An integral representation theorem of g-expectations

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Abstract

There are two classes of nonlinear expectations, one is the Choquet expectation given by Choquet (1955), the other is the Peng's g-expectation given by Peng (1997) via backward differential equations (BSDE). Recently, Peng raised the following question: can a g-expectation be represented by a Choquet expectation? In this paper, we provide a necessary and sufficient condition on g-expectations under which Peng's g-expectation can be represented by a Choquet expectation for some random variables (Markov processes). It is well known that Choquet expectation and g-expectation (also BSDE) have been used extensively in the pricing of options in finance and insurance. Our result also addresses the following open question: given a BSDE (g-expectation), is there a Choquet expectation operator such that both BSDE pricing and Choquet pricing coincide for all European options? Furthermore, the famous Feynman-Kac formula shows that the solutions of a class of (linear) partial differential equations (PDE) can be represented by (linear) mathematical expectations. As an application of our result, we obtain a necessary and sufficient condition under which the solutions of a class of nonlinear PDE can be represented by nonlinear Choquet expectations.

Keywords: g-expectation, Choquet expectation, integral representation theorem.

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1 Introduction

The concept of expectation is crucial in probability theory. Given a probability measure P, the expectation $E\xi$ of a random variable ξ under P can be calculated by

$$E\xi = \int_{-\infty}^{0} [P(\xi \geq t) - 1]dt + \int_{0}^{\infty} P(\xi \geq t)dt.$$

One of the important properties of mathematical expectations is its linearity, that is

$$E(\xi + \eta) = E\xi + E\eta$$

when both sides are finite. Choquet [Ch] introduced a definition of nonlinear expectations of ξ under a non-additive probability measure V as

$$C(\xi) := \int_{-\infty}^{0} [V(\xi \geq t) - 1]dt + \int_{0}^{\infty} V(\xi \geq t)dt.$$

This expectation, usually called the Choquet expectation, has no longer the linearity property because of the non-additivity of V in the sense that $V(A+B) \not\equiv V(A) + V(B)$, even if $A \cap B = \emptyset$.

The Choquet expectation was originally motivated by potential theory in physics, but it has found many applications in various fields. In particular, owing to Schmeidler's work [Sc2] the Choquet expectation has become an important tool in describing individuals' behavior under uncertainty in economics.

Peng [P1, P2] introduced the notion of g-expectation via a class of nonlinear backward stochastic differential equations (BSDEs). He showed that g-expectations preserve many of the basic properties of mathematical expectations except linearity (see [P1, P2, CP] for details). A natural question is to ask for which class of random variables does a Peng's g-expectation can be represented by a Choquet expectation? In this paper, we discuss this issue and provide a necessary and sufficient condition under which a g-expectation can be represented by a Choquet expectation for some random variables.

The paper is organized as follows: In Section 2, we recall briefly some notions of BSDEs and related g-expectations. In Section 3, we give a necessary and sufficient condition under which a g-expectation can be represented as a Choquet expectation. In Section 4 we consider an extension to multiple dimensions. As an application, in Section 5, we consider a relation between nonlinear PDE and Choquet expectations, which implies that the famous Feynman-Kac formula can be extended to nonlinear case. For the reader's convenience, Section 6 is an appendix which contains several lemmas which are used in this paper.

2 BSDEs and g-expectations

Pardoux and Peng [PP1] showed an existence and uniqueness theorem for nonlinear BSDEs. Furthermore, Peng [P1] introduced the notion of g-expectation via this kind of BSDEs. This section gives a brief review of BSDEs and related g-expectations.

Fix $T \in [0, \infty)$, let $(W_t)_{0\le t\le T}$ be a d-dimensional standard Brownian motion defined on a completed probability space $(\Omega, \mathcal{F}, P)$. Suppose $\{\mathcal{F}_t\}_{0\le t\le T}$ is the natural filtration generated by $(W_t)_{0\le t\le T}$, i.e.

$$\mathcal{F}_t = \sigma\{W_s;\ s \le t\}.$$

We also assume $\mathcal{F}_T = \mathcal{F}$.

Set

$$L^2(0,T) := \left\{V;\ V \text{ is a } (\mathcal{F}_t)\text{-adapted process with } [E\int_0^T |V(s)|^2 ds]^{\frac{1}{2}} < \infty\right\};$$
$$L^2(\Omega,\mathcal{F},P) := \{\xi; \xi \text{ is a } \mathcal{F}\text{-measurable random variable with } E|\xi|^2 < \infty\}.$$

Let g be a function from $\mathbf{R} \times \mathbf{R}^d \times [0, T]$ into $\mathbf{R}$ such that

(H1) For any $(y, z) \in \mathbf{R} \times \mathbf{R}^d$, $g(y, z, t)$ is continuous in t and $\int_0^T |g(y, z, s)|^2 ds < \infty$.

(H2) **Lipschitz condition:** There exists a constant $\mu > 0$ such that $|g(y_1, z_1, t) - g(y_2, z_2, t)| \le \mu(|y_1 - y_2| + |z_1 - z_2|]), \forall (y_i, z_i) \in \mathbf{R} \times \mathbf{R}^d,\ i = 1, 2$.

(H3) $g(y, 0, t) = 0, \forall (y, t) \in \mathbf{R} \times [0, T]$.

By Pardoux and Peng's Theorem [PP1], for any $\xi \in L^2(\Omega, \mathcal{F}, P)$, there exists a unique pair of adapted processes $(y, z) \in L^2(0, T) \times L^2(0, T)$ satisfying the BSDE

$$y_t = \xi + \int_t^T g(y_s, z_s, s)ds - \int_t^T z_s dW_s\ ,\ 0 \le t \le T\ . \tag{1}$$

When $d > 1$ we interpret $z_s dW_s$ as a matrix product. Furthermore, Peng [P1] introduced the notion of g-expectation via the BSDE (1).

Definition 1 *Suppose that g satisfies (H1), (H2) and (H3). For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, let (y_t, z_t) be the solution of BSDE (1).*

(1) *We call $\mathcal{E}_g[\xi]$ defined by*

$$\mathcal{E}_g[\xi] := y_0$$

the g-expectation of the random variable ξ.

(2) *We call $\mathcal{E}_g[\xi|\mathcal{F}_t]$ defined by*

$$\mathcal{E}_g[\xi] := y_t, \quad t \in [0, T)$$

the conditional g-expectation of the random variable ξ.

(3) *Let 1_A be the indicator function of the set A. For any $A \in \mathcal{F}$, $P_g(A)$ is called the g-probability of the event A, where*

$$P_g(A) := \mathcal{E}_g[1_A] \; .$$

Remark 1 *If $g = 0$, then $\mathcal{E}_g[\xi|\mathcal{F}_t] = E[\xi|\mathcal{F}_t]$, $\mathcal{E}_g[\xi] = E\xi$ and $P_g(A) = P(A)$. See [P1] for details.*

Peng [P1] showed that g-expectations and conditional g-expectations preserve many of the basic properties of mathematical expectations except for linearity. Some of these important properties are given below.

Property 1 *(i) $\mathcal{E}_g[c] = c, \forall c \in \boldsymbol{R}$;*

(ii) $\mathcal{E}_g[\xi] = \mathcal{E}_g[\mathcal{E}_g[\xi|\mathcal{F}_t]]$;

(iii) $\mathcal{E}_g[\mathcal{E}_g[\xi|\mathcal{F}_t]|\mathcal{F}_r] = \mathcal{E}_g[\xi|\mathcal{F}_{r\wedge t}]$;

(iv) $\mathcal{E}_g[\xi + \eta|\mathcal{F}_t] = \xi + \mathcal{E}_g[\eta|\mathcal{F}_t]$, if ξ is $\mathcal{F}_t$-measurable;

(v) If g is deterministic and ξ is independent of $\mathcal{F}_t$, then $\mathcal{E}_g[\xi|\mathcal{F}_t] = \mathcal{E}_g[\xi]$.

(iv) $\eta := \mathcal{E}_g[\xi|\mathcal{F}_t]$ is the unique $\mathcal{F}_t$-measurable random variable satisfying the following equation

$$\mathcal{E}_g[\xi 1_A] = \mathcal{E}_g[\eta 1_A] \; , \text{ for all } A \in \mathcal{F}_t \; .$$

We now recall briefly the notions of capacity and Choquet expectation.

Definition 2 *(1) A real valued set function $V : \mathcal{F} \to [0, 1]$ is called a capacity if*

(i) $V(\emptyset) = 0, \quad V(\Omega) = 1$

(ii) $V(A) \leq V(B)$ for any $A \subset B$ where $A, B \in \mathcal{F}$.

(2) **Choquet expectation***: Let V be a capacity. For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, the functional $C(\xi)$ defined by*

$$C(\xi) := \int_{-\infty}^{0} (V(\xi \geq t) - 1) \, dt + \int_{0}^{\infty} V(\xi \geq t) dt$$

is called the Choquet expectation of ξ with respect to V.

(3) Random variables ξ and η are called comonotonic if

$$[\xi(\omega) - \xi(\omega')][\eta(\omega) - \eta(\omega')] \geq 0 \; , \; \forall \omega, \omega' \in \Omega \; .$$

(4) A real function F defined on $L^2(\Omega, \mathcal{F}, P)$ is called comonotonic additive if for all comonotonic ξ and η

$$F(\xi + \eta) = F(\xi) + F(\eta) \ .$$

The results of Dellacherie [De] and Schmeidler [Sc1] can be rewritten as the following Lemma.

Lemma 1 *Suppose F is a real continuous functional on $L^2(\Omega, \mathcal{F}, P)$. Then F can be represented by a Choquet expectation for all random variables in $L^2(\Omega, \mathcal{F}, P)$ if and only if F is comonotonic additive.*

3 Main Result

In order to simplify the notation, in this section, we shall discuss our main result under the assumption that Brownian motion $\{W_t\}$ is 1-dimensional, that is $d = 1$. The case of a multidimensional Brownian motion will be discussed in Section 4.

Let $b(t, x)$ and $\sigma(t, x)$ be continuous in t and Lipschitz continuous in x, then the following SDE has a unique solution $\{X_t^{a,\sigma}\}$ (shortly $\{X_t\}$) which depends on coefficients a, σ:

$$\begin{cases} dX_s &= b(s, X_s)ds + \sigma(s, X_s)dW_s, \quad 0 \le s \le T, \\ X_0 &= x \in \mathbf{R} \end{cases} \tag{2}$$

Define H as

$$H := \{X_T^{a,\sigma} \in L^2(\Omega, \mathcal{F}, P) : b, \sigma \text{ are continuous in } t \text{ and Lipschitz continuous in } x\} \ .$$

Let $\Phi(x)$ be an increasing function such that $\Phi(X_T) \in L^2(\Omega, \mathcal{F}, P)$.

We consider the question: under which condition on g is there a Choquet expectation operator $C(\cdot)$ such that

$$\mathcal{E}_g[\Phi(X_T)] = C[\Phi(X_T)], \forall X_T \in H?$$

In this case, we say that g-expectation can be represented by a Choquet expectation on H. This question is answered in Theorem 1.

Remark 2 *The question can be explained in finance as follows:*

Suppose $\{X_t\}$ is the price of a stock. Let $\Phi(X_T) = (X_T - k)^+$ be the value of a European option at exercise time T. Then by the viewpoint of El Karoui, Peng and Quenz [KPQ], $\mathcal{E}_g[\Phi(X_T)]$ is the price of an European option with payoff $\Phi(X_T)$. If g is linear, then the corresponding market model is a complete market model and $\mathcal{E}_g[\Phi(X_T)]$ is the classical mathematical expectation, that is the Black-Scholes formula. If g is nonlinear, *then the corresponding market is incomplete. Our question is the following: for which class of incomplete market models can the price of an European option be obtained by a Choquet expectation? Similarly to the Black-Scholes formula which shows that the price of a contingent claim in a complete market is a mathematical expectation, our result shows that the price of a contingent claim in some incomplete market model is a Choquet expectation.*

The main result in this section is

Theorem 1 *Suppose that g satisfies (H1), (H2) and (H3). $\mathcal{E}_g[\xi]$ can be represented as a Choquet expectation for any $\xi \in H$ if and only if there exist two continuous functions $\{\alpha(t)\}$, $\{\beta(t)\}$ such that g does not depend on y and has the form*

$$g(y, z, t) = \alpha(t)|z| + \beta(t)z. \tag{3}$$

To investigate a necessary condition, we need the following lemma (see [BCMP]):

Lemma 2 *Suppose that $\{X_t\}$ is of the form*

$$X_t = x + \int_0^t \sigma_s dW_s \ , \quad 0 \le t \le T$$

where $\{\sigma_t\}$ is a continuous bounded process. Consider the BSDE (1) with a given function g that satisfies conditions (H1), (H2) and (H3), and for which $\xi = X_T$. Then

(i) $\mathcal{E}_g[X_\tau|\mathcal{F}_t] \to X_t, \tau \to t$

(ii) $\lim_{\tau\to t^+} \frac{\mathcal{E}_g[X_\tau|\mathcal{F}_t]-E[X_\tau|\mathcal{F}_t]}{\tau-t} = g(X_t, \sigma_t, t)$

where the limits are in the sense of $L^2(\Omega, \mathcal{F}, P)$.

The proof of necessary condition in Theorem 1:

If for any $\xi \in H$, $\mathcal{E}_g[\xi]$ can be represented by a Choquet expectation, by Dellacherie's Theorem (Lemma 1), then $\mathcal{E}_g[\cdot]$ is comonotonic additive, that is whenever ξ and η are comonotonic then

$$\mathcal{E}_g[\xi + \eta] = \mathcal{E}_g[\xi] + \mathcal{E}_g[\eta] \ . \tag{4}$$

Choose constants (y_1, z_1, t), $(y_2, z_2, t) \in \mathbf{R}^2 \times [0, T]$ with $z_1 z_2 \ge 0$. For any $\tau \in [t, T]$ the random variables $\xi_\tau = y_1 + z_1(W_\tau - W_t)$ and $\eta_\tau = y_2 + z_2(W_\tau - W_t)$ are comonotonic and independent of $\mathcal{F}_t$.

Recall in (1) that $g : \mathbf{R} \times \mathbf{R} \times [0, T]$ is a non-random function, and also that y_i and z_i $(i = 1, 2)$ are constants. Applying Property 1(v),

$$\mathcal{E}_g[\xi_\tau|\mathcal{F}_t] = \mathcal{E}_g[\xi_\tau] \ , \quad \mathcal{E}_g[\eta_\tau|\mathcal{F}_t] = \mathcal{E}_g[\eta_\tau] \ , \quad \mathcal{E}_g[\xi_\tau + \eta_\tau|\mathcal{F}_t] = \mathcal{E}_g[\xi_\tau + \eta_\tau].$$

This together with (4) implies

$$\frac{\mathcal{E}_g[\xi_\tau+\eta_\tau|\mathcal{F}_t]-E[\xi_\tau+\eta_\tau|\mathcal{F}_t]}{\tau-t}=\frac{\mathcal{E}_g[\xi_\tau|\mathcal{F}_t]-E[\xi_\tau|\mathcal{F}_t]}{\tau-t}+\frac{\mathcal{E}_g[\eta_\tau|\mathcal{F}_t]-E[\eta_\tau|\mathcal{F}_t]}{\tau-t} \ . \tag{5}$$

Taking limits as $\tau\to t$ on both sides of (5), by Lemma 2(ii), we obtain

$$g(y_1+y_2,z_1+z_2,t)=g(y_1,z_1,t)+g(y_2,z_2,t)\ ,\ \forall z_1z_2\geq 0\ ,\quad y_1,y_2\in\mathbf{R}\ , \tag{6}$$

which then implies that g is linear with respect to y in $\mathbf{R}$ and z in $\mathbf{R}_+$ (or $\mathbf{R}_-$). Applying (6), it is easy to check that for any $a>0$, $g(0,a,t)=ag(0,1,t)$.

Thus, for any $(y,z,t)\in\mathbf{R}^2\times[0,T]$, by assumption (H3), that is $g(y,0,t)=0$, and by the fact that $z1_{[z\leq 0]}\cdot z1_{[z\geq 0]}=0$, then (6) implies

$$\begin{aligned}
g(y,z,t) &= g\left(y+0,z1_{[z\geq 0]}+z1_{[z\leq 0]},t\right)\\
&= g\left(y,z1_{[z\geq 0]},t\right)\ +\ g\left(0,z1_{[z\leq 0]},t\right)\\
&= g\left(y+0,0+z1_{[z\geq 0]},t\right)\ +\ g\left(0,-(-z)1_{[z\leq 0]},t\right)\\
&= g(y,0,t)\ +\ g\left(0,z1_{[z\geq 0]},t\right)\ +\ g\left(0,-(-z)1_{[z\leq 0]},t\right)\\
&= g(0,1,t)z1_{[z\geq 0]}\ -\ g(0,-1,t)z1_{[z\leq 0]}\\
&= g(0,1,t)\tfrac{|z|+z}{2}\ +\ g(0,-1,t)\tfrac{|z|-z}{2}\\
&= \tfrac{g(0,1,t)+g(0,-1,t)}{2}|z|\ +\ \tfrac{g(0,1,t)-g(0,-1,t)}{2}z\ .
\end{aligned}$$

Set $\alpha(t):=\frac{g(0,1,t)+g(0,-1,t)}{2}$ and $\beta(t):=\frac{g(0,1,t)-g(0,-1,t)}{2}$ to complete the proof of the necessary condition. $\square$

Before proving the sufficient condition part of Theorem 1, we state and prove Lemmas 3, 4 and 5.

Consider the SDE (2) and its solution $\{X_t\}$. Then $X_T\in H$. We will also be interested in random variables $\xi=\Phi(X_T)$ for various functions Φ.

Lemma 3 *Suppose that $\{X_s\}$ is the solution of the SDE (2). Suppose also that Φ and Ψ are two increasing functions such that $\Phi(X_T)$, $\Psi(X_T)\in L^2(\Omega,\mathcal{F},P)$. Let (y_t^Φ,z_t^Φ) and (y_t^Ψ,z_t^Ψ) be the solutions of the BSDE (1) with terminal values $\xi=\Phi(X_T)$ and $\xi=\Psi(X_T)$ respectively. If b and σ in (2), the function g in (1), Φ and Ψ are assumed to be C^3, then*

$$z_t^\Phi z_t^\Psi\geq 0\ ,\quad a.e.\ t\in[0,T]\ .$$

Proof. Let $\{X_s^{t,x}\}$ be the solution of the SDE:

$$\begin{cases} dX_s^{t,x}=b(s,X_s^{t,x})ds+\sigma(s,X_s^{t,x})dW_s\ ,\\ X_t=x\ ,\quad s\in[t,T]\ .\end{cases}$$

Obviously, the solution $\{X_s^{0,x}\}$ of the above SDE with $t=0$ is indeed the solution $\{X_s\}$ of SDE (2). Let $(y_s^{t,x,\Phi},z_s^{t,x,\Phi})$ and $(y_s^{t,x,\Psi},z_s^{t,x,\Psi})$ be the solutions of the BSDE (1) corresponding to terminal values $\xi=\Phi(X_T^{t,x})$ and $\xi=\Psi(X_T^{t,x})$, respectively, then

(i) Since $X_T = X_T^{0,x}$, thus $z_s^\Phi = z_s^{0,x,\Phi}$, $z_s^\Psi = z_s^{0,x,\Psi}$; $s \in [0,T]$.

(ii) Since Φ, Ψ are increasing, applying the Comparison Theorem for SDEs and BSDEs, we get that $y_t^{t,x,\Phi}$ and $y_t^{t,x,\Psi}$ are increasing in x.

Let $u(t,x) := y_t^{t,x,\Phi}$ and $v(t,x) := y_t^{t,x,\Psi}$. By Lemma 7(ii) in Section 6

$$\begin{cases} z_s^{t,x,\Phi} = \sigma(s, X_s^{t,x})\partial_x u(s, X_s^{t,x}) \ , \text{a.e. } s \in [t,T] \ , \\ z_s^{t,x,\Psi} = \sigma(s, X_s^{t,x})\partial_x v(s, X_s^{t,x}) \ , \text{a.e. } s \in [t,T] \ . \end{cases} \tag{7}$$

This together with (ii) above implies

$$z_s^{t,x,\Phi} z_s^{t,x,\Psi} = \sigma^2(s, X_s^{t,x})\partial_x u(s, X_s^{t,x})\partial_x v(s, X_s^{t,x}) \geq 0 \ , \ \text{a.e. } \ s \in [t,T] \ .$$

Letting $t = 0$ and applying (i), it then follows that

$$\begin{aligned} z_s^\Phi z_s^\Psi &= z_s^{0,x,\Phi} z_s^{0,x,\Psi} \\ &= \sigma(s, X_s)\partial_x u(s, X_s)\sigma(s, X_s)\partial_x v(s, X_s) \\ &= \sigma^2(s, X_s)\partial_x u(s, X_s)\partial_x v(s, X_s) \geq 0 \ , \quad \text{a.e.} \quad s \in [0,T]. \end{aligned} \tag{8}$$

The proof is complete. $\square$

We next consider the case where Φ and Ψ are indicator functions.

Suppose that $\{X_t\}$ is the solution of SDE (2). For given constants $\alpha, c \in R$ and $\alpha \leq c$, set $B = \{X_T \geq \alpha\}$ and $C = \{X_T \geq c\}$ and let (y^B, z^B) and (y^C, z^C) be the solutions of the BSDE (1) corresponding to terminal values $\xi = 1_B$ and $\xi = 1_C$ respectively. Clearly $C \subset B$.

Lemma 4 *Suppose that the functions b and σ in (2), and the function g in (1) satisfy the assumptions of Lemma 3. Then*

$$z_t^C z_t^B \geq 0, \qquad a.e. \ t \in [0,T] \ . \tag{9}$$

Proof. Indeed for the indicator functions $1_{(x \geq \alpha)}$ and $1_{(x \geq c)}$, we can construct a sequence of C^3-increasing functions $\Phi_n(\cdot, \alpha), \Phi_n(\cdot, c)$ such that

$$\Phi_n(x, \alpha) \to 1_{(x \geq \alpha)}, \quad \Phi_n(x, c) \to 1_{(x \geq c)} \text{ as } n \to \infty.$$

For example, for any $n = 1, 2, \cdots$ define

$$\Phi_n(x, \alpha) := e^{-nd^{(\alpha)}(x)} \ , \quad \Phi_n(x, c) := e^{-nd^{(c)}(x)}$$

where

$$d^{(\alpha)}(x) = \begin{cases} (\alpha - x)^3 & \text{if } x < \alpha \\ 0 & \text{if } x \geq \alpha \end{cases} \ ; \ d^{(c)}(x) = \begin{cases} (c - x)^3 & \text{if } x < c \\ 0 & \text{if } x \geq c \ . \end{cases}$$

Let $(y^{n,\alpha}, z^{n,\alpha})$ and $(y^{n,c}, z^{n,c})$ be the solutions of the BSDE (1) corresponding to $\xi = \Phi_n(X_T, \alpha)$ and $\xi = \Phi_n(X_T, c)$ respectively. Applying Lemma 3, we have

$$z_s^{n,\alpha} z_s^{n,c} \geq 0 \ , \quad \text{a.e. } s \in [0,T] \ .$$

Note that $\Phi_n(X_T, \alpha) \to 1_B$ and $\Phi_n(X_T, c) \to 1_C$ as $n \to \infty$ in $L^2(\Omega, \mathcal{F}, P)$. By Lemma 6 in Section 6, then $z^{n,\alpha} \to z^B$ and $z^{n,c} \to z^C$ as $n \to \infty$ in $L^2(0,T)$.
The proof is complete. $\square$

Note that if $\Phi_n(x, \alpha)$ is the function defined above, and Φ is an increasing function, then $\Phi_n(\Phi(x), \alpha)$ is increasing, and we get immediatly

Remark 3 *Suppose Φ is an increasing function, and let $B = \{\Phi(X_T) \geq \alpha\}$, $C = \{\Phi(X_T) \geq c\}$, then Lemma 4 is still true.*

We next consider the case where g is of the form:

$$g(y, z, t) = a(t)|z| \tag{10}$$

where a is a continuous function bounded by μ. In order to signify the dependence on a when g is of the form (10) we rewrite $\mathcal{E}_g[\xi]$, $\mathcal{E}_g[\xi|\mathcal{F}_t]$ and $P_g(\cdot)$ as $\mathcal{E}_a[\xi]$, $\mathcal{E}_a[\xi|\mathcal{F}_t]$ and $P_a(\cdot)$.

Lemma 5 *Suppose that $X_T \in H$ and $k_1 \leq k_2 \leq \cdots k_n$ is a sequence in $\boldsymbol{R}$. Let $A_i := \{\omega : X_T(\omega) \geq k_i\}$, $i = 1, 2, \cdots, n$. Then for any sequence of positive constants $\{b_i\}_{i=1}^n$, we have*

$$\mathcal{E}_a[\sum_{i=1}^n b_i 1_{A_i}] = \sum_{i=1}^n b_i P_a[A_i] \ .$$

Proof. For any $\epsilon > 0$, the function g_ϵ defined by $g_\epsilon(y, z, t) := a(t)\sqrt{|z|^2 + \epsilon}$ is differentiable with respect to z and its derivative is uniformly bounded.

Let $(y_t^{i,\epsilon}, z_t^{i,\epsilon})$ be the solution of the BSDE

$$y_t = 1_{A_i} + \int_t^T a(s)\sqrt{(|z_s|^2 + \epsilon)}ds - \int_t^T z_s dW_s \ , \ i = 1, 2, \cdots, n \ .$$

Applying Lemma 6 of Section 6, then $(y_t^{i,\epsilon}, z_t^{i,\epsilon}) \to (y^i, z^i)$ as $\epsilon \to 0$ in $L^2(0,T) \times L^2(0,T)$, where (y^i, z^i) are the solutions of the BSDE:

$$y_t^i = 1_{A_i} + \int_t^T a(s)|z_s^i|ds - \int_t^T z_s^i dW_s \ , \ i = 1, 2, \cdots, n \ . \tag{11}$$

Applying Lemma 4, we have for any $i, j = 1, 2, \cdots, n$

$$z_t^{i,\epsilon} z_t^{j,\epsilon} \geq 0 \ .$$

Hence,

$$z_t^i z_t^j \geq 0\ ,\quad \text{a.e. } t \in [0,T], \quad i,j = 1,2,\cdots,n$$

which implies

$$|\sum_{i=1}^n z_t^i| = \sum_{i=1}^n |z_t^i| \quad \text{a.e. } t \in [0,T]\ . \tag{12}$$

Multiplying both sides of BSDE (11) by b_i, summing from $i=1$ to n and applying (12) with $b_i \geq 0$, we obtain

$$\sum_{i=1}^n b_i y_t^i = \sum_{i=1}^n b_i 1_{A_i} + \int_t^T a(s)|\sum_{i=1}^n b_i z_s^i| ds - \int_t^T \sum_{i=1}^n b_i z_s^i dW_s\ .$$

Therefore $(\sum_{i=1}^n b_i y_t^i\ , \sum_{i=1}^n b_i z_t^i)$ is the solution of the BSDE

$$y_t = \sum_{i=1}^n b_i 1_{A_i} + \int_t^T a(s)|z_s| ds - \int_t^T z_s dW_s\ .$$

By uniqueness of the solution of the BSDE, we get

$$\mathcal{E}_a[\sum_{i=1}^n b_i 1_{A_i}|\mathcal{F}_t] = \sum_{i=1}^n b_i y_t^i\ ,\ t \in [0,T]\ .$$

By the definition of g-expectation, we have from (11),

$$\sum_{i=1}^n b_i y_t^i = \sum_{i=1}^n b_i \mathcal{E}_a[1_{A_i}|\mathcal{F}_t] = \sum_{i=1}^n b_i P_a[A_i|\mathcal{F}_t]\ ,\ t \in [0,T]\ .$$

Thus

$$\mathcal{E}_a[\sum_{i=1}^n b_i 1_{A_i}|\mathcal{F}_t] = \sum_{i=1}^n b_i P_a[A_i|\mathcal{F}_t]\ ,\ t \in [0,T]\ .$$

In particular, taking $t=0$, we obtain the conclusion of Lemma 5.

□

Remark 4 *Lemma 4 and Lemma 5 can be modified to remove the assumption that g, Φ and Ψ are C^3-functions. The proofs would be modified to construct a sequence of C^3-functions g_n, Φ_n and Ψ_n such that $g_n \to g, \Phi_n \to \Phi$ and $\Psi_n \to \Psi$, as $n \to \infty$.*

We now prove the sufficiency part of Theorem 1, that is if g is of the form (3), then $\mathcal{E}_g[\xi]$ can be represented by a Choquet expectation. Without loss of generality, we can assume β in (3) satisfies $\beta \equiv 0$. This can be seen as follows. If $y_t := \mathcal{E}_g[\xi|\mathcal{F}_t]$ is the solution of the BSDE (1) then

$$\begin{aligned} y_t &= \xi + \int_t^T [\alpha(s)|z_s| + \beta(s)z_s]ds - \int_t^T z_s dW_s \\ &= \xi + \int_t^T \alpha(s)|z_s|ds - \int_t^T z_s d\overline{W}_s \end{aligned}$$

where $\overline{W}_t := W_t - \int_0^t \beta(s)ds$ is a Q-Brownian motion under Q defined by

$$\frac{dQ}{dP} = \exp\{-\frac{1}{2}\int_0^T \beta^2(s)ds + \int_0^T \beta(s)dW_s\} \ .$$

Thus we can consider BSDE (1) with $g(y, z, t) = \alpha(t)|z|$ on the probability space $(\Omega, \mathcal{F}, Q)$.

The proof of the sufficiency of Theorem 1 now follows from Theorem 2 which shows that a Choquet representation holds in the special case where g is of the form (10).

$\square$.

Theorem 2 *Assume g is of the form (10), then, for any $\xi \in H$, $\mathcal{E}_a[\xi]$ can be represented as a Choquet expectation.*

Proof. The proof is divided into two steps.
Step 1: Assume that ξ is strictly bounded by N, that is $|\xi| < N$.

First, assume $\xi \geq 0$. Set

$$\xi_-^{(n)} := \sum_{i=0}^{2^n-1} \frac{iN}{2^n} 1_{(\frac{iN}{2^n} \leq \xi < \frac{(i+1)N}{2^n})} \ ; \quad \xi_+^{(n)} := \sum_{i=0}^{2^n-1} \frac{(i+1)N}{2^n} 1_{(\frac{iN}{2^n} \leq \xi < \frac{(i+1)N}{2^n})} \ .$$

Then

(i) : $0 \leq \xi_-^{(n)} \leq \xi \leq \xi_+^{(n)}$

(ii) : $\xi_-^{(n)} \to \xi, \quad \xi_+^{(n)} \to \xi$ as $n \to \infty$ in $L^2(\Omega, \mathcal{F}, P)$.

Thus (i) and the comparison theorem of BSDEs give

$$\mathcal{E}_a[\xi_-^{(n)}] \leq \mathcal{E}_a[\xi] \leq \mathcal{E}_a[\xi_+^{(n)}],$$

and from (ii) we have

$$\lim_{n\to\infty} \mathcal{E}_a[\xi_-^{(n)}] = \lim_{n\to\infty} \mathcal{E}_a[\xi_+^{(n)}] = \mathcal{E}_a[\xi] \ .$$

Note that it is easy to rewrite $\xi_-^{(n)}$ and $\xi_+^{(n)}$ as

$$\xi_-^{(n)} = \frac{N}{2^n}\sum_{i=1}^{2^n-1} 1_{(\xi \geq \frac{iN}{2^n})} \ , \quad \xi_+^{(n)} = \frac{N}{2^n}\sum_{i=1}^{2^n} 1_{(\xi \geq \frac{(i-1)N}{2^n})} \ .$$

Applying Lemma 5, and recalling $P_a(A) = \mathcal{E}_a[1_A]$, we obtain

$$\mathcal{E}_a[\xi_-^{(n)}] = \sum_{i=1}^{2^n-1} \frac{N}{2^n} P_a(\xi \geq \frac{iN}{2^n}) \ , \quad \mathcal{E}_a[\xi_+^{(n)}] = \sum_{i=1}^{2^n} \frac{N}{2^n} P_a(\xi \geq \frac{(i-1)N}{2^n}) \ .$$

Since

$$\sum_{i=1}^{2^n-1} \frac{N}{2^n} P_a(\xi \geq \frac{iN}{2^n}) \leq \int_0^N P_a(\xi \geq t)dt \leq \sum_{i=1}^{2^n} \frac{N}{2^n} P_a(\xi \geq \frac{(i-1)N}{2^n})$$

hence

$$\mathcal{E}_a[\xi_-^{(n)}] \leq \int_0^N P_a(\xi \geq t)dt \leq \mathcal{E}_a[\xi_+^{(n)}] \ .$$

Letting $n \to \infty$

$$\mathcal{E}_a[\xi] = \int_0^N P_a(\xi \geq t)dt \ . \tag{13}$$

Thus $\mathcal{E}_a[\xi]$ can be represented by the Choquet expectation.

Secondly if ξ is not positive, let $\overline{\xi} = \xi + N$, then $0 \leq \overline{\xi} < 2N$. Applying (13) gives

$$\mathcal{E}_a[\overline{\xi}] = \int_0^{2N} P_a(\overline{\xi} \geq t)dt = \int_{-N}^N P_a(\xi \geq t)dt \ .$$

By Property 1(iv), we have

$$\mathcal{E}_a[\overline{\xi}] = \mathcal{E}_a[\xi + N] = \mathcal{E}_a[\xi] + N.$$

Consequently,

$$\mathcal{E}_a[\xi] = \mathcal{E}_a[\overline{\xi}] - N = \int_{-N}^0 [P_a(\xi \geq t) - 1]dt + \int_0^N P_a(\xi \geq t)dt. \tag{14}$$

Step 2: For sufficient large number $N > 0$, let $\Phi(x) = x \vee (-N+1) \wedge (N-1)$. Then Φ is increasing in x and for any $\xi \in H$, $\xi^N := \Phi(\xi)$ is strictly bounded by N, that is $|\Phi(\xi)| < N$. By Step 1, $\mathcal{E}_a[\xi^N]$ satisfies (14).

Letting $N \to \infty$ and noting that $\lim_{N\to\infty} \mathcal{E}_a[\xi^N] = \mathcal{E}_a[\xi]$ we then obtain

$$\mathcal{E}_a[\xi] = \int_{-\infty}^0 (P_a(\xi \geq t) - 1)\, dt + \int_0^\infty P_a(\xi \geq t)dt \ .$$

This concludes the proof of Theorem 2. □

It is interesting to note that the necessary and sufficient condition for the Choquet representation is related to Jensen's inequality.

The following is a combination of Theorem 1 and Jensen's inequality in [CKL].

Corollary 1 *Suppose that g satisfies (H1), (H2) and (H3). If g is convex in z, then the following statements are equivalent.*

(i) the g-expectation satisfies Jensen's inequality, e.g. for any convex function f

$$f\left(\mathcal{E}_g[\xi]\right) \leq \mathcal{E}_g[f(\xi)], \text{ whenever } f(\xi) \text{ and } \xi \in L^2(\Omega, \mathcal{F}, P) \ ;$$

(ii) g is of the form (3), e.g there exist two continuous functions $\alpha \geq 0, \beta$ such that

$$g(y, z, t) = \alpha(t)|z| + \beta(t)z;$$

(iii) the g-expectation $\mathcal{E}_g[\xi]$ is a Choquet expectation for all $\xi \in H$.

4 Extension to multiple dimensional Brownian motion

In Section 3, we have proven our theorems under the assumption that the Brownian motion $\{W_t\}$ is 1−dimensional, that is $d = 1$. For the multiple Brownian motion, that is when $\{W_t\} := \{W_t^1, W_t^2, \cdots, W_t^d\}^*$ with $d > 1$, the results still hold. Here $\{\cdots\}^*$ denotes transpose.

Indeed, let H be the set of all $\mathcal{F}$-measurable random variables $X_T \in L^2(\Omega, \mathcal{F}, P)$, where X_T is the value of the solution $\{X_t\}$ of the following SDE at time T

$$\begin{cases} dX_s = b(s, X_s)ds + \sigma(s, X_s) \cdot dW_s \ , \quad 0 \leq s \leq T \ , \\ X_0 = x \in \mathbf{R} \ . \end{cases}$$

Here $\sigma(t, x) : [0, T] \times \mathbf{R}^{1\times d} \to \mathbf{R}^d$, $b(t, x) : \mathbf{R} \to \mathbf{R}$ are continuous in (t, x) and uniformly Lipschitz in x. The notation $x \cdot y$ is the inner product of $x, y \in \mathbf{R}^d$. Furthermore, we assume that $d > 1$ and that there exist g_i such that

$$g(y, z, t) = \sum_{i=1}^{d} g_i(y, z^i, t) \tag{15}$$

where z^i is the i-th component of z.

Theorem 1 can then be extended as follows.

Theorem 3 *Suppose g is of the form (15) and that g_i satisfies (H1), (H2) and (H3) for each i. Then $\mathcal{E}_g[\xi]$ can be represented as a Choquet expectation for any $\xi \in H$ if and only if there exist two sequences of continuous functions $\{\alpha_i(t)\}$ and $\{\beta_i(t)\}$ such that g is of the form*

$$g(y,z,t) = \sum_{i=1}^{d} \alpha_i(t)|z^i| + \sum_{i=1}^{d} \beta_i(t) z^i \ .$$

Proof. The proof is similar to the case when $\{W_t\}$ is $1-$dimensional and so we only sketch the proof.
Necessary condition: This is analogous to the proof of necessary condition in Section 3.

For each $i = 1, 2, \cdots, d$ choose (y_1, z_1^i, t) , $(y_2, z_2^i, t) \in \mathbf{R} \times \mathbf{R} \times [0, T]$ with $z_1^i z_2^i \geq 0$. For any $\tau \in [t, T]$, let $\xi = y_1 + z_1^i (W_\tau^i - W_t^i)$ and $\eta = y_2 + z_2^i (W_\tau^i - W_t^i)$.

Let (y, z) be the solution of the BSDE with multiple Brownian motion,

$$y_t = \xi + \int_t^T g(y_s, z_s, s) ds - \int_t^T z_s \cdot dW_s$$

where $\{W_t\}$ is $d-$dimensional, and $(\overline{y}, \overline{z})$ be the solution of BSDE with $1-$dimensional Brownian motion

$$y_t = \xi + \int_t^T g_i(y_s, z_s^i, s) ds - \int_t^T z_s^i dW_s^i \ , i = 1, 2, \ldots, d$$

where $\{W_t^i\}$ is the i-th component of d-dimensional Brownian motion $\{W_t\}$.

It is easy to check that

$$\begin{cases} y_t &= \overline{y}_t \ , \\ z_t &= (z_t^1, \cdots, z_t^d), \text{ where } \quad z_t^i = \overline{z}_t, z_t^j = 0, j \neq i, j = 1, 2, \cdots, d \end{cases}$$

where $z^k, k = 1, 2, \cdots, d$ is the k-th component of z.

A corresponding result can be obtained for η, thus by the necessary condition in Theorem 1, there exists $\alpha_i(t)$ and $\beta_i(t)$ such that

$$g_i(y, z^i, t) = \alpha_i(t)|z^i| + \beta_i(t) z^i \ .$$

It then follows by the assumption $g(y,z,t) = \sum_{i=1}^d g_i(y, z^i, t)$ that the necessary condition can be proved by the necessary condition in Theorem 2.
Sufficient condition: Note that all proofs of in Section 3 can be adapted to the case where $\{W_t\}$ is $d-$dimensional Brownian motion.

5 A Generalized Feynman-Kac Formula

In this section, we will consider the application of our result in partial differential equations. We only consider a 1-dimensional PDE.

Let u be the solution of the partial differential equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} \\ u(0,x) = f(x), \quad t \geq 0, \quad x \in \mathbf{R} \end{cases} \tag{16}$$

where f is a bounded function. By the Feynman-Kac formula, there exists a probability measure such that the solution $u(t,x)$ of PDE (16) can be represented by a mathematical expectation

$$u(t,x) = Ef(W_t + x) \tag{17}$$

where $\{W_t\}$ is a $1-$dimensional standard Brownian motion.

The Feynman-Kac formula (17) implies that under some conditions the solution of a class of linear PDEs can be represented by a (linear) mathematical expectation, which make it possible to solve a linear PDE using Monte Carlo methods (the Law of Large Numbers for additive probabilities). A natural question is for which class of nonlinear PDEs, can their solutions be represented by nonlinear Choquet expectations? If this is feasible, then an application of the Law of Large Numbers for non-additive probabilities [D, M] would suggest that a Monte Carlo-like method could be used to solve non-linear PDEs. It is interesting that our result [Theorem 1] gives an answer for a class of nonlinear PDEs.

For convenience in the exposition we now consider the following simple nonlinear PDE. Let u be the solution of PDE

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + g\left(u, \frac{\partial u(t,x)}{\partial x}\right) \\ u(0,x) = f(x)\ , \quad t \geq 0, \end{cases} \tag{18}$$

where g is a function satisfying (H1), (H2) and (H3) in Section 2 and $f(x)$ is increasing in x.

Theorem 4 *For any bounded increasing function f, the solution $u(t,x)$ of PDE (18) can be represented by a Choquet expectation if and only if there exist constants α and β such that g is of the form*

$$g(y,z) = \alpha|z| + \beta z\ .$$

Proof: Let $\{W_t\}$ be a $1-$dimensional Brownian motion. Denote by $v(s,x) = u(t-s,x)$. Then $v(t,x) = u(0,x) = f(x)$ and from (18)

$$\frac{\partial v(s,x)}{\partial s} + \frac{1}{2}\frac{\partial^2 v(s,x)}{\partial x^2} = -g\left(v(s,x), \frac{\partial v(s,x)}{\partial x}\right)\ , \quad s \in [0,t]\ .$$

Applying Itô's formula for $v(s, W_s + x)$, we get

$$\begin{aligned} dv(s, W_s + x) &= \left(\frac{\partial v(s,W_s+x)}{\partial s} + \frac{1}{2} \frac{\partial^2 v(s,W_s+x)}{\partial x^2} \right) ds + \frac{\partial v(s,W_s+x)}{\partial x} dW_s \\ &= -g\left(v(s, W_s + x), \frac{\partial v(s,W_s+x)}{\partial x} \right) ds + \frac{\partial v(s,W_s+x)}{\partial x} dW_s \ , \quad s \in [0, t] \end{aligned}$$

with boundary condition $v(t, W_t + x) = f(W_t + x)$.

This implies that $y_s = v(s, W_s + x), z_t = \frac{\partial v(s,W_s+x)}{\partial x}$ is the solution of the BSDE

$$y_s = f(W_t + x) + \int_s^t g(y_r, z_r)dr - \int_s^t z_r dW_r \ , \quad s \in [0, t] \ .$$

By the definition of g-expectation, $v(0, x) = y_0 = \mathcal{E}_g[f(W_t + x)$.

On the other hand, by the definition of v, $v(0, x) = u(t, x)$ and thus

$$u(t, x) = \mathcal{E}_g[f(W_t + x)] \ .$$

Since g does not depend on t, $\alpha(t) = \alpha$ and $\beta(t) = \beta$ in Theorem 1 and the proof of Theorem 4 is complete. □

Remark 5 *Our result shows that one can find a nonlinear function g such that both g-expectation and Choquet expectation coincide in H. A natural question is that can one find a nonlinear function g such that both g-expectation and Choquet expectation coincide in $L^2(\Omega, \mathcal{F}, P)$? Unfortunately, a recent result by Chen et.al. [CD] shows it is impossible.*

6 Appendix: Lemmas

The following lemmas have been used in this paper. Lemma 6 can be found in [KPQ].

Lemma 6 *Suppose that g_1 and g_2 satisfy (H1) and (H2). For any ξ_1, $\xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$, let (y^i, z^i) $(i = 1, 2)$ be the solutions of BSDE (1) corresponding to $\xi = \xi_1$ and $g = g_1$, $\xi = \xi_2$ and $g = g_2$ respectively. Then there exists a constant $c > 0$ such that*

$$E\left[\sup_{t \le s \le T} |y_s^1 - y_s^2|^2 + \int_t^T |z_s^1 - z_s^2|^2 ds | \mathcal{F}_t \right] \le cE\left[|\xi_1 - \xi_2|^2 + \left(\int_t^T |\overline{g}_s| ds \right)^2 |\mathcal{F}_t \right]$$

where $\overline{g}_s := g_1(y_s^1, z_s^1, s) - g_2(y_s^1, z_s^1, s)$.

Remark 6 *Lemma 6 implies that if ξ_2 converges to ξ_1 in $L^2(\Omega, \mathcal{F}, P)$ and $g_1(y^1, z^1, \cdot)$ converges to $g_2(y^1, z^1, \cdot)$ in $L^2(0, T)$, then (y^2, z^2) converges to (y^1, z^1) in $L^2(0, T) \times L^2(0, T)$.*

Let $b(t,x) : [0,T] \times \mathbf{R} \to \mathbf{R}$, $\sigma(t,x) : [0,T] \times \mathbf{R} \to \mathbf{R}$ be continuous in (t,x) and uniformly Lipschitz continuous in x.

By the existence theorem for stochastic differential equations, there exists a unique strong solution $\{X_s^{t,x}\}$ satisfying the SDE

$$\begin{cases} dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_t = x, \ s \in [t,T] \ . \end{cases} \tag{19}$$

Let $\Phi(x)$ be a continuous function defined on $\mathbf{R}$ such that $\Phi(X_T^{t,x}) \in L^2(\Omega, \mathcal{F}, P)$ and $(y^{t,x}, z^{t,x})$ be the solution of the BSDE (1) with $\xi = \Phi(X_T^{t,x})$.

$$y_s = \Phi(X_T^{t,x}) + \int_s^T g(y_r, z_r, r)dr - \int_s^T z_r dW_r \ , \quad s \in [0,T]. \tag{20}$$

The following Lemma can be found in Pardoux and Peng [PP2] or in Ma, Potter and Yong [MPY].

Lemma 7 *Let $(y^{t,x}, z^{t,x})$ be the solution of the BSDE (1) with $\xi = \Phi(X_T^{t,x})$, where $\{X_t\}$ is the solution of (19). Suppose b, σ of (19), g of (1) and Φ are C^3. Then*

(i) $u(t,x) := y_t^{t,x} \in C^{1,2}([0,T] \times R)$ is the unique solution of the following partial differential equation (PDE):

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}u(t,x) + g(t, u(t,x), \sigma(t,x)\partial_x u(t,x)) = 0, \\ u(T,x) = \Phi(x), \end{cases} \tag{21}$$

where $\mathcal{L}u(t,x) := \frac{1}{2}\sigma^2(t,x)\partial_x^2 u(t,x) + b(t,x)\partial_x u(t,x)$,

(ii) $z_s^{t,x} = \sigma(s, X_s^{t,x})\partial_x u(s, X_s^{t,x})$, a.e. $s \in [t,T]$, where $\partial_x u$ is the partial derivative of u.

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