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AMERICAN OPTIONS IN AN IMPERFECT COMPLETE MARKET WITH DEFAULT

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Abstract **WARNING: — An English abstract is mandatory! —**

Abstract. We study pricing and hedging for American options in an imperfect market model with default, where the imperfections are taken into account via the nonlinearity of the wealth dynamics. The payoff is given by an RCLL adapted process (ξ_t). We define the *seller's price* of the American option as the minimum of the initial capitals which allow the seller to build up a (super)hedging portfolio. We prove that this price coincides with the value function of an optimal stopping problem with a nonlinear expectation $\mathcal{E}^g$ (induced by a BSDE), which corresponds to the solution of a nonlinear reflected BSDE with obstacle (ξ_t). Moreover, we show the existence of a (super)hedging portfolio strategy. We then consider the *buyer's price* of the American option, which is defined as the supremum of the initial prices which allow the buyer to select an exercise time τ and a portfolio strategy φ so that he/she is superhedged. We show that the *buyer's price* is equal to the value function of an optimal stopping problem with a nonlinear expectation, and that it can be characterized via the solution of a reflected BSDE with obstacle (ξ_t). Under the additional assumption of left upper semicontinuity along stopping times of (ξ_t), we show the existence of a super-hedge (τ, φ) for the buyer.

Key-words: American options, imperfect markets, nonlinear expectation, superhedging, default, reflected backward stochastic differential equations

1. Introduction

We consider an American option associated with a terminal time T and a payoff given by an RCLL adapted process (ξ_t). The case of a classical perfect market has been largely studied in the literature (see e.g. [16, 18]). Recall that the *seller's price* (called also fair price in the literature), denoted by u_0, is classically defined as the minimal initial capital which enables the seller to invest in a portfolio which covers his liability to pay to the buyer up to T no matter what the exercise time chosen by the buyer. Moreover, this price is equal to the value function of the following optimal stopping time problem

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_Q(\tilde{\xi}_\tau), \tag{1}$$

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where $\mathcal{T}$ is a set of stopping times valued in $[0,T]$. Here, $\tilde{\xi}_t$ denotes the discounted value of ξ_t, equal to $e^{-rt}\xi_t$ in the Black and Scholes model, where r is the instantaneous interest rate. Moreover, $\mathbb{E}_Q$ denotes the expectation under the unique martingale probability measure Q of the market model. In [9], the seller's price is characterized via a reflected BSDE with lower obstacle (ξ_t).

The aim of the present paper is to study pricing and hedging issues for American options in the case of imperfections in the market model taken into account via the nonlinearity of the dynamics of the wealth (or equivalently, of the porfolio's values), which are modeled via a nonlinear driver g. We moreover include the possibility of a default. The market model we consider here is *complete*, in the sense that for each European option, there exists a unique portfolio such that its value at the exercise time is equal to the payoff. A large class of imperfect market models can fit in our framework, like different borrowing and lending interest rates, and the case when the seller of the option is a "large trader" whose hedging strategy may affect the market prices and even the default probability.

We provide a characterization of the seller's price u_0 as the value of a corresponding *optimal stopping problem* with a nonlinear expectation, more precisely

$$u_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^g(\xi_\tau), \tag{2}$$

where $\mathcal{E}^g$ is the g-evaluation (also called g-expectation) induced by a nonlinear BSDE with default jump solved under the primitive probability measure P with driver g. Note that in the particular case of a perfect market, the driver g is linear and (2) reduces to (1). We also show that the seller's price can be characterized via the solution of the reflected BSDE with driver g and lower obstacle (ξ_t), as well as the existence of a (super)hedging portfolio strategy for the seller.

We then consider the *buyer's price* of the American option, denoted by v_0, defined as the supremum of the initial prices which allow the buyer to select an exercise time τ and a portfolio strategy φ so that he/she is (super)hedged. We provide a characterization of the buyer's price v_0 as the value of a corresponding *optimal stopping problem* with nonlinear expectation. More precisely, we prove that

$$v_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau),$$

where $\mathcal{E}^{\tilde{g}}$ is the $\tilde{g}$-expectation, associated with driver $\tilde{g}(t,y,z,k) := -g(t,-y,-z,-k)$. Moreover, we show that the buyer's price can also be characterized via the solution of a nonlinear reflected BSDE with obstacle (ξ_t) and driver $\tilde{g}$. Under the additional assumption of left upper-semicontinuity along stopping times of (ξ_t), we show the existence of a super-hedge $(\tilde{\tau}, \tilde{\varphi})$ for the buyer.

Note that in the classical case of a perfect market, the buyer's price is equal to the seller's price, that is $v_0 = u_0$, since, in this case, $\tilde{g} = g$.

When $-g(t,-y,-z,-k) \leq g(t,y,z,k)$, then $v_0 \leq u_0$. The interval $[v_0, u_0]$ can then be seen as a non-arbitrage interval for the price of the American option in the sense of [15]. In the example of a higher interest rate for borrowing, this result corresponds to the one shown in [15] by a dual approach.

The paper is organized as follows: in Section 2, we introduce our imperfect market model with default and nonlinear wealth dynamics. In Section 3, we study pricing and (super)hedging of American options from the seller's point of view. In Section 4, we address the pricing and (super)hedging problem from the buyer's point of view.

2. Imperfect market model with default

2.1. Market model with default

Let $(\Omega, \mathcal{G}, P)$ be a complete probability space equipped with two stochastic processes: a unidimensional standard Brownian motion W and a jump process N defined by $N_t = \mathbf{1}_{\vartheta \leq t}$ for all $t \in [0,T]$, where ϑ is a random variable which models a default time. We assume that this default can appear at any time, that is

$P(\vartheta \geq t) > 0$ for all $t \geq 0$. We denote by $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ the *augmented filtration* generated by W and N (in the sense of [4, IV-48]). We suppose that W is a $\mathbb{G}$-Brownian motion. We denote by $\mathcal{P}$ the $\mathbb{G}$-predictable σ-algebra. Let (Λ_t) be the predictable compensator of the nondecreasing process (N_t). Note that $(\Lambda_{t\wedge\vartheta})$ is then the predictable compensator of $(N_{t\wedge\vartheta}) = (N_t)$. By uniqueness of the predictable compensator, $\Lambda_{t\wedge\vartheta} = \Lambda_t$, $t \geq 0$ a.s. We assume that Λ is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a nonnegative process λ, called the intensity process, such that $\Lambda_t = \int_0^t \lambda_s ds$, $t \geq 0$. Since $\Lambda_{t\wedge\vartheta} = \Lambda_t$, λ vanishes after ϑ. The compensated martingale is given by

$$M_t := N_t - \int_0^t \lambda_s ds \,.$$

Let $T > 0$ be the terminal time. We define the following sets:

- S^2 is the set of $\mathbb{G}$-adapted RCLL processes φ such that $\mathbb{E}[\sup_{0\leq t\leq T} |\varphi_t|^2] < +\infty$.
- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.
- $\mathbb{H}^2$ is the set of $\mathbb{G}$-predictable processes Z such that $\|Z\|^2 := \mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] < \infty$.
- $\mathbb{H}^2_\lambda := L^2(\Omega \times [0,T], \mathcal{P}, \lambda_t \, dP \otimes dt)$, equipped with scalar product $\langle U, V\rangle_\lambda := \mathbb{E}\Big[\int_0^T U_t V_t \lambda_t dt\Big]$, for all U, V in $\mathbb{H}^2_\lambda$. For all $U \in \mathbb{H}^2_\lambda$, we set $\|U\|^2_\lambda := \mathbb{E}\Big[\int_0^T |U_t|^2 \lambda_t dt\Big] < \infty$.

Since λ vanishes after ϑ, we can suppose that for each U in $\mathbb{H}^2_\lambda = L^2(\Omega \times [0,T], \mathcal{P}, \lambda_t \, dP \otimes dt)$, U (or its representant still denoted by U) vanishes after ϑ.
Moreover, $\mathcal{T}$ denotes the set of stopping times τ such that $\tau \in [0,T]$ a.s. and for each S in $\mathcal{T}$, $\mathcal{T}_S$ is the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

Recall that in this setup, we have a martingale representation theorem with respect to W and M (see e.g. Lemma 1 in [7]).

We consider a financial market which consists of one risk-free asset, with price process S^0 satisfying $dS_t^0 = S_t^0 r_t dt$, and two risky assets with price processes S^1, S^2 evolving according to the following equations:

$$\begin{cases} dS_t^1 = S_t^1[\mu_t^1 dt + \sigma_t^1 dW_t] \\ dS_t^2 = S_{t^-}^2[\mu_t^2 dt + \sigma_t^2 dW_t - dM_t]. \end{cases}$$

The process $S^0 = (S_t^0)_{0\leq t\leq T}$ corresponds to the price of a non risky asset with interest rate process $r = (r_t)_{0\leq t\leq T}$, $S^1 = (S_t^1)_{0\leq t\leq T}$ to a non defaultable risky asset, and $S^2 = (S_t^2)_{0\leq t\leq T}$ to a defaultable asset with total default. The price process S^2 vanishes after ϑ.

All the processes σ^1, σ^2, r, μ^1, μ^2 are predictable (that is $\mathcal{P}$-measurable). We suppose that the coefficients $\sigma^1, \sigma^2 > 0$, and r, σ^1, σ^2, μ^1, μ^2, λ, λ^{-1},$(\sigma^1)^{-1}$, $(\sigma^2)^{-1}$ are bounded.

We consider an investor, endowed with an initial wealth equal to x, who can invest his wealth in the three assets of the market. At each time $t < \vartheta$, he chooses the amount φ_t^1 (resp. φ_t^2) of wealth invested in the first (resp. second) risky asset. However, after time ϑ, he cannot invest his wealth in the defaultable asset since its price is equal to 0, and he only chooses the amount φ_t^1 of wealth invested in the first risky asset. Note that the process φ^2 can be defined on the whole interval $[0,T]$ by setting $\varphi_t^2 = 0$ for each $t \geq \vartheta$. A process $\varphi_\cdot = (\varphi_t^1, \varphi_t^2)'_{0\leq t\leq T}$ is called a *risky assets strategy* if it belongs to $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$. The value of the associated portfolio (also called *wealth*) at time t is denoted by $V_t^{x,\varphi}$ (or simply V_t).
The perfect market model. In the classical case of a perfect market model, the wealth process and the strategy satisfy the self financing condition:

$$dV_t = (r_t V_t + \varphi_t^1(\mu_t^1 - r_t) + \varphi_t^2(\mu_t^2 - r_t))dt + (\varphi_t^1\sigma_t^1 + \varphi_t^2\sigma_t^2)dW_t - \varphi_t^2 dM_t. \tag{3}$$

Setting $K_t := -\varphi_t^2$, and $Z_t := \varphi_t^1\sigma_t^1 + \varphi_t^2\sigma_t^2$, we get

$$dV_t = (r_t V_t + Z_t\theta_t^1 + K_t\theta_t^2\lambda_t)dt + Z_t dW_t + K_t dM_t,$$

where $\theta_t^1 := \dfrac{\mu_t^1 - r_t}{\sigma_t^1}$ and $\theta_t^2 := \dfrac{\sigma_t^2\theta_t^1 - \mu_t^2 + r_t}{\lambda_t}\mathbf{1}_{\{t\leq\vartheta\}}$.

Consider a European contingent claim with maturity $T > 0$ and $\mathcal{G}_T$-measurable payoff ξ in L^2. The problem is to price and hedge this claim by constructing a replicating portfolio. From [7, Proposition 2.6], there exists an unique process $(X, Z, K) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ solution of the following BSDE with default jump:

$$-dX_t = -(r_t X_t + Z_t\theta_t^1 + K_t\theta_t^2\lambda_t)dt - Z_t dW_t - K_t dM_t\,; \quad X_T = \xi. \tag{4}$$

The solution (X, Z, K) provides the replicating portfolio. More precisely, the process X corresponds to its value, and the hedging risky assets stategy φ in $\mathbb{H}^2_\lambda$ is given by $\varphi = \Phi(Z, K)$, where Φ is the one-to-one map defined on $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ by:

Definition 2.1. *Let $\Phi : \mathbb{H}^2 \times \mathbb{H}^2_\lambda \to \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ be the one-to-one map defined for each $(Z, K) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ by $\Phi(Z, K) := \varphi$, where $\varphi = (\varphi^1, \varphi^2)$ is given by*

$$\varphi_t^2 = -K_t \;\; ; \;\; \varphi_t^1 = \frac{Z_t + \sigma_t^2 K_t}{\sigma_t^1},$$

which is equivalent to $K_t = -\varphi_t^2\,;\;\; Z_t = \varphi_t^1\sigma_t^1 + \varphi_t^2\,\sigma_t^2 = {\varphi_t}'\sigma_t$.

Note that the processes φ^2 and K, which belong to $\mathbb{H}^2_\lambda$, both vanish after time ϑ.

The process X coincides with $V^{X_0,\varphi}$, the value of the portfolio associated with initial wealth $x = X_0$ and portfolio strategy φ. Since $X_T = \xi$ a.s. , we derive that $V_T^{X_0,\varphi} = \xi$ a.s.[1] From the seller's point of view, this portfolio is a hedging portfolio. Indeed, by investing at initial time 0 the initial amount X_0 in the reference assets along the strategy φ, the seller can pay the amount ξ to the buyer at time T (and similarly at each initial time t with initial amount X_t). We derive that X_t is the price at time t of the option, called *hedging price*, and denoted by $X_t(\xi)$. By the representation property of the solution of a λ-linear BSDE with default jump (see [7, Theorem 2.13]), the solution X of BSDE (4) is given by:

$$X_t(\xi) = \mathbb{E}[e^{-\int_t^T r_s ds}\zeta_{t,T}\xi \,|\, \mathcal{G}_t], \tag{5}$$

where $\zeta_{t,\cdot}$ satisfies $d\zeta_{t,s} = \zeta_{t,s^-}[-\theta_s^1 dW_s - \theta_s^2 dM_s]$ with $\zeta_{t,t} = 1$. This defines a *linear* price system X: $\xi \mapsto X(\xi)$. Suppose now that

$$\theta_t^2 < 1, \; 0 \leq t \leq \vartheta \;\; dt \otimes dP - a.s. \tag{6}$$

Then $\zeta_{t,\cdot} > 0$. Let Q be the probability measure which admits $\zeta_{0,T}$ as density on $\mathcal{G}_T$. Using Girsanov's theorem, it can be shown that Q is the unique martingale probability measure. In this case, the price system X is increasing and corresponds to the classical arbitrage free price system (see [2, 14]).

The imperfect market model $\mathcal{M}^g$. From now on, we assume that there are imperfections in the market which are taken into account via the *nonlinearity* of the dynamics of the wealth. More precisely, the dynamics of the wealth V associated with strategy $\varphi = (\varphi^1, \varphi^2)$ can be written via a *nonlinear* driver, defined as follows:

Definition 2.2 (Driver, λ-*admissible* driver)**.** *A function g is said to be a* driver *if $g : [0, T] \times \Omega \times \mathbb{R}^3 \to \mathbb{R}$; $(\omega, t, y, z, k) \mapsto g(\omega, t, y, z, k)$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)-$ measurable, and such that $g(., 0, 0, 0) \in \mathbb{H}^2$. A driver g is called a* λ-admissible driver *if moreover there exists a constant $C \geq 0$ such that $dP \otimes dt$-a.s. , for each (y_1, z_1, k_1), (y_2, z_2, k_2),*

$$|g(t, y_1, z_1, k_1) - g(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda_t}|k_1 - k_2|). \tag{7}$$

A nonnegative constant C which satisfies this inequality is called a λ-constant *associated with driver g.*

[1]This property holds for any payoff $\xi \in L^2(\mathcal{F}_T)$, which corresponds to the so-called *completeness* property of the perfect market.

Note that condition (7) implies that for each $t > \vartheta$, since $\lambda_t = 0$, g does not depend on k. In other terms, $dP \otimes dt$-a.s. , for all (y, z, k), we have: $g(t, y, z, k) = g(t, y, z, 0)$, $t > \vartheta$.

Let $x \in \mathbb{R}$ be the initial wealth and let $\varphi = (\varphi^1, \varphi^2)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ be a portfolio strategy. We suppose now that the associated *wealth* process $V_t^{x,\varphi}$ (or simply V_t) satisfies the dynamics:

$$-dV_t = g(t, V_t, {\varphi_t}'\sigma_t, -\varphi_t^2)dt - {\varphi_t}'\sigma_t dW_t + \varphi_t^2 dM_t. \tag{8}$$

with $V_0 = x$.[2] Equivalently, setting $Z_t = {\varphi_t}'\sigma_t$ and $K_t = -\varphi_t^2$, the dynamics (8) of the wealth process V_t can be written as follows:

$$-dV_t = g(t, V_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t. \tag{9}$$

In the following, our imperfect market model is denoted by $\mathcal{M}^g$.
Note that in the case of a perfect market (see (3)), we have:

$$g(t, y, z, k) = -r_t y - \theta_t^1 z - \theta_t^2 k \lambda_t, \tag{10}$$

which is a λ-admissible driver since by the assumptions on the coefficients of the model, the processes θ^1 and θ^2 are bounded.

2.2. Nonlinear pricing system $\mathcal{E}^g$ (for the seller)

Pricing and hedging European options in the imperfect market $\mathcal{M}^g$ leads to BSDEs with nonlinear driver g and a default jump. By [7, Proposition 2.6], we have

Proposition 2.3. *Let g be a λ-admissible driver, let $\xi \in L^2(\mathcal{G}_T)$. There exists an unique solution $(X(T,\xi), Z(T,\xi), K(T,\xi))$ (denoted simply by (X, Z, K)) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the following BSDE:*

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t; \quad X_T = \xi. \tag{11}$$

Let us consider a European option with maturity T and terminal payoff ξ in $L^2(\mathcal{G}_T)$ in this market model. Let $x \in \mathbb{R}$ and and let $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$. The process φ is called a *hedging risky-assets portfolio strategy for the seller* (associated with initial capital x) if the value of the associated portfolio is equal to ξ at time T, that is $V_T^{x,\varphi} = \xi$ a.s.

Let (X, Z, K) be the solution of BSDE (11). The process X is equal to the value of the portfolio associated with initial capital $x = X_0$ and the risky assets strategy $\varphi = \Phi(Z, K)$ (where Φ is defined in Definition 2.1), that is $X = V^{X_0,\varphi}$. Since $X_T = \xi$, the process φ is thus a *hedging risky-assets strategy for the seller* associated with initial capital $x = X_0$.

Remark 2.4 (*Completeness* property of the market $\mathcal{M}^g$)**.** *By the above observations, we derive that for each $\xi \in L^2(\mathcal{G}_T)$, there exists a unique initial capital $x \in \mathbb{R}$ and a unique risky-assets strategy φ such that the value of the associated portfolio is equal to ξ at time T. By analogy with the classical case of a perfect market, this property is called* completeness property *of the market $\mathcal{M}^g$.*

Using the previous notation, the initial value $X_0 = X_0(T, \xi)$ of the hedging portfolio is thus a sensible price (at time 0) of the claim ξ for the seller since this amount is the unique initial capital which allows him/her to build a *hedging risky-assets strategy*. Similarly, $X_t = X_t(T, \xi)$ satisfies an analogous property at time t, and is called the *hedging price* (for the seller) at time t. This leads to a *nonlinear pricing* system, first introduced in [11] (also called *g-evaluation* in [19]) and denoted by $\mathcal{E}^g$. For each $S \in [0, T]$, for each $\xi \in L^2(\mathcal{G}_S)$ the associated g-evaluation is defined by $\mathcal{E}^g_{t,S}(\xi) := X_t(S, \xi)$ for each $t \in [0, S]$.

In order to ensure the (strict) monotonicity and the no arbitrage property of the nonlinear pricing system $\mathcal{E}^g$, we make the following assumption (see [7, Section 3.3]).

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[2]Note that since g is Lipschitz continuous with respect to y, the process $(V_t^{x,\varphi})$ is well defined as the unique solution of the forward differential equation (8) with initial condition $V_0^{x,\varphi} = x$.

Assumption 2.5. *Assume that there exists a bounded map*

$$\gamma : [0,T] \times \Omega \times \mathbb{R}^4 \to \mathbb{R}\,;\ (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^4)$*-measurable and satisfying* $dP \otimes dt$*-a.s. , for each* $(y, z, k_1, k_2) \in \mathbb{R}^4$*,*

$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \gamma_t^{y,z,k_1,k_2}(k_1 - k_2)\lambda_t, \tag{12}$$

and P*-a.s. , for each* $(y, z, k_1, k_2) \in \mathbb{R}^4$*,* $\gamma_t^{y,z,k_1,k_2} > -1$*.*

This assumption is satisfied e.g. when g is $\mathcal{C}^1$ in k with $\partial_k g(t,\cdot) > -\lambda_t$ on $\{t \leq \vartheta\}$. In the special case of a perfect market, g is given by (10), which implies that $\partial_k g(t,\cdot) = -\theta_t^2 \lambda_t$. Assumption 2.5 is then equivalent to $\theta_t^2 < 1$ (which corresponds to the usual assumption (6)).

Remark 2.6. *Assume that* $g(t, 0, 0, 0) = 0$ $dP \otimes dt$*-a.s. Then for all* $S \in [0,T]$*,* $\mathcal{E}^g_{\cdot,S}(0) = 0$ *a.s. Moreover, by the comparison theorem for BSDEs with default jump (see [7, Theorem 2.17]), the nonlinear pricing system* $\mathcal{E}^g$ *is nonnegative, that is, for all* $S \in [0,T]$*, for all* $\xi \in L^2(\mathcal{G}_S)$*, if* $\xi \geq 0$ *a.s., then* $\mathcal{E}^g_{\cdot,S}(\xi) \geq 0$ *a.s.*

Definition 2.7. *Let* $Y \in S^2$*. The process* (Y_t) *is said to be a strong* $\mathcal{E}$*-supermartingale (resp. martingale) if* $\mathcal{E}_{\sigma,\tau}(Y_\tau) \leq Y_\sigma$ *(resp.* $= Y_\sigma$*) a.s. on* $\sigma \leq \tau$*, for all* $\sigma, \tau \in \mathcal{T}_0$*.*

Note that, by the flow property of BSDEs, for each $S \in [0,T]$ and for each $\xi \in L^2(\mathcal{G}_S)$, the g-evaluation $\mathcal{E}^g_{\cdot,S}(\xi)$ is an $\mathcal{E}^g$-martingale. Moreover, since $V_t^{x,\varphi} = \mathcal{E}^g_{t,T}(V_T^{x,\varphi})$, we have:

Proposition 2.8. *For each* $x \in \mathbb{R}$ *and each portfolio strategy* $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$*, the associated wealth process* $V^{x,\varphi}$ *is an* $\mathcal{E}^g$*-martingale.*

Example 2.9 (Large investor seller). *Suppose that the seller of the option is a large trader whose hedging strategy* φ *and its associated value* V *may influence the market prices (see e.g. [1, 3]). Taking into account the possible feedback effects in the market model, the large trader-seller may consider that the coefficients are of the form* $\sigma_t(\omega) = \bar{\sigma}(\omega, t, V_t, \varphi_t)$ *where* $\bar{\sigma} : \Omega \times [0,T] \times \mathbb{R}^3 \mapsto \mathbb{R}^2\,;\ (\omega, t, x, z, k) \mapsto \bar{\sigma}(\omega, t, x, z, k)$ *is a* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)/\mathcal{B}(\mathbb{R}^2)$*-measurable map, and similarly for the other coefficients* r*,* μ^1*,* μ^2*. This leads to a driver of the form:*

$$g(t, V_t, \varphi_t' \bar{\sigma}_t(t, V_t, \varphi_t), -\varphi_t^2) = -\bar{r}(t, V_t, \varphi_t)V_t - \varphi_t^1\,(\bar{\mu}_t^1 - \bar{r}_t)(t, V_t, \varphi_t) - \varphi_t^2(\bar{\mu}_t^2 - \bar{r}_t)(t, V_t, \varphi_t).$$

Here, the map $\Psi : (\omega, t, y, \varphi) \mapsto (z, k)$ *with* $z = \varphi' \bar{\sigma}_t(\omega, t, y, \varphi)$ *and* $k = -\varphi^2$ *is assumed to be one-to-one with respect to* φ*, and such that its inverse* Ψ_φ^{-1} *is* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)/\mathcal{B}(\mathbb{R}^2)$*-measurable.*

We can also consider the case when the seller strategy influences the default probability via the default intensity by taking a driver of the form [3]

$$g(t, V_t, \varphi_t' \bar{\sigma}_t(t, V_t, \varphi_t), -\varphi_t^2) - \gamma(t, V_t, \varphi_t)\lambda_t \varphi_t^2,$$

where $\gamma : \Omega \times [0,T] \times \mathbb{R}^3 \mapsto \mathbb{R}^2$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)/\mathcal{B}(\mathbb{R}^2)$*-measurable.*

For other examples such that the case of taxes and the case of different borrowing and lending interest rates, the reader is referred to [8] or [10].

Note that when the market is perfect, the prices S^0, S^1 and S^2 are $\mathcal{E}^g$-martingales.[4] This is also true in the examples of the large investor and different borrowing and lending interest rates. In the case of taxes, this property is not necessarily satisfied.

[3]For details, see in [7] the example of the large investor seller, in particular equation (3.12).

[4]This corresponds to the well-kown property that the discounted prices of the reference assets are martingales under the martingale probability measure Q.

3. American option pricing from the seller's point of view

Let us consider an American option associated with horizon $T > 0$ and a payoff given by an RCLL adapted process $(\xi_t, 0 \leq t \leq T)$. At time 0, it consists in the selection of a stopping time $\tau \in \mathcal{T}$ and the payment of the payoff ξ_τ from the seller to the buyer.

The *seller's price* of the American option at time 0, denoted by u_0, is classically defined as the minimal initial capital which enables the seller to invest in a portfolio which covers his liability to pay to the buyer up to T no matter what the exercise time chosen by the buyer. More precisely, for each initial wealth x, we denote by $\mathcal{A}(x)$ the set of all portfolio strategies $\varphi \in \mathbb{H}^2$ such that $V_t^{x,\varphi} \geq \xi_t$, $0 \leq t \leq T$ a.s. The *seller's price* of the American option is thus defined by

$$u_0 := \inf\{x \in \mathbb{R},\ \exists \varphi \in \mathcal{A}(x)\}.$$

Remark 3.1. *Suppose that $g(t,0,0,0) = 0$. From Remark 2.6, we derive that if $\xi_\cdot \geq 0$, the infimum in the definition of u_0 can be taken only over nonnegative initial wealths, that is, $u_0 := \inf\{x \geq 0,\ \exists \varphi \in \mathcal{A}(x)\}$.*

We define the *g-value* associated with the American option as the value function (at time 0) of the *$\mathcal{E}^g$-optimal stopping problem* associated with payoff (ξ_t), that is

$$\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau). \tag{13}$$

Proposition 3.2 (Characterization of the g-value)**.** *There exists a unique process (Y, Z, K, A) solution of the reflected BSDE (RBSDE) associated with driver g and obstacle ξ in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2$, that is*

$$-\,dY_t = g(t, Y_t, Z_t, K_t)dt + dA_t - Z_t dW_t - K_t dM_t;\ \ Y_T = \xi_T,\ \text{with} \tag{14}$$

$$Y \geq \xi\,,$$

$$\int_0^T (Y_t - \xi_t)dA_t^c = 0 \ \text{a.s. and}\ \Delta A_t^d = \Delta A_t^d \mathbf{1}_{\{Y_{t^-} = \xi_{t^-}\}}, \tag{15}$$

where A^c denotes the continuous part of A and A^d its discontinuous part. Moreover, for each $S \in \mathcal{T}$, we have

$$Y_S = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \ \mathcal{E}^g_{S,\tau}(\xi_\tau) \quad \text{a.s.} \tag{16}$$

In particular, Y_0 is equal to the g-value $\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau)$.

Note that this result was shown in [11] in a Brownian framework with a continuous obstacle, and generalized in [20] to the RCLL case with jumps.

Proof. Let us first show that there exists a unique solution of RBSDE (14). As usual, we first consider the case when the driver $g(t)$ does not depend on the solution. By using the representation property of $\mathbb{G}$-martingales (see e.g. Lemma 1 in [7]) and some results of optimal stopping theory, one can show, proceeding as in [9] (see also [13] and [20]), that there exists a unique solution of the associated RBSDE (14). The proof in the general case is the same as for non reflected BSDEs with default jump (see the proof of Proposition 2.6 in [7]). It is based on a fixed point argument and the a priori estimates for RBSDEs with default given in the Appendix (see Lemma 5.1).

Proceeding as in the proof of [20, Theorem 3.3] which was given in the framework of a random Poisson measure, we can prove equality (16) in our framework, which completes the proof. □

Lemma 3.3. *If ξ is left-u.s.c. along stopping time, then A is continuous.*

Proof. Let τ be a predictable stopping time. By (14), we have $\Delta A_\tau = (\Delta Y_\tau)^-$. Using the Skorokhod conditions (15), we get $\Delta A_\tau = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(Y_\tau - Y_{\tau^-})^- = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(Y_\tau - \xi_{\tau^-})^-$ a.s. Now, since by assumption, $\xi_{\tau^-} \leq \xi_\tau$ a.s., we have $Y_\tau - \xi_{\tau^-} \geq Y_\tau - \xi_\tau \geq 0$ a.s. We derive that $\Delta A_\tau = 0$ a.s. It follows that A is continuous. □

We now provide two characterizations of the seller's price, which generalize those provided in the literature in the case of a perfect market (see [9]) to the case of an imperfect market.

Theorem 3.4 (*Seller's price* of the American option)**.** *The* seller's price u_0 *of the American option is equal to the g*-value*, that is*

$$u_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau). \tag{17}$$

Moreover, we have

$$u_0 = Y_0,$$

where (Y, Z, K, A) *is the solution of the* nonlinear *reflected BSDE* (14) *and the portfolio strategy* $\varphi^* := \Phi(Z, K)$ *(where* Φ *is defined in Definition 2.1) is a superhedging strategy for the seller.*

Note that in the case of a perfect market, equality (17) reduces to the well-known characterization of the price of the American option as the value function of a classical optimal stopping problem, and the equality $u_0 = Y_0$ corresponds to the well-known characterization of this price as the solution of the *linear* reflected BSDE associated with the *linear* driver (10) (see [9]).

Proof. The proof is based on the characterization of the *g-value* as the solution of the reflected BSDE (14) (see Proposition 3.2). It is sufficient to show that $u_0 = Y_0$ and $\varphi^* \in \mathcal{A}(u_0)$.
Let $\mathcal{H}$ be the set of initial capitals which allow the seller to be "super-hedged", that is $\mathcal{H} = \{x \in \mathbb{R} : \exists \varphi \in \mathcal{A}(x)\}$. Note that $u_0 = \inf \mathcal{H}$.

Let us first show that

$$\varphi^* \in \mathcal{A}(Y_0). \tag{18}$$

By (8)-(9), for almost every ω, the trajectory of the value of this portfolio $t \mapsto V_t^{Y_0,\varphi^*}(\omega)$ satisfies the following forward differential equation:

$$V_t^{Y_0,\varphi^*}(\omega) = Y_0 - \int_0^t g(s, \omega, V_s^{Y_0,\varphi^*}(\omega), Z_s(\omega), K_s(\omega))ds + f_t^1(\omega),\ 0 \le t \le T, \tag{19}$$

where $f_t^1 := \int_0^t Z_s dW_s + \int_0^t K_s dM_s$. Moreover, since Y is the solution of the reflected BSDE (14), for almost every ω, the function $t \mapsto Y_t(\omega)$ satisfies:

$$Y_t(\omega) = Y_0 - \int_0^t g(s, \omega, Y_s(\omega), Z_s(\omega), K_s(\omega))ds + f_t^2(\omega),\ 0 \le t \le T, \tag{20}$$

where $f_t^2 := f_t^1 - A_t$. Since $f_t^1 \ge f_t^2$, by a comparison result for forward differential equations (see e.g. [8] in the Appendix), we derive that $V_t^{Y_0,\varphi^*}(\omega) \ge Y_t(\omega)$, $0 \le t \le T$ for almost every ω. Since $Y_t \ge \xi_t$, $0 \le t \le T$ a.s., we thus have $V_t^{Y_0,\varphi^*} \ge \xi_t$, $0 \le t \le T$ a.s., which implies the desired property (18). It follows that $Y_0 \ge u_0$.

Let us show the converse inequality. Let $x \in \mathcal{H}$. There exists $\varphi \in \mathcal{A}(x)$ such that $V_t^{x,\varphi} \ge \xi_t$, $0 \le t \le T$ a.s. For each $\tau \in \mathcal{T}$ we thus have $V_\tau^{x,\varphi} \ge \xi_\tau$ a.s. By taking the $\mathcal{E}^g$-evaluation in this inequality, using the monotonicity of $\mathcal{E}^g$ and the $\mathcal{E}^g$-martingale property of the wealth process $V^{x,\varphi}$, we obtain $x = \mathcal{E}^g_{0,\tau}[V_\tau^{x,\varphi}] \ge \mathcal{E}^g_{0,\tau}[\xi_\tau]$. By arbitrariness of $\tau \in \mathcal{T}$, we get $x \ge \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}[\xi_\tau]$, which holds for any $x \in \mathcal{H}$. By taking the infimum over $x \in \mathcal{H}$, we obtain $u_0 \ge Y_0$. We derive that $u_0 = Y_0$. By (18), we thus have $\varphi^* \in \mathcal{A}(u_0)$, which ends the proof. $\square$

Remark 3.5. *In general, except when g does not depend on y, by* (20)*, we have*

$$Y_. = Y_0 - \int_0^. g(s, Y_s, Z_s, K_s)ds + \int_0^. Z_s dW_s + \int_0^. K_s dM_s - A_. \not\equiv V_.^{Y_0,\varphi^*} - A_..$$

Remark 3.6. *The seller's price u_0 is clearly an* upper bound *of the possible prices for the American option. Indeed, no rational agent would pay more than u_0 since there is a cheaper way to achieve at least the same payoff, whatever the exercise time is. Indeed, by investing the amount $u_0+\varepsilon$ and following the strategy φ^*, whatever the exercise time τ is, he will make the gain $V_\tau^{u_0+\varepsilon,\varphi^*} > V_\tau^{u_0,\varphi^*} (\geq \xi_\tau)$ a.s. by a strict comparison property for deterministic differential equations (see [8] in the Appendix). Moreover, if the price of the option is equal to $u_0+\varepsilon$, then, by investing this amount following the strategy φ^*, whatever the exercise time τ is, the seller will make a gain $V_\tau^{u_0+\varepsilon,\varphi^*} - \xi_\tau > 0$ a.s. Hence if the price is strictly greater than u_0, there exists an arbitrage for the seller.*

Remark 3.7. *In [15], it is proved that the* seller's price *of the American option is equal to the g-*value *in the case of a higher interest rate for borrowing, by using a dual approach. This approach relies on the convexity properties of the driver and cannot be adapted to our case, except when g is convex with respect to (y,z,k).*

We define now the seller's price of the American option at each time/stopping time $S \in \mathcal{T}$. We first define, for each initial wealth $X \in L^2(\mathcal{F}_S)$, a *super-hedge* against the American option as a portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that $V_t^{S,X,\varphi} \geq \xi_t$, $S \leq t \leq T$ a.s., where $V^{S,X,\varphi}$ denotes the wealth process associated with initial time S and initial condition X. The *seller's price at time S* is defined by the random variable

$$u(S) := \operatorname{ess\,inf}\{X \in L^2(\mathcal{F}_S),\ \exists \varphi \in \mathcal{A}_S(X)\},$$

where $\mathcal{A}_S(X)$ is the set of all super-hedges associated with initial time S and initial wealth X. Using equality (16) and similar arguments to those used in the proof of Theorem 3.4 above, one can show the following result, which generalizes the results of Theorem 3.4 to any time $S \in \mathcal{T}$.

Proposition 3.8 (*Seller's price process* and characterization)**.** *For each time $S \in \mathcal{T}$, the* seller's price $u(S)$ at time S *of the American option satisfies the equalities*

$$u(S) = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S,\tau}(\xi_\tau) = Y_S \quad \text{a.s.},$$

where (Y,Z,K,A) is the solution of reflected BSDE (14)*.*
In other terms, the seller's price family of random variables $(u(S), S \in \mathcal{T})$ can be aggregated by an RCLL adapted process, which we call the seller's price process *of the American option. Moreover, this price process is both characterized as the value function process of the $\mathcal{E}^g$-optimal stopping problem with payoff (ξ_t), and also as the solution (Y_t) of the reflected BSDE* (14)*.*

Suppose now that the buyer has bought the American option at the selling price $u_0 = Y_0$. We address the problem of the choice of her/his exercise time. We introduce the following definition.

Definition 3.9. *A stopping time $\tau \in \mathcal{T}$ is called a* rational exercise time *for the buyer of the American option if it is optimal for Problem* (17)*, that is if it satisfies* $\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau) = \mathcal{E}^g_{0,\tau}(\xi_\tau)$*.*

By the optimality criterium provided in [20] (see Proposition 3.5), we have:

Proposition 3.10. *(Characterization of* rational exercise times*) Let $\tau \in \mathcal{T}$. Then, τ is a a* rational exercise time *for the buyer if and only if $Y_\tau = \xi_\tau$ a.s. and $A_\tau = 0$ a.s., where (Y,Z,K,A) is the solution of the reflected BSDE* (14)*.*

Suppose now that the price of the American option is equal to the seller's price not only at time 0 but at all time $S \in \mathcal{T}$, that is, the price at time S is equal to $u(S) = Y_S$ (see Proposition 3.8). Suppose that the buyer buys the American option at time 0 (at price $u_0 = Y_0$). Let us show that it is not profitable for him to exercise his option at a stopping time τ which is not a rational exercise time. First, it is not in his interest to exercise at a time t such that $Y_t > \xi_t$ since he would loose a financial asset (the option) with value Y_t and only receive the lower amount ξ_t. Second, it is not in his interest to exercise at a stopping time τ greater than $\bar{\tau}$, defined by $\bar{\tau} := \inf\{s \geq 0,\ A_s \neq 0\}$. Let us show that $Y_{\bar{\tau}} = V_{\bar{\tau}}^{Y_0,\varphi^*}$ a.s. Note that by definition of $\bar{\tau}$, $A_{\bar{\tau}} = 0$ a.s. Hence, for a.e. ω, the trajectories $t \mapsto Y_t(\omega)$ and $t \mapsto V_t^{Y_0,\varphi^*}(\omega)$ are solutions on $[0, \bar{\tau}(\omega)]$ of the same differential equation (with initial value Y_0), which implies that they are equal, by uniqueness of the solution. Hence, $Y_{\bar{\tau}} = V_{\bar{\tau}}^{Y_0,\varphi^*}$ a.s. Without loss of generality, we can suppose that for each ω, we have $Y_{\bar{\tau}}(\omega) = V_{\bar{\tau}}^{Y_0,\varphi^*}(\omega)$. Let $\tau \geq \bar{\tau}$. Let $B := \{\tau > \bar{\tau}\}$. Suppose that $P(B) > 0$. Hence, $A_\tau > 0$ a.s. on B. Then, by exercising the option at time $\bar{\tau}$, the option holder receives the amount $Y_{\bar{\tau}}$ which he can invest in the market along the strategy φ^*. Since $Y_{\bar{\tau}} = V_{\bar{\tau}}^{Y_0,\varphi^*}$, by the flow property of the forward differential equation (8) with $\varphi = \varphi^*$ and $x = Y_0$, the value of the associated portfolio is equal to $V_\tau^{Y_{\bar{\tau}},\varphi^*} = V_\tau^{V_{\bar{\tau}}^{Y_0,\varphi^*},\varphi^*} = V_\tau^{Y_0,\varphi^*}$ at time τ. Since $A_\tau > 0$ on B, by the strict comparison result for forward differential equations applied to (19) and (20) (see [8] in the Appendix), we get $V_\tau^{Y_0,\varphi^*} > Y_\tau$ a.s. on B, which implies $V_\tau^{Y_{\bar{\tau}},\varphi^*} = V_\tau^{Y_0,\varphi^*} > \xi_\tau$ a.s. on B. We thus have $V_\tau^{Y_{\bar{\tau}},\varphi^*} \geq \xi_\tau$ a.s. with $P(V_\tau^{Y_{\bar{\tau}},\varphi^*} > \xi_\tau) > 0$. Hence, at time $\bar{\tau}$, it is more interesting for the buyer to exercise immediately than later.

Proposition 3.11. *(Existence of* rational exercise times*) Suppose that the payoff ξ is left u.s.c. along stopping times. Let (Y, Z, K, A) be the solution of the reflected BSDE* (14).
Let $\tau^* := \inf\{s \geq 0,\ Y_s = \xi_s\}$ and $\bar{\tau} := \inf\{s \geq 0,\ A_s \neq 0\}$.[5]
The stopping time τ^* (resp. $\bar{\tau}$) is the minimal (resp. maximal) rational exercise time.

Proof. The right continuity of (Y_t) and (ξ_t) ensures that $Y_{\tau^*} = \xi_{\tau^*}$ a.s. By definition of τ^*, we have $Y_t > \xi_t$ a.s. on $[0, \tau^*[$. Hence the process A is constant on $[0, \tau^*[$ and even on $[0, \tau^*]$ because A is continuous (see Lemma 3.3).

By Proposition 3.10, τ^* is thus a rational exercise time and is the minimal one. From the definition of $\bar{\tau}$, and the continuity of A, we have $A_{\bar{\tau}} = 0$ a.s. Also, we have a.s. for all $t > \bar{\tau}$, $A_t > A_{\bar{\tau}} = 0$. Since A increases only on the set $\{Y_\cdot = \xi_\cdot\}$, it follows that $Y_{\bar{\tau}} = \xi_{\bar{\tau}}$. By Proposition 3.10, $\bar{\tau}$ is a rational exercise time and is the maximal one. □

When ξ is only RCLL, there does not exist necessarily an *rational* exercise time for the buyer. However, we have the following result.

Proposition 3.12. *(Existence of ε-*rational *exercise time) Suppose ξ is RCLL. For each $\varepsilon > 0$, the stopping time $\tau_\varepsilon := \inf\{t \geq 0 : \ Y_t \leq \xi_t + \varepsilon\}$ satisfies*

$$\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau) \ \leq \ \mathcal{E}^g_{0,\tau_\varepsilon}(\xi_{\tau_\varepsilon}) + K\varepsilon \quad \text{a.s.}, \tag{21}$$

*where K is a constant which only depends on T and the λ-constant C of g. In other words, $\tau_\varepsilon := \inf\{t \geq 0 : Y_t \leq \xi_t + \varepsilon\}$ is a $K\varepsilon$-*rational *exercise time.*

The proof, which relies on similar arguments as those used in the proof of Theorem 3.2 in [20], is left to the reader.

4. The buyer's point of view

Let us consider the pricing and hedging problem of a European option with maturity T and payoff $\xi \in L^2(\mathcal{G}_T)$ from the buyer's point of view. Supposing the initial price of the option is z, he starts with the amount $-z$ at time $t = 0$, and wants to find a risky-assets strategy $\tilde{\varphi}$ such that the payoff that he receives at time T allows him to recover the debt he incurred at time $t = 0$ by buying the option, that is such that $V_T^{-z,\tilde{\varphi}} + \xi = 0$ a.s. or equivalently, $V_T^{-z,\tilde{\varphi}} = -\xi$ a.s.

The buyer's price of the option is thus equal to the opposite of the seller's price of the option with payoff $-\xi$, that is $-\mathcal{E}^g_{0,T}(-\xi) = -\tilde{X}_0$, where $(\tilde{X}, \tilde{Z}, \tilde{K})$ is the solution of the BSDE associated with driver g and terminal condition $-\xi$. Let us specify the hedging strategy for the buyer. Suppose that the initial price of the option is $z := -\tilde{X}_0$. The process $\tilde{X}$ is equal to the value of the portfolio associated with initial value $-z = \tilde{X}_0$ and strategy

[5] Note that by Proposition 3.8, the process (Y_t) is equal to the *seller's price process* of the American option.

$\tilde{\varphi} := \Phi(\tilde{Z}, \tilde{K})$ (where Φ is defined in Definition 2.1) that is $\tilde{X} = V^{\tilde{X}_0, \tilde{\varphi}} = V^{-z, \tilde{\varphi}}$. Hence, $V_T^{-z,\tilde{\varphi}} = \tilde{X}_T = -\xi$ a.s. , which yields that $\tilde{\varphi}$ is the hedging risky-assets strategy for the buyer. Similarly, $-\mathcal{E}^g_{t,T}(-\xi) = -\tilde{X}_t$ satisfies an analogous property at time t, and is called the *hedging price for the buyer* at time t.

Let us now introduce the λ-*admissible* driver $\tilde{g}$ defined by

$$\tilde{g}(t, y, z, k) := -g(t, -y, -z, -k). \tag{22}$$

Note that for all $\xi \in L^2(\mathcal{G}_T)$, we have $-\mathcal{E}^g_{\cdot,T}(-\xi) = \mathcal{E}^{\tilde{g}}_{\cdot,T}(\xi)$.

This leads to the *nonlinear pricing system* $\mathcal{E}^{\tilde{g}}$ *relative to the buyer* (corresponding to the $\tilde{g}$-*evaluation*), defined for each $(S, \xi) \in [0, T] \times L^2(\mathcal{G}_S)$ by $\mathcal{E}^{\tilde{g}}_{\cdot,S}(\xi)$, which is equal to the solution of the BSDE with driver $\tilde{g}$, terminal time S and terminal condition ξ.

Remark 4.1. *If we suppose that $-g(t, -y, -z, -k) \le g(t, y, z, k)$ (which is satisfied if, for example, g is convex with respect to (y, z, k)), then, by the comparison theorem for BSDEs, we have $\mathcal{E}^{\tilde{g}}_{\cdot,S}(\xi) \le \mathcal{E}^g_{\cdot,S}(\xi)$ for each $(S, \xi) \in [0, T] \times L^2(\mathcal{G}_S)$. In other terms, the hedging price of a European option for the buyer is always smaller than the one for the seller.*

Moreover, when $-g(t, -y, -z, -k) = g(t, y, z, k)$ (which is satisfied if, for example, g is linear with respect to (y, z, k), as in the perfect market case), then the pricing system for the buyer is equal the one for the seller, that is, $\mathcal{E}^{\tilde{g}} = \mathcal{E}^g$.

Note that by the flow property of BSDEs, the first coordinate of the solution of a BSDE with driver $\tilde{g}$ is an $\mathcal{E}^{\tilde{g}}$-martingale.

Moreover, the opposite of a wealth process is an $\mathcal{E}^{\tilde{g}}$-martingale, that is

Proposition 4.2. *For each $x \in \mathbb{R}$ and each portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$, the process $(-V_t^{x,\varphi})$ is an $\mathcal{E}^{\tilde{g}}$-martingale.*

Proof. Since $g(t, y, z, k) := -\tilde{g}(t, -y, -z, -k)$, we get that the wealth process $(V_t^{x,\varphi})$ satisfies:

$$dV_t^{x,\varphi} = \tilde{g}(t, -V_t^{x,\varphi}, -Z_t, -K_t)dt + Z_t dW_t + K_t dM_t, \tag{23}$$

where $(Z, K) = \phi^{-1}(\varphi)$. Hence, $(-V^{x,\varphi}, -Z, -K)$ is the solution of the BSDE associated with driver $\tilde{g}$, terminal time T and terminal conditon $-V_T^{x,\varphi}$. The result follows. $\square$

Let us consider the case of the American option from the point of view of the buyer. Supposing the initial price of the American option is z, he starts with the amount $-z$ at time $t = 0$, and wants to find a *super-hedge*, that is an exercise time τ and a risky-assets strategy φ, such that the payoff that he receives allows him to recover the debt he incurred at time $t = 0$ by buying the American option. This notion of *super-hedge for the buyer* can be defined more precisely as follows.

Definition 4.3. *A* super-hedge for the buyer *against the American option with initial price $z \in \mathbb{R}$ is a pair (τ, φ) of a stopping time $\tau \in \mathcal{T}$ and a risky-assets strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that*

$$V_\tau^{-z,\varphi} + \xi_\tau \ge 0 \ a.s.$$

We denote by $\mathcal{B}(z)$ the set of all *super-hedges for the buyer* associated with initial price $z \in \mathbb{R}$.

We now define the *buyer's price* v_0 of the American option as the supremum of the initial prices which allow the buyer to be super-hedged, that is [6]

$$v_0 = \sup\{z \in \mathbb{R}, \ \exists (\tau, \phi) \in \mathcal{B}(z)\}. \tag{24}$$

[6] We have $(0, 0) \in \mathcal{B}(\xi_0)$. Hence, $\tilde{u}_0 \ge \xi_0$. Moreover, if $g(t, 0, 0, 0) = 0$ and $\xi_0 \ge 0$, then $\tilde{u}_0 = \sup\{z \ge 0, \ \exists (\tau, \varphi) \in \mathcal{B}(z)\}$.

We first consider the simpler case when ξ is left-u.s.c. along stopping times. In this case, we prove below that the infimum in (24) is attained, which implies that the *buyer's price* v_0 allows the buyer to build a super-hedge. Moreover, a super-hedge strategy is provided via the solution of a reflected BSDE associated with driver $\tilde{g}$. More precisely, we have

Theorem 4.4 (*Buyer's price* and *super-hedge*)**.** *Suppose that (ξ_t) is left upper-semicontinuous along stopping times. The buyer's price v_0 of the American option satisfies:*

$$v_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau). \tag{25}$$

Moreover, we have

$$v_0 = \tilde{Y}_0, \tag{26}$$

where $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{A})$ is the solution of the reflected BSDE associated with driver $\tilde{g}(t,y,z,k) := -g(t,-y,-z,-k)$ and lower obstacle ξ, that is,

$$\begin{aligned} & -d\tilde{Y}_t = \tilde{g}(t, \tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)dt + d\tilde{A}_t - \tilde{Z}_t dW_t - \tilde{K}_t dM_t; \;\; \tilde{Y}_T = \xi_T, \;\; \textit{with} \\ & \tilde{Y} \geq \xi \,, \;\; \int_0^T (\tilde{Y}_t - \xi_t) d\tilde{A}_t = 0 \;\; \textit{a.s.} \;\; , \end{aligned} \tag{27}$$

where the non decreasing process $\tilde{A}$ is continuous.

Let $\tilde{\tau} := \inf\{t \geq 0 : \;\; \tilde{Y}_t = \xi_t\}$ and $\tilde{\varphi} := \Phi(-\tilde{Z}, -\tilde{K})$ (where Φ is defined in Definition 2.1). The pair $(\tilde{\tau}, \tilde{\varphi})$ is a super-hedge for the buyer, that is $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(v_0)$.

Proof. Note first that, by Remark 3.3 applied to the above reflected BSDE (27), since the obstacle (ξ_t) is supposed to be left upper semicontinuous along stopping times, the process $\tilde{A}$ is continuous.

Now, by Proposition 3.2 (applied to the driver $\tilde{g}$ instead of g), we have $\sup_\tau \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau) = \tilde{Y}_0$. In order to show equality (25), it is thus sufficient to show that $v_0 = \tilde{Y}_0$.

Set $\mathcal{S} := \{z \in \mathbb{R} : \exists (\tau, \varphi) \in \mathcal{B}(z)\}$.

Let us first show that $\tilde{Y}_0 \leq v_0$. Since $v_0 = \sup \mathcal{S}$, it is sufficient to show that $\tilde{Y}_0 \in \mathcal{S}$. To this aim, we prove that

$$(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{Y}_0). \tag{28}$$

By definition of $\tilde{\tau}$, we have that $\tilde{Y}_t > \xi_t$ on $[0, \tilde{\tau}[$, which implies that the process $\tilde{A}$ is constant on $[0, \tilde{\tau}[$. Since $\tilde{A}$ is continuous, we derive that $\tilde{A}$ is equal to 0 on $[0, \tilde{\tau}]$. By equation (27), we thus get

$$\tilde{Y}_t = \tilde{Y}_0 - \int_0^t \tilde{g}(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{K}_s) ds + \int_0^t \tilde{Z}_s dW_s + \int_0^t \tilde{K}_s dM_s, \;\; 0 \leq t \leq \tilde{\tau} \;\; \text{a.s.} \tag{29}$$

We now consider the portfolio associated with the initial capital $-\tilde{Y}_0$ and the strategy $\tilde{\varphi} = \Phi(-\tilde{Z}, -\tilde{K})$. Using the equality $g(t,y,z,k) = -\tilde{g}(t,-y,-z,-k)$, we derive that its value $V^{-\tilde{Y}_0, \tilde{\varphi}}$ satisfies

$$-V_t^{-\tilde{Y}_0, \tilde{\varphi}} = \tilde{Y}_0 - \int_0^t \tilde{g}(s, -V_s^{-\tilde{Y}_0, \tilde{\varphi}}, \tilde{Z}_s, \tilde{K}_s) ds + \int_0^t \tilde{Z}_s dW_s + \int_0^t \tilde{K}_s dM_s, \;\; 0 \leq t \leq T \;\; \text{a.s.} \tag{30}$$

Hence, $-V^{-\tilde{Y}_0, \tilde{\varphi}}$ and $\tilde{Y}$ satisfy the same forward differential equation on $[0, \tilde{\tau}]$ with the same initial condition. By uniqueness of the solution, they coincide. We thus get the equality $V_{\tilde{\tau}}^{-\tilde{Y}_0, \tilde{\varphi}} = -\tilde{Y}_{\tilde{\tau}}$ a.s. Moreover, the definition of the stopping time $\tilde{\tau}$ together with the right-continuity of the processes $(\tilde{Y}_t)$ and (ξ_t) yield the equality $\tilde{Y}_{\tilde{\tau}} = \xi_{\tilde{\tau}}$ a.s. We thus have the equality $V_{\tilde{\tau}}^{-\tilde{Y}_0, \tilde{\varphi}} = -\xi_{\tilde{\tau}}$ a.s. The desired property (28) follows. Hence, we have $\tilde{Y}_0 \in \mathcal{S}$, and thus $\tilde{Y}_0 \leq v_0$.

It remains to prove that $v_0 \leq \tilde{Y}_0$. Let $z \in \mathcal{S}$. By definition of $\mathcal{S}$, there exists $(\tau, \varphi) \in \mathcal{B}(z)$ such that $-V_\tau^{-z,\varphi} \leq \xi_\tau$ a.s. By taking the $\mathcal{E}^{\tilde{g}}$-evaluation, using and the $\mathcal{E}^{\tilde{g}}$-martingale property of the process $(-V_t^{-z,\varphi})$ (see Proposition 4.2) and the monotonicity of the $\tilde{g}$-expectation $\mathcal{E}^{\tilde{g}}$, we derive that $z = \mathcal{E}^{\tilde{g}}_{0,\tau}(-V_\tau^{-z,\varphi}) \leq \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau)$, which implies the inequality

$$z \leq \sup_{\tau \in \mathcal{T}} \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau) = \tilde{Y}_0.$$

Since this inequality holds for any $z \in \mathcal{S}$, by taking the supremum over $z \in \mathcal{S}$, we get $v_0 \leq \tilde{Y}_0$. It follows that $v_0 = \tilde{Y}_0$. By (28), we get $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(v_0)$, which completes the proof. □

Remark 4.5. *Note that the above characterizations of the buyer's price cannot be directly derived from the characterizations of the seller's price given in Theorem 4.4, contrary to the case of game options. Indeed, concerning the pricing and hedging problem of game options, there is in a way a symmetry between the buyer and the seller (see [8]), which is not the case for American options.*

We now consider the general case when ξ is only RCLL. In this case, the characterizations (25) and (26) of the buyer's price v_0 still hold. However, the supremum in (24) is not necessarily attained; in other words, the price v_0 does not necessarily allow the buyer to build a *super-hedge* against the American option.

We introduce the definition of an ε-*super-hedge for the buyer*:

Definition 4.6. *For each initial price z and for each $\varepsilon > 0$, an ε-*super-hedge for the buyer *against the American option is a pair (τ, φ) of a stopping time $\tau \in \mathcal{T}$ and a risky-assets strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that*

$$V_\tau^{-z,\varphi} + \xi_\tau \geq -\varepsilon \quad \text{a.s.}$$

Theorem 4.7 (*Buyer's price* and ε-*super-hedge*)**.** *Suppose that ξ is only RCLL.*
The buyer's price v_0 *of the American option satisfies*

$$v_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau) = \tilde{Y}_0,$$

where $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{A})$ is the solution of the reflected BSDE associated with driver $\tilde{g}(t, y, z, k) := -g(t, -y, -z, -k)$ and lower obstacle ξ, that is,

$$\begin{aligned} & -d\tilde{Y}_t = \tilde{g}(t, \tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)dt + d\tilde{A}_t - \tilde{Z}_t dW_t - \tilde{K}_t dM_t; \;\; \tilde{Y}_T = \xi_T, \; \textit{with} \\ & \tilde{Y} \geq \xi \,, \; \int_0^T (\tilde{Y}_t - \xi_t) d\tilde{A}^c_t = 0 \; \textit{a.s. and} \; \Delta \tilde{A}^d_t = \Delta \tilde{A}^d_t \mathbf{1}_{\{\tilde{Y}_{t^-} = \xi_{t^-}\}}. \end{aligned} \tag{31}$$

Let $\tilde{\varphi} := \Phi(-\tilde{Z}, -\tilde{K})$ and for each $\varepsilon > 0$, let

$$\tau_\varepsilon := \inf\{t \geq 0 : \;\; \tilde{Y}_t \leq \xi_t + \varepsilon\}. \tag{32}$$

*The pair $(\tau_\varepsilon, \tilde{\varphi})$ is an ε-*super-hedge for the buyer *(associated with the initial price v_0).*

Proof. Let $\varepsilon > 0$. We have $\tilde{Y}_\cdot \geq \xi_\cdot + \varepsilon$ on $[0, \tau_\varepsilon[$. Since $\tilde{A}$ satisfies the Skorohod condition, it follows that almost surely, $\tilde{A}$ is constant on $[0, \tau_\varepsilon[$. Also, $\tilde{Y}_{(\tau^\varepsilon)^-} \geq \xi_{(\tau^\varepsilon)^-} + \varepsilon$ a.s., which implies that $\Delta \tilde{A}_{\tau_\varepsilon} = 0$ a.s. Hence, $\tilde{A}_{\tau^\varepsilon} = 0$ a.s. It follows that $\tilde{Y}_\cdot(\omega)$ satisfies the *forward* differential equation (29) on $[0, \tau_\varepsilon]$. Now, the wealth $-V_\cdot^{-\tilde{Y}_0, \tilde{\varphi}}$ is the solution of the *forward* differential equation (30). Hence, $\tilde{Y}$ satisfies the same *forward* differential equation on $[0, \tau_\varepsilon]$ as $-V^{-\tilde{Y}_0, \tilde{\varphi}}$, with the same initial condition $\tilde{Y}_0$. By uniqueness of the solution, they coincide, which implies in particular that $V_{\tau_\varepsilon}^{-\tilde{Y}_0, \tilde{\varphi}} = -\tilde{Y}_{\tau_\varepsilon}$ a.s. Moreover, by definition of the stopping time τ_ε and the right-continuity of $(\tilde{Y}_t)$ and (ξ_t), we have

$$\tilde{Y}_{\tau_\varepsilon} \leq \xi_{\tau_\varepsilon} + \varepsilon \quad \text{a.s.} \tag{33}$$

We thus get the inequality $V_{\tau_\varepsilon}^{-\tilde{Y}_0,\tilde{\varphi}} + \xi_{\tau_\varepsilon} \geq -\varepsilon$ a.s. Hence, the pair $(\tau_\varepsilon, \tilde{\varphi})$ is an *ε-super-hedge for the buyer* associated with the initial price $\tilde{Y}_0$.

Let us now show that $\tilde{Y}_0 = v_0$. First, by Proposition 3.2, we have $\sup_\tau \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau) = \tilde{Y}_0$. Using this property together with the same arguments as those used in the second part of the proof of Theorem 4.4 (which do not require the continuity of $\tilde{A}$), we obtain that $v_0 \leq \tilde{Y}_0$.

It remains to prove that $v_0 \geq \tilde{Y}_0$. Let $\varepsilon > 0$. Let (Y', Z', K') be the solution of the BSDE associated with driver $\tilde{g}$, terminal time τ_ε and terminal condition $\xi_{\tau_\varepsilon} \wedge \tilde{Y}_{\tau_\varepsilon}$. Now, since $\tilde{A}_{\tau^\varepsilon} = 0$, the process $(\tilde{Y}, \tilde{Z}, \tilde{K})$ is the solution of the BSDE associated with driver $\tilde{g}$, terminal time τ_ε and terminal condition $\tilde{Y}_{\tau_\varepsilon}$. By an a priori estimate on BSDEs with default jump (see [7, Proposition 2.4]), since by (33) we have $\tilde{Y}_{\tau_\varepsilon} \leq \xi_{\tau_\varepsilon} \wedge \tilde{Y}_{\tau_\varepsilon} + \varepsilon$, we derive that $\tilde{Y}_0 \leq Y'_0 + K\varepsilon$, where K is a constant which only depends on T and where C is a λ-constant associated with driver $\tilde{g}$ (or equivalently with driver g). Moreover, since by assumption $Y'_{\tau_\varepsilon} = \xi_{\tau_\varepsilon} \wedge \tilde{Y}_{\tau_\varepsilon}$, we have $Y'_{\tau_\varepsilon} \leq \xi_{\tau_\varepsilon}$. Now, one can show that $Y' = -V^{-Y'_0,\varphi'}$, where $\varphi' := \Phi(-Z', -K')$. We thus get $-V_{\tau_\varepsilon}^{-Y'_0,\varphi'} \leq \xi_{\tau_\varepsilon}$ We derive that $(\tau_\varepsilon, \varphi')$ is a super-hedge (for the buyer) associated with initial price Y'_0. The initial price Y'_0 $(\geq \tilde{Y}_0 - K\varepsilon)$ thus allows the buyer to be super-hedged. By definition of v_0, we derive that $v_0 \geq Y'_0 \geq \tilde{Y}_0 - K\varepsilon$. We thus get $v_0 \geq \tilde{Y}_0 - K\varepsilon$ for each $\varepsilon > 0$. Hence, $v_0 \geq \tilde{Y}_0$, which completes the proof. □

Note that if the price of the option is equal to $v_0 - \varepsilon$, then, by investing the amount $-v_0 + \varepsilon$ following the strategy $\tilde{\varphi}$, whatever the exercise time τ chosen, the buyer will make a gain $V_\tau^{-v_0+\varepsilon,\tilde{\varphi}} + \xi_\tau > V_\tau^{-v_0,\tilde{\varphi}} + \xi_\tau \geq 0$ a.s. Hence if the price is strictly smaller than v_0, then there exists an arbitrage for the buyer.

Moreover, if $-g(t, -y, -z, -k) \leq g(t, y, z, k)$, then, by Remark 4.1, we get $v_0 \leq u_0$. The previous observations together with Remark 3.6 yield that, in this case, the interval $[v_0, u_0]$ is a non-arbitrage interval for the price of the American option in the sense of [15]. In the example of a higher interest rate for borrowing, this result corresponds to the one shown in [15] by a dual approach.[7]

5. Appendix

We give here some a priori estimates for RBSDEs with default jump.

Lemma 5.1 (A priori estimate for RBSDEs). *Let f^1 and f^2 be two λ-admissible drivers. Let C be a λ-constant associated with f^1. Let ξ be an adapted RCLL processes. For $i = 1, 2$, let (Y^i, Z^i, K^i, A^i) be a solution of the RBSDE associated with terminal time T, driver f^i and obstacle ξ. Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$ and $\eta \leq \frac{1}{C^2}$.*
Let $\bar{f}(s) := f^1(s, Y^2_s, Z^2_s, K^2_s) - f^2(s, Y^2_s, Z^2_s, K^2_s)$. For each $t \in [0, T]$, we then have

$$e^{\beta t}(Y^1_s - Y^2_s)^2 \leq \eta\, \mathbb{E}[\int_t^T e^{\beta s} \bar{f}(s)^2 ds \mid \mathcal{G}_t] \quad \text{a .s.} \tag{34}$$

Moreover, $\|\bar{Y}\|^2_\beta \leq T\eta \|\bar{f}\|^2_\beta$, and if $\eta < \frac{1}{C^2}$, we then have $\|\bar{Z}\|^2_\beta + \|\bar{K}\|^2_{\lambda,\beta} \leq \frac{\eta}{1-\eta C^2} \|\bar{f}\|^2_\beta$.

The proof is similar to the one given for DRSBDEs in the same framework with default (see the proof of Proposition 6.1 in the Appendix in [8]), and left to the reader.

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