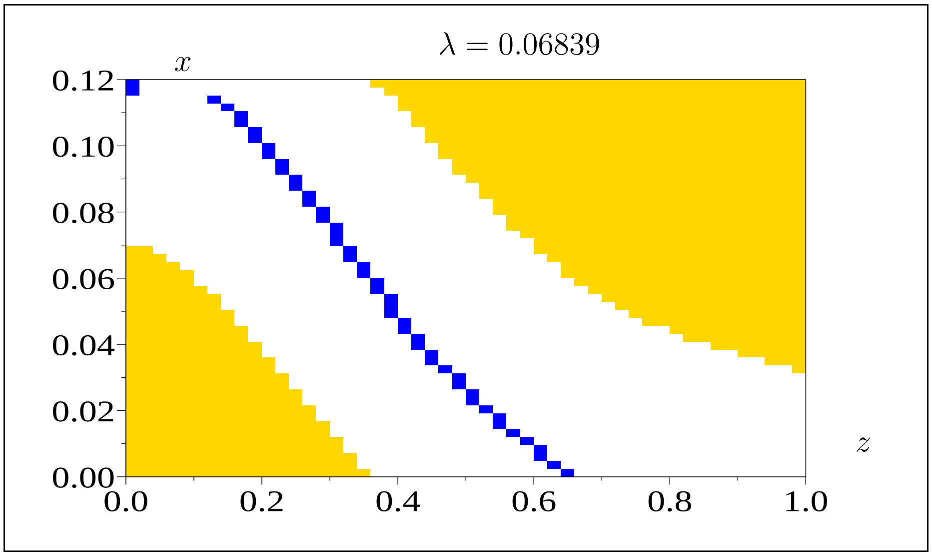

Figure 5: $\alpha = 0.010$, $\theta = 0.10$, $N_x = 50$, $N_z = 50$

θ	$\underline{z}$	$z^\star$	$\overline{z}$	δ_1	λ
0.100	0.082	0.367	0.673	0.0204	0.06839
2.000	–	0.184	0.449	0.0204	0.06343
4.000	–	0.102	0.327	0.0204	0.06201

Table 2: $\underline{z}$,$z^\star$ and $\overline{z}$ for $x = 0.06$ and $N_x = N_z = 50$

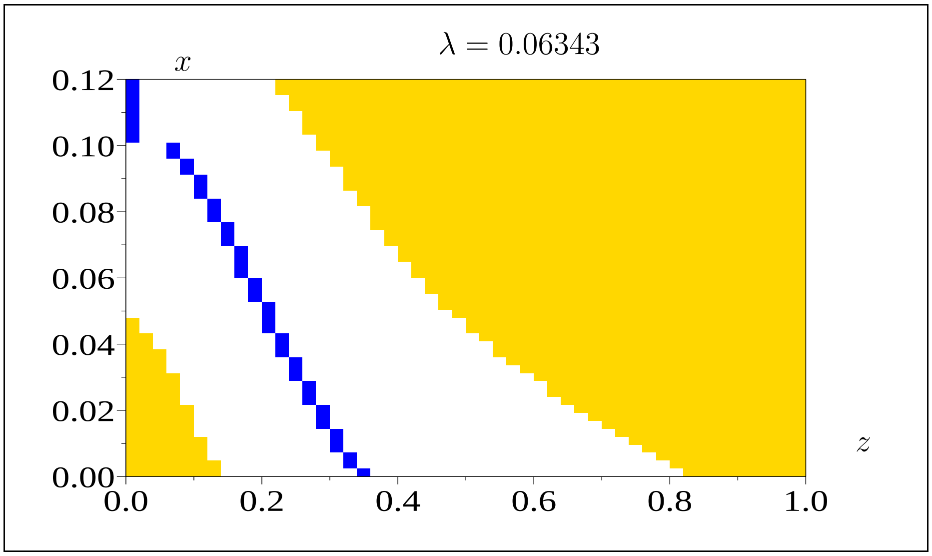

Figure 6: $\alpha = 0.010$, $\theta = 2.00$, $N_x = 50$, $N_z = 50$

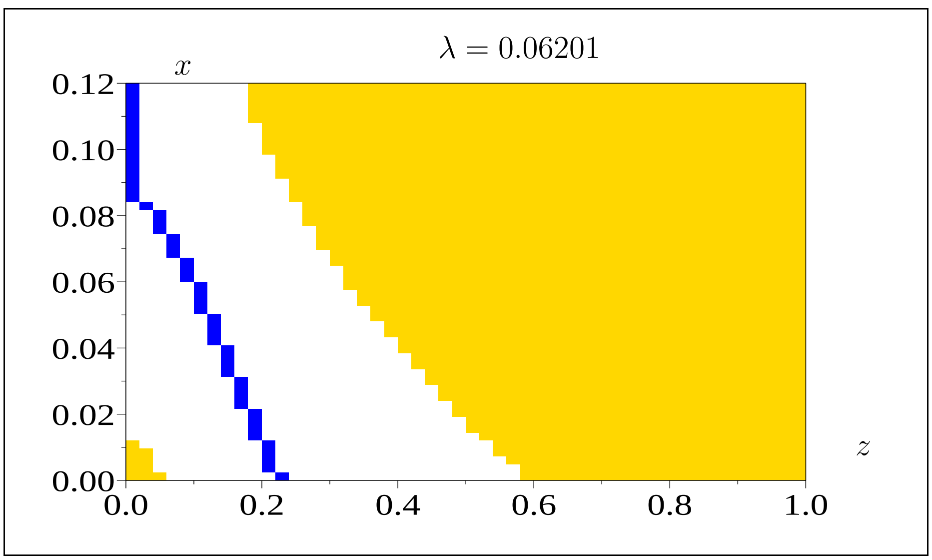

Figure 7: $\alpha = 0.010$, $\theta = 4.00$, $N_x = 50$, $N_z = 50$

θ	$\underline{z}$	$z^\star$	$\overline{z}$	δ_1	λ
0.100	0.069	0.345	0.655	0.0345	0.06840
2.000	–	0.172	0.448	0.0345	0.06340
4.000	–	0.103	0.310	0.0345	0.06199

Table 3: $\underline{z}$,$z^\star$ and $\overline{z}$ for $x = 0.06$ and $N_x = N_z = 30$

50). As expected, as the transaction cost α increases, the no-transaction region increases; see Figures 8, 9, and 10 and Tables 4 and 5.

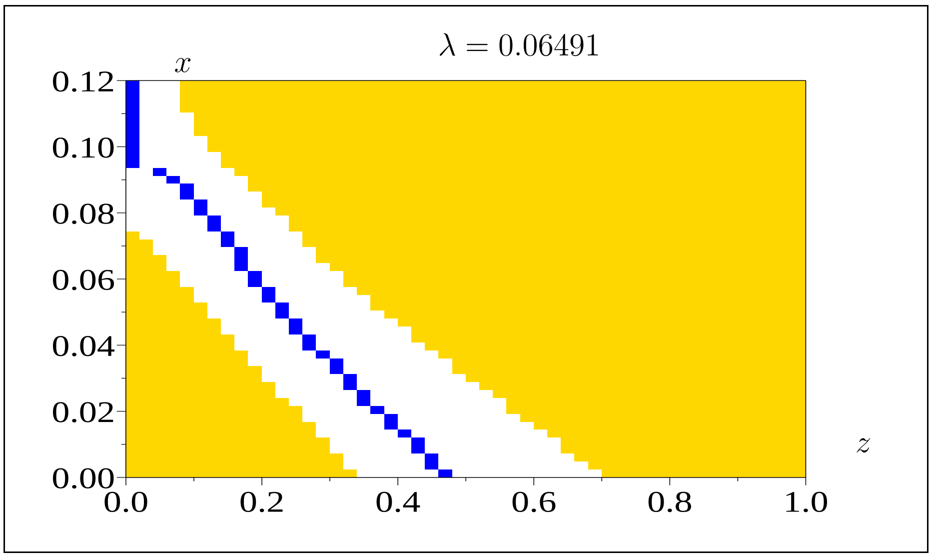

Figure 8: $\alpha = 0.001$, $\theta = 2.00$, $N_x = 50$, $N_z = 50$

Figure 11 displays the evolution of the computed value λ_k with respect to the iteration index k of the algorithm for the six examples given in Figures 5-10. We see that only a few iterations is needed for the policy iteration algorithm to converge.

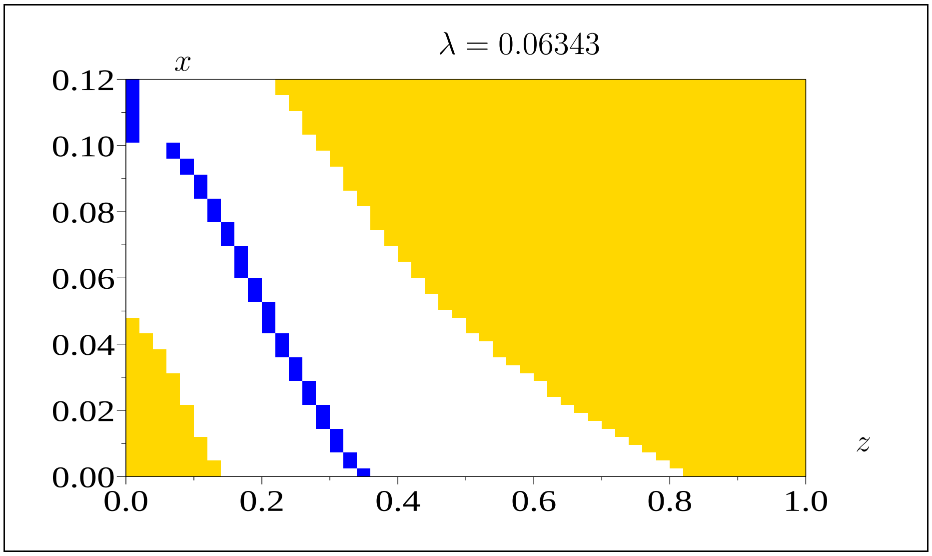

Figure 9: $\alpha = 0.010$, $\theta = 2.00$, $N_x = 50$, $N_z = 50$

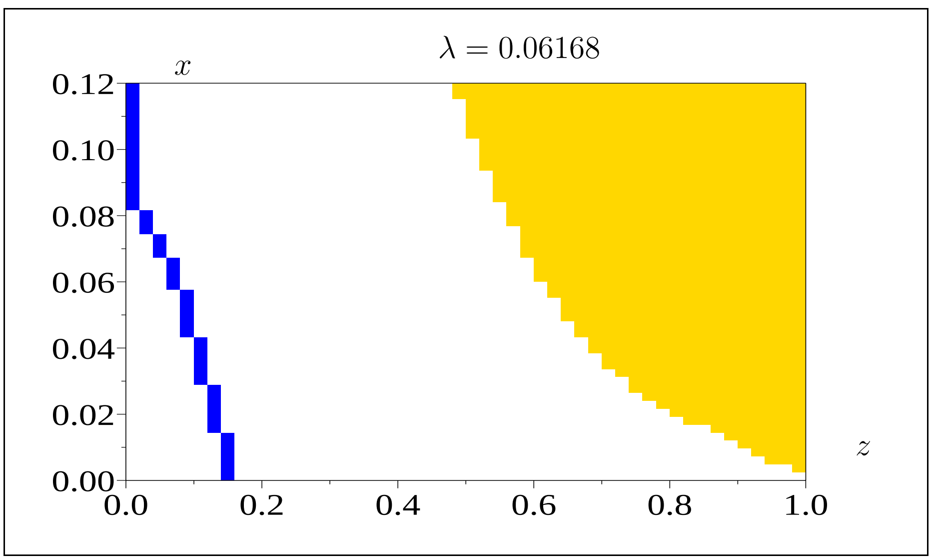

Figure 10: $\alpha = 0.100$, $\theta = 2.00$, $N_x = 50$, $N_z = 50$

α	$\underline{z}$	$z^\star$	$\overline{z}$	δ_1	λ
0.001	0.061	0.184	0.327	0.0204	0.06491
0.010	–	0.184	0.449	0.0204	0.06343
0.100	–	0.061	0.633	0.0204	0.06168

Table 4: $\underline{z}$,$z^\star$ and $\overline{z}$ for $x = 0.06$ and $N_x = N_z = 50$

α	$\underline{z}$	$z^\star$	$\overline{z}$	δ_1	λ
0.001	0.069	0.172	0.310	0.0345	0.06493
0.010	–	0.172	0.448	0.0345	0.06340
0.100	–	0.069	0.621	0.0345	0.06165

Table 5: $\underline{z}$,$z^\star$ and $\overline{z}$ for $x = 0.06$ and $N_x = N_z = 30$

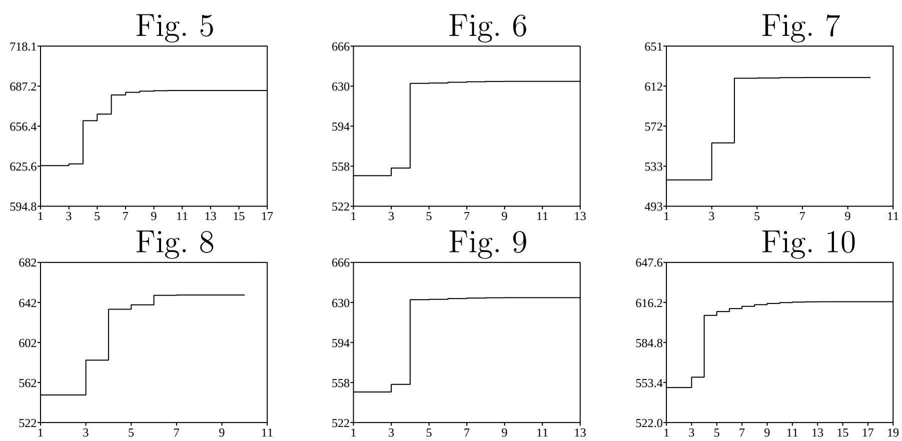

Figure 11: Evolution of λ_k with respect to the iteration index k

7 Concluding Remarks: a Vanishing Transaction Cost

Not surprisingly, taking $\alpha = 0$ in our RSQVI (3.2) we reproduce the no-transaction cost results of Bielecki and Pliska [3], [5]. To see this, with $\alpha = 0$ we get $\mathcal{M}\psi(z,x) = \sup_{z' \in [0,1]} \Psi(z',x) \leq \Psi(z,x)$ for all z. Consequently, equation (3.2) becomes

$$\begin{cases} \Psi(z,x) \equiv \psi(x) \\ b_2\psi'(x) + d_{22}\psi''(x) - \frac{\theta}{4}(\lambda_1^2 + \lambda_2^2)\psi'(x)^2 - \lambda + \bar{f}(z,x) \leq 0 \quad \text{in } \mathcal{D}, \end{cases} \tag{7.15}$$

It can be proven that the solution (ψ, λ) of the control problem is the minimal solution of Eq. (7.15). Consequently, (ψ, λ) must satisfy:

$$\max_{z \in [0,1]} \{b_2\psi'(x) + d_{22}\psi''(x) - \frac{\theta}{4}\lambda_1^2\psi'(x)^2 - \frac{\theta}{4}\lambda_2^2\psi'(x)^2 - \lambda + \bar{f}(z,x)\} = 0.$$

This suggests that the optimal investment policy must satisfy:

$$z^*(x) = \frac{a_1 - a_0 + (A_1 - A_0)x - \theta/2\lambda_1\sigma_1\psi'(x)}{\sigma_1^2(\theta/2 + 1)}.$$

Substituting this in the preceding equation, it is apparent that the function ψ satisfies:

$$b_2\psi'(x) + d_{22}\psi''(x) - \frac{\theta}{4}\lambda_1^2\psi'(x)^2 - \frac{\theta}{4}\lambda_2^2\psi'(x)^2 - \lambda + \bar{f}(z^*(x),x) = 0.$$

It turns out that the solution $\psi(x)$, defined within an additive constant, is a quadratic function of x, so we shall set

$$\psi'(x) := \mu x + \nu$$

for scalar parameters μ and ν. Substituting this in the preceding equation and then doing some algebra, one obtains the final solution, namely,

$$\mu = \frac{Q - B - \sqrt{(Q-B)^2 - \frac{P\,(A_1 - A_0)^2}{\xi}}}{P}$$

$$\nu = \frac{-\mu\, b + \mu\,\theta\,\lambda_1\,\sigma_1\,(a_1 - a_0)/(2\,) - (A_1 - A_0)\,(a_1 - a_0)/\xi - A_0}{\mu\, P + B - Q}$$

and

$$\lambda = \nu\, b + d_{22}\, \mu - \frac{\theta\, d_{22}\, \nu^2}{2} + a_0 + \frac{((a_1 - a_0) - \theta\, \lambda_1\, \sigma_1\, \nu/2)^2}{2\, \xi}$$

where

$$P := -\frac{\theta}{2}(\lambda_1^2 + \lambda_2^2) + \frac{\theta^2}{4}(\frac{\lambda_1^2}{\theta/2 + 1}), \quad \xi = {\sigma_1}^2\, (\frac{\theta}{2} + 1)$$

and

$$Q := \frac{(A_1 - A_0)\theta\lambda_1}{2\sigma_1(\theta/2 + 1)}.$$

The function ψ is obviously $\mathcal{C}^2$. We can thus check by a verification theorem that λ is indeed the optimal performance and ψ is the potential function of our control problem.

We see numerically that the optimal policy converges when α goes to zero to the solution of the no-transaction cost problem. Figure 12 displays the optimal policy for $\alpha = 10^{-8}, \theta = 2$ and the same other values of the parameters.

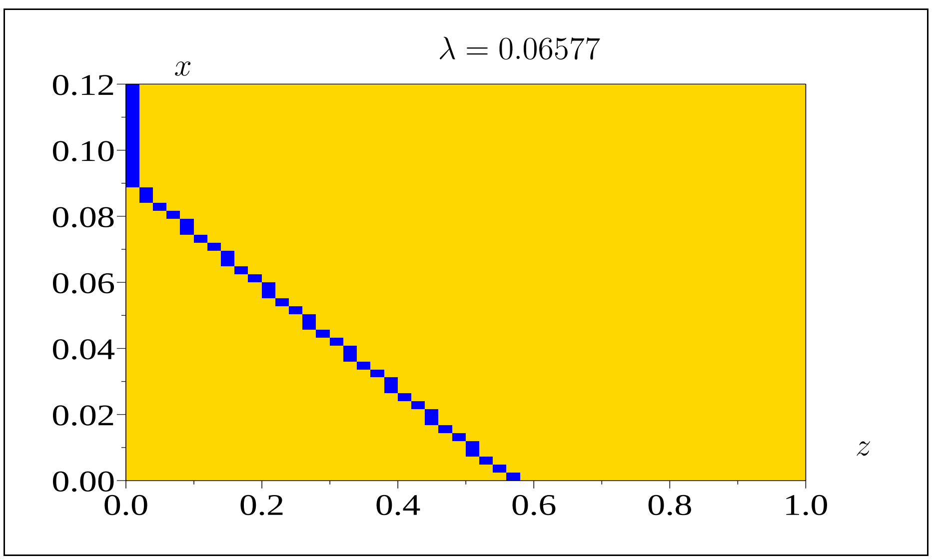

Figure 12: Optimal investment strategy for vanishing transaction cost $\alpha = 1.e - 8$, $\theta = 2$, $N_x = 50$, $N_z = 50$

We can identify $\psi(x) = 3.07x^2 + 0.25x - 0.029$ as a quadratic function of x which leads to $z^*(x)$ as an affine function of x, as can be seen on figure 12.

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