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Optimal Stopping for Dynamic Risk Measures with Jumps and Obstacle Problems

Roxana Dumitrescu · **Marie-Claire Quenez** · **Agnès Sulem**

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Abstract We study the optimal stopping problem for a monotonous dynamic risk measure induced by a Backward Stochastic Differential Equation with jumps in the Markovian case. We show that the value function is a viscosity solution of an obstacle problem for a partial integro-differential variational inequality, and we provide an uniqueness result for this obstacle problem.

Keywords Dynamic risk-measures · optimal stopping · obstacle problem · reflected backward stochastic differential equations with jumps · viscosity solution · comparison principle · partial integro-differential variational inequality

Mathematics Subject Classification (2000) 93E20 · 60J60 · 47N10

1 Introduction

In the last years, there has been several studies on dynamic risk measures and their links with nonlinear backward stochastic differential equations (BSDEs). We recall that nonlinear BSDEs have been introduced in [1] in a Brownian framework, in order to provide a probabilistic representation of semilinear parabolic partial-differential equations. BSDEs with jumps and their links with partial integro-differential equations are studied in [2]. A comparison theorem is established in [3] and generalized in [4], where properties of dynamic risk measures induced by BSDEs with jumps are also provided. An optimal stopping problem for such risk measures is addressed in [5], and the value function is characterized as the solution of a reflected BSDE with jumps and RCLL obstacle process.

Roxana Dumitrescu

CEREMADE, Université Paris 9 Dauphine, Place de Maréchal de Lattre de Tassigny, 75016 Paris, France, and INRIA Paris-Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, Le Chesnay Cedex, 78153, France, E-mail: roxana@ceremade.dauphine.fr

Marie-Claire Quenez

LPMA, Université Paris 7 Denis Diderot, Boite courrier 7012, 75251 Paris Cedex 05, France, E-mail: quenez@math.univ-paris-diderot.fr

Agnès Sulem (Corresponding author)

INRIA Paris-Rocquencourt, Domaine de Voluceau, Rocquencourt, BP 105, Le Chesnay Cedex, 78153, France, and Université Paris-Est, 77455 Marne-la-Vallée, France, E-mail: agnes.sulem@inria.fr

In the present paper, we focus on the optimal stopping problem for dynamic risk measures induced by BSDEs with jumps in a Markovian framework. In this case the driver of the BSDE depends on a given state process X, which can represent, for example, an index or a stock price. This process will be assumed to be driven by a Brownian motion and a Poisson random measure.

Our main contribution consists in establishing the link between the value function of our optimal stopping problem and parabolic partial integro-differential variational inequalities (PIDVIs). We prove that the minimal risk measure, which corresponds to the solution of a reflected BSDE with jumps, is a viscosity solution of a PIDVI. This provides an existence result for the obstacle problem under relatively weak assumptions. Our result generalizes a result of [6] obtained in the Brownian case. The proof was based on a penalization method via non-reflected BSDEs. We provide here instead a direct and shorter proof.

Furthermore, under some additional assumptions, we prove a comparison theorem in the class of bounded continuous functions, relying on a non-local version of Jensen-Ishii Lemma (see [7]), from which the uniqueness of the viscosity solution follows. We point out that our problem is not covered by the study in [7], since we are dealing with nonlinear BSDEs, and this leads to a more complex integro-differential operator in the associated PDE.

The paper is organized as follows: In Section 2 we give the formulation of our optimal stopping problem.
In Section 3, we prove that the value function is a solution of an obstacle problem for a PIDVI in the viscosity sense. In Section 4, we establish an uniqueness result. In the Appendix, we prove some estimates, from which we derive that the value function is continuous and has polynomial growth and provide some complementary results.

2 Optimal Stopping Problem for Dynamic Risk Measures with Jumps in the Markovian Case

Let $(\Omega,\mathscr{F},P)$ be a probability space. Let W be a one-dimensional Brownian motion and $N(dt,du)$ be a Poisson random measure with compensator $\nu(du)dt$ such that ν is a σ-finite measure on $\mathbb{R}^*$ equipped with its Borel field $\mathscr{B}(\mathbb{R}^*)$, and satisfies $\int_{\mathbb{R}^*}(1\wedge e^2)\nu(de)<\infty$. Let $\tilde{N}(dt,du)$ be its compensated process. Let $I\!F=\{\mathscr{F}_t, t\geq 0\}$ be the natural filtration associated with W and N.

We consider a state process X which may be interpreted as an index, an interest rate process, an economic factor, an indicator of the market or the value of a portfolio, which has an influence on the risk measure and the position. For each initial time $t\in[0,T]$ and each condition $x\in\mathbb{R}$, let $X^{t,x}$ be the solution of the following stochastic differential equation (SDE):

$$X_s^{t,x}=x+\int_t^s b(X_r^{t,x})dr+\int_t^s \sigma(X_r^{t,x})dW_r+\int_t^s\int_{\mathbb{R}^*}\beta(X_{r^-}^{t,x},e)\tilde{N}(dr,de), \tag{1}$$

where $b,\sigma:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous, and $\beta:\mathbb{R}\times\mathbb{R}^*\to\mathbb{R}$ is a measurable function such that for some non negative real C, and for all $e\in\mathbb{R}$

$$|\beta(x,e)|\leq C(1\wedge|e|),\;\; x\in\mathbb{R}$$

$$|\beta(x,e)-\beta(x',e)|\leq C|x-x'|(1\wedge|e|),\;\; x,x'\in\mathbb{R}.$$

We introduce a dynamic risk measure ρ induced by a BSDE with jumps. For this, we consider two functions γ and f satisfying the following assumption:

Assumption 2.1 • *$\gamma : \mathbb{R}\times\mathbb{R}^* \to \mathbb{R}$ is $\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^*)$-measurable,*

$|\gamma(x,e)-\gamma(x',e)| < C|x-x'|(1\wedge|e|), x,x' \in \mathbb{R}, e\in\mathbb{R}^*$

$-1 \le \gamma(x,e) \le C(1\wedge|e|)$, $e\in\mathbb{R}^*$

- *$f : [0,T]\times\mathbb{R}^3\times L^2_\nu \to \mathbb{R}$ is continuous in t uniformly with respect to x,y,z,k, and continuous in x uniformly with respect to y,z,k.*

  (i) $|f(t,x,0,0,0)| \le C(1+x^p)$, $\forall x\in\mathbb{R}$

  (ii) $|f(t,x,y,z,k)-f(t,x',y',z',k')| \le C(|y-y'|+|z-z'|+\|k-k'\|_{L^2_\nu})$, $\forall\, t\in[0,T]$, $y,y',z,z'\in\mathbb{R}$, $k,k'\in L^2_\nu$

  (iii) $f(t,x,y,z,k_1)-f(t,x,y,z,k_2) \ge <\gamma(x,\cdot),k_1-k_2>_\nu, \forall t,x,y,z,k_1,k_2.$

Here, L^2_ν denotes the set of Borelian functions $\ell:\mathbb{R}^*\to\mathbb{R}$ such that $\|\ell\|^2_\nu := \int_{\mathbb{R}^*}|\ell(u)|^2\nu(du) < +\infty$. It is a Hilbert space equipped with the scalar product $\langle\delta,\ell\rangle_\nu := \int_{\mathbb{R}^*}\delta(e)\ell(e)\nu(de)$ for all $\delta,\ell\in L^2_\nu\times L^2_\nu$.

We also introduce the set $\mathbb{H}^2$ (resp. $\mathbb{H}^2_\nu$) of predictable processes (π_t) (resp. $(l_t(\cdot))$) such that $\mathbb{E}\int_0^T\pi_s^2ds<\infty$ (resp. $\mathbb{E}\int_0^T\|l_s\|^2_{L^2_\nu}ds<\infty$); the set $\mathscr{S}^2$ of real-valued RCLL adapted processes (φ_s) with $\mathbb{E}[\sup_s\varphi_s^2]<\infty$, and the set $L^2(\mathscr{F}_T)$ of $\mathscr{F}_T$-measurable and square-integrable random variables.

Let (t,x) be a fixed intial condition. For each maturity S in $[t,T]$ and each position ζ in $L^2(\mathscr{F}_S)$, the associated risk measure at time $s\in[t,S]$ is defined by

$$\rho_s^{t,x}(\zeta,S) := -\mathscr{E}^{t,x}_{s,S}(\zeta),\ t\le s\le S, \tag{2}$$

where $\mathscr{E}^{t,x}_{\cdot,S}(\zeta)$ denotes the f-conditional expectation, starting at (t,x), defined as the solution in $\mathscr{S}^2$ of the BSDE with Lipschitz driver $f(s,X_s^{t,x},y,z,k)$, terminal condition ζ and terminal time S, that is the solution $(\mathscr{E}_s^{t,x})$ of

$$-d\mathscr{E}_s = f(s,X_s^{t,x},\mathscr{E}_s,\pi_s,l_s(\cdot))ds - \pi_s dW_s - \int_{\mathbb{R}^*} l_s(u)\tilde{N}(dt,du)\,;\ \mathscr{E}_S=\zeta, \tag{3}$$

where (π_s), (l_s) are the associated processes, which belong to $\mathbb{H}^2$ and $\mathbb{H}^2_\nu$ respectively.
The functional $\rho:(\zeta,S)\to\rho_\cdot(\zeta,S)$ defines then a dynamic risk measure induced by the BSDE with driver f (see [4]). Assumption 2.1 implies that the driver $f(s,X_s^{t,x},y,z,k)$ satisfies Assumption 3.1 in [5], which ensures the monotonocity property of ρ with respect to ζ. More precisely, for each maturity S and for each positions ζ_1, $\zeta_2\in L^2(\mathscr{F}_S)$, with $\zeta_1\le\zeta_2$ a.s., we have $\rho_s^{t,x}(\zeta_1,S)\ge\rho_s^{t,x}(\zeta_2,S)$ a.s.

We now formulate our optimal stopping problem for dynamic risk measures. For each $(t,x)\in[0,T]\times\mathbb{R}$, we consider a dynamic financial position given by the process $\left(\xi_s^{t,x}, t\le s\le T\right)$, defined via the state process $(X_s^{t,x})$ and two functions g and h such that

- $g\in\mathscr{C}(\mathbb{R})$ with at most polynomial growth at infinity,

- $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous in t, x and there exist $p \in \mathbb{N}$ and a real constant C, such that

$$|h(t,x)| \leq C(1+|x|^p), \forall t \in [0,T], x \in \mathbb{R}, \tag{4}$$

- $h(T,x) \leq g(x), \ \forall x \in \mathbb{R}$.

For each initial condition $(t,x) \in [0,T] \times \mathbb{R}$, the dynamic position is then defined by:

$$\begin{cases} \xi_s^{t,x} := h(s, X_s^{t,x}), \ s < T \\ \xi_T^{t,x} := g(X_T^{t,x}). \end{cases}$$

Let $t \in [0,T]$ be the initial time and let $x \in \mathbb{R}$ be the initial condition. The minimal risk measure at time t is given by:

$$\operatorname*{ess\,inf}_{\tau \in \mathscr{T}_t} \rho_t^{t,x}(\xi_\tau^{t,x}, \tau) = -\operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \mathscr{E}_{t,\tau}^{t,x}(\xi_\tau^{t,x}). \tag{5}$$

Here $\mathscr{T}_t$ denotes the set of stopping times with values in $[t,T]$.

By Th. 3.2 in [5], the minimal risk measure is characterized via the solution $Y^{t,x}$ in $\mathscr{S}^2$ of the following reflected BSDE (RBSDE) associated with driver f and obstacle ξ:

$$\begin{cases} Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, K_r^{t,x}(\cdot))dr + A_T^{t,x} - A_s^{t,x} \\ \qquad - \int_s^T Z_r^{t,x} dW_r - \int_s^T \int_{\mathbb{R}^*} K^{t,x}(r,e) \tilde{N}(dr,de) \\ Y_s^{t,x} \geq \xi_s^{t,x}, 0 \leq s \leq T \ \text{ a.s.} \\ A^{t,x} \text{ is a nondecreasing, continuous predictable process in } \mathscr{S}^2 \text{ with } A_t^{t,x} = 0 \text{ and such that} \\ \int_t^T (Y_s^{t,x} - \xi_s^{t,x}) dA_s^{t,x} = 0 \text{ a.s. }, \end{cases} \tag{6}$$

with $Z^{t,x}, K^{t,x} \in \mathbb{H}^2$ (resp. $\mathbb{H}_\nu^2$). Note that by the assumptions made on h and g, the obstacle $(\xi_s^{t,x})_{s \geq t}$ is continuous except at the inaccessible jump times of the Poisson measure, and at time T with $\Delta \xi_T^{t,x} \leq 0$ a.s., and this implies the continuity of $A^{t,x}$ by Th. 2.6 in [5]. Moreover, Th. 3.2 in [5] ensures that

$$Y_t^{t,x} = \operatorname*{ess\,sup}_{\tau \in \mathscr{T}_t} \mathscr{E}_{t,\tau}^{t,x}(\xi_\tau^{t,x}) \quad \text{a.s.} \tag{7}$$

The SDE (1) and the RBSDE (6) can be solved with respect to the translated Brownian motion $(W_s - W_t)_{s \geq t}$. Hence $Y_t^{t,x}$ is constant for each t,x. We can thus define a deterministic function u called *value function* of our optimal stopping problem by setting for each t,x

$$u(t,x) := Y_t^{t,x}. \tag{8}$$

By Lemma A.1 and Lemma A.2 given in Appendix, the function u is continuous and has at most polynomial growth. The continuity of u implies that $Y_s^{t,x} = u(s, X_s^{t,x})$, $t \leq s \leq T$ a.s.

Moreover, the stopping time $\tau^{*,t,x}$ (also denoted by τ^*), defined by

$$\tau^* := \inf\{s \geq t,\ Y_s^{t,x} = \xi_s^{t,x}\} = \inf\{s \geq t,\ u(s, X_s^{t,x}) = \bar{h}(s, X_s^{t,x})\}$$

is an optimal stopping time for (5) (see Th. 3.6 in [5]). Here, the function $\bar{h}$ is defined by $\bar{h}(t,x) := h(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T}$, so that $\xi_s^{t,x} = \bar{h}(s, X_s^{t,x})$, $0 \leq t \leq T$ a.s.

In the next section, we prove that the value function is a viscosity solution of an obstacle problem.

3 The Value Function, Viscosity Solution of an Obstacle Problem

We consider the following related obstacle problem for a parabolic PIDE:

$$\begin{cases} \min(u(t,x) - h(t,x), \\ \quad -\dfrac{\partial u}{\partial t}(t,x) - Lu(t,x) - f(t,x,u(t,x),(\sigma\dfrac{\partial u}{\partial x})(t,x), Bu(t,x)) = 0, \quad (t,x) \in [0,T[\times\mathbb{R} \\ u(T,x) = g(x),\ x \in \mathbb{R} \end{cases} \tag{9}$$

where

$$\begin{aligned} &L := A + K, \\ &A\phi(t,x) := \frac{1}{2}\sigma^2(x)\frac{\partial^2\phi}{\partial x^2}(t,x) + b(x)\frac{\partial\phi}{\partial x}(t,x), \\ &K\phi(t,x) := \int_{\mathbb{R}^*}\left(\phi(t,x+\beta(x,e)) - \phi(t,x) - \frac{\partial\phi}{\partial x}(t,x)\beta(x,e)\right)\nu(de), \\ &B\phi(t,x)(\cdot) := \phi(t,x+\beta(x,\cdot)) - \phi(t,x) \in L^2_\nu. \end{aligned} \tag{10}$$

The operator B and K are well defined for $\phi \in C^{1,2}([0,T]\times\mathbb{R})$. Indeed, since β is bounded, we have $|\phi(t,x+\beta(x,e)) - \phi(t,x)| \leq C|\beta(x,e)|$ and

$$|\phi(t,x+\beta(x,e)) - \phi(t,x) - \frac{\partial\phi}{\partial x}(t,x)\beta(x,e)| \leq C\beta(x,e)^2.$$

We prove below that the value function u defined by (8) is a viscosity solution of the above obstacle problem.

Definition 3.1 • A continuous function u is said to be a *viscosity subsolution* of (9) iff $u(T,x) \leq g(x), x \in \mathbb{R}$, and iff for any point $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and for any $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its minimum at (t_0,x_0), we have

$$\begin{aligned} &\min(u(t_0,x_0) - h(t_0,x_0), \\ &-\frac{\partial\phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0), B\phi(t_0,x_0)) \leq 0. \end{aligned}$$

In other words, if $u(t_0,x_0) > h(t_0,x_0)$, then

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)) \leq 0.$$

• A continuous function u is said to be a *viscosity supersolution* of (9) iff $u(T,x) \geq g(x), x \in \mathbb{R}$, and iff for any point $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and for any $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its maximum at (t_0,x_0), we have

$$\begin{aligned}&\min(u(t_0,x_0) - h(t_0,x_0),\\&-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)) \geq 0.\end{aligned}$$

In other words, we have both $u(t_0,x_0) \geq h(t_0,x_0)$, and

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,u(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)) \geq 0.$$

Theorem 3.1 *The function u, defined by* (8)*, is a viscosity solution (i.e. both a viscosity sub- and supersolution) of the obstacle problem* (9)*.*

Proof • We first prove that u is a subsolution of (9).
Let $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ be such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi(t,x) \geq u(t,x), \forall (t,x) \in [0,T]\times\mathbb{R}$.
Suppose by contradiction that $u(t_0,x_0) > h(t_0,x_0)$ and that

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f(t_0,x_0,\phi(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)) > 0.$$

By continuity of $K\phi$ (which can be shown using Lebesgue's theorem) and that of $B\phi : [0,T]\times\mathbb{R} \to L^2_\nu$, we can suppose that there exists $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that:
$\forall (t,x)$ such that $t_0 \leq t \leq t_0 + \eta_\varepsilon < T$ and $|x - x_0| \leq \eta_\varepsilon$, we have: $u(t,x) \geq h(t,x) + \varepsilon$ and

$$-\frac{\partial \phi}{\partial t}(t,x) - L\phi(t,x) - f(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x),B\phi(t,x)) \geq \varepsilon. \tag{11}$$

Note that $Y_s^{t_0,x_0} = Y_s^{s,X_s^{t_0,x_0}} = u(s,X_s^{t_0,x_0})$ a.s. because X^{t_0,x_0} is a Markov process and u is continuous. We define the stopping time θ as:

$$\theta := (t_0 + \eta_\varepsilon) \wedge \inf\{s \geq t_0, |X_s^{t_0,x_0} - x_0| > \eta_\varepsilon\}. \tag{12}$$

By definition of the stopping time θ,

$$u(s,X_s^{t_0,x_0}) \geq h(s,X_s^{t_0,x_0}) + \varepsilon > h(s,X_s^{t_0,x_0}), t_0 \leq s < \theta \text{ a.s.}$$

This means that for a.e. ω the process $(Y_s^{t_0,x_0}(\omega), s \in [t_0, \theta(\omega)[)$ stays strictly above the barrier. It follows that for a.e. ω, the function $s \to A_s^c(\omega)$ is constant on $[t_0, \theta(\omega)]$. In other words, $Y_s^{t_0,x_0} = \mathscr{E}_{s,\theta}^{t_0,x_0}(Y_\theta)$, $t_0 \leq s \leq \theta$ $a.s$, that is $(Y_s^{t_0,x_0}, s \in [t_0, \theta])$ is the solution of the classical BSDE associated with driver f, terminal time θ and terminal value $Y_\theta^{t_0,x_0}$. Applying Itô's lemma to $\phi(t, X_t^{t_0,x_0})$, we get:

$$\begin{aligned}\phi(t, X_t^{t_0,x_0}) = \phi(\theta, X_\theta^{t_0,x_0}) - \int_t^\theta \psi(s, X_s^{t_0,x_0}) ds - \int_t^\theta (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}) dW_s \\ - \int_t^\theta \int_{\mathbb{R}^*} B\phi(s, X_{s^-}^{t_0,x_0}) \tilde{N}(ds, de) \qquad (13)\end{aligned}$$

where $\psi(s,x) := \dfrac{\partial \phi}{\partial s}(s,x) + L\phi(s,x)$.
Note that $(\phi(s, X_s^{t_0,x_0}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_{s^-}^{t_0,x_0}); s \in [t_0, \theta])$ is the solution of the BSDE associated to terminal time θ, terminal value $\phi(\theta, X_\theta^{t_0,x_0})$ and driver process $-\psi(s, X_s^{t_0,x_0})$.

By (11) and the definition of the stopping time θ, we have a.s. that for each $s \in [t_0, \theta]$:

$$\begin{aligned}&-\frac{\partial \phi}{\partial t}(s, X_s^{t_0,x_0}) - L\phi(s, X_s^{t_0,x_0}) \\ &\quad - f\left(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_s^{t_0,x_0})\right) \geq \varepsilon. \qquad (14)\end{aligned}$$

Using the definition of the function ψ, (14) can be rewritten: for all $s \in [t_0, \theta]$,

$$-\psi(s, X_s^{t_0,x_0}) - f\left(s, X_s^{t_0,x_0}, \phi(s, X_s^{t_0,x_0}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{t_0,x_0}), B\phi(s, X_s^{t_0,x_0})\right) \geq \varepsilon.$$

This gives a relation between the drivers $-\psi(s, X_s^{t_0,x_0})$ and $f(s, X_s^{t_0,x_0}, \cdot)$ of the two BSDEs.
Also, $\phi(\theta, X_\theta^{t_0,x_0}) \geq u(\theta, X_\theta^{t_0,x_0}) = Y_\theta^{t_0,x_0}$ a.s.
Consequently, the extended comparison result for BSDEs with jumps given in the Appendix (see Proposition A.3) implies that:

$$\phi(t_0, x_0) = \phi(t_0, X_{t_0}^{t_0,x_0}) > Y_{t_0}^{t_0,x_0} = u(t_0, x_0),$$

which leads to a contradiction.

• We now prove that u is a viscosity supersolution of (9).
Let $(t_0, x_0) \in [0, T[\times \mathbb{R}$ and $\phi \in C^{1,2}([0,T] \times \mathbb{R})$ be such that $\phi(t_0, x_0) = u(t_0, x_0)$ and $\phi(t,x) \leq u(t,x)$, $\forall (t,x) \in [0,T] \times \mathbb{R}$.
Since the solution $(Y_s^{t_0,x_0})$ stays above the obstacle, we have:

$$u(t_0, x_0) \geq h(t_0, x_0).$$

We must prove that:

$$-\frac{\partial \phi}{\partial t}(t_0, x_0) - L\phi(t_0, x_0) - f\left(t_0, x_0, \phi(t_0, x_0), (\sigma \frac{\partial \phi}{\partial x})(t_0, x_0), B\phi(t_0, x_0)\right) \geq 0.$$

Suppose by contradiction that:

$$-\frac{\partial \phi}{\partial t}(t_0,x_0) - L\phi(t_0,x_0) - f\left(t_0,x_0,\phi(t_0,x_0),(\sigma\frac{\partial \phi}{\partial x})(t_0,x_0),B\phi(t_0,x_0)\right) < 0.$$

By continuity, we can suppose that there exists $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that for each (t,x) such that $t_0 \leq t \leq t_0+\eta_\varepsilon < T$ and $|x-x_0| \leq \eta_\varepsilon$, we have:

$$-\frac{\partial \phi}{\partial t}(t,x) - L\phi(t,x) - f\left(t,x,\phi(t,x),(\sigma\frac{\partial \phi}{\partial x})(t,x),B\phi(t,x)\right) \leq -\varepsilon. \tag{15}$$

We define the stopping time θ as:

$$\theta := (t_0+\eta_\varepsilon) \wedge \inf\{s \geq t_0 / |X_s^{t_0,x_0} - x_0| > \eta_\varepsilon\}.$$

Applying as above Itô's lemma to $\phi(s,X_s^{t_0,x_0})$, we get that $(\phi(s,X_s^{t_0,x_0}),(\sigma\frac{\partial \phi}{\partial x})(s,X_s^{t_0,x_0}),B\phi(s,X_{s^-}^{t_0,x_0}); s \in [t_0,\theta])$ is the solution of the BSDE associated with terminal value $\phi(\theta,X_\theta^{t_0,x_0})$ and driver $-\psi(s,X_s^{t_0,x_0})$.
The process $(Y^{t_0,x_0}, s \in [t_0,\theta])$ is the solution of the classical BSDE associated with terminal condition $Y_\theta^{t_0,x_0} = u(\theta,X_\theta^{t_0,x_0})$ and generalized driver

$$f(s,X_s^{t_0,x_0},y,z,q)ds + dA_s^{t_0,x_0}.$$

By (15) and the definition of the stopping time θ, we have :

$$\begin{aligned}(-\frac{\partial \phi}{\partial t}(s,X_s^{t_0,x_0}) - L\phi(s,X_s^{t_0,x_0}) - f(s,X_s^{t_0,x_0},\phi(s,X_s^{t_0,x_0}),\\ (\sigma\frac{\partial \phi}{\partial x})(s,X_s^{t_0,x_0}),B\phi(s,X_s^{t_0,x_0})))ds - dA_s^{t_0,x_0} \leq -\varepsilon\, ds, \quad t_0 \leq s \leq \theta \text{ a.s.}\end{aligned}$$

or, equivalently,

$$\begin{aligned}-\psi(s,X_s^{t_0,x_0})ds \leq (f(s,X_s^{t_0,x_0},\phi(s,X_s^{t_0,x_0}),(\sigma\frac{\partial \phi}{\partial x})(s,X_s^{t_0,x_0}),B\phi(s,X_s^{t_0,x_0})))ds\\ + dA_s^{t_0,x_0} - \varepsilon\, ds, \quad t_0 \leq s \leq \theta \text{ a.s.}\end{aligned}$$

This gives a relation between the drivers of the two BSDEs.
Also, $\phi(\theta,X_\theta^{t_0,x_0}) \leq u(\theta,X_\theta^{t_0,x_0}) = Y_\theta^{t_0,x_0}$ a.s. Consequently, Proposition A.3 in the Appendix implies that:

$$\phi(t_0,x_0) = \phi(t_0,X_{t_0}^{t_0,x_0}) < Y_{t_0}^{t_0,x_0} = u(t_0,x_0),$$

which leads to a contradiction. □

4 Uniqueness Result for the Obstacle Problem

We provide a uniqueness result for (9) in the particular case when for each $\phi \in C^{1,2}([0,T]\times\mathbb{R})$, $B\phi$ is a map valued in $\mathbb{R}$ instead of L^2_ν. More precisely,

$$B\phi(t,x) := \int_{\mathbb{R}^*} (\phi(t,x+\beta(x,e)) - \phi(t,x))\gamma(x,e)\nu(de), \tag{16}$$

which is well defined since $|\phi(t,x+\beta(x,e))-\phi(t,x)| \leq C|\beta(x,e)|$.

We suppose that Assumption 2.1 holds and we make the additional assumption:

Assumption 4.1

1. $f(s,X^{t,x}_s(\omega),y,z,k) := \overline{f}\left(s,X^{t,x}_s(\omega),y,z,\int_{\mathbb{R}^*} k(e)\gamma(X^{t,x}_s(\omega),e)\nu(de)\right)\mathbf{1}_{s\geq t}$,

where $\overline{f}:[0,T]\times\mathbb{R}^4\to\mathbb{R}$ is continuous in t uniformly with respect to x,y,z,k, continuous in x uniformly with respect to y,z,k, and satisfies:

(i) $|\overline{f}(t,x,0,0,0)| \leq C$, *for all* $t\in[0,T], x\in\mathbb{R}$.

(ii) $|\overline{f}(t,x,y,z,k)-\overline{f}(t,x',y',z',k')| \leq C(|y-y'|+|z-z'|+|k-k'|)$, *for all* $t\in[0,T]$, $y,y',z,z',k,k'\in\mathbb{R}$.

(iii) $k\mapsto \overline{f}(t,x,y,z,k)$ *is non-decreasing, for all* $t\in[0,T]$, $x,y,z\in\mathbb{R}$.

2. *For each* $R>0$*, there exists a continuous function* $m_R:\mathbb{R}_+\to\mathbb{R}_+$ *such that* $m_R(0)=0$ *and*
$|\overline{f}(t,x,v,p,q)-\overline{f}(t,y,v,p,q)| \leq m_R(|x-y|(1+|p|))$, *for all* $t\in[0,T]$, $|x|,|y|\leq R, |v|\leq R$, $p,q\in\mathbb{R}$.
3. $|\gamma(x,e)-\gamma(y,e)| \leq C|x-y|(1\wedge e^2)$ *and* $0\leq\gamma(x,e)\leq C(1\wedge|e|)$, *for all* $x,y\in\mathbb{R}, e\in\mathbb{R}^*$.
4. *There exists* $r>0$ *such that for all* $t\in[0,T]$, $x,u,v,p,l\in\mathbb{R}$*:*

$$\overline{f}(t,x,v,p,l)-\overline{f}(t,x,u,p,l) \geq r(u-v) \text{ when } u\geq v.$$

5. $|h(t,x)|+|g(x)|\leq C$, *for all* $t\in[0,T]$, $x\in\mathbb{R}$.

To simplify notation, $\overline{f}$ is denoted by f in the sequel.

We state below a comparison theorem, which uses results of three lemmas. The proofs of these lemmas are given in Subsection 4.1.

Theorem 4.1 (Comparison principle) *Under the above hypotheses, if U is a bounded continuous viscosity subsolution and V is a bounded continuous viscosity supersolution of the obstacle problem (9), then $U(t,x)\leq V(t,x)$, for each $(t,x)\in[0,T]\times\mathbb{R}$.*

Proof Set

$$M := \sup_{[0,T]\times\mathbb{R}} (U-V).$$

It is sufficient to prove that $M\leq 0$. For each $\varepsilon,\eta>0$, we introduce the function:

$$\psi^{\varepsilon,\eta}(t,s,x,y) := U(t,x)-V(s,y)-\frac{(x-y)^2}{\varepsilon^2}-\frac{(t-s)^2}{\varepsilon^2}-\eta^2(x^2+y^2),$$

for t,s,x,y in $[0,T]^2\times\mathbb{R}^2$. Let

$$M^{\varepsilon,\eta}:=\max_{[0,T]^2\times\mathbb{R}^2}\psi^{\varepsilon,\eta}.$$

This supremum is reached at some point $(t^{\varepsilon,\eta},s^{\varepsilon,\eta},x^{\varepsilon,\eta},y^{\varepsilon,\eta})$.

Using that $\psi^{\varepsilon,\eta}(t^{\varepsilon,\eta},s^{\varepsilon,\eta},x^{\varepsilon,\eta},y^{\varepsilon,\eta})\geq\psi^{\varepsilon,\eta}(0,0,0,0)$, we obtain:

$$\begin{aligned}U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-\frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2}-\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}\\ -\eta^2((x^{\varepsilon,\eta})^2+(y^{\varepsilon,\eta})^2)\geq U(0,0)-V(0,0),\end{aligned}\tag{17}$$

or, equivalently,

$$\begin{aligned}\frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2}+\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}+\eta^2((x^{\varepsilon,\eta})^2+(y^{\varepsilon,\eta})^2)\\ \leq\|U\|_\infty+\|V\|_\infty-U(0,0)-V(0,0).\end{aligned}\tag{18}$$

Consequently, we can find a constant C such that:

$$|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|+|t^{\varepsilon,\eta}-s^{\varepsilon,\eta}|\leq C\varepsilon\tag{19}$$

$$|x^{\varepsilon,\eta}|\leq\frac{C}{\eta},|y^{\varepsilon,\eta}|\leq\frac{C}{\eta}.\tag{20}$$

Extracting a subsequence if necessary, we may suppose that for each η the sequences $(t^{\varepsilon,\eta})_\varepsilon$ and $(s^{\varepsilon,\eta})_\varepsilon$ converge to a common limit t^η when ε tends to 0, and from (19) and (20) we may also suppose, extracting again, that for each η, the sequences $(x^{\varepsilon,\eta})_\varepsilon$ and $(y^{\varepsilon,\eta})_\varepsilon$ converge to a common limit x^η.

Lemma 4.1 *We have:*

$$\lim_{\varepsilon\to0}\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}=0;\quad\lim_{\varepsilon\to0}\frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2}=0$$

$$\lim_{\eta\to0}\lim_{\varepsilon\to0}M^{\varepsilon,\eta}=M.$$

We now introduce the functions:

$$\Psi_1(t,x):=V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})+\frac{(x-y^{\varepsilon,\eta})^2}{\varepsilon^2}+\frac{(t-s^{\varepsilon,\eta})^2}{\varepsilon^2}+\eta^2(x^2+(y^{\varepsilon,\eta})^2)$$

$$\Psi_2(s,y):=U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-\frac{(x^{\varepsilon,\eta}-y)^2}{\varepsilon^2}-\frac{(t^{\varepsilon,\eta}-s)^2}{\varepsilon^2}-\eta^2((x^{\varepsilon,\eta})^2+y^2).$$

As $(t,x)\to(U-\Psi_1)(t,x)$ reaches its maximum at $(t^{\varepsilon,\eta},x^{\varepsilon,\eta})$ and U is a subsolution we have two cases:

- $t^{\varepsilon,\eta}=T$ and then $U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\leq g(x^{\varepsilon,\eta})$,

- $t^{\varepsilon,\eta} \neq T$ and then

$$\min\Bigg(U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-h(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),\ \frac{\partial \Psi_1}{\partial t}(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-L\Psi_1(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-$$
$$-f\left(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(\sigma\frac{\partial \Psi_1}{\partial x})(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),B\Psi_1(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\right)\Bigg)\leq 0. \tag{21}$$

As $(s,y)\to(\Psi_2-V)(s,y)$ reaches its maximum at $(s^{\varepsilon,\eta},y^{\varepsilon,\eta})$ and V is a supersolution we have the two following cases:

- $s^{\varepsilon,\eta}=T$ and then $V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\geq g(y^{\varepsilon,\eta})$,
- $s^{\varepsilon,\eta}\neq T$ and then

$$\min(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-h(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),$$
$$\frac{\partial \Psi_2}{\partial t}(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-L\Psi_2(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-f(s^{\varepsilon,\eta},y^{\varepsilon,\eta},V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),(\sigma\frac{\partial \Psi_2}{\partial x})(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),B\Psi_2(s^{\varepsilon,\eta},y^{\varepsilon,\eta}))\geq 0. \tag{22}$$

We now prove that $M\leq 0$. Three cases are possible.

1st case: There exists a subsequence of (t^η) such that $t^\eta=T$ for all η (of this subsequence). As U is continuous, for all η and for ε small enough

$$U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\leq U(t^\eta,x^\eta)+\eta\leq g(x^\eta)+\eta,$$

and as V is continuous, for all η and for ε small enough

$$V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\geq V(t^\eta,x^\eta)-\eta\geq g(x^\eta)-\eta.$$

Hence

$$U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\leq 2\eta$$

and

$$M^{\varepsilon,\eta}=U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}-\frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2}$$
$$-\eta^2((x^{\varepsilon,\eta})^2+(y^{\varepsilon,\eta})^2)\leq U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\leq 2\eta.$$

Letting $\varepsilon\to 0$ and then $\eta\to 0$ one gets, using Lemma 4.1, that $M\leq 0$.

2nd case: There exists a subsequence such that $t^\eta\neq T$, and for all η belonging to this subsequence, there exists a subsequence of $(x^{\varepsilon,\eta})_\eta$ such that

$$U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-h(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\leq 0.$$

As from (22) one has

$$V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-h(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\geq 0,$$

it comes that

$$M^{\varepsilon,\eta} \leq U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}) \leq h(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - h(s^{\varepsilon,\eta},y^{\varepsilon,\eta}).$$

Letting $\varepsilon \to 0$ and then $\eta \to 0$, using the equality $\lim_{\eta\to 0}\lim_{\varepsilon\to 0} M^{\varepsilon,\eta} = M$ (see Lemma 4.1), we derive that $M \leq 0$.

Last case: We are left with the case when, for a subsequence of η, we have $t^{\eta} \neq T$ and for all η belonging to this subsequence there exists a subsequence of $(x^{\varepsilon,\eta})_{\varepsilon}$ such that:

$$U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - h(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) > 0.$$

Set

$$\varphi(t,s,x,y) := \frac{(x-y)^2}{\varepsilon^2} + \frac{(t-s)^2}{\varepsilon^2} + \eta^2(x^2+y^2). \tag{23}$$

The maximum of the function $\psi^{\varepsilon,\eta}(t,s,x,y) := U(t,x) - V(s,y) - \varphi(t,s,x,y)$ is reached at the point $(t^{\varepsilon,\eta},s^{\varepsilon,\eta},x^{\varepsilon,\eta},y^{\varepsilon,\eta})$. We apply the non-local version of Jensen Ishii's lemma [7] and we obtain that there exist:

$$(a,\overline{p},X) \in \mathscr{P}^{2,+}U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),\ (b,\overline{q},Y) \in \mathscr{P}^{2,-}V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})$$

such that

$$\begin{cases} \overline{p} = p + 2\eta^2 x^{\varepsilon,\eta};\ \ \overline{q} = p - 2\eta^2 y^{\varepsilon,\eta};\ \ p = \frac{2(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})}{\varepsilon^2} \\ a = b = \frac{2(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})}{\varepsilon^2} \\ \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{2}{\varepsilon^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + 2\eta^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Here, $\mathscr{P}^{2,+}$ (resp. $\mathscr{P}^{2,-}$) is the set of superjets (resp. subjets) defined in [7] (see Definition 3). Since $(t^{\varepsilon,\eta},s^{\varepsilon,\eta},x^{\varepsilon,\eta},y^{\varepsilon,\eta})$ is a global maximum of $\psi^{\varepsilon,\eta}$,we have:

$$\begin{aligned} &\psi^{\varepsilon,\eta}(t^{\varepsilon,\eta},s^{\varepsilon,\eta},x^{\varepsilon,\eta}+\beta(x^{\varepsilon,\eta},e),y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e)) \leq \psi^{\varepsilon,\eta}(t^{\varepsilon,\eta},s^{\varepsilon,\eta},x^{\varepsilon,\eta},y^{\varepsilon,\eta}) \\ &\quad \Leftrightarrow U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}+\beta(x^{\varepsilon,\eta},e)) - V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e)) \\ &\quad - \frac{(x^{\varepsilon,\eta}+\beta(x^{\varepsilon,\eta},e)-y^{\varepsilon,\eta}-\beta(y^{\varepsilon,\eta},e))^2}{\varepsilon^2} \\ &\quad - \frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2} - \eta^2((x^{\varepsilon,\eta}+\beta(x^{\varepsilon,\eta},e))^2 + (y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e))^2) \\ &\quad \leq U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}) - \frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2} - \frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2} - \eta^2((x^{\varepsilon,\eta})^2+(y^{\varepsilon,\eta})^2). \end{aligned}$$

Consequently, we get:

$$
\begin{aligned}
&U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}+\beta(x^{\varepsilon,\eta},e))-U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\leq V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e))\\
&-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})+\frac{(\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e))^2}{\varepsilon^2}+p(\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e))\\
&+\eta^2(\beta^2(x^{\varepsilon,\eta},e)+2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)+2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)+\beta^2(y^{\varepsilon,\eta},e)).
\end{aligned}
\tag{24}
$$

Let us fix $\delta>0$ and consider the ball $\mathscr{B}_\delta=\mathscr{B}(0,\delta)$. We introduce the operators K^δ, $\tilde{K}^\delta, B^\delta, \tilde{B}^\delta$ corresponding to the operators K and B defined in (10) and (16), but integrating on $\mathscr{B}_\delta$ or $\mathbb{R}\backslash\mathscr{B}_\delta$ (also denoted by $\mathscr{B}_\delta^c$) only.
They are defined respectively for all $\phi\in C^{1,2}$, $\Phi\in\mathscr{C}$ by

$$
K^\delta[t,x,\phi]:=\int_{\mathscr{B}_\delta}\left(\phi(t,x+\beta(x,e))-\phi(t,x)-\frac{\partial\phi}{\partial x}(t,x)\beta(x,e)\right)\nu(de) \tag{25}
$$

$$
\tilde{K}^\delta[t,x,\pi,\Phi]:=\int_{\mathscr{B}_\delta^c}\left(\Phi(t,x+\beta(x,e))-\Phi(t,x)-\pi\beta(x,e)\right)\nu(de). \tag{26}
$$

$$
B^\delta[t,x,\phi]:=\int_{\mathscr{B}_\delta}\left(\phi(t,x+\beta(x,e))-\phi(t,x)\right)\gamma(x,e)\nu(de) \tag{27}
$$

$$
\tilde{B}^\delta[t,x,\Phi]:=\int_{\mathscr{B}_\delta^c}\left(\Phi(t,x+\beta(x,e))-\Phi(t,x)\right)\gamma(x,e)\nu(de) \tag{28}
$$

Here $\mathscr{C}$ denotes the set of bounded continuous functions.

By approaching U by a sequence (ϕ_k) in $C^{1,2}$, and passing to the limit in the operators, one can show that in the definition of a sub-solution (see Definition 3.1), $B(\phi)(t,x)$ can be replaced by $B^\delta[t,x,\phi]+\tilde{B}^\delta[t,x,U]$. A similar property holds for a super-solution and the operator K. We then can use the alternative definition for sub-superviscosity solutions in terms of sub-superjets (see Definition 4 in [7]). Since U is a subviscosity solution and V is superviscosity solution, we have:

$$
\begin{cases}
F(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),a,\overline{p},X,K^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x]\\
\qquad+\tilde{K}^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\overline{p},U],B^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x]+\tilde{B}^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},U])\leq 0\\
F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,K^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y]\\
\qquad+\tilde{K}^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},\overline{q},V],B^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y]+\tilde{B}^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},V])\geq 0
\end{cases}
\tag{29}
$$

where

$$
F(t,x,u,a,p,X,l_1,l_2):=-a-\frac{1}{2}\sigma^2(x)X-b(x)p-l_1-f(t,x,u,p\sigma(x),l_2). \tag{30}
$$

We denote by φ_x the function $(t,x)\mapsto\varphi(t,x,s^{\varepsilon,\eta},y^{\varepsilon,\eta})$ and by φ_y the function $(s,y)\mapsto\varphi(t^{\varepsilon,\eta},x^{\varepsilon,\eta},s,y)$.
The two following lemmas hold.

Lemma 4.2 *Let*

$$\begin{aligned} l_K &:= K^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] + \tilde{K}^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \overline{p}, U] \\ l'_K &:= K^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + \tilde{K}^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, \overline{q}, V]. \end{aligned} \tag{31}$$

We have

$$l_K \le l'_K + O(\frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(\eta^2) + (\frac{1}{\varepsilon^2} + \eta^2)O(\delta). \tag{32}$$

Lemma 4.3 *Let*

$$\begin{aligned} l_B &:= B^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, \varphi_x] + \tilde{B}^{\delta}[t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U] \\ l'_B &:= B^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, -\varphi_y] + \tilde{B}^{\delta}[s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V]. \end{aligned} \tag{33}$$

We have

$$l_B \le l'_B + (\eta^2 + \frac{1}{\varepsilon^2})O(\delta) + O(\frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|) + O(\eta^2). \tag{34}$$

We argue now by contradiction by assuming that

$$M > 0. \tag{35}$$

Using Point 4 of Assumption 4.1, we get

$$\begin{aligned} 0 &< \frac{r}{2}M \le rM_{\varepsilon,\eta} \le r(U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})) \\ &\le F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{q}, Y, l'_K, l'_B) - F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l'_K, l'_B) \\ &= F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{q}, Y, l'_K, l'_B) - F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l'_K, l'_B) \\ &\quad + F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l'_K, l'_B) - F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_K, l_B) \\ &\quad + F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_K, l_B) - F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{p}, X, l_K, l_B) \\ &\quad + F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}), a, \overline{p}, X, l_K, l_B) - F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, V(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l'_K, l'_B) \\ &\le K|U(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}) - U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta})| + F(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}, U(s^{\varepsilon,\eta}, y^{\varepsilon,\eta}), a, \overline{q}, Y, l_K, l_B) \\ &\quad - F(t^{\varepsilon,\eta}, x^{\varepsilon,\eta}, U(t^{\varepsilon,\eta}, X^{\varepsilon,\eta}), a, \overline{p}, X, l_K, l_B) \\ &\quad + (\eta^2 + \frac{1}{\varepsilon^2})O(\delta) + O(\frac{(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|) + O(\eta^2). \end{aligned} \tag{36}$$

We have used here the (nonlocal) ellipticity of F, the Lipschitz property of F, (29) and the estimates proven in Lemma 4.2 and Lemma 4.3. From the hypothesis on b and σ, we have:

$$\begin{aligned} \sigma^2(x^{\varepsilon,\eta})X - \sigma^2(y^{\varepsilon,\eta})Y &\le \frac{C(x^{\varepsilon,\eta} - y^{\varepsilon,\eta})^2}{\varepsilon^2} + O(\eta^2), \\ b(x^{\varepsilon,\eta})\overline{p} - b(y^{\varepsilon,\eta})\overline{q} &\le \frac{C|x^{\varepsilon,\eta} - y^{\varepsilon,\eta}|}{\varepsilon^2} + O(\eta^2). \end{aligned}$$

We thus obtain the inequality:

$$
\begin{aligned}
&F(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),a,\overline{q},Y,l_K,l_B)-F(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),a,\overline{p},X,l_K,l_B)\\
&\leq \frac{C(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}+O(\eta^2)\\
&+f(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(p+2\eta^2)\sigma(x^{\varepsilon,\eta}),l_B)\\
&-f(s^{\varepsilon,\eta},y^{\varepsilon,\eta},U(s^{\varepsilon,\eta},y^{\varepsilon,\eta}),(p-2\eta^2)\sigma(y^{\varepsilon,\eta}),l_B)\\
&\leq f(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(p+2\eta^2)\sigma(x^{\varepsilon,\eta}),l_B)\\
&-f(s^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(p+2\eta^2)\sigma(x^{\varepsilon,\eta}),l_B)\\
&+m_R(|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|(1+(p+2\eta^2)\sigma(x^{\varepsilon,\eta})))\\
&+K|U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-U(s^{\varepsilon,\eta},y^{\varepsilon,\eta})|+O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2})+O(\eta^2). \qquad (37)
\end{aligned}
$$

The last equality is obtained by some computations similar to those in (36). From (36), (37) we get

$$
\begin{aligned}
0<\frac{r}{2}M\leq rM^{\varepsilon,\eta}&\leq f(t^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(p+2\eta^2)\sigma(x^{\varepsilon,\eta}),l_B)\\
&-f(s^{\varepsilon,\eta},x^{\varepsilon,\eta},U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}),(p+2\eta^2)\sigma(x^{\varepsilon,\eta}),l_B)\\
&+m_R(|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|(1+(p+2\eta^2)\sigma(x^{\varepsilon,\eta}))\\
&+K|U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-U(s^{\varepsilon,\eta},y^{\varepsilon,\eta})|+\\
&+O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2})+O(|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|)+(\eta^2+\frac{1}{\varepsilon^2})O(\delta)+O(\eta^2). \qquad (38)
\end{aligned}
$$

By Lemma 4.1, letting successively δ,ε and η tend to 0 in (38) we obtain that $0<\frac{r}{2}M\leq 0$. Hence, the assumption $M>0$ made above (see (35)) is wrong. This ends the proof of Theorem 4.1. □

Corollary 4.1 (Uniqueness) *Under the additional Assumption 4.1, the value function is the unique solution of the obstacle problem* (9) *in the class of bounded continuous functions.*

4.1 Proofs of the lemmas

Proof of Lemma 4.1. For $\eta>0$, we introduce the functions $\tilde{U}^\eta(t,x)=U(t,x)-\eta^2x^2$ and $\tilde{V}^\eta(t,x)=V(t,x)+\eta^2x^2$. Set

$$M^\eta := \sup_{[0,T]\times\mathbb{R}} (\tilde{U}^\eta-\tilde{V}^\eta).$$

The maximum M^η is reached at some point $(\hat{t}^\eta,\hat{x}^\eta)$. From the form of $\psi^{\varepsilon,\eta}$, we have that for fixed η, there exists a subsequence $(t^{\varepsilon,\eta},s^{\varepsilon,\eta},x^{\varepsilon\eta},y^{\varepsilon,\eta})_\varepsilon$ which converges to some point $(t^\eta,s^\eta,x^\eta,y^\eta)$ when ε tends to 0.

Since $M^{\varepsilon,\eta}$ is reached at $(t^{\varepsilon,\eta},s^{\varepsilon,\eta},x^{\varepsilon,\eta},y^{\varepsilon,\eta})$, we have:

$$
\begin{aligned}
(\tilde{U}^\eta-\tilde{V}^\eta)(\hat{t}^\eta,\hat{x}^\eta) &= (U-V)(\hat{t}^\eta,\hat{x}^\eta)-\eta^2((\hat{x}^\eta)^2+(\hat{y}^\eta)^2)\le M^{\varepsilon,\eta}\\
&= U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})-\frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2}\\
&\quad -\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}-\eta^2((x^{\varepsilon,\eta})^2+(y^{\varepsilon,\eta})^2).
\end{aligned}
$$

Setting

$$
\bar{l}_\eta := \limsup_{\varepsilon\to 0}\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}, \quad \underline{l}_\eta := \liminf_{\varepsilon\to 0}\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}
$$

we get

$$
0\le \underline{l}_\eta \le \bar{l}_\eta \le (\tilde{U}^\eta-\tilde{V}^\eta)(t^\eta,x^\eta)-(\tilde{U}^\eta-\tilde{V}^\eta)(\hat{t}^\eta,\hat{x}^\eta)\le 0. \tag{39}
$$

We derive that, up to a subsequence, $\lim_{\varepsilon\to 0}\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}=0$ and $\lim_{\varepsilon\to 0}M^{\varepsilon,\eta}=M^\eta$.
Similarly, we get $\lim_{\varepsilon\to 0}\frac{(t^{\varepsilon,\eta}-s^{\varepsilon,\eta})^2}{\varepsilon^2}=0$.
Let us prove that $\lim_{\eta\to 0}M^\eta = M$. First, note that $M^\eta\le M$, for all η. By definition of M, for all $\delta>0$ there exists $(t_\delta,x_\delta)\in[0,T]\times\mathbb{R}$ such that $M-\delta\le (U-V)(t_\delta,x_\delta)$. Consequently, we get

$$
M-2\eta^2x_\delta^2-\delta\le (U-V)(t_\delta,x_\delta)-2\eta^2x_\delta^2=(\tilde{U}^\eta-\tilde{V}^\eta)(t_\delta,x_\delta)\le M^\eta\le M.
$$

By letting η and then δ tend to 0, the result follows. □

Proof of Lemma 4.2. We have:

$$
K^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x]=\int_{\mathscr{B}_\delta}(\frac{1}{\varepsilon^2}+\eta^2)\beta^2(x^{\varepsilon,\eta},e)\nu(de) \tag{40}
$$

$$
K^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y]=\int_{\mathscr{B}_\delta}(-\frac{1}{\varepsilon^2}-\eta^2)\beta^2(y^{\varepsilon,\eta},e)\nu(de). \tag{41}
$$

Equations (40) and (41) imply:

$$
\begin{aligned}
&K^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x]\le K^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y]+(\frac{1}{\varepsilon^2}+\eta^2)\int_{\mathscr{B}_\delta}\beta^2(y^{\varepsilon,\eta},e)\nu(de)\\
&+(\frac{1}{\varepsilon^2}+\eta^2)\int_{\mathscr{B}_\delta}\beta^2(x^{\varepsilon,\eta},e)\nu(de)\le K^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y]+(\frac{1}{\varepsilon^2}+\eta^2)O(\delta).
\end{aligned} \tag{42}
$$

Using inequality (24) and integrating on $\mathscr{B}_\delta^c$, we obtain:

$$
\begin{aligned}
&\tilde{K}^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\overline{p},U] = \int_{\mathscr{B}_\delta^c}\Big(U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}+\beta(x^{\varepsilon,\eta},e)) - U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}) - (p+2\eta^2x^{\varepsilon,\eta})\beta(x^{\varepsilon,\eta},e)\Big)\nu(de)\\
&\leq \int_{\mathscr{B}_\delta^c}\Big(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e)) - V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}) - (p-2\eta^2y^{\varepsilon,\eta})\beta(y^{\varepsilon,\eta},e)\Big)\nu(de)\\
&\quad + \int_{\mathscr{B}_\delta^c}\frac{(\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e))^2}{\varepsilon^2}\nu(de) + \eta^2\int_{\mathscr{B}_\delta^c}(\beta^2(x^{\varepsilon,\eta},e)+\beta^2(y^{\varepsilon,\eta},e))\nu(de)\\
&\leq \tilde{K}^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},\overline{q},V] + O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(\eta^2).
\end{aligned}
$$

Using (31) and (42), we derive (32), which ends the proof of Lemma 4.2. □

Proof of Lemma 4.3. From (27), we derive that:

$$
\begin{aligned}
B^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x] &= \int_{\mathscr{B}_\delta}\Big((\eta^2+\frac{1}{\varepsilon^2})\beta^2(x^{\varepsilon,\eta},e) + \frac{2\beta(x^{\varepsilon,\eta},e)}{\varepsilon^2}(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})\\
&\qquad + 2\eta^2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e)\nu(de) \qquad (43)\\
B^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y] &= \int_{\mathscr{B}_\delta}\Big((-\eta^2-\frac{1}{\varepsilon^2})\beta^2(y^{\varepsilon,\eta},e) + \frac{2\beta(y^{\varepsilon,\eta},e)}{\varepsilon^2}(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})\\
&\qquad - 2\eta^2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)\Big)\gamma(y^{\varepsilon,\eta},e)\nu(de). \qquad (44)
\end{aligned}
$$

After some computations, we obtain:

$$
\begin{aligned}
&\Big((\eta^2+\frac{1}{\varepsilon^2})\beta^2(x^{\varepsilon,\eta},e) + \frac{2\beta(x^{\varepsilon,\eta},e)}{\varepsilon^2}(x^{\varepsilon,\eta}-y^{\varepsilon,\eta}) + 2\eta^2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e)\\
&= (-\eta^2-\frac{1}{\varepsilon^2})\beta^2(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e) + \frac{2\beta(y^{\varepsilon,\eta},e)}{\varepsilon^2}(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})\gamma(y^{\varepsilon,\eta},e)\\
&- 2\eta^2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e) + (\eta^2+\frac{1}{\varepsilon^2})\Big(\beta^2(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e) + \beta^2(x^{\varepsilon,\eta},e)\gamma(x^{\varepsilon,\eta},e)\Big)\\
&+ \frac{2}{\varepsilon^2}(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})\Big(\beta(x^{\varepsilon,\eta},e)\gamma(x^{\varepsilon,\eta},e) - \beta(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e)\Big)\\
&+ 2\eta^2\Big(x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)\gamma(x^{\varepsilon,\eta},e) + y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)\gamma(y^{\varepsilon,\eta},e)\Big). \qquad (45)
\end{aligned}
$$

From (43), (44), (45) and using the hypothesis on β and γ, we get:

$$
B^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},\varphi_x] \leq B^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},-\varphi_y] + (\eta^2+\frac{1}{\varepsilon^2})O(\delta) + O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2}) + O(\eta^2). \qquad (46)
$$

We now estimate the operator $\tilde{B}^\delta$. Inequality (24) implies:

$$\begin{aligned}
&\Big(U(t^{\varepsilon,\eta},x^{\varepsilon,\eta}+\beta(x^{\varepsilon,\eta},e))-U(t^{\varepsilon,\eta},x^{\varepsilon,\eta})\Big)\gamma(x^{\varepsilon,\eta},e)\\
&\quad\leq\Big(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e))-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\\
&\quad+\frac{|\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e)|^2}{\varepsilon^2}+p(\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e))\\
&\quad+\eta^2(\beta^2(x^{\varepsilon,\eta},e)+2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)+2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)+\beta^2(y^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e)\\
&\quad=\Big(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e))-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\Big)\gamma(y^{\varepsilon,\eta},e)\\
&\quad+\Big(V(s^{\varepsilon,\eta},y^{\varepsilon,\eta}+\beta(y^{\varepsilon,\eta},e))-V(s^{\varepsilon,\eta},y^{\varepsilon,\eta})\Big)\Big(\gamma(x^{\varepsilon,\eta},e)-\gamma(y^{\varepsilon,\eta},e)\Big)\\
&\quad+\frac{|\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e)|^2}{\varepsilon^2}\gamma(x^{\varepsilon,\eta},e)+p\Big(\beta(x^{\varepsilon,\eta},e)-\beta(y^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e)\\
&\quad+\eta^2\Big(\beta^2(x^{\varepsilon,\eta},e)+2x^{\varepsilon,\eta}\beta(x^{\varepsilon,\eta},e)+2y^{\varepsilon,\eta}\beta(y^{\varepsilon,\eta},e)+\beta^2(y^{\varepsilon,\eta},e)\Big)\gamma(x^{\varepsilon,\eta},e).
\end{aligned}$$

Now, by (20), we have $|x^{\varepsilon,\eta}|\leq\dfrac{C}{\eta}$ and $|y^{\varepsilon,\eta}|\leq\dfrac{C}{\eta}$. Hence, using the hypothesis on β,γ and integrating on $\mathscr{B}_\delta^c$, we get

$$\tilde{B}^\delta[t^{\varepsilon,\eta},x^{\varepsilon,\eta},U]\leq\tilde{B}^\delta[s^{\varepsilon,\eta},y^{\varepsilon,\eta},V]+O(|x^{\varepsilon,\eta}-y^{\varepsilon,\eta}|)+O(\frac{(x^{\varepsilon,\eta}-y^{\varepsilon,\eta})^2}{\varepsilon^2})+O(\eta^2). \tag{47}$$

Finally, from (46), (33) and (47), we derive inequality (34). □

5 Conclusions

In this paper, we have studied the optimal stopping problem for a monotonous dynamic risk measure defined by a Markovian BSDE with jumps. We have shown that, under relatively weak hypotheses, the value function is a viscosity solution of an obstacle problem for a partial integro-differential variational inequality. Recall that in the Brownian case, this existence result was proven in [6] by using an approximation method via penalized BSDEs. Note that this method could also be adapted to our case with jumps, but would involve heavy computations in order to prove the convergence of the solutions of the penalized BSDEs to the solution of the reflected BSDE. It would also require some convergence results of the viscosity solutions theory in the integro-differential case. We have adopted instead a direct method allowing us to give a shorter proof. Moreover, using a nonlocal version of the Jensen Ishii Lemma, we have proven a comparison theorem which extends some results established in [7] (Section 5.1, Th.3) to the case of a nonlinear BSDE.

The links given in this paper between optimal stopping problems for BSDEs and obstacle problems for PDEs can be extended to a larger class of problems. Among them, we can mention generalized Dynkin games with nonlinear expectation (see [8]), and mixed optimal stopping/stochastic control problems (see [9]). However, the latter case requires to establish a dynamic programming principle, which does not follow from the flow property of reflected BSDEs only, and needs rather sophisticated techniques.

A Appendix

A.1 Some Useful Estimates

Let $T>0$ be a fixed terminal time. A map $f:[0,T]\times\Omega\times\mathbb{R}^2\times L^2_\nu\to\mathbb{R};(t,\omega,y,z,k)\mapsto f(t,\omega,y,z,k)$ is said to be a *Lipschitz driver* if it is predictable, uniformly Lipchitz with respect to y,z,k and such that $f(t,0,0,0)\in\mathbb{H}^2$.

Let $\xi^1_t,\xi^2_t\in\mathscr{S}^2$. Let f^1,f^2 be two admissible Lipschitz drivers with Lipchitz constant C. For $i=1,2$, let $\mathscr{E}^i$ be the f^i-conditional expectation associated with driver f^i, and let (Y^i_t) be the adapted process defined for each $t\in[0,T]$,

$$Y^i_t := \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_t}\mathscr{E}^i_{t,\tau}(\xi^i_\tau). \tag{48}$$

Proposition A.1 *For $s\in[0,T]$, denote $\overline{Y}_s=Y^1_s-Y^2_s$, $\overline{\xi}_s=\xi^1_s-\xi^2_s$ and $\overline{f}_s=\sup_{y,z,k}|f^1(s,y,z,k)-f^2(s,y,z,k)|$. Let $\eta,\beta>0$ be such that $\beta\geq\frac{3}{\eta}+2C$ and $\eta\leq\frac{1}{C^2}$. Then for each t, we have:*

$$e^{\beta t}\overline{Y}_t^2\leq e^{\beta T}(\mathbb{E}[\sup_{s\geq t}\overline{\xi}_s^2|\mathscr{F}_t]+\eta\mathbb{E}[\int_t^T\overline{f}_s^2ds|\mathscr{F}_t])\ a.s. \tag{49}$$

Proof For $i=1,2$ and for each $\tau\in\mathscr{T}_0$, let $(X^{i,\tau},\pi^{i,\tau}_s,l^{i,\tau}_s)$ be the solution of the BSDE associated with driver f^i, terminal time τ and terminal condition ξ^i_τ. Set $\overline{X}^\tau_s=X^{1,\tau}_s-X^{2,\tau}_s$.

By a priori estimate on BSDEs (see Proposition $A.4$ in [5]), we have:

$$\begin{aligned}e^{\beta t}(\overline{X}^\tau_t)^2\leq e^{\beta T}\mathbb{E}[\overline{\xi}^2_\tau|\mathscr{F}_t]+\eta\mathbb{E}[\int_t^T e^{\beta s}(f^1(s,X^{2,\tau}_s,\pi^{2,\tau}_s,l^{2,\tau}_s)\\ -f^2(s,X^{2,\tau}_s,\pi^{2,\tau}_s,l^{2,\tau}_s))^2ds|\mathscr{F}_t]\quad\text{a.s.}\end{aligned} \tag{50}$$

from which we derive that

$$e^{\beta t}(\overline{X}^\tau_t)^2\leq e^{\beta T}(\mathbb{E}[\sup_{s\geq t}\overline{\xi}^2_s|\mathscr{F}_t]+\eta\mathbb{E}[\int_t^T\overline{f}^2_sds|\mathscr{F}_t]). \tag{51}$$

Now, by definition of Y^i, we have $Y^i_t=\operatorname{ess\,sup}_{\tau\geq t}X^{i,\tau}_t$ a.s. for $i=1,2$. We thus get $|\overline{Y}_t|\leq\operatorname{ess\,sup}_{\tau\geq t}|\overline{X}^\tau_t|$ a.s. The result follows. □

Let $\xi_t\in\mathscr{S}^2$. Let f be a Lipschitz driver with Lipschitz constant $C>0$. Set

$$Y_t := \operatorname*{ess\,sup}_{\tau\in\mathscr{T}_t}\mathscr{E}_{t,\tau}(\xi_\tau) \tag{52}$$

where $\mathscr{E}$ is the f-conditional expectation associated with driver f.

Proposition A.2 *Let $\eta,\beta>0$ be such that $\beta\geq\frac{3}{\eta}+2C$ and $\eta\leq\frac{1}{C^2}$. Then for each t, we have:*

$$e^{\beta t}Y_t^2\leq e^{\beta T}(\mathbb{E}[\sup_{s\geq t}{\xi_s}^2|\mathscr{F}_t]+\eta\mathbb{E}[\int_t^T f(s,0,0,0)^2ds|\mathscr{F}_t])\ a.s. \tag{53}$$

Proof Let X^τ_t be the solution of the BSDE associated with driver f, terminal time τ and terminal condition ξ_τ. By applying inequality (50) with $f^1=f$, $\xi_1=\xi$, $f^2=0$ and $\xi^2=0$, we get:

$$e^{\beta t}(X^\tau_t)^2\leq e^{\beta T}\mathbb{E}[\xi^2_\tau|\mathscr{F}_t]+\eta\mathbb{E}[\int_t^T e^{\beta s}(f(s,0,0,0))^2|\mathscr{F}_t]. \tag{54}$$

The result follows.

Remark A.1 If the drivers satisfy Assumption 3.1 in [5], then Y (resp. Y^i) is the solution of the RBSDE associated with driver f (resp.f^i) and obstacle ξ (resp. ξ^i). Hence the above estimates provide some new estimates on RBSDEs. Note that η and β are universal constants, i.e. they do not depend on $T, \xi, \xi^1, \xi^2, f, f^1, f^2$. This was not the case for the estimates given in the previous literature (see e.g. [6]).

A.2 Some Properties of the Value Function u

We prove below the continuity and polynomial growth of the function u defined by (8).

Lemma A.1 *The function u is continuous in (t,x).*

Proof It is sufficient to show that, when $(t_n,x_n) \to (t,x)$, $|u(t_n,x_n) - u(t,x)| \to 0$.
Let $\bar{h}$ be the map defined by $\bar{h}(t,x) = h(t,x)$ for $t < T$ and $\bar{h}(T,x) = g(x)$, so that, for each (t,x), we have $\xi_s^{t,x} = \bar{h}(s, X_s^{t,x})$, $0 \leq s \leq T$ a.s.
By applying Proposition A.1 with $X_s^1 = X_s^{t_n,x_n}$, $X_s^2 = X_s^{t,x}$, $f^1(s,\omega,y,z,q) := \mathbf{1}_{[t,T]}(s) f(s, X_s^{t,x}(\omega), y,z,q)$ and $f^2(s,\omega,y,z,q) := \mathbf{1}_{[t_n,T]}(s) f(s, X_s^{t_n,x_n}(\omega), y,z,q)$, we obtain:

$$|u(t_n,x_n) - u(t,x)|^2 \leq K_{C,T} \mathbb{E}[\sup_{0 \leq s \leq T} |\overline{h}(s, X_s^{t_n,x_n}) - \overline{h}(s, X_s^{t,x})|^2 + \int_0^T (\overline{f}_s^n)^2],$$

where

$$\begin{cases} K_{C,T} := e^{(3C^2+2C)T} \max(1, \frac{1}{C^2}) \\ \overline{f}_s^n(\omega) := \sup_{y,z,q} |\mathbf{1}_{[t,T]} f(s, X_s^{t,x}(\omega), y,z,q) - \mathbf{1}_{[t_n,T]} f(s, X_s^{t_n,x_n}(\omega), y,z,q)|. \end{cases}$$

The continuity of u is then a consequence of the following convergences as $n \to \infty$:

$$\begin{aligned} &\mathbb{E}(\sup_{0 \leq s \leq T} |\overline{h}(s, X_s^{t,x}) - \overline{h}(s, X_s^{t_n}(x_n))|^2) \to 0 \\ &\mathbb{E}[\int_0^T (\overline{f}_s^n)^2 ds] \to 0, \end{aligned}$$

which follow from the Lebesgue's theorem, using the continuity assumptions and polynomial growth of f and h . □

Lemma A.2 *The function u has at most polynomial growth at infinity.*

Proof By applying Prop. A.2 , we obtain the following estimate:

$$u(t,x)^2 \leq K_{C,T} (\mathbb{E}(\int_0^T f(s, X_s^{t,x}, 0,0,0)^2 ds + \sup_{0 \leq s \leq T} \overline{h}(s, X_s^{t,x})^2). \tag{55}$$

Using now the hypothesis of polynomial growth on f, h, g and the standard estimate

$$\mathbb{E}[\sup_{0 \leq s \leq T} |X_s^{t,x}|^2] \leq C'(1+x^2),$$

we derive that there exist $\bar{C} \in \mathbb{R}$ and $p \in \mathbb{N}$ such that $|u(t,x)| \leq \bar{C}(1+x^p)$, $\forall t \in [0,T]$, $\forall x \in \mathbb{R}$. □

Remark A.2 By (55), if $(t,x) \mapsto f(t,x,0,0)$, h and g are bounded, then u is bounded.

A.3 An Extension of the Comparison Result for BSDEs with Jumps

We provide here an extension of the comparison theorem for BSDEs given in [4] which formally states that if two drivers f_1, f_2 satisfy $f_1 \geq f_2 + \varepsilon$, then the associated solutions X^1 and X^2 satisfy $X_0^1 > X_0^2$.

Proposition A.3 *Let $t_0 \in [0,T]$ and let θ be a stopping time such that $\theta > t_0$ a.s.*
Let ξ_1 and $\xi_2 \in L^2(\mathscr{F}_\theta)$. Let f_1 be a driver. Let f_2 be a Lipschitz driver. For $i = 1,2$, let (X_t^i, π_t^i, l_t^i) be a solution in $S^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ of the BSDE

$$-dX_t^i = f_i(t, X_t^i, \pi_t^i, l_t^i)dt - \pi_t^i dW_t - \int_{\mathbb{R}^*} l_t^i(u)\tilde{N}(dt,du); \quad X_\theta^i = \xi_i. \tag{56}$$

Assume that there exists a bounded predictable process (γ_t) such that $dt \otimes dP \otimes \nu(de)$-a.s. $\gamma_t(e) \geq -1$ and $|\gamma_t(e)| \leq C(1 \wedge |e|)$, and such that

$$f_2(t, X_t^2, \pi_t^2, l_t^1) - f_2(t, X_t^2, \pi_t^2, l_t^2) \geq \langle \gamma_t \,, l_t^1 - l_t^2 \rangle_\nu, \;\; t_0 \leq t \leq \theta, \;\; dt \otimes dP \text{ a.s.} \tag{57}$$

Suppose also that

$$\begin{aligned} &\xi_1 \geq \xi_2 \text{ a.s.} \\ &f_1(t, X_t^1, \pi_t^1, l_t^1) \geq f_2(t, X_t^1, \pi_t^1, l_t^1) + \varepsilon, \;\; t_0 \leq t \leq \theta, \;\; dt \otimes dP \text{ a.s.} \end{aligned}$$

where ε is a real constant. Then,

$$X_{t_0}^1 - X_{t_0}^2 \geq \varepsilon \alpha \quad \text{a.s.}$$

where α is a non negative $\mathscr{F}_{t_0}$-measurable r.v. which does not depend on ε, with $P(\alpha > 0) > 0$.

Proof From inequality (4.22) in the proof of the Comparison Theorem in [4], we derive that

$$X_{t_0}^1 - X_{t_0}^2 \geq e^{-CT}\mathbb{E}\left[\int_{t_0}^\theta H_{t_0,s}\, \varepsilon\, ds | \mathscr{F}_{t_0}\right] \quad \text{a.s.},$$

where C is the Lipschitz constant of f_2, and $(H_{t_0,s})_{s \in [t_0,T]}$ is the square integrable non negative martingale satisfying

$$dH_{t_0,s} = H_{t_0,s^-}\left[\beta_s dW_s + \int_{\mathbb{R}^*} \gamma_s(u)\tilde{N}(ds,du)\right]; \;\; H_{t_0,t_0} = 1,$$

(β_s) being a predictable process bounded by C. We get

$$X_{t_0}^1 - X_{t_0}^2 \geq e^{-CT}\varepsilon\, \mathbb{E}\left[H_{t_0,\theta}\,(\theta - t_0) | \mathscr{F}_{t_0}\right] \quad \text{a.s.}$$

Since $\theta > t_0$ a.s. , we have $H_{t_0,\theta}\,(\theta - t_0) \geq 0$ a.s. and $P(H_{t_0,\theta}\,(\theta - t_0) > 0) > 0$. Setting $\alpha := e^{-CT}\,\mathbb{E}\left[H_{t_0,\theta}\,(\theta - t_0)|\mathscr{F}_{t_0}\right]$, the result follows. □

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