Anticipative stochastic control for Lévy processes with application to insider trading

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11th October, 2007.

Abstract

An insider is an agent who has access to larger information than the one given by the development of the market events and who takes advantage of this in optimizing his position in the market . In this paper we consider the optimization problem of an insider who is so influential in the market to affect the price dynamics: in this sense he is called a "large" insider. The optimal portfolio problem for a general utility function is studied for a financial market driven by a Lévy process in the framework of forward anticipating calculus.

Key words and phrases: forward integral, insider trading, large trader, utility function, enlargement of filtration.
AMS (2000) Classification: primary 91B28; secondary 60H05.

1 Introduction.

The modeling of insider trading is a challenge that recently has been taken up by many scientists with the aim of understanding the behavior and quantifying the gain of a dealer who takes advantage of some extra information, i.e. not deducible from the market behavior itself, that he may happen to have at his disposal.

Thus in a market model on the probability space $(\Omega, \mathcal{F}, P)$ with two investment possibilities such as

- a bond with price $S_0(t)$, $t \in [0,T]$,
- a stock with price $S_1(t)$, $t \in [0,T]$,

an "honest" agent is taking decisions relying only on the flow of information

$$\mathbb{F} := \{\mathcal{F}_t \subset \mathcal{F}, 0 \le t \le T\}$$

given by the development of the market events, while an "insider" would rely on the flow of information

$$\mathbb{H} := \{\mathcal{H}_t \subset \mathcal{F}, 0 \le t \le T\} : \quad \mathcal{H}_t \supset \mathcal{F}_t.$$

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Therefore the insider's portfolios are in general stochastic processes adapted to $\mathbb{H}$.

Different aspects of the insider trading have been considered and with different approaches. It is rather hard to mention all past and recent achievements, so we will restrict ourselves to the papers that have mostly inspired the present work.

The subject we are dealing with is the optimization problem

$$\max_{\pi \in \mathcal{A}} E\big[U(X_\pi(T))\big] \tag{1.1}$$

of an insider who wants to maximize the expected utility of his final wealth $X_\pi(T)$ given by the dynamics

$$dX_\pi(t) = \big(1 - \pi(t)\big) X_\pi(t) dS_0(t) + \pi(t) X_\pi(t) dS_1(t), \qquad X_\pi(0) > 0,$$

over all admissible choices of portfolios $\pi \in \mathcal{A}$. See Section 3.

Optimization problems of this kind have been studied widely. Here we mention the pivotal work of Karatzas and Pikovski [KP]. They were considering the problem (1.1) for a market driven by a Brownian motion and a logarithmic utility function in the framework of classical enlargement of filtrations. This framework applies under the *a priori* assumption that the $\mathbb{F}$-adapted Brownian motion driving the market is a semimartingale with respect to $\mathbb{H}$. This assumption is often difficult if not impossible to be verified since it depends on the kind of information $\mathbb{H}$ available to the insider.

In [BØ] a general approach is suggested to the modeling of insider trading that overcomes the need of the above assumption in the framework of forward anticipating calculus. In this setting the authors give a solution to problem (1.1) for a general utility function. However, they restrict themselves to the case of markets driven by Brownian motion only.

Remark.
The reasons for taking this approach into account can be summarized in the following points:

(a) The forward integral provides the natural interpretation of the gains from the trade process. Indeed, suppose that a trader buys one stock at a random time τ_1 and keeps it until the random time $\tau_2 > \tau_1$. When he sells it, the gain obtained is $S_1(\tau_2) - S_1(\tau_1) = \int 1_{(\tau_1,\tau_2]}(t) d^- S_1(t)$, where the integral is a forward stochastic integral

(b) If the integrand is càglàd (i.e. left continuous and with right sided limits), the forward integral may be regarded as the limit of Riemann sums, see e.g. [BØ], [KS1]

(c) If the stochastic process driving the market happens to be a semimartingale with respect to the insider filtration $\mathbb{H}$, then the corresponding stochastic integral coincide with the forward integral.

In [ØS] the forward integral calculus and anticipative calculus is used to study the optimal portfolio problem with logarithmic utiliy for a trader with partial information in a (Lévy-Brownian type) anticipative market (e.g. a market influenced by insiders).

In [KS1] the study of [ØS] is extended to cover the case when there are no a priori assumptions about the relation between the information available to the trader and the information generated by the possibly anticipative market. Here the market is assumed to be driven by Brownian motion and the utility function is logarithmic.

In [DMØP1] and [DMØP2] the authors extend the forward integration to the case of compensated Poisson random measures and thus to more general Lévy processes and solve problem (1.1) in the case of a logarithmic utility function. This extension of framework to Lévy processes is motivated by the ongoing discussion on the better fitting of these models to real financial markets than the ones driven only by Brownian motion. Here we can refer to [B-N], [CT], [ER] and [Sc], for example.

In the same line of [BØ] and relying on the achievements in [DMØP1] and [DMØP2], we now solve problem (1.1) for a general utility function and for a general Lévy process. This represents the major contribution of this paper.

Besides there is also another element of novelty. In fact, inspired by [CC] and [KS1], we consider the problem (1.1) from the point of view of a trader so influential in the market that his decisions effect the price process dynamics. In this sense our dealer is called "large" trader. In [CC] the impact of the trader's positions on the prices is exogenously specified. In our paper we chose to use a similar approach - see (3.1)-(3.2). This visible impact of a large trader on the price dynamics may arise because of the volumes traded or also because the other market investors may suppose, though without certainty, that the large trader is an insider. Note that actually in [CC] the large trader is not an insider. On the other hand paper [KS1] considers a similar model for prices, but extends the analysis to the cases in which the large trader is truly an insider. The analysis in [KS1] is however restricted to the case of logarithmic utility and Brownian motion driven dynamics.

In this present paper, as said, the major concern is the solution of an optimal portfolio problem from the point of view of a "large insider" and we do not attempt to discuss here price formation. This would require a study of equilibria under asymmetric information. For this we can refer to the seminal paper [Ky], [Ba] and the recent literature in this line.

This paper is organized as follows. In Section 2 we recall the basic tools of forward calculus for Lévy processes and in particular the Itô formula (see Theorem 2.6), which are then applied in Section 3 where criteria for the existence of the solution of the "large" insider's portfolio optimization problem (1.1) are given. In Section 4 some examples are considered.

For related works in the context of insider modeling and portfolio optimization see also [EJ], [EGK], [KS2], [KY1], [KY2], [Ku] and [Ø], for example.

2 Framework: forward anticipating calculus.

In this section we briefly recall some properties of the forward integral. Our presentation is already in the form we are going to apply later. We can refer to e.g. [BØ], [NP], [RV1], [RV2], [RV3] for information on the forward integration with respect to the Brownian motion and to e.g. [DMØP1] for the integration with respect to the compensated Poisson random measure.

As announced in the introduction we are interested in a Lévy process

$$\eta(t) = \sigma B(t) + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz), \qquad t \in [0, T], \tag{2.1}$$

on the complete filtered probability space $(\Omega, \mathcal{F}, P)$, $\mathbb{F} = \{\mathcal{F}_t \subset \mathcal{F}, 0 \le t \le T\}$ ($\mathcal{F}_0$ trivial) with a finite time horizon $T > 0$. In the Itô representation (2.1) (see [I]) of the Lévy process we can distinguish the standard Brownian motion $B(t)$, $t \in [0,T]$ ($B(0) = 0$), the constant $\sigma \in \mathbb{R}$ and the compensated Poisson random measure

$$\widetilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt.$$

Here $\nu(dz)$, $z \in \mathbb{R}_0$, is a σ-finite Borel measure on $\mathbb{R}_0 = (-\infty, 0) \cup (0, \infty)$ such that

$$\int_{\mathbb{R}_0} z^2 \nu(dz) < \infty.$$

Then $E[\eta^2(t)] = t\big(\sigma^2 + \int_{\mathbb{R}_0} z^2 \nu(dz)\big) < \infty$ for all $t \in [0,T]$. For more information on Lévy processes we can refer to e.g. [A], [Be], [P], [Sa].

The following definition is due to [RV2].

Definition 2.1 *We say that the (measurable) stochastic process* $\varphi = \varphi(t)$, $t \in [0,T]$, *is* forward integrable *over the interval* $[0,T]$ *with respect to the Brownian motion if there exists a process* $I(t)$, $t \in [0,T]$, *such that*

$$\sup_{t \in [0,T]} \Big| \int_0^t \varphi(s) \frac{B(s+\varepsilon) - B(s)}{\varepsilon} ds - I(t) \Big| \longrightarrow 0, \quad \varepsilon \to 0, \tag{2.2}$$

in probability. Then, for any $t \in [0,T]$,

$$I(t) = \int_0^t \varphi(s) d^- B(s)$$

is called the forward integral *of* φ *with respect to the Brownian motion on* $[0,t]$.

The corresponding definition of forward integral with respect to the compensated Poisson random measure is due to [DMØP1]. Here a modified version of what is suggested in [DMØP1] is actually given to be in the line with the definition suggested in [RV2]. Note that these definitions are such that the Itô formulae for forward integrals with respect to the Brownian motion and the compensated Poisson random measure hold true, see [RV2] and [DMØP1].

Definition 2.2 *We say that the (measurable) random field* $\psi = \psi(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}_0$, *is* forward integrable *over* $[0,T]$ *with respect to the compensated Poisson random measure if there exists a process* $J(t)$, $t \in [0,T]$, *such that*

$$\sup_{t \in [0,T]} \Big| \int_0^t \int_{\mathbb{R}_0} \psi(s,z) 1_{U_n}(z) \widetilde{N}(ds, dz) - J(t) \Big| \longrightarrow 0, \quad n \to \infty, \tag{2.3}$$

in probability. Here U_n, $n = 1, 2, ...$, *is an increasing sequence of compact sets* $U_n \subseteq \mathbb{R}_0$ *with* $\nu(U_n) < \infty$ *such that* $\bigcup_n U_n = \mathbb{R}_0$. *Then, for any* $t \in [0,T]$,

$$J(t) = \int_0^t \int_{\mathbb{R}_0} \psi(s,z) \widetilde{N}(d^- s, dz)$$

is called the forward integral *of* ψ *with respect to the compensated Poisson random measure on* $[0,t]$.

Remark 2.3
i) If the integrands in the above definitions are adapted to the filtration $\mathbb{F}$, then the limits (2.1) and (2.2) coincide with the Itô integral. In particular, if we consider the stronger convergence in $L_2(P)$ in the above definitions we obtain an extension of the classical Itô integral. This is useful for the forthcoming applications and is the case we take into account in the sequel.
ii) If G is a random variable then

$$\begin{aligned} G \cdot \Big[\int_0^T \varphi(t) d^- B(t) + \int_0^T \int_{\mathbb{R}_0} \psi(t,z) \widetilde{N}(d^-t, dz) \Big] \\ = \int_0^T G\varphi(t) d^- B(t) + \int_0^T \int_{\mathbb{R}_0} G\psi(t,z) \widetilde{N}(d^-t, dz). \end{aligned} \tag{2.4}$$

Note that this property does not hold in general for the Itô integral. □

Definition 2.4 *A* forward process *is a measurable stochastic function $X(t) = X(\omega, t)$, $\omega \in \Omega$, $t \in [0,T]$, that admits the representation*

$$X(t) = x + \int_0^t \alpha(s) ds + \int_0^t \varphi(s) d^- B(s) + \int_0^t \int_{\mathbb{R}_0} \psi(s,z) \widetilde{N}(d^-s, dz), \tag{2.5}$$

where $x = X(0)$ is a constant. A shorthand notation for (2.5) is

$$d^- X(t) = \alpha(t) dt + \varphi(t) d^- B(t) + \int_{\mathbb{R}_0} \psi(t,z) \widetilde{N}(d^-t, dz), \qquad X(0) = x. \tag{2.6}$$

We call $d^- X(t)$ the forward differential *of $X(t)$, $t \in [0,T]$.*

Remark 2.5 There is a relation between the forward integral and the Skorohod integral, see [DMØP1], Lemma 4.1. Using this we can see that under mild conditions there is a càdlàg version of the process $X(t)$, $t \in [0,T]$. From now on we will consider and use this càdlàg version. □

We can now state the *Itô formula for forward integrals*. See e.g. [RV2], [RV3], for the Brownian motion case, and [DMØP1], for the compensated Poisson random measure case.

Theorem 2.6 *Let $X(t)$, $t \in [0,T]$, be a forward process of the form (2.5) and assume that $\psi(\omega, t, z)$ is continuous in z around zero for (ω, t)-a.a. and $\int_0^T \int_{\mathbb{R}_0} \psi(t,z)^2 \nu(dz) dt < \infty$ ω-a.e. Let $f \in C^2(\mathbb{R})$. Then the forward differential of $Y(t) = f\big(X(t)\big)$, $t \in [0,T]$, is given by the following formula:*

$$\begin{aligned} d^- Y(t) = \Big[& f'\big(X(t)\big)\alpha(t) + \frac{1}{2} f''(X(t))\varphi^2(t) \\ & + \int_{\mathbb{R}_0} \Big(f\big(X(t^-) + \psi(t,z)\big) - f\big(X(t^-)\big) - f'\big(X(t^-)\big)\psi(t,z) \Big) \nu(dz) \Big] dt \\ & + f'(X(t))\varphi(t) d^- B(t) + \int_{\mathbb{R}_0} \Big(f\big(X(t^-) + \psi(t,z)\big) - f\big(X(t^-)\big) \Big) \widetilde{N}(d^-t, dz), \end{aligned} \tag{2.7}$$

where $f'(x) = \frac{d}{dx} f(x)$ and $f''(x) = \frac{d^2}{dx^2} f(x)$, $x \in \mathbb{R}$.

3 Optimal portfolio problem for a "large" insider.

In this section we study the existence of an optimal portfolio for the problem (1.1).

Let us consider the following market model with a finite time horizon $T > 0$ and two investment possibilities:

- a bond with price dynamics

$$
\begin{cases} dS_0(t) = r(t)S_0(t)dt, & t \in (0,T], \\ S_0(0) = 1 \end{cases} \tag{3.1}
$$

- a stock with price dynamics

$$
\begin{cases} dS_1(t) = S_1(t^-)\big[\mu(t,\pi(t))dt + \sigma(t)d^-B(t) + \int_{\mathbb{R}_0}\theta(t,z)\widetilde{N}(d^-t,dz)\big], t \in (0,T], \\ S_1(0) > 0 \end{cases} \tag{3.2}
$$

on the complete probability space $(\Omega, \mathcal{F}, P)$. The stochastic coefficients $r(t)$, $\mu(t,\pi)$, $\sigma(t)$ and $\theta(t,z)$, $t \in [0,T]$, $z \in \mathbb{R}_0$, are measurable, càglàd processes with respect to the parameter t, adapted to some given filtration $\mathbb{G}$, for each constant value of π. Here $\mathbb{G} := \{\mathcal{G}_t \subset \mathcal{F}, t \in [0,T]\}$ is a filtration with

$$
\mathcal{G}_t \supset \mathcal{F}_t, \quad t \in [0,T].
$$

We also assume that $\theta(t,z) > -1$, $dt \times \nu(dz)$-a.e. and that

$$
E\int_0^T \Big\{|r(t)| + |\mu(t)| + \sigma^2(t) + \int_{\mathbb{R}_0} \theta^2(t,z)\nu(dz)\Big\}dt < \infty.
$$

We recall that $\mathbb{F} := \{\mathcal{F}_t \subset \mathcal{F}, t \in [0,T]\}$ is the filtration generated by the development of the market events, i.e. the driving processes $B(t)$ and $\widetilde{N}(t,z)$, $t \in [0,T], z \in \mathbb{R}_0$.

In this model the coefficient $\mu(t)$, $t \in [0,T]$, is depending on the portfolio choice $\pi(t)$, $t \in [0,T]$, of an insider who has access to the information represented by the filtration $\mathbb{H} := \{\mathcal{H}_t \subset \mathcal{F}, t \in [0,T]\}$ with

$$
\mathcal{H}_t \supset \mathcal{G}_t \supset \mathcal{F}_t, \quad t \in [0,T].
$$

Accordingly the insider's portfolio $\pi = \pi(t)$, $t \in [0,T]$, is a stochastic process adapted to $\mathbb{H}$. With the above conditions on μ, we intend to model a possible situation in which an insider is so influencial in the market to affect the prices with his choices. In this sense we talk about a "large" insider.

This exogenous model for the price dynamics (3.1)-(3.2) is in line with [CC]. In [CC] a dependence of the coefficient r on the portfolio π is also considered. In our paper, this can also be mathematically carried through without substantial change, however the assumption that the return of the bond depends on the agent's portfolio could be considered unrealistic.

We consider the *insider's wealth process* to be given by

$$dX_\pi(t) = X_\pi(t^-)\Big\{\big[r(t) + \big(\mu(t,\pi(t)) - r(t)\big)\pi(t)\big]dt \\ + \pi(t)\sigma(t)d^-B(t) + \pi(t)\int_{\mathbb{R}_0} \theta(t,z)\widetilde{N}(d^-t,dz)\Big\}, \tag{3.3}$$

with initial capital $X_\pi(0) = x > 0$. In the sequel we put $x = 1$ for simplicity in notation. By the Itô formula for forward integrals, see Theorem 2.6, the final wealth of the admissible portfolio π is the solution of equation (3.3):

$$X_\pi(t) = \exp\Big\{\int_0^t \Big[r(s) + (\mu(s,\pi(s)) - r(s))\pi(s) \\ - \frac{1}{2}\sigma^2(s)\pi^2(s)\Big]ds - \int_0^t\int_{\mathbb{R}_0}\Big[\pi(s)\theta(s,z) - \ln\big(1+\pi(s)\theta(s,z)\big)\Big]\nu(dz)ds \\ + \int_0^t \pi(s)\sigma(s)d^-B(s) + \int_0^t\int_{\mathbb{R}_0} \ln\big(1+\pi(s)\theta(s,z)\big)\widetilde{N}(d^-s,dz)\Big\}. \tag{3.4}$$

Taking the point of view of an insider, with the only purpose of understanding his opportunities in the market, we are interested in solving the optimization problem

$$\Phi := \sup_{\pi\in\mathcal{A}} E\left[U(X_\pi(T))\right] = E\left[U(X_{\pi^*}(T))\right], \tag{3.5}$$

for the given *utility function*

$$U : [0,\infty) \longrightarrow [-\infty,\infty)$$

that is a non-decreasing, concave and lower semi-continuous function which we assume to be continuously differentiable on $(0,\infty)$. Here the controls belong to the set $\mathcal{A}$ of admissible portfolios characterized as follows.

Definition 3.1 *The set* $\mathcal{A}$ *of* admissible portfolios *consists of all processes* $\pi = \pi(t)$, $t \in [0,T]$*, such that*

(3.6) π *is càglàd and adapted to the filtration* $\mathbb{H}$;

(3.7) $\pi(t)\sigma(t),\ t \in [0,T],$ *is forward integrable with respect to* $d^-B(t)$;

(3.8) $\pi(t)\theta(t,z),\ t\in[0,T], z\in\mathbb{R}_0,$ *is forward integrable with respect to* $\widetilde{N}(d^-t,dz)$;

(3.9) $\pi(t)\theta(t,z) > -1 + \epsilon_\pi$ *for a.a.* (t,z) *with respect to* $dt\times\nu(dz)$*, for some* $\epsilon_\pi \in (0,1)$ *depending on* π;

$$E\int_0^T\Big\{|\mu(s,\pi(s))-r(s)||\pi(s)|+\sigma^2(s)\pi^2(s)+\int_{\mathbb{R}_0}\pi^2(s)\theta^2(s,z)\nu(dz)\Big\}ds<\infty \tag{3.10}$$
$$\textit{and } E\Big[\exp\Big\{K\int_0^T|\pi(s)|ds\Big\}\Big]<\infty \textit{ for all } K>0;$$

$$\ln\big(1+\pi(t)\theta(t,z)\big) \textit{ is forward integrable with respect to } \widetilde{N}(d^-t,dz); \tag{3.11}$$

$$E\big[U(X_\pi(T))\big]<\infty \textit{ and } 0<E\big[U'(X_\pi(T))X_\pi(T)\big]<\infty, \tag{3.12}$$
$$\textit{where } U'(w)=\frac{d}{dw}U(w), w\geq 0.$$

Moreover we assume that for all $\pi,\beta\in\mathcal{A}$, with β bounded, there exists a $\zeta>0$ such that the family

$$\big\{U'(X_{\pi+\delta\beta}(T))X_{\pi+\delta\beta}(T)\big|M_{\pi+\delta\beta}(T)\big|\big\}_{\delta\in(-\zeta,\zeta)} \tag{3.13}$$

is uniformly integrable. Note that, for $\pi\in\mathcal{A}$ and $\beta\in\mathcal{A}$ bounded, $\pi+\delta\beta\in\mathcal{A}$ for any $\delta\in(-\zeta,\zeta)$ with ζ small enough. Here the stochastic process $M_\pi(t)$, $t\in[0,T]$, is defined as

$$\begin{aligned}M_\pi(t):=&\int_0^t\Big\{\mu(s,\pi(s))-r(s)+\mu'(s,\pi(s))\pi(s)\\&-\sigma^2(s)\pi(s)-\int_{\mathbb{R}_0}\frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)}\nu(dz)\Big\}ds\\&+\int_0^t\sigma(s)d^-B(s)+\int_0^t\int_{\mathbb{R}_0}\frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}\widetilde{N}(d^-s,dz)\end{aligned} \tag{3.14}$$

where $\mu'(s,\pi)=\frac{\partial}{\partial\pi}\mu(s,\pi)$.

Remark 3.2 Condition (3.13) may be difficult to verify. Here we give some examples of conditions under which it holds.

First, consider $M(\delta):=M_{\pi+\delta\beta}(T)$. The uniformly integrability of $\{M(\delta)\}_{\delta\in(-\zeta,\zeta)}$ is assured by

$$\sup_{\delta\in(-\zeta,\zeta)}E\big[|M|^p(\delta)\big]<\infty \qquad \text{for some } p>1.$$

Observe that, from $\pi,\beta\in\mathcal{A}$, we have that $1+\big(\pi(s)+\delta\beta(s)\big)\theta(s,z)\geq\epsilon_\pi-\zeta$ $dt\times\nu(dz)$-a.e. for some $\zeta\in(0,\epsilon_\pi)$. Moreover, for $\epsilon>0$,

$$\int_0^T\int_{|z|\geq\epsilon}\frac{\theta(s,z)}{1+(\pi(s)+\delta\beta(s))\theta(s,z)}\widetilde{N}(d^-s,dz)=\int_0^T\int_{|z|\geq\epsilon}\frac{\theta(s,z)}{1+(\pi(s)+\delta\beta(s))\theta(s,z)}\widetilde{N}(ds,dz).$$

Thus we have that

$$E\Big[\Big(\int_0^T\int_{|z|\geq\epsilon}\frac{\theta(s,z)}{1+(\pi(s)+\delta\beta(s))\theta(s,z)}\widetilde{N}(d^-s,dz)\Big)^2\Big]\leq\frac{1}{(\epsilon_\pi-\zeta)^2}E\Big[\Big(\int_0^T\int_{|z|\geq\epsilon}\theta^2(s,z)\nu(dz)ds\Big]<\infty.$$

So, if

$$E\Big[\Big(\int_0^T\sigma(s)d^-B(s)\Big)^2\Big]<\infty \text{ and } E\Big[\Big(\int_0^T\int_{|z|<\epsilon}|\theta(s,z)|\widetilde{N}(d^-s,dz)\Big)^2\Big]<\infty$$

(see Remark 2.3 (i)), we have that $E[M^2(\delta)] < \infty$ uniformly in $\delta \in (-\zeta, \zeta)$ if, for example, the coefficients μ, μ', r, σ are bounded. This shows that (3.13) holds if $U'(x)x$ is uniformly bounded for $x \in (0, \infty)$. This is the case, for example, of $U(x) = \ln x$ and $U(x) = -\exp\{-\lambda x\}$ $(\lambda > 0)$.

Similarly, in the case of power utility function

$$U(x) = \frac{1}{\gamma} x^\gamma, \quad x > 0 \qquad \text{for some } \gamma \in (0,1),$$

we see that $U'(X_{\pi+\delta\beta}(T)) X_{\pi+\delta\beta}(T)|M(\delta)| = X^\gamma_{\pi+\delta\beta}(T)|M(\delta)|$ and condition (3.13) would be satisfied if

$$\sup_{\delta\in(-\zeta,\zeta)} E\big[(X^\gamma_{\pi+\delta\beta}(T)|M(\delta)|)^p\big] < \infty \qquad \text{for some } p > 1.$$

Note that we can write

$$X_{\pi+\delta\beta}(T) = X_\pi(T) N(\delta),$$

where

$$\begin{aligned} N(\delta) := \exp\Big\{ \int_0^T & \big[(\mu(s,\pi(s)+\delta\beta(s)) - r(s))\delta\beta(s) + (\mu(s,\pi(s)+\delta\beta(s)) - \mu(s,\pi(s))\pi(s) \\ & - \sigma^2(s)\delta\beta(s)\pi(s) - \frac{1}{2}\sigma^2(s)\delta^2\beta^2(s)\big] ds + \int_0^T \delta\sigma(s)\beta(s) d^- B(s) \\ & + \int_0^T\int_{\mathbb{R}_0} \big[\ln(1+(\pi(s)+\delta\beta(s))\theta(s,z)) - \ln(1+\pi(s)\theta(s,z)) - \delta\beta(s)\theta(s,z)\big]\nu(dz)ds \\ & + \int_0^T\int_{\mathbb{R}_0} \big[\ln(1+(\pi(s)+\delta\beta(s))\theta(s,z)) - \ln(1+\pi(s)\theta(s,z))\big]\widetilde{N}(d^-s,dz)\Big\}. \end{aligned}$$

From the iterated application of the Hölder inequality we have

$$\begin{aligned} & E\big[(X^\gamma_{\pi+\delta\beta}(T)|M(\delta)|)^p\big] \\ & \le \big(E\big[\big(X_\pi(T)\big)^{\gamma p a_1 b_1}\big]\big)^{\frac{1}{a_1 b_1}} \big(E\big[\big(N(\delta)\big)^{\gamma p a_1 b_2}\big]\big)^{\frac{1}{a_1 b_2}} \big(E\big[\big(|M(\delta)|\big)^{p a_2}\big]\big)^{\frac{1}{a_2}}, \end{aligned}$$

where a_1, a_2: $\frac{1}{a_1} + \frac{1}{a_2} = 1$ and b_1, b_2: $\frac{1}{b_1} + \frac{1}{b_2} = 1$. Then we can choose $a_1 = \frac{2}{2-p}$, $a_2 = \frac{2}{p}$ and also $b_1 = \frac{2-p}{\gamma p}$, $b_2 = \frac{2-p}{2-p-\gamma p}$ for some $p \in (1, \frac{2}{\gamma+1})$. Hence

$$\begin{aligned} & E\big[(X^\gamma_{\pi+\delta\beta}(T)|M(\delta)|)^p\big] \\ & \le \big(E\big[\big(X_\pi(T)\big)^2\big]\big)^{\frac{\gamma p}{2}} \big(E\big[\big(N(\delta)\big)^{\frac{2\gamma p}{2-p-\gamma p}}\big]\big)^{\frac{2-p-\gamma p}{2}} \big(E\big[\big(|M(\delta)|\big)^2\big]\big)^{\frac{p}{2}}. \end{aligned}$$

If the value $X_\pi(T)$ in (3.4) satisfies

$$E\big[\big(X_\pi(T)\big)^2\big] < \infty, \tag{3.15}$$

then the condition (3.13) holds if

$$\sup_{\delta\in(-\zeta,\zeta)} E\big[\big(N(\delta)\big)^{\frac{2\gamma p}{2-p-\gamma p}}\}\big] < \infty.$$

Since (3.10) holds, it is enough, e.g., that μ, μ', r, σ are bounded to have $E\big[(N(\delta))^{\frac{2\gamma p}{2-p-\gamma p}}\}\big] < \infty$ uniformly in $\delta \in (-\zeta, \zeta)$. Note that condition (3.15) is verified, for example, if for all $K > 0$

$$E\Big[\exp\Big\{K\Big(\int_0^T |\pi(s)|ds + \Big|\int_0^T \pi(s)\sigma(s)d^-B(s)\Big| + \Big|\int_0^T \int_{\mathbb{R}_0} \ln(1+\pi(s)\theta(s,z))\widetilde{N}(d^-s,dz)\Big|\Big)\Big\}\Big] < \infty.$$

By similar arguments we can also treat the case of a utility function such with $U'(x)$ is uniformly bounded for $x \in (0, \infty)$. We omit the details. $\square$

The forward stochastic calculus gives an adequate mathematical framework in which we can proceed to solve the optimization problem (3.5). Define

$$J(\pi) := E\big[U\big(X_\pi(T)\big)\big], \quad \pi \in \mathcal{A}.$$

First, let us suppose that π is optimal for the insider. Choose $\beta \in \mathcal{A}$ bounded, then $\pi + \delta\beta \in \mathcal{A}$ for all δ small enough. Since the function $J(\pi + \delta\beta)$ is maximal at π, by (3.13) and (2.4), we have that

$$\begin{aligned} 0 = & \frac{d}{d\delta} J(\pi + \delta\beta)_{|\delta=0} \\ = & E\Big[U'(X_\pi(T))X_\pi(T)\Big\{\int_0^T \beta(s)\big[\mu(s,\pi(s)) - r(s) \\ & + \mu'(s,\pi(s))\pi(s) - \sigma^2(s)\pi(s) \\ & - \int_{\mathbb{R}_0} \big\{\theta(s,z) - \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}\big\}\nu(dz)\big]ds \\ & + \int_0^T \beta(s)\sigma(s)d^-B(s) + \int_0^T \int_{\mathbb{R}_0} \frac{\beta(s)\theta(s,z)}{1+\pi(s)\theta(s,z)}\widetilde{N}(d^-s,dz)\Big\}\Big]. \end{aligned} \tag{3.16}$$

Now let us fix $t \in [0, T)$ and $h > 0$ such that $t + h \le T$. We can choose $\beta \in \mathcal{A}$ of the form

$$\beta(s) = \alpha\chi_{(t,t+h]}(s), \quad 0 \le s \le T,$$

where α is an arbitrary bounded $\mathcal{H}_t$-measurable random variable. Then (3.16) gives

$$\begin{aligned} 0 = & E\Big[U'(X_\pi(T))X_\pi(T)\Big\{\int_t^{t+h} \big[\mu(s,\pi(s)) - r(s) \\ & + \mu'(s,\pi(s))\pi(s) - \sigma^2(s)\pi(s) \\ & - \int_{\mathbb{R}_0} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)}\nu(dz)\big]ds \\ & + \int_t^{t+h} \sigma(s)d^-B(s) + \int_t^{t+h} \int_{\mathbb{R}_0} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)}\widetilde{N}(d^-s,dz)\Big\} \cdot \alpha\Big]. \end{aligned} \tag{3.17}$$

Since this holds for all such α we can conclude that

$$E\big[F_\pi(T)\big(M_\pi(t+h) - M_\pi(t)\big)|\mathcal{H}_t\big] = 0 \tag{3.18}$$

where

$$F_\pi(T) := \frac{U'(X_\pi(T))X_\pi(T)}{E\big[U'(X_\pi(T))X_\pi(T)\big]} \tag{3.19}$$

and

$$\begin{aligned} M_\pi(t) := \int_0^t \Big\{ & \mu(s,\pi(s)) - r(s) + \mu'(s,\pi(s))\pi(s) \\ & - \sigma^2(s)\pi(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\theta^2(s,z)}{1+\pi(s)\theta(s,z)}\nu(dz)\Big\} ds \\ & + \int_0^t \sigma(s) d^- B(s) + \int_0^t\int_{\mathbb{R}_0} \frac{\theta(s,z)}{1+\pi(s)\theta(s,z)} \widetilde{N}(d^-s,dz), \quad t\in[0,T] \end{aligned} \tag{3.20}$$

- cf. (3.14). Define the probability measure Q_π on $(\Omega,\mathcal{H}_T)$ by

$$Q_\pi(d\omega) := F_\pi(T)P(d\omega) \tag{3.21}$$

and denote the expectation with respect to the measure Q_π by E_{Q_π}. Then, by (3.19), we have

$$E_{Q_\pi}\big[M_\pi(t+h) - M_\pi(t)|\mathcal{H}_t\big] = \frac{E\big[F_\pi(T)\big(M_\pi(t+h) - M_\pi(t)\big)|\mathcal{H}_t\big]}{E\big[F_\pi(T)|\mathcal{H}_t\big]} = 0.$$

Hence the process $M_\pi(t)$, $t\in[0,T]$ is a $(\mathbb{H},Q_\pi)$-*martingale* (i.e. a martingale with respect to the filtration $\mathbb{H}$ and under the probability measure Q_π).
On the other hand, the argument can be reversed as follows. If $M_\pi(t)$, $t\in[0,T]$, is a $(\mathbb{H},Q_\pi)$-martingale, then

$$E\big[F_\pi(T)\big(M_\pi(t+h) - M_\pi(t)\big)|\mathcal{H}_t\big] = 0,$$

for all $h>0$ such that $0\le t<t+h\le T$, which is (3.18). Or equivalently,

$$E\big[\alpha\, F_\pi(T)\big(M_\pi(t+h) - M_\pi(t)\big)\big] = 0$$

for all bounded $\mathcal{H}_t$-measurable $\alpha\in\mathcal{A}$. Hence (3.17) holds for all such α. Taking linear combinations we see that (3.16) holds for all caglad step processes $\beta\in\mathcal{A}$. By our assumptions (3.7) and (3.8) on $\mathcal{A}$ and using that the forward integral of a caglad process is the limit of Riemann sums (see Remark (b) in Section 1) we get, by an approximation argument, that (3.16) holds for all $\beta\in\mathcal{A}$. If the function $g(\delta) := E\big[U(X_{\pi+\delta\beta}(T))\big]$, $\delta\in(-\zeta,\zeta)$, is concave for each $\beta\in\mathcal{A}$, we conclude that its maximum is achieved at $\delta=0$. Hence we have proved the following result.

Theorem 3.3 *(i) If the stochastic process $\pi\in\mathcal{A}$ is optimal for the problem* (3.5)*, then the stochastic process $M_\pi(t)$, $t\in[0,T]$, is an $(\mathbb{H},Q_\pi)$-martingale.*
(ii) Conversely, if the function $g(\delta) := E\big[U(X_{\pi+\delta\beta}(T))\big]$, $\delta\in(-\zeta,\zeta)$, is concave for each $\beta\in\mathcal{A}$ and $M_\pi(t)$, $t\in[0,T]$, is an $(\mathbb{H},Q_\pi)$-martingale, then $\pi\in\mathcal{A}$ is optimal for the problem (3.5).

Remark.
Since the composition of a concave increasing function with a concave function is concave, we can see that a sufficient condition for the function $g(\delta)$, $\delta \in (-\zeta, \zeta)$, to be concave is that the function

$$\Lambda(s) : \pi \longrightarrow r(s) + (\mu(s,\pi) - r(s))\pi - \frac{1}{2}\sigma^2(s)\pi^2 \tag{3.22}$$

is concave for all $s \in [0,T]$. For this it is sufficient that $\mu(s,\cdot)$ are C^2 for all s and that

$$\mu''(s,\pi)\pi + 2\mu'(s,\pi) - \sigma^2 \leq 0 \tag{3.23}$$

for all s, π. Here we have set $\mu' = \frac{\partial \mu}{\partial \pi}$ and $\mu'' = \frac{\partial^2 \mu}{\partial \pi^2}$.

Moreover, we also obtain the following result

Theorem 3.4 *(i) A stochastic process $\pi \in \mathcal{A}$ is optimal for the problem* (3.5) *only if the process*

$$\hat{M}_\pi(t) := M_\pi(t) - \int_0^t \frac{d[M_\pi, Z_\pi](s)}{Z_\pi(s)}, \quad t \in [0,T], \tag{3.24}$$

is an $(\mathbb{H}, P)$-martingale (i.e. a martingale with respect to the filtration $\mathbb{H}$ and under the probability measure P). Here

$$Z_\pi(t) := E_{Q_\pi}\Big[\frac{dP}{dQ_\pi}|\mathcal{H}_t\Big] = \big(E\big[F_\pi(T)|\mathcal{H}_t\big]\big)^{-1}, \quad t \in [0,T]. \tag{3.25}$$

(ii) Conversely, if $g(\delta) := E\big[U(X_{\pi+\delta\beta}(T))\big]$, $\delta \in (-\zeta, \zeta)$, is concave and (3.24) *is an $(\mathbb{H}, P)$-martingale, then $\pi \in \mathcal{A}$ is optimal for the problem* (3.5).

Proof. If $\pi \in \mathcal{A}$ is an optimal portfolio for an insider, then by Theorem 3.3 we know that $M_\pi(t)$, $t \in [0,T]$, is an $(\mathbb{H}, Q_\pi)$-martingale. Applying the Girsanov theorem (see e.g. [P] Theorem III.35) we obtain that

$$\hat{M}_\pi(t) := M_\pi(t) - \int_0^t \frac{d[M_\pi, Z_\pi](s)}{Z_\pi(s)}, \quad t \in [0,T],$$

is an $(\mathbb{H}, P)$-martingale with

$$Z_\pi(t) = E_{Q_\pi}\Big[\frac{dP}{dQ_\pi}|\mathcal{H}_t\Big] = E\Big[(F_\pi(T))^{-1}\frac{F_\pi(T)}{E\big[F_\pi(T)|\mathcal{H}_t\big]}|\mathcal{H}_t\Big] = \big(E\big[F_\pi(T)|\mathcal{H}_t\big]\big)^{-1}.$$

Conversely, if $\hat{M}_\pi(t)$, $t \in [0,T]$, is $(\mathbb{H}, P)$-martingale, then $M_\pi(t)$, $t \in [0,T]$, is an $(\mathbb{H}, Q_\pi)$-martingale. Hence π is optimal by Theorem 3.3. ■

4 Examples.

In this section we give some examples to illustrate the contents of the main results in Section 3.

Example A. Suppose that

$$\sigma(t) \neq 0,\ \theta = 0 \ \text{ and } \ \mathcal{H}_t = \mathcal{F}_t \vee \sigma(B(T_0)), \text{ for all } t \in [0,T] \text{ (for some } T_0 > T\text{)}, \tag{4.1}$$

i.e. we consider a market driven by the Brownian motion only and where the insider's filtration is a classical example of enlargement of the filtration $\mathbb{F}$ by the knowledge derived from the value of the Brownian motion at some future time $T_0 > T$. Then we obtain the following result.

Theorem 4.1 *Suppose that the function Λ in (3.22) is concave for all $s \in [0,T]$. A portfolio $\pi \in \mathcal{A}$ is optimal for the problem (3.5) if and only if $d[M_\pi, Z_\pi](t)$ is absolutely continuous with respect to the Lebesgue measure dt and*

$$\begin{aligned} &\mu'(t,\pi(t))\pi(t) + \mu(t,\pi(t)) - r(t) \\ &- \sigma^2(t)\pi(t) + \sigma(t)\Big[\frac{B(T_0)-B(t)}{T_0-t} - \frac{1}{Z_\pi(t)}\frac{d}{dt}[B,Z_\pi](t)\Big] = 0. \end{aligned} \tag{4.2}$$

Proof. By Theorem 3.4 the portfolio $\pi \in \mathcal{A}$ is optimal for the problem (3.5) if and only if the process

$$\begin{aligned} \hat{M}_\pi(t) = \int_0^t &\{\mu'(s,\pi(s))\pi(s) \\ &+ \mu(s,\pi(s)) - r(s) - \sigma^2(s)\pi(s)\}ds + \int_0^t \sigma(s)d^-B(s) - \int_0^t \frac{d[M_\pi, Z_\pi](s)}{Z_\pi(s)} \end{aligned} \tag{4.3}$$

is an $(\mathbb{H}, P)$-martingale. Since $\hat{M}_\pi(t)$ is continuous and has quadratic variation

$$[\hat{M}_\pi, \hat{M}_\pi](t) = \int_0^t \sigma^2(s)ds$$

we conclude that $\hat{M}_\pi(t)$ can be written

$$\hat{M}_\pi(t) = \int_0^t \sigma(s)d\hat{B}(s) \tag{4.4}$$

for some $(\mathbb{H}, P)$-Brownian motion $\hat{B}$.

On the other hand, by a result of Itô [I] we know that $B(t)$ is a semimartingale with respect to $(\mathbb{H}, P)$ with decomposition

$$B(t) = \tilde{B}(t) + \int_0^t \frac{B(T_0)-B(s)}{T_0-s}ds, \qquad 0 \leq t \leq T, \tag{4.5}$$

for some $(\mathbb{H}, P)$-Brownian motion $\tilde{B}(t)$. Combining (4.3), (4.4) and (4.5) we get

$$
\begin{aligned}
\sigma(t)d\hat{B}(t) = d\hat{M}_\pi(t) = \big\{\mu'(t,\pi(t))\pi(t) \\
+ \mu(t,\pi(t)) - r(t) - \sigma^2(t)\pi(t)\big\}dt + \sigma(t)d\tilde{B}(t) \\
+ \sigma(t)\frac{B(T_0)-B(t)}{T_0-t}dt - \frac{d[M_\pi, Z_\pi](t)}{Z_\pi(t)}.
\end{aligned} \tag{4.6}
$$

By uniqueness of the semimartingale decomposition of $\hat{M}_\pi(t)$ with respect to $(\mathbb{H}, P)$ we conclude that $\hat{B}(t) = \tilde{B}(t)$ and

$$
\begin{aligned}
\big\{\mu'(t,\pi(t))\pi(t) + \mu(t,\pi(t)) - r(t) - \sigma^2(t)\pi(t) \\
\sigma(t)\frac{B(T_0)-B(t)}{T_0-t}\big\}dt - \frac{d[M_\pi, Z_\pi](t)}{Z_\pi(t)} = 0.
\end{aligned} \tag{4.7}
$$

From this we deduce that $d[M_\pi, Z_\pi](t) = \sigma(t)d[B, Z_\pi](t)$ is absolutely continuous with respect to dt and (4.2) follows. ■

Corollary 4.2 *Assume that* (4.1) *holds and, in addition, that*

$$
\mu(t,\pi) = \mu_0(t) + a(t)\pi \tag{4.8}
$$

for some $\mathbb{F}$-adapted processes μ_0 and a with $0 \le a(t) \le \frac{1}{2}\sigma^2(t)$, $t \in [0,T]$, which do not depend on π. Then $\pi \in \mathcal{A}$ is optimal if and only if $d[M_\pi, Z_\pi](t)$ is absolutely continuous with respect to dt and

$$
\big(\sigma^2(t) - 2a(t)\big)\pi(t) = \mu_0(t) - r(t) + \sigma(t)\Big[\frac{B(T_0)-B(t)}{T_0-t} - \frac{1}{Z_\pi(t)}\frac{d[B, Z_\pi](t)}{dt}\Big]. \tag{4.9}
$$

Proof. In this case we have that $\mu'(t,\pi(t)) = a(t)$. Therefore the function Λ defined in (3.22) is concave (by (3.23)) and the result follows from Theorem 4.1. ■

Next we give an example for a pure jump financial market.

Example B. Suppose that

$$
\sigma(t) = 0 \quad \text{and} \quad \theta(t,z) = \beta z, \tag{4.10}
$$

where $\beta z > -1$ $\nu(dz)$-a.e. $(\beta > 0)$ and that

$$
\mathcal{H}_t = \mathcal{F}_t \vee \sigma(\eta(T_0)) \;\text{ for some } T_0 > T, \tag{4.11}
$$

where

$$
\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds,dz)
$$

(i.e. the insider's filtration is the enlargement of $\mathbb{F}$ by the knowledge derived from some future value $\eta(T_0)$ of the market driving process). Then by a result of Itô, as extended by Kurtz (see [P] p. 256), the process

$$
\hat{\eta}(t) := \eta(t) - \int_0^t \frac{\eta(T_0)-\eta(s)}{T_0-s}ds \tag{4.12}
$$

is an $(\mathbb{H}, P)$-martingale. By Proposition 5.2 in [DMØP2] the $\mathbb{H}$-compensating measue $\nu_{\mathbb{H}}$ of the jump measure N is given by

$$
\begin{aligned}
(4.13) \qquad \nu_{\mathbb{H}}(ds, dz) &= \nu_{\mathbb{F}}(dz)ds + E\Big[\frac{1}{T_0 - s}\int_s^{T_0} \widetilde{N}(dr, dz)\big|\mathcal{H}_s\Big]ds \\
&= E\Big[\frac{1}{T_0 - s}\int_s^{T_0} N(dr, dz)\big|\mathcal{H}_s\Big]ds.
\end{aligned}
$$

where $\nu_{\mathbb{F}} = \nu$. This implies that the $\mathbb{H}$-compensated random measure $\widetilde{N}_{\mathbb{H}}$ is related to $\widetilde{N}_{\mathbb{F}} = \widetilde{N}$ by

$$
(4.14) \quad \widetilde{N}_{\mathbb{H}}(ds, dz) = N(ds, dz) - \nu_{\mathbb{H}}(ds, dz) = \widetilde{N}(ds, dz) - E\Big[\frac{1}{T_0 - s}\int_s^{T_0} \widetilde{N}(dr, dz)\big|\mathcal{H}_s\Big]ds.
$$

Hence, directly from the definition of the forward integral, we have

$$
\begin{aligned}
(4.15) \qquad \int_0^t \int_{\mathbb{R}_0} \frac{\beta z}{1 + \pi(s)\beta z}\widetilde{N}(d^- s, dz) &= \int_0^t \int_{\mathbb{R}_0} \frac{\beta z}{1 + \pi(s)\beta z}\widetilde{N}_{\mathbb{H}}(ds, dz) \\
+ \int_0^t \int_{\mathbb{R}_0} \frac{\beta z}{1 + \pi(s)\beta z} & E\Big[\frac{1}{T_0 - s}\int_s^{T_0} \widetilde{N}(dr, dz)\big|\mathcal{H}_s\Big]ds.
\end{aligned}
$$

By Theorem 3.4 a portfolio $\pi \in \mathcal{A}$ is optimal if and only if the process

$$
\begin{aligned}
(4.16) \qquad \hat{M}_\pi(t) = \int_0^t & \big\{\mu(s, \pi(s)) - r(s) + \mu'(s, \pi(s))\pi(s) \\
&- \int_{\mathbb{R}_0} \frac{\beta^2 z^2 \pi(s)}{1 + \pi(s)\beta z}\nu(dz)\big\}ds \\
&+ \int_0^t \int_{\mathbb{R}_0} \frac{\beta z}{1 + \pi(s)\beta z}\widetilde{N}(d^- s, dz) - \int_0^t \frac{d[M_\pi, Z_\pi](s)}{Z_\pi(s)}
\end{aligned}
$$

is an $(\mathbb{H}, P)$-martingale. Therefore, if we put

$$
\begin{aligned}
(4.17) \qquad G_\pi(s) := & \mu(s, \pi(s)) - r(s) + \mu'(s, \pi(s))\pi(s) \\
&- \int_{\mathbb{R}_0} \frac{\beta^2 z^2 \pi(s)}{1 + \pi(s)\beta z}\nu(dz) \\
&+ \int_{\mathbb{R}_0} \frac{\beta z}{1 + \pi(s)\beta z} E\Big[\frac{1}{T_0 - s}\int_s^{T_0} \widetilde{N}(dr, dz)\big|\mathcal{H}_s\Big],
\end{aligned}
$$

and combine (4.15) and (4.16), we obtain that the process

$$
\hat{M}_\pi(t) = \int_0^t G_\pi(s)ds - \int_0^t \frac{d[M_\pi, Z_\pi](s)}{Z_\pi(s)} + \int_0^t \int_{\mathbb{R}_0} \frac{\beta z}{1 + \pi(s)\beta z}\widetilde{N}_{\mathbb{H}}(ds, dz)
$$

is an $(\mathbb{H}, P)$-martingale. This is possible if and only if

$$
\int_0^t G_\pi(s)ds - \int_0^t \frac{d[M_\pi, Z_\pi](s)}{Z_\pi(s)} = 0, \quad \text{for all } t \in [0, T].
$$

This implies that $d[M_\pi, Z_\pi](t)$ is absolutely continuous with respect to the Lebesgue measure dt. We have thus proved the following statement.

Theorem 4.3 *Assume that* (4.10) *and* (4.11) *hold. Then* $\pi \in \mathcal{A}$ *is optimal if and only if* $d[M_\pi, Z_\pi](t)$ *is absolutely continuous with respect to the Lebesgue measure* dt *and*

$$G_\pi(t) = \frac{1}{Z_\pi(t)} \frac{d}{dt}[M_\pi, Z_\pi](t) \quad \text{for almost all } t \in [0,T] \tag{4.18}$$

where G_π *is given by* (4.17).

In analogy with Corollary 4.2 we get the following result in the special case when the influence of the trader on the market is given by (4.8).

Corollary 4.4 *Assume that* (4.10) *and* (4.11) *hold and, in addition, that also* (4.8) *holds. Then* $\pi \in \mathcal{A}$ *is optimal if and only if* $d[M_\pi, Z_\pi](t)$ *is absolutely continuous with respect to the Lebesgue measure* dt *and*

$$\begin{aligned} &\pi(s) \int_{\mathbb{R}_0} \frac{\beta^2 z^2}{1+\pi(s)\beta z} \nu(dz) - \int_{\mathbb{R}_0} \frac{\beta z}{1+\pi(s)\beta z} E\Big[\frac{1}{T_0 - s} \int_s^{T_0} \widetilde{N}(dr, dz) \big| \mathcal{H}_s\Big] \\ &- 2a(s)\pi(s) = \mu_0(s) - r(s) - \frac{1}{Z_\pi(s)} \frac{d}{ds}[M_\pi, Z_\pi](s). \end{aligned} \tag{4.19}$$

Corollary 4.5 *Suppose that* (4.8), (4.10) *and* (4.11) *hold and that*

$$U(x) = \ln x, \quad x \geq 0. \tag{4.20}$$

Then $\pi \in \mathcal{A}$ *is optimal if and only if*

$$\begin{aligned} \pi(s) \int_{\mathbb{R}_0} \frac{\beta^2 z^2}{1+\pi(s)\beta z} \nu(dz) - \int_{\mathbb{R}_0} \frac{\beta z}{1+\pi(s)\beta z} E\Big[\frac{1}{T_0 - s} \int_s^{T_0} \widetilde{N}(dr, dz) \big| \mathcal{H}_s\Big] - 2a(s)\pi(s) \\ = \mu_0(s) - r(s). \end{aligned}$$

Proof. If $U(x) = \ln x$ then $F_\pi(T) = 1 = Z_\pi(t)$, $t \in [0,T]$. Hence $[M_\pi, Z_\pi] = 0$. ■

Acknowledgment. The authors would like to thank Terje Bjuland for his useful comments.

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