Mixed generalized Dynkin game and stochastic control in a Markovian framework

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Abstract

We introduce a mixed *generalized* Dynkin game/stochastic control with $\mathcal{E}^f$-expectation in a Markovian framework. We study both the case when the terminal reward function is Borelian only and when it is continuous. By using the characterization of the value function of a *generalized* Dynkin game via an associated doubly reflected BSDEs (DRBSDE) first provided in [16], we obtain that the value function of our problem coincides with the value function of an optimization problem for DRBSDEs. Using this property, we establish a weak dynamic programming principle by extending some results recently provided in [17]. We then show a strong dynamic programming principle in the continuous case, which cannot be derived from the weak one. In particular, we have to prove that the value function of the problem is continuous with respect to time t, which requires some technical tools of stochastic analysis and new results on DRBSDEs. We finally study the links between our mixed problem and generalized Hamilton–Jacobi–Bellman variational inequalities in both cases.

Key-words: generalized Dynkin games, Markovian stochastic control, mixed stochastic control/Dynkin game with nonlinear expectation, doubly reflected BSDEs, dynamic programming principles, generalized Hamilton-Jacobi-Bellman variational inequalities, viscosity solution.

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1 Introduction

In this paper, we study a new mixed stochastic control/optimal stopping game in a Markovian framework which can be formulated as follows. We consider two actors A and B. Actor A, called "controller/stopper", can control a state process X^α through the selection of a control process α, which impacts both the drift and the volatility, and can also choose the duration of the "game" via a stopping time τ. Actor B, called "stopper", can only decide when to stop the game via an another stopping time σ.

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We denote by $\mathcal{T}$ the set of stopping times with values in $[0,T]$, where $T>0$ is a fixed terminal time, and by $\mathcal{A}$ the set of admissible control processes α. For each $\alpha \in \mathcal{A}$, let (X_s^α) be a jump diffusion process of the form

$$X_s^\alpha = x + \int_0^s b(X_u^\alpha, \alpha_u)du + \int_0^s \sigma(X_u^\alpha, \alpha_u)dW_u + \int_0^s \int_{\mathbf{E}} \beta(X_u^\alpha, \alpha_u, e)\tilde{N}(du, de).$$

If A chooses a strategy (α, τ) and B chooses a stopping time σ, the associated cost (or gain), is defined by $g(X_T^\alpha)$ if A and B decide to stop at terminal time T, $h_1(X_\tau^\alpha)$ if A stops before B, and $h_2(X_\sigma^\alpha)$ otherwise. More precisely, the cost is given by

$$I^\alpha(\tau,\sigma) := h_1(X_\tau^\alpha)\mathbf{1}_{\tau\leq\sigma, \tau<T} + h_2(X_\sigma^\alpha)\mathbf{1}_{\sigma<\tau} + g(X_T^\alpha)\mathbf{1}_{\tau=\sigma=T}. \tag{1.1}$$

If the strategy of A is given by (α, τ), the stopper B chooses σ in order to minimize the expected cost evaluated under a nonlinear expectation. This nonlinear expectation denoted by $\mathcal{E}^\alpha$ is defined via a BSDE with jumps with a driver f^α which may depend on the control α. The minimal expected cost for B is then given by $\inf_{\sigma\in\mathcal{T}} \mathcal{E}^\alpha_{0,\tau\wedge\sigma}[I^\alpha(\tau,\sigma)]$. The aim of actor A is to maximize this quantity over all choices of (α,τ), which leads to the following mixed optimization problem:

$$\sup_{(\alpha,\tau)\in\mathcal{A}\times\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \mathcal{E}^\alpha_{0,\tau\wedge\sigma}[I^\alpha(\tau,\sigma)]. \tag{1.2}$$

In the special case when the first player can only act on the duration τ of the game (i.e. when there is no control α), this problem reduces to a *generalized* Dynkin game that we have introduced in [16]. It is proved there that the value function at 0, given by

$$\sup_{\tau\in\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \mathcal{E}_{0,\tau\wedge\sigma}[h_1(X_\tau)\mathbf{1}_{\tau\leq\sigma,\tau<T} + h_2(X_\sigma)\mathbf{1}_{\sigma<\tau} + g(X_T)\mathbf{1}_{\tau=\sigma=T}], \tag{1.3}$$

is characterized via a doubly reflected BSDE.

Note that Problem (1.2) can be seen as a mixed *generalized* Dynkin game/Stochastic control problem since

$$\sup_{(\alpha,\tau)\in\mathcal{A}\times\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \mathcal{E}^\alpha_{0,\tau\wedge\sigma}[I^\alpha(\tau,\sigma)] = \sup_{\alpha\in\mathcal{A}} \big(\sup_{\tau\in\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \mathcal{E}^\alpha_{0,\tau\wedge\sigma}[I^\alpha(\tau,\sigma)] \big). \tag{1.4}$$

The control α can then be interpreted as an ambiguity parameter on the model which affects both the drift and the volatility of the underlying state process.

Using the characterization of the solution of a DRBSDE as the value function of a *generalized* Dynkin game that we have provided in [16], Problem (1.4) corresponds to an optimization problem on DRBSDEs with jumps.

In this paper, we first give a *weak* dynamic programming principle for Problem (1.2) when g is assumed to be Borelian only, which follows from some results recently obtained in [17] together with some properties of DRBSDEs. We then focus on the continuous case for which we prove a *strong* dynamic programming principle. We stress that it cannot be directly derived from the *weak* dynamic programming principle. To this aim, we show in particular the continuity of the value function with respect to time t, which requires some refined properties of doubly reflected BSDEs with jumps. The last part of the paper is devoted to the relation of the value function with generalized Hamilton–Jacobi–Bellman variational inequalities (HJBVIs). We prove that the value function is a *weak* viscosity solution of the HJBVIs in the irregular case, and a *classical* viscosity solution in the continuous case. Uniqueness is obtained under additional assumptions. Finally, using approximation methods as well as the previous results (in particular the comparison principle obtained in the regular case), we show some additional properties of the value function in the irregular case. Note that our results provide a probabilistic interpretation of nonlinear HJBVIs in terms of game problems.

This work completes the one of Buckdahn-Li [10], who studies a related optimization problem for doubly reflected BSDEs (of the form $\sup_\alpha \inf_\beta$) in the case of a Brownian filtration and a continuous reward g. Unlike our approach, the latter does not use a dynamic programming principle to show that the value function is a viscosity solution of a generalized HJB equation.

The paper is organized as follows. In Section 2, we introduce the mixed *generalized* Dynkin game/stochastic control problem. In Section 3, we provide some preliminary properties for doubly reflected BSDEs with jumps. In Section 4, we prove the dynamic programming principles both in the irregular and the regular case. In Section 5, we derive that the value function of our problem is a weak (respectively classical) viscosity solution of a generalized HJBVIs in the irregular (respectively regular) case. In Section 6, using the results obtained in the previous sections, we give some additional results for the value function in the discontinuous case. In the Appendix, we provide some complementary properties which are used in the paper.

Related literature on optimal stopping games and mixed control problems with stopping times. Classical Markovian Dynkin games (with linear expectation) have been studied in particular by Bensoussan-Friedman [4], and by Bismut [6], Alario-Nazaret et al.[1], Kobylanski et al [28] in a non Markovian framework. See also e.g. Hamadène-Lepeltier [21] for the study of mixed classical Dynkin games. Links between Dynkin games and doubly reflected BSDEs have been provided in the classical case (see e.g. Cvitanić-Karatzas [13], Hamadène and Hassani [20], Hamadène and Wang [23], Hamadène and Ouknine [22]), and extended to *generalized* Dynkin games (that is with nonlinear expectation) in [16].

Controller/stopper games are special cases of Problem (1.2) when player A is only a controller (no stopping time τ). They have been studied in the case of linear expectation by e.g. Karatzas-Zamfirescu [25], Bayraktar-Huang [2] and Choukroun et al. [11].

Finally, mixed optimal stopping/stochastic control problems are special cases of Problem (1.2) when there is no player B. There is then no more game aspect. In the particular case of linear expectation, we refer to Bensoussan-Lions [3], Øksendal-Sulem [29], and Bouchard-Touzi [7] who provided a weak dynamic programming principle when the value function is irregular. This result has been extended to the $\mathcal{E}^f$-expectation case in [17], where links between the value function and generalized HJBIs are provided under very weak assumptions on the terminal cost (or reward) function.

We have seen above that Problem (1.4) is related to an optimization problem on DRBSDEs. When the terminal reward map g is continuous, let us mention the study of some optimization problems relative to BSDEs (see e.g. Peng [30]), to RBSDEs (see Buckdahn-Li [9]) or DRBSDEs (Buckdahn-Li [10] in the Brownian case).

Motivating applications in mathematical finance. Links between classical Dynkin games and game options have been provided by e.g. Kifer [26], Kifer and Yu [27], Hamadène [19]. Recall that a game contingent claim is a contract between a seller and a buyer which allows the seller to cancel it at a stopping time time $\sigma \in \mathcal{T}$ and the buyer to exercise it at any time $\tau \in \mathcal{T}$. The process X may be interpreted as the price process of the underlying asset. If the buyer (resp. the seller) exercises (resp. cancels) at maturity time T, then the seller pays the amount $g(X_T)$ to the buyer. If the buyer exercises at time $\tau < T$ before the seller cancels, then the seller pays the buyer the amount $h_1(X_\tau)$, but if the seller cancels before the buyer exercises, then he pays the amount $h_2(X_\sigma)$ to the buyer at the cancellation time σ. The difference $h_2(X_\sigma) - h_1(X_\sigma) \geq 0$ is interpreted as a penalty that the seller pays to the buyer for the cancellation of the contract. In other terms, if the seller (resp. the buyer) selects a cancellation time σ (resp. an exercise time τ), the seller pays to the buyer at time $\tau \wedge \sigma$ the *payoff*

$$I(\tau, \sigma) := h_1(X_\tau)\mathbf{1}_{\tau \leq \sigma, \tau < T} + h_2(X_\sigma)\mathbf{1}_{\sigma < \tau} + g(X_T)\mathbf{1}_{\tau = \sigma = T}].$$

In a perfect market model, there exists an (unique) superhedging price for a game option, which can be characterized as the value function of a Dynkin game of the form

$$\sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} I(\tau, \sigma) = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathbf{E}[h_1(X_\tau)\mathbf{1}_{\tau \leq \sigma, \tau < T} + h_2(X_\sigma)\mathbf{1}_{\sigma < \tau} + g(X_T)\mathbf{1}_{\tau = \sigma = T}],$$

where the expectation $\mathbf{E}$ is taken under the risk-neutral measure, and X may be interpreted as the price process of the underlying asset (see [26, 27, 19]). Recently, we have generalized this result to the case of nonlinear pricing by using the results on *generalized* Dynkin games provided in [16] (see [18]). The superhedging price of the game option is then of the form (1.3).

In the presence of constraints and ambiguity on the model represented by an "ambiguity" parameter $\alpha \in \mathcal{A}$, the controlled payoff $I^\alpha(\tau, \sigma)$ is given by (1.1), and the nonlinear expectation by $\mathcal{E}^\alpha$. In this case, there is a set of possible superhedging prices for the game option which can be written as

$$\sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^\alpha_{0,\tau\wedge\sigma}[I^\alpha(\tau, \sigma)] = \sup_{\tau \in \mathcal{T}} \inf_{\sigma \in \mathcal{T}} \mathcal{E}^\alpha_{0,\tau\wedge\sigma}[h_1(X^\alpha_\tau)\mathbf{1}_{\tau \leq \sigma, \tau < T} + h_2(X^\alpha_\sigma)\mathbf{1}_{\sigma < \tau} + g(X^\alpha_T)\mathbf{1}_{\tau = \sigma = T}],$$

where $\alpha \in \mathcal{A}$. The supremum of these possible prices over $\mathcal{A}$ then corresponds to the value function of the mixed *generalized* Dynkin game/stochastic control problem (1.4).

2 Mixed stochastic control/generalized Dynkin game

Let $T > 0$ be fixed. We consider the product space $\Omega := \Omega_W \otimes \Omega_N$, where $\Omega_W := \mathcal{C}([0, T])$ is the Wiener space, that is the set of continuous functions ω^1 from $[0, T]$ into $\mathbb{R}^p$ such that $\omega^1(0) = 0$, and $\Omega_N := \mathbb{D}([0, T])$ is the Skorohod space of right-continuous with left limits (RCLL) functions ω^2 from $[0, T]$ into $\mathbb{R}^d$, such that $\omega^2(0) = 0$. Recall that Ω is a Polish space for the topology of Skorohod. Here $p, d \geq 1$, but, for notational simplicity, we shall consider only $\mathbb{R}$-valued functions, that is the case $p = d = 1$.

Let $B = (B^1, B^2)$ be the canonical process defined for each $t \in [0, T]$ and each $\omega = (\omega^1, \omega^2)$ by $B^i_t(\omega) = B^i_t(\omega^i) := \omega^i_t$, for $i = 1, 2$. Let us denote the first coordinate process B^1 by W. Let P^W be the probability measure on $(\Omega_W, \mathcal{B}(\Omega_W))$ such that W is a Brownian motion. Here $\mathcal{B}(\Omega_W)$ denotes the Borelian σ-algebra on Ω_W.

Set $\mathbf{E} := \mathbb{R}^n \backslash \{0\}$ equipped with its Borelian σ-algebra $\mathcal{B}(\mathbf{E})$, where $n \geq 1$, and a σ-finite positive measure ν such that $\int_{\mathbf{E}}(1 \wedge |e|)\nu(de) < \infty$. We define the jump random measure N as follows: for each $t > 0$ and each $\mathbf{B} \in \mathcal{B}(\mathbf{E})$,

$$N(\omega, [0,t] \times \mathbf{B}) = N(\omega^2, [0,t] \times \mathbf{B}) := \sum_{0<s\leq t} \mathbf{1}_{\{\Delta\omega_s^2 \in \mathbf{B}\}}. \tag{2.1}$$

Let P^N be the probability measure on $(\Omega_N, \mathcal{B}(\Omega_N))$ such that N is a Poisson random measure with compensator $\nu(de)dt$ and such that $B_t^2 = \sum_{0<s\leq t} \Delta B_s^2$ a.s. Note that the sum of jumps is well defined up to a P^N-null set. We denote by $\tilde{N}(dt, de) := N(dt, de) - \nu(de)dt$ the compensated Poisson measure. The space Ω is equipped with its Borelian σ-algebra $\mathcal{B}(\Omega)$ and the probability measure $P := P^W \otimes P^N$. Let $\mathbb{F} := (\mathcal{F}_t)_{t\leq T}$ be the filtration generated by W and N *completed* with respect to $\mathcal{B}(\Omega)$ and P (see e.g. [24] p.3 for the definition of a *completed* filtration). Note that $\mathcal{F}_T$ is equal to the *completion* of the σ-algebra $\mathcal{B}(\Omega)$ with respect to P, and $\mathcal{F}_0$ is the σ-algebra generated by P-null sets. Let $\mathcal{P}$ be the predictable σ-algebra on $\Omega \times [0,T]$ associated with the filtration $\mathbb{F}$.

We introduce the following spaces :

- $\mathbb{H}_T^2$ (also denoted $\mathbb{H}^2$) := the set of real-valued predictable processes (Z_t) with $\mathbb{E}\int_0^T Z_s^2 ds < \infty$.
- $\mathcal{S}^2$:= the set of real-valued RCLL adapted processes (φ_s) with $\mathbb{E}[\sup_{0\leq s\leq T} \varphi_s^2] < \infty$.
- L_ν^2 := the set of measurable functions $l : (\mathbf{E}, \mathcal{K}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $\|l\|_\nu^2 := \int_{\mathbf{E}} l^2(e)\nu(de) < \infty$. The set L_ν^2 is a Hilbert space equipped with the scalar product $\langle l,\, l'\rangle_\nu := \int_{\mathbf{E}} l(e)l'(e)\nu(de)$ for all $l,\, l' \in L_\nu^2 \times L_\nu^2$.
- $\mathbb{H}_\nu^2$: the set of predictable real-valued processes $(k_t(\cdot))$ with $\mathbb{E}\int_0^T \|k_s\|_{L_\nu^2}^2 ds < \infty$.

Let $\mathbf{A}$ be a nonempty closed subset of $\mathbb{R}^p$. Let $\mathcal{A}$ be the set of controls, defined as the set of predictable processes α valued in $\mathbf{A}$. For each $\alpha \in \mathcal{A}$, initial time $t \in [0,T]$ and initial condition x in $\mathbb{R}$, let $(X_s^{\alpha,t,x})_{t\leq s\leq T}$ be the unique $\mathbb{R}$-valued solution in $\mathcal{S}^2$ of the stochastic differential equation:

$$X_s^{\alpha,t,x} = x + \int_t^s b(X_r^{\alpha,t,x}, \alpha_r)dr + \int_t^s \sigma(X_r^{\alpha,t,x}, \alpha_r)dW_r + \int_t^s \int_{\mathbf{E}} \beta(X_{r^-}^{\alpha,t,x}, \alpha_r, e)\tilde{N}(dr, de), \tag{2.2}$$

where $b,\ \sigma : \mathbb{R} \times \mathbf{A} \to \mathbb{R}$, are Lipschitz continuous with respect to x and α, and $\beta : \mathbb{R} \times \mathbf{A} \times \mathbf{E} \to \mathbb{R}$ is a bounded measurable function such that for some constant $C \geq 0$, and for all $e \in \mathbb{R}$

$$\begin{aligned}
&|\beta(x, \alpha, e)| \leq C\,\Psi(e), \quad x \in \mathbb{R}, \alpha \in \mathbf{A} \\
&|\beta(x, \alpha, e) - \beta(x', \alpha', e)| \leq C(|x - x'| + |\alpha - \alpha'|)\Psi(e), \quad x, x' \in \mathbb{R}, \alpha, \alpha' \in \mathbf{A},
\end{aligned}$$

where $\Psi \in L_\nu^2 \cap L_\nu^1$.

The criterion of our mixed control problem, depending on α, is defined via a BSDE with a Lipschitz driver function f satisfying the following conditions:

(i) $f : \mathbf{A} \times [0,T] \times \mathbb{R}^3 \times L_\nu^2 \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is $\mathcal{B}(\mathbf{A}) \otimes \mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^3) \otimes \mathcal{B}(L_\nu^2)$-measurable

(ii) $|f(\alpha, t, x, 0, 0, 0)| \leq C(1 + |x|^p), \forall \alpha \in \mathbf{A}, t \in [0,T], x \in \mathbb{R}$, where $p \in \mathbb{N}^*$.

(iii) $|f(\alpha, t, x, y, z, k) - f(\alpha', t, x', y', z', k')| \leq C(|\alpha - \alpha'| + |x - x'| + |y - y'| + |z - z'| + \|k - k'\|_{L_\nu^2})$, $\forall t \in [0,T],\ x, x', y, y', z, z' \in \mathbb{R},\ k, k' \in L_\nu^2, \alpha, \alpha' \in \mathbf{A}$.

(iv) $f(\alpha, t, x, y, z, k_2) - f(\alpha, t, x, y, z, k_1) \geq < \gamma(\alpha, t, x, y, z, k_1, k_2), k_2 - k_1 >_\nu, \forall t, x, y, z, k_1, k_2, \alpha,$

where $\gamma : \mathbf{A}\times[0,T]\times\mathbb{R}^3\times(L^2_\nu)^2 \to L^2_\nu$ is $\mathcal{B}(\mathbf{A})\otimes\mathcal{B}([0,T])\otimes\mathcal{B}(\mathbb{R}^3)\otimes\mathcal{B}((L^2_\nu)^2)/\mathcal{B}(L^2_\nu)$-measurable, $|\gamma(.)(e)| \leq \Psi(e)$ and $\gamma(.)(e) \geq -1$ $d\nu(e)$-a.s. where $\Psi \in L^2_\nu$.

Condition (iv) allows us to apply the comparison theorem for non reflected BSDEs, reflected and doubly reflected BSDEs with jumps (see [31], [32] and [16]).

Let $(t,x) \in [0,T]\times\mathbb{R}$ and $\alpha \in \mathcal{A}$. Let $\mathcal{E}^{f^{\alpha,t,x}}$ (also denoted by $\mathcal{E}^{\alpha,t,x}$) be the nonlinear conditional expectation associated with $f^{\alpha,t,x}$, defined for each stopping time S and for each $\zeta \in L^2(\mathcal{F}_S)$ as:

$$\mathcal{E}^{\alpha,t,x}_{r,S}[\zeta] := y^{\alpha,t,x}_r, \;\; t \leq r \leq S,$$

where $(y^{\alpha,t,x}_r)_{t\leq r\leq S}$ is the solution in $\mathcal{S}^2$ of the BSDE associated with driver $f^{\alpha,t,x}(r,y,z,k) := f(\alpha_r, r, X^{\alpha,t,x}_r, y, z, k)$, terminal time S and terminal condition ζ, that is satisfying the dynamics

$$-\,dy^{\alpha,t,x}_r = f(\alpha_r, r, X^{\alpha,t,x}_r, y^{\alpha,t,x}_r, z^{\alpha,t,x}_r, k^{\alpha,t,x}_r)dr - z^{\alpha,t,x}_r dW_r - \int_{\mathbf{E}} k^{\alpha,t,x}_r(e)\tilde{N}(dr,de), \tag{2.3}$$

with $y^{\alpha,t,x}_S = \zeta$, where $z^{\alpha,t,x}_{.}, k^{\alpha,t,x}_{.}$ are the associated processes in $\mathbb{H}^2$ and $\mathbb{H}^2_\nu$ respectively.

For all $(t,x) \in [0,T]\times\mathbb{R}$ and all control $\alpha \in \mathcal{A}$, we define the barriers for $i = 1,2$ by $h_i(s, X^{\alpha,t,x}_s)$, for $t \leq s < T$, and the terminal condition by $g(X^{\alpha,t,x}_T)$, where

(i) $g : \mathbb{R} \to \mathbb{R}$ is Borelian,

(ii) $h_1 : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $h_2 : [0,T]\times\mathbb{R} \to \mathbb{R}$ are functions which are Lipschitz continuous with respect to x uniformly in t, and continuous with respect to t on $[0,T]$, with $h_1 \leq h_2$,

(iii) h_1 (or h_2) is $\mathcal{C}^{1,2}$ with bounded derivatives,

(iv) $|h_1(t,x)| + |h_2(t,x)| + |g(x)| \leq C(1+|x|^p), \forall t \in [0,T], x \in \mathbb{R},$ with $p \in \mathbb{N}$.

Let $\mathcal{T}$ be the set of stopping times with values in $[0,T]$. Suppose the initial time is equal to 0. For each initial condition $x \in \mathbb{R}$, we consider the following mixed *generalized* Dynkin game and stochastic control problem:

$$u(0,x) := \sup_{\alpha\in\mathcal{A}} \sup_{\tau\in\mathcal{T}} \inf_{\sigma\in\mathcal{T}} \mathcal{E}^{\alpha,0,x}_{0,\tau\wedge\sigma}\left[h_1(\tau, X^{\alpha,0,x}_\tau)\mathbf{1}_{\tau\leq\sigma,\tau<T} + h_2(\sigma, X^{\alpha,0,x}_\sigma)\mathbf{1}_{\sigma<\tau} + g(X^{\alpha,0,x}_T)\mathbf{1}_{\tau\wedge\sigma=T}\right]. \tag{2.4}$$

We now make the problem dynamic. We define, for $t \in [0,T]$ and each $\omega \in \Omega$ the t-translated path $\omega^t = (\omega^t_s)_{s\geq t} := (\omega_s - \omega_t)_{s\geq t}$. Note that $(\omega^{1,t}_s)_{s\geq t} := (\omega^1_s - \omega^1_t)_{s\geq t}$ corresponds to the realizations of the translated Brownian motion $W^t := (W_s - W_t)_{s\geq t}$ and that the translated Poisson random measure $N^t := N(]t,s],.)_{s\geq t}$ can be expressed in terms of $(\omega^{2,t}_s)_{s\geq t} := (\omega^2_s - \omega^2_t)_{s\geq t}$ similarly to (2.1). Let $\mathbb{F}^t = (\mathcal{F}^t_s)_{t\leq s\leq T}$ be the filtration generated by W^t and N^t *completed with respect to* $\mathcal{B}(\Omega)$ *and* P. Note that for each $s \in [t,T]$, $\mathcal{F}^t_s$ is the σ-algebra generated by W^t_r, N^t_r, $t \leq r \leq s$ and $\mathcal{F}_0$. Recall also that we have a martingale representation theorem for $\mathbb{F}^t$-martingales as stochastic integrals with respect to W^t and $\tilde{N}^t$.

Let us denote by $\mathcal{T}^t_t$ the set of stopping times with respect to $\mathbb{F}^t$ with values in $[t,T]$. Let $\mathcal{P}^t$ be the predictable σ-algebra on $\Omega\times[t,T]$ equipped with the filtration $\mathbb{F}^t$.
We introduce the following spaces of processes. Let $t \in [0,T]$.
Let $\mathbb{H}^2_t$ be the $\mathcal{P}^t$-measurable processes Z on $\Omega\times[t,T]$ such that $\|Z\|_{\mathbb{H}^2_t} := \mathbb{E}[\int_t^T Z_u^2 du] < \infty$.
Let $\mathbb{H}^2_{t,\nu}$ be the set of $\mathcal{P}^t$-measurable processes K on $\Omega\times[t,T]$ with $\|K\|_{\mathbb{H}^2_{t,\nu}} := \mathbb{E}[\int_t^T \|K_u\|^2_\nu du] < \infty$.
Let $\mathcal{S}^2_t$ be the set of $\mathbb{R}$-valued RCLL processes φ on $\Omega\times[t,T]$, $\mathbb{F}^t$-adapted, with $\mathbb{E}[\sup_{t\leq s\leq T}\varphi_s^2] < \infty$.

Let $\mathcal{A}_t^t$ be the set of controls $\alpha : \Omega \times [t,T] \mapsto \mathbf{A}$, which are $\mathcal{P}^t$-measurable. For each initial time t and each initial condition x, the value function is defined by:

$$u(t,x) := \sup_{\alpha \in \mathcal{A}_t^t} \sup_{\tau \in \mathcal{T}_t^t} \inf_{\sigma \in \mathcal{T}_t^t} \mathcal{E}_{t,\tau\wedge\sigma}^{\alpha,t,x} \left[h_1(\tau, X_\tau^{\alpha,t,x}) \mathbf{1}_{\tau \leq \sigma, \tau < T} + h_2(\sigma, X_\sigma^{\alpha,t,x}) \mathbf{1}_{\sigma < \tau} + g(X_T^{\alpha,t,x}) \mathbf{1}_{\tau \wedge \sigma = T} \right]. \tag{2.5}$$

Note that since α, τ and σ depend only on ω^t, the SDE (2.2) and the BSDE (2.3) can be solved with respect to the translated Brownian motion $(W_s - W_t)_{s \geq t}$ and the translated Poisson random measure $N(]t,s], \cdot)_{s \geq t}$. Hence the function u is well defined as a deterministic function of t and x.

For each $\alpha \in \mathcal{A}_t^t$, we introduce the function u^α defined as

$$u^\alpha(t,x) := \sup_{\tau \in \mathcal{T}_t^t} \inf_{\sigma \in \mathcal{T}_t^t} \mathcal{E}_{t,\tau\wedge\sigma}^{\alpha,t,x} \left[h_1(\tau, X_\tau^{\alpha,t,x}) \mathbf{1}_{\tau \leq \sigma, \tau < T} + h_2(\sigma, X_\sigma^{\alpha,t,x}) \mathbf{1}_{\sigma < \tau} + g(X_T^{\alpha,t,x}) \mathbf{1}_{\tau \wedge \sigma = T} \right].$$

We thus get

$$u(t,x) = \sup_{\alpha \in \mathcal{A}_t^t} u^\alpha(t,x). \tag{2.6}$$

Note that for all α, x and $t < T$, we have $h_1(t,x) \leq u^\alpha(t,x) \leq h_2(t,x)$, and hence $h_1(t,x) \leq u(t,x) \leq h_2(t,x)$. Moreover, $u^\alpha(T,x) = u(T,x) = g(x)$.

By Assumption (iii), h_1 (or h_2) is $\mathcal{C}^{1,2}$ with bounded derivatives. It follows from Proposition A.1 in the Appendix, that for each $t \in [0,T]$ and for each $\alpha \in \mathcal{A}_t^t$, the processes $\xi_s^{\alpha,t,x} := h_1(s, X_s^{\alpha,t,x}) \mathbf{1}_{s<T} + g(X_T^{\alpha,t,x}) \mathbf{1}_{s=T}$ and $\zeta_s^{\alpha,t,x} := h_2(s, X_s^{\alpha,t,x}) \mathbf{1}_{s<T} + g(X_T^{\alpha,t,x}) \mathbf{1}_{s=T}$ satisfy Mokobodzki's condition, that is there exist two nonnegative $\mathbb{F}^t$-supermartingales $H^{\alpha,t,x}$ and $H'^{\alpha,t,x}$ in $\mathcal{S}_t^2$ such that

$$\xi_s^{\alpha,t,x} \leq H_s^{\alpha,t,x} - H_s'^{\alpha,t,x} \leq \zeta_s^{\alpha,t,x}, \quad t \leq s \leq T \quad \text{a.s.} \tag{2.7}$$

By Theorem 4.7 in [16], for each α, the value function u^α of the above *generalized* Dynkin game is characterized as the solution of the doubly reflected BSDE associated with driver $f^{\alpha,t,x}$, barriers $\xi_s^{\alpha,t,x} = h_1(s, X_s^{\alpha,t,x})$, $\zeta_s^{\alpha,t,x} = h_2(s, X_s^{\alpha,t,x})$ for $s < T$, and terminal condition $g(X_T^{\alpha,t,x})$, that is

$$u^\alpha(t,x) = Y_t^{\alpha,t,x}, \tag{2.8}$$

where $(Y^{\alpha,t,x}, Z^{\alpha,t,x}, K^{\alpha,t,x}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ is the solution of the doubly reflected BSDE:

$$\begin{cases} Y_s^{\alpha,t,x} = g(X_T^{\alpha,t,x}) + \displaystyle\int_s^T f(\alpha_r, r, X_r^{\alpha,t,x}, Y_r^{\alpha,t,x}, Z_r^{\alpha,t,x}, K_r^{\alpha,t,x}(\cdot)) dr \\ \qquad + A_T^{1,\alpha,t,x} - A_s^{1,\alpha,t,x} - A_T^{2,\alpha,t,x} + A_s^{2,\alpha,t,x} - \displaystyle\int_s^T Z_r^{\alpha,t,x} dW_r - \int_s^T \int_{\mathbf{E}} K^{\alpha,t,x}(r,e) \tilde{N}(dr,de) \\ \xi_s^{\alpha,t,x} \leq Y_s^{\alpha,t,x} \leq \zeta_s^{\alpha,t,x}, t \leq s < T \;\; \text{a.s.}\;, \\ A^{1,\alpha,t,x}, A^{2,\alpha,t,x} \text{ are RCLL nondecreasing predictable processes with } A_t^{1,\alpha,t,x} = A_t^{2,\alpha,t,x} = 0 \text{ and} \\ \displaystyle\int_t^T (Y_s^{\alpha,t,x} - \xi_s^{\alpha,t,x}) dA_s^{1,\alpha,t,x,c} = 0 \text{ a.s. and } \Delta A_s^{1,\alpha,t,x,d} = -\Delta A_s^{1,\alpha,t,x} \mathbf{1}_{\{Y_{s^-}^{\alpha,t,x} = \xi_{s^-}^{\alpha,t,x}\}} \quad \text{a.s.} \\ \displaystyle\int_t^T (\zeta_s^{\alpha,t,x} - Y_s^{2,\alpha,t,x}) dA_s^{2,\alpha,t,x,c} = 0 \text{ a.s. and } \Delta A_s^{2,\alpha,t,x,d} = -\Delta A_s^{2,\alpha,t,x} \mathbf{1}_{\{Y_{s^-}^{\alpha,t,x} = \zeta_{s^-}^{\alpha,t,x}\}} \quad \text{a.s.} \end{cases} \tag{2.9}$$

Here $A^{1,\alpha,t,x,c}$ (resp. $A^{2,\alpha,t,x,c}$) denotes the continuous part of A^1 (resp. A^2) and $A^{1,\alpha,t,x,d}$ (resp.$A^{2,\alpha,t,x,d}$) the discontinuous part. In the particular case when $h_1(T,x) \leq g(x) \leq h_2(T,x)$, then the obstacles $\xi^{\alpha,t,x}$ and $\zeta^{\alpha,t,x}$ satisfy for all predictable stopping time τ, $\xi_{\tau^-} \leq \xi_\tau$ and $\zeta_{\tau^-} \geq \zeta_\tau$ a.s. which implies the continuity of $A^{1,\alpha,t,x}$ and $A^{2,\alpha,t,x}$ (see [16]).

Note that the doubly reflected BSDE (2.9) can be solved in $\mathcal{S}^2 \times \mathbb{H}_t^2 \times \mathbb{H}_{t,\nu}^2$ with respect to the t-translated Brownian motion and the t-translated Poisson random measure.

In the following, for each $\alpha \in \mathcal{A}_t^t$, $Y_s^{\alpha,t,x}$ will be also denoted by $Y_{s,T}^{\alpha,t,x}[g(X_T^{\alpha,t,x})]$.

Using (2.8), our initial optimization problem (2.5) can thus be reduced to an optimal control problem for doubly reflected BSDEs:

$$u(t,x) = \sup_{\alpha \in \mathcal{A}_t^t} Y_t^{\alpha,t,x} = \sup_{\alpha \in \mathcal{A}_t^t} Y_{t,T}^{\alpha,t,x}[g(X_T^{\alpha,t,x})].$$

We now provide some new results on doubly reflected BSDEs, which will be used to prove the dynamic programming principles.

3 Preliminary properties for doubly reflected BSDEs

We show in a general non Markovian setting a continuity property and a Fatou lemma for doubly reflected BSDEs, where the limit involves both terminal condition and terminal time.

A function f is said to be a *Lipschitz driver* if
$f : [0,T] \times \Omega \times \mathbb{R}^2 \times L_\nu^2 \to \mathbb{R}$ $(\omega,t,y,z,k(\cdot)) \mapsto f(\omega,t,y,z,k(\cdot))$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(L_\nu^2)-$ measurable, uniformly Lipschitzian with respect to $y, z, k(\cdot)$ and such that $f(.,0,0,0) \in \mathbb{H}^2$.

A Lipschitz driver f is said to satisfy Assumption 3.1 if the following holds:

Assumption 3.1. *$dP \otimes dt$-a.s for each $(y,z,k_1,k_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$,*

$$f(t,y,z,k_1) - f(t,y,z,k_2) \geq \langle \gamma_t^{y,z,k_1,k_2}\,,\, k_1 - k_2 \rangle_\nu,$$

with $\gamma : [0,T] \times \Omega \times \mathbb{R}^2 \times (L_\nu^2)^2 \to L_\nu^2$; $(\omega,t,y,z,k_1,k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega,.)$, supposed to be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L_\nu^2)^2)$-measurable, and satisfying $dP \otimes dt$-a.s. , for each $(y,z,k_1,k_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$, $\gamma_t^{y,z,k_1,k_2}(\cdot) \geq -1$ and $\|\gamma_t^{y,z,k_1,k_2}(\cdot)\|_\nu \leq K$, where K is a positive constant.

This assumption ensures the comparison theorem for BSDEs, reflected BSDEs and doubly reflected BSDEs with jumps (see [31], [32] and [16]). It is satisfied if, for example, f is of class $\mathcal{C}^1$ with respect to k such that $\nabla_k f$ is bounded (in L_ν^2) and $\nabla_k f \geq -1$ (see [16] Proposition A.2).

We extend the definition of reflected and doubly reflected BSDEs when the terminal time is a stopping time $\theta \in \mathcal{T}$ and the terminal condition is a random variable ξ in $L^2(\mathcal{F}_\theta)$. let f be a given Lipschitz driver. Let (η_t) be a given obstacle RCLL process in $\mathcal{S}^2$. The solution, denoted $(Y_{.,\theta}(\xi), Z_{.,\theta}(\xi), k_{.,\theta}(\xi))$, of the reflected BSDEs associated with terminal time θ, driver f, obstacle $(\eta_s)_{s<\theta}$, and terminal condition ξ is defined as the unique solution in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2$ of the reflected BSDE with terminal time T, driver $f(t,y,z,k)\mathbf{1}_{\{t\leq\theta\}}$, terminal condition ξ and obstacle $\eta_t \mathbf{1}_{t<\theta} + \xi \mathbf{1}_{t\geq\theta}$. Note that $Y_{t,\theta}(\xi) = \xi, Z_{t,\theta}(\xi) = 0, k_{t,\theta}(\xi) = 0$ for $t \geq \theta$. Similarly, let (η_t) and (ζ_t) be given RCLL processes in $\mathcal{S}^2$ satisfying Mokobodzki's condition, that is, there exist two nonnegative supermartingales H and H' in $\mathcal{S}^2$ such that

$$\eta_s \leq H_s - H'_s \leq \zeta_s, \;\; 0 \leq s \leq T \quad \text{a.s.}$$

The solution, denoted $(Y_{.,\theta}(\xi), Z_{.,\theta}(\xi), k_{.,\theta}(\xi))$ of the doubly reflected BSDEs associated with terminal stopping time θ, driver f, barriers $(\eta_s)_{s<\theta}$ and $(\zeta_s)_{s<\theta}$, and terminal condition ξ, is defined as the unique solution in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ of the doubly reflected BSDE with driver $f(t,y,z,k)\mathbf{1}_{\{t \leq \theta\}}$, terminal time T, terminal condition ξ and barriers $\eta_t \mathbf{1}_{t<\theta} + \xi \mathbf{1}_{t \geq \theta}$ and $\zeta_t \mathbf{1}_{t<\theta} + \xi \mathbf{1}_{t \geq \theta}$. Note that $Y_{t,\theta}(\xi) = \xi, Z_{t,\theta}(\xi) = 0, k_{t,\theta}(\xi) = 0$ for $t \geq \theta$.

We first prove a continuity property for doubly reflected BSDEs where the limit involves both terminal condition and terminal time.

Proposition 3.2 (A continuity property for doubly reflected BSDEs)**.** *Let $T > 0$. Let f be a Lipschitz driver satisfying Assumption 3.1. Let (η_t), (ζ_t) be two RCLL processes in $\mathcal{S}^2$, with $\eta_. \leq \zeta_.$. Let f be a given Lipschitz driver. Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}$, converging a.s. to $\theta \in \mathcal{T}$ as n tends to ∞. Let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $E[\operatorname{ess\,sup}_n (\xi^n)^2] < +\infty$, and for each n, ξ^n is $\mathcal{F}_{\theta^n}$-measurable. Suppose that for each n, the processes $\eta_t \mathbf{1}_{\{t<\theta^n\}} + \xi^n \mathbf{1}_{\{t\geq\theta^n\}}$ and $\zeta_t \mathbf{1}_{\{t<\theta^n\}} + \xi^n \mathbf{1}_{\{t\geq\theta^n\}}$ satisfy Mokobodzki's condition. Suppose that ξ^n converges a.s. to an $\mathcal{F}_\theta$-measurable random variable ξ as n tends to ∞. Suppose that*

$$\eta_\theta \leq \xi \leq \zeta_\theta \quad \text{a.s.}\,, \tag{3.1}$$

and that the processes $\eta_t \mathbf{1}_{\{t<\theta\}} + \xi \mathbf{1}_{\{t\geq\theta\}}$ and $\zeta_t \mathbf{1}_{\{t<\theta\}} + \xi \mathbf{1}_{\{t\geq\theta\}}$ satisfy Mokobodzki's condition.

Let $Y_{.,\theta^n}(\xi^n)$; $Y_{.,\theta}(\xi)$ be the solutions of the doubly reflected BSDEs associated with driver f, barriers $(\eta_s)_{s<\theta^n}$ and $(\zeta_s)_{s<\theta^n}$(resp. $(\eta_s)_{s<\theta}$ and $(\zeta_s)_{s<\theta}$) , terminal time θ^n (resp. θ), terminal condition ξ^n (resp. ξ). Then, for each stopping time τ with $\tau \leq \theta$ a.s.,

$$Y_{\tau,\theta}(\xi) = \lim_{n\to+\infty} Y_{\tau,\theta^n}(\xi^n) \quad a.s.$$

When for each n, $\theta_n = \theta$ a.s., the result still holds without Assumption (3.1).

Remark 3.3. *As in the case of reflected BSDEs (see Proposition A.6 in the Appendix), there is an extra difficulty due to the presence of the barriers (and the variation of the terminal time). The additional assumption (3.1) on the obstacle is here required to obtain the result.*

Proof. We first consider the simpler case when for each n, $\theta_n = \theta$ a.s. By the a priori estimates on doubly reflected BSDEs provided in [16, Proposition 5.3] and the convergence of ξ_n to ξ, we derive that $Y_{\tau,\theta}(\xi) = \lim_{n\to+\infty} Y_{\tau,\theta}(\xi^n)$ a.s.
We now turn to the general case. In this case, Proposition 5.3 in [16] does not give the result. By the flow property for doubly reflected BSDEs (or "semigroup property", see [10]), we have:

$$Y_{\tau,\theta_n}(\xi^n) = Y_{\tau,\theta}(Y_{\theta,\theta_n}(\xi^n)).$$

By the first step, it is thus sufficient to show that $\lim_{n\to} Y_{\theta,\theta_n}(\xi^n) = \xi$ a.s.
Since the solution of the doubly reflected BSDE associated with terminal condition ξ^n and terminal time θ_n is smaller than the solution $\tilde{Y}_{.,\theta_n}(\xi^n)$ of the reflected BSDE associated with (one) obstacle $(\xi_t)_{t<\theta_n}$, terminal condition ξ^n and terminal time θ_n , we have:

$$Y_{\theta,\theta_n}(\xi^n) \leq \tilde{Y}_{\theta,\theta_n}(\xi^n) \text{ a.s.} \tag{3.2}$$

By the continuity property of reflected BSDEs with respect to terminal time and terminal condition (Lemma A.6), we have $\lim_{n\to\infty} \tilde{Y}_{\theta,\theta_n}(\xi^n) = \tilde{Y}_{\theta,\theta}(\xi) = \xi$ a.s. Taking the lim sup in (3.2), we get

$$\limsup_{n\to\infty} Y_{\theta,\theta_n}(\xi^n) \leq \lim_{n\to\infty} \tilde{Y}_{\theta,\theta_n}(\xi^n) = \xi \quad \text{a.s.}$$

It remains to show that

$$\liminf_{n\to\infty} Y_{\theta,\theta_n}(\xi^n) \geq \xi \quad \text{a.s.} \tag{3.3}$$

Since the solution of the doubly reflected BSDE associated with terminal condition ξ^n and terminal time θ_n is greater than the solution $\bar{Y}_{.,\theta_n}(\xi^n)$ of the reflected BSDE associated with the upper obstacle $(\zeta_t)_{t<\theta_n}$, terminal condition ξ^n and terminal time θ_n, we have:

$$Y_{\theta,\theta_n}(\xi^n) \geq \bar{Y}_{\theta,\theta_n}(\xi^n) \text{ a.s.} \tag{3.4}$$

By Lemma A.6, we derive that the solution of a reflected BSDEs with upper barrier is continuous with respect to both terminal time and terminal condition. Hence, $\lim_{n\to\infty} \bar{Y}_{\theta,\theta_n}(\xi^n) = \xi$ a.s. Taking the lim inf in (3.4), we derive inequality (3.3). □

Using Proposition 3.2 together with the monotonicity property of the solution of a doubly reflected BSDEs with respect to terminal condition, one can derive the following Fatou lemma.

Lemma 3.4 (A Fatou lemma for doubly reflected BSDEs)**.** *Let $T > 0$. Let (η_t), (ζ_t) be two RCLL processes in $\mathcal{S}^2$, with $\eta_. \leq \zeta_.$, and satisfying Mokobodzki's condition. Let f be a Lipschitz driver satisfying Assumption 3.1. Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}$, converging a.s. to $\theta \in \mathcal{T}$ as n tends to ∞. Let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $E[\sup_n(\xi^n)^2] < +\infty$, and for each n, ξ^n is $\mathcal{F}_{\theta^n}$-measurable. Suppose that for each n, the processes $\eta_t \mathbf{1}_{\{t<\theta^n\}} + \xi^n \mathbf{1}_{\{t\geq\theta^n\}}$ and $\zeta_t \mathbf{1}_{\{t<\theta^n\}} + \xi^n \mathbf{1}_{\{t\geq\theta^n\}}$ satisfy Mokobodzki's condition.*

Let $\underline{\xi} := \liminf_{n\to+\infty} \xi^n$. Suppose that the processes $\eta_t \mathbf{1}_{\{t<\theta\}} + \underline{\xi} \mathbf{1}_{\{t\geq\theta\}}$ and $\zeta_t \mathbf{1}_{\{t<\theta\}} + \underline{\xi} \mathbf{1}_{\{t\geq\theta\}}$ satisfy Mokobodzki's condition, and that

$$\eta_\theta \leq \underline{\xi} \leq \zeta_\theta \quad \text{a.s.} \tag{3.5}$$

Let $Y_{.,\theta^n}(\xi^n)$ (resp. $Y_{.,\theta}(\underline{\xi})$) be the solution(s) of the doubly reflected BSDEs associated with driver f, barriers $(\eta_s)_{s<\theta^n}$ and $(\zeta_s)_{s<\theta^n}$ (resp. $(\eta_s)_{s<\theta}$ and $(\zeta_s)_{s<\theta}$), terminal time θ^n (resp. θ), terminal condition ξ^n (resp. $\underline{\xi}$).

Then, for each stopping time τ with $\tau \leq \theta$ a.s.,

$$Y_{\tau,\theta}(\underline{\xi}) \leq \liminf_{n\to+\infty} Y_{\tau,\theta^n}(\xi^n) \quad \text{a.s.} \tag{3.6}$$

When for each n, $\theta_n = \theta$ a.s., the result still holds without Assumption (3.5).

Remark 3.5. *Under the same assumptions with $\underline{\xi}$ replaced by $\bar{\xi}$, the result also holds for $\bar{\xi} :=$* $\limsup_{n\to+\infty} \xi^n$ *instead of $\underline{\xi}$ with the converse inequality in (3.6), and with* lim inf *replaced by* lim sup.

The proof, which is very similar to that of the Fatou lemma for classical BSDEs (see Lemma A.5), is left to the reader. This lemma will be used to obtain a weak dynamic principle.

4 Dynamic programming principles

We provide now two dynamic programming principles. When the terminal reward g is Borelian (resp. continuous), we show a *weak* (resp. *strong*) dynamic programming principle. We stress that in the continuous case, the *strong* dynamic principle cannot be directly deduced from the weak one.

4.1 A *weak* dynamic programming principle in the irregular case

Measurability properties of the functions u and u^α. We first show some measurability properties of the functions $u^\alpha(t,x)$ with respect to control α and initial condition x.

By the a priori estimates on doubly reflected BSDEs provided in [16, Proposition 5.3], we derive the following measurability property of u^α.

Lemma 4.1 (A measurability property of u^α)**.** *Let $s \in [0,T]$.*

The map $(\alpha, x) \mapsto u^\alpha(s,x)$; $(\mathcal{A}_s^s \times \mathbb{R}\,,\,\mathcal{B}'(\mathcal{A}_s^s)\otimes\mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable. Here $\mathcal{B}'(\mathcal{A}_s^s)$ denotes the σ-algebra induced by $\mathcal{B}(\mathbb{H}_s^2)$ on $\mathcal{A}_s^s$.

Proof. Let $x^1, x^2 \in \mathbb{R}$, and $\alpha^1, \alpha^2 \in \mathcal{A}_s^s$. By classical results on diffusion processes, we have

$$\mathbb{E}[\sup_{r\geq s} |X_r^{\alpha^1,s,x^1} - X_r^{\alpha^2,s,x^2}|^2] \leq C(\|\alpha^1-\alpha^2\|^2_{\mathbb{H}_s^2} + |x^1-x^2|^2). \tag{4.1}$$

Let $Y_{s,T}^{\alpha,s,x}[\xi_\cdot,\zeta_\cdot,\eta]$ be the solution at time s of the doubly reflected BSDE associated with driver $f^{\alpha,s,x} := (f(\alpha_r, r, X_r^{\alpha,s,x}, .)\mathbf{1}_{r\geq s})$, barriers $\xi_r \leq \zeta_r$, $s \leq r < T$, terminal condition η. We suppose here that $\bar{\xi}_r := \xi_r\mathbf{1}_{r<T} + \eta\mathbf{1}_{r=T}$ and $\bar{\zeta}_r := \zeta_r\mathbf{1}_{r<T} + \eta\mathbf{1}_{r=T}$ satisfy Mokobodzki's condition (see (2.7) or [16, Definition 3.9]). Using the Lipschitz property of f with respect to x, α, we have

$$\|\sup_{y,z}|f(\alpha^1, X^{\alpha^1,s,x^1}, y, z) - f(\alpha^2, X^{\alpha^2,s,x^2}, y, z)|\,\|_{\mathbb{H}^2} \leq C(\|\alpha^1-\alpha^2\|_{\mathbb{H}^2} + \|X^{\alpha^1,s,x^1} - X^{\alpha^2,s,x^2}\|_{\mathcal{S}^2}).$$

By the estimates on doubly reflected BSDEs with universal constants provided in [16, Proposition 5.3], we obtain that for all $x^1,x^2 \in \mathbb{R}$, $\alpha^1,\alpha^2 \in \mathcal{A}_s^s$, barriers $\xi_\cdot^1, \zeta_\cdot^1 \in S_s^2$, $\xi_\cdot^2, \zeta_\cdot^2 \in S_s^2$, and terminal conditions η^1, $\eta^2 \in L_s^2$,

$$\begin{aligned} &|Y_s^{\alpha^1,s,x^1}[\xi_\cdot^1,\zeta_\cdot^1,\eta^1] - Y_s^{\alpha^2,s,x^2}[\xi_\cdot^2,\zeta_\cdot^2,\eta^2]|^2 \\ &\leq C'(\|\alpha^1-\alpha^2\|^2_{\mathbb{H}_s^2} + |x^1-x^2|^2 + \|\xi_\cdot^1-\xi_\cdot^2\|^2_{S_s^2} + \|\zeta_\cdot^1-\zeta_\cdot^2\|^2_{S_s^2} + \|\eta^1-\eta^2\|^2_{L_s^2}), \end{aligned} \tag{4.2}$$

where C' is a constant depending only on T and the Lipschitz constant C of the map f.

Let $\Phi : \mathcal{A}_s^s \times \mathbb{R} \times (\mathcal{S}_s^2)^2 \times L_s^2 \to \mathcal{S}_s^2$; $(\alpha, x, \xi_\cdot, \zeta_\cdot, \eta) \mapsto Y_{s,T}^{\alpha,s,x}[\xi_\cdot,\zeta_\cdot,\eta]$. By (4.2), the map Φ is Lipschitz-continuous with respect to the norm $\|\,.\,\|^2_{\mathbb{H}_s^2} + |\,.\,|^2 + \|\,.\,\|^2_{\mathcal{S}_s^2} + \|\,.\,\|^2_{\mathcal{S}_s^2} + \|\,.\,\|^2_{L_s^2}$.
Now, by [17, Proposition 9], the map $L_s^{2p} \cap L_s^2 \to L_s^2$, $\xi \mapsto g(\xi)$ is Borelian. Hence, the map $(\alpha, x) \mapsto (\alpha, x, h_1(., X_\cdot^{\alpha,s,x}), h_2(., X_\cdot^{\alpha,s,x}), g(X_T^{\alpha,s,x}))$ defined on $\mathcal{A}_s^s \times \mathbb{R}$ and valued in $\mathcal{A}_s^s \times \mathbb{R} \times (\mathcal{S}_s^2)^2 \times L_s^2$ is $\mathcal{B}(\mathcal{A}_s^s)\otimes\mathcal{B}(\mathbb{R}) /\ \mathcal{B}(\mathcal{A}_s^s)\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}((\mathcal{S}_s^2)^2)\otimes\mathcal{B}(L_s^2)$-measurable.
By composition, it follows that the map $(\alpha,x) \mapsto Y_{s,T}^{\alpha,s,x}[g(X_T^{\alpha,t,x})] = u^\alpha(s,x)$ is Borelian. □

Using the a priori estimates on doubly reflected BSDEs and standard arguments, we derive the following result.

Lemma 4.2. *For each $t \in [0,T]$, for each $\alpha \in \mathcal{A}_t^t$, the map u^α has at most polynomial growth with respect to x. The property still holds for the value function u.*

We introduce the upper semicontinuous envelope u^* and the lower semicontinuous envelope u_* of the value function u, defined by

$$u^*(t,x) := \limsup_{(t',x')\to(t,x)} u(t',x'); \quad u_*(t,x) := \liminf_{(t',x')\to(t,x)} u(t',x') \quad \forall (t,x) \in [0,T]\times\mathbb{R}.$$

We also define the maps $\bar{u}^*$ and $\bar{u}_*$ for each $(t,x) \in [0,T] \times \mathbb{R}$ by

$$\bar{u}^*(t,x) := u^*(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T}; \quad \bar{u}_*(t,x) := u_*(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T}.$$

The functions $\bar{u}^*$ and $\bar{u}_*$ are Borelian. We have $\bar{u}_* \leq u \leq \bar{u}^*$ and $\bar{u}_*(T,.) = u(T,.) = \bar{u}^*(T,.) = g(.)$. Note that $\bar{u}^*$ (resp. $\bar{u}_*$) is not necessarily upper (resp. lower) semicontinuous on $[0,T] \times \mathbb{R}$, since the terminal reward g is only Borelian.

For each $\theta \in \mathcal{T}$ and each ξ in $L^2(\mathcal{F}_\theta)$, we denote by $(Y^{\alpha,t,x}_{.,\theta}(\xi), Z^{\alpha,t,x}_{.,\theta}(\xi), k^{\alpha,t,x}_{.,\theta}(\xi))$ the unique solution in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\nu$ of the doubly reflected BSDE with driver $f^{\alpha,t,x}\mathbf{1}_{\{s \leq \theta\}}$, terminal time T, terminal condition ξ and barriers $h_i(r, X^{\alpha,t,x}_r)\mathbf{1}_{r<\theta} + \xi\mathbf{1}_{r \geq \theta}$.

We now state a weak dynamic programming principle for our mixed game problem, which can be seen as the analogous of the one shown in [17] for a mixed optimal stopping/stochastic control problem.

Theorem 4.3 (Weak dynamic programming principle)**.** *Suppose the subset* $\mathbf{A}$ *of* $\mathbb{R}$ *is compact. We have the following* sub–optimality principle of dynamic programming*:*
for each $t \in [0,T]$ *and for each stopping time* $\theta \in \mathcal{T}^t_t$*,*

$$u(t,x) \leq \sup_{\alpha \in \mathcal{A}^t_t} Y^{\alpha,t,x}_{t,\theta}\left[\bar{u}^*(\theta, X^{\alpha,t,x}_\theta)\right]. \tag{4.3}$$

We also have the super–optimality principle of dynamic programming*:*
for each $t \in [0,T]$ *and for each stopping time* $\theta \in \mathcal{T}^t_t$*,*

$$u(t,x) \geq \sup_{\alpha \in \mathcal{A}^t_t} Y^{\alpha,t,x}_{t,\theta}\left[\bar{u}_*(\theta, X^{\alpha,t,x}_\theta)\right]. \tag{4.4}$$

These results still hold with θ *replaced by* θ^α *in inequalities* (4.3) *and* (4.4)*, given a family of stopping times indexed by controls* $\{\theta^\alpha, \alpha \in \mathcal{A}^t_t\}$*.*

Proof. Without loss of generality, we can suppose that $t = 0$. Let us show inequality (4.3). For each $(t,x) \in [0,T] \times \mathbb{R}$, we set $\bar{h}_i(t,x) := h_i(t,x)\mathbf{1}_{t<T} + g(x)\mathbf{1}_{t=T}$. Let $\theta \in \mathcal{T}$. For each $n \in \mathbb{N}$, we define $\theta^n := \sum_{k=0}^{2^n-1} t_k \mathbf{1}_{A_k} + T\mathbf{1}_{\theta=T}$, where $t_k := \frac{(k+1)T}{2^n}$ and $A_k := \{\frac{kT}{2^n} \leq \theta < \frac{(k+1)T}{2^n}\}$. Note that $\theta^n \in \mathcal{T}$ and $\theta^n \downarrow \theta$.
Let $n \in \mathbb{N}$. Let $\alpha \in \mathcal{A}$. By the flow property for doubly reflected BSDEs, we get $Y^{\alpha,0,x}_{0,T} = Y^{\alpha,0,x}_{0,\theta^n}[Y^{\alpha,\theta^n,X^{\alpha,0,x}_{\theta^n}}_{\theta^n,T}]$. For each $s \in [0,T]$, for each $\omega \in \Omega$, set ${}^s\omega := (\omega_{r\wedge s})_{0 \leq r \leq T}$. Note that for each $\omega \in \Omega$, we have $\omega = {}^s\omega + \omega^s\mathbf{1}_{]s,T]}$. In the sequel, we identify ω with $({}^s\omega, \omega^s)$.

By the splitting property for doubly reflected BSDEs (see Proposition A.2), there exists a P-null set $\mathcal{N}$ such that for each k and for each $\omega \in A_k \cap \mathcal{N}^c$, we have

$$Y^{\alpha,\theta^n,X^{\alpha,0,x}_{\theta^n}}_{\theta^n,T}(\omega) = Y^{\alpha,t_k,X^{\alpha,0,x}_{t_k}}_{t_k,T}({}^{t_k}\omega) = Y^{\alpha({}^{t_k}\omega,\cdot),t_k,X^{\alpha,0,x}_{t_k}({}^{t_k}\omega)}_{t_k,T} = u^{\alpha({}^{t_k}\omega,\cdot)}(t_k, X^{\alpha,0,x}_{t_k}({}^{t_k}\omega)),$$

where the last equality follows from the definition of $u^{\alpha({}^{t_k}\omega,\cdot)}$. Now, by definition of u, we have $u^{\alpha({}^{t_k}\omega,\cdot)}(t_k, X^{\alpha,0,x}_{t_k}({}^{t_k}\omega)) \leq u(t_k, X^{\alpha,0,x}_{t_k}({}^{t_k}\omega))$. Since $u \leq \bar{u}^*$, we derive that $Y^{\alpha,\theta^n,X^{\alpha,0,x}_{\theta^n}}_{\theta^n,T} \leq \bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n})$ a.s. Hence, using the comparison theorem for doubly reflected BSDEs, we obtain $Y^{\alpha,0,x}_{0,T} = Y^{\alpha,0,x}_{0,\theta^n}[Y^{\alpha,\theta^n,X^{\alpha,0,x}_{\theta^n}}_{\theta^n,T}] \leq Y^{\alpha,0,x}_{0,\theta^n}[\bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n})]$. By taking the limsup, we thus get

$$Y^{\alpha,0,x}_{0,T} \leq \limsup_{n\to\infty} Y^{\alpha,0,x}_{0,\theta^n}[\bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n})]. \tag{4.5}$$

Let us now show that the assumptions of the Fatou lemma for doubly reflected BSDEs (see Lemma 3.4 and Remark 3.5) are satisfied. Recall that $u \leq h_2$ on $[0,T[\times\mathbb{R}$. Since h_2 is continuous, we get $u^* \leq h_2$ on $[0,T[\times\mathbb{R}$. On $\{\theta < T\}$, we thus have

$$\limsup_{n\to\infty} \bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n}) \leq \limsup_{n\to\infty} \bar{h}_2(\theta^n, X^{\alpha,0,x}_{\theta^n}) = h_2(\theta, X^{\alpha,0,x}_{\theta}) \quad \text{a.s.}$$

where the last equality follows from the continuity property of h_2 on $[0,T[\times\mathbb{R}$.
Now, on $\{\theta = T\}$, $\theta^n = T$. Hence, we have

$$\bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n}) = \bar{u}^*(T, X^{\alpha,0,x}_{T}) = g(X^{\alpha,0,x}_{T}) = \bar{h}_2(T, X^{\alpha,0,x}_{T}).$$

We thus get $\limsup_{n\to+\infty} \bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n}) \leq \bar{h}_2(\theta, X^{\alpha,0,x}_{\theta})$ a.s.
Similarly, one can show that $\limsup_{n\to+\infty} \bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n}) \geq \bar{h}_1(\theta, X^{\alpha,0,x}_{\theta})$ a.s. We thus have

$$\bar{h}_1(\theta, X^{\alpha,0,x}_{\theta}) \leq \limsup_{n\to+\infty} \bar{u}_*(\theta^n, X^{\alpha,0,x}_{\theta^n}) \leq \bar{h}_2(\theta, X^{\alpha,0,x}_{\theta}) \text{ a.s.}$$

Condition (3.5) (with limsup instead of liminf) is thus satisfied with $\xi^n = \bar{u}_*(\theta^n, X^{\alpha,0,x}_{\theta^n})$ and $\eta_t = \bar{h}_1(t, X^{\alpha,0,x}_t)$ and $\zeta_t = \bar{h}_2(t, X^{\alpha,0,x}_t)$.

Since h_1 is $\mathcal{C}^{1,2}$, by Proposition A.1, for each n, $h_1(s, X^{\alpha,t,x}_s)\mathbf{1}_{\{s<\theta^n\}} + \xi^n \mathbf{1}_{\{s=\theta^n\}}$ and $h_2(s, X^{\alpha,t,x}_s)\mathbf{1}_{\{s<\theta^n\}} + \xi^n \mathbf{1}_{\{s=\theta^n\}}$ satisfy Mokobodzki 's condition.

Moreover, setting $\bar{\xi} := \limsup_{n\to+\infty} \bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n})$, the processes $h_1(s, X^{\alpha,t,x}_s)\mathbf{1}_{\{s<\theta\}} + \bar{\xi}\mathbf{1}_{\{s=\theta\}}$ and $h_2(s, X^{\alpha,t,x}_s)\mathbf{1}_{\{s<\theta\}} + \bar{\xi}\mathbf{1}_{\{s=\theta\}}$ also satisfy Mokobodzki 's condition. We can thus apply the Fatou lemma for doubly reflected BSDEs (Lemma 3.4). Using (4.5), we then get:

$$Y^{\alpha,0,x}_{0,T} \leq \limsup_{n\to\infty} Y^{\alpha,0,x}_{0,\theta^n}[\bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n})] \leq Y^{\alpha,0,x}_{0,\theta}[\limsup_{n\to\infty} \bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n})].$$

Using the upper semicontinuity property of $\bar{u}^*$ on $[0,T[\times\mathbb{R}$ and $\bar{u}^*(T,x) = g(x)$, we obtain

$$Y^{\alpha,0,x}_{0,T} \leq Y^{\alpha,0,x}_{0,\theta}[\limsup_{n\to\infty} \bar{u}^*(\theta^n, X^{\alpha,0,x}_{\theta^n})] \leq Y^{\alpha,0,x}_{0,\theta}[\bar{u}^*(\theta, X^{\alpha,0,x}_{\theta})],$$

and this holds for each $\alpha \in \mathcal{A}$. Taking the supremum over $\alpha \in \mathcal{A}$, we get (4.3).
The proof of inequality (4.4) relies on similar arguments as above as well as on an existence result of ε-optimal controls satisfying appropriate measurability properties (see Proposition A.3). For more details on this last argument, we refer to the proof of the weak dynamic DPP for a mixed optimal stopping/control problem with $\mathcal{E}^f$-expectations (in [17, Theorem 17]). □

4.2 A *strong* dynamic programming principle in the continuous case

In this section, the set $\mathbf{A}$, where the controls are valued, is a nonempty closed subset of $\mathbb{R}^p$.
We suppose here that Assumption 4.4 holds.

Assumption 4.4. *The terminal reward map g is Lipschitz-continuous and the barriers maps h_1, h_2 are Lipschitz-continuous with respect to x uniformly in t. Moreover, we have*

$$h_1(T,x) \leq g(x) \leq h_2(T,x), \;\; \forall x \in \mathbb{R}. \tag{4.6}$$

Under this assumption, we show that u is continuous with respect to x, and that u^α is continuous with respect to (α, x).

Lemma 4.5 (A continuity property of u^α and u)**.** *Suppose Assumption 4.4 holds. Then for each $s \in [0,T]$, the map $(\alpha, x) \mapsto u^\alpha(s,x)$ is continuous and the value function $x \mapsto u(s,x)$ is continuous.*

Proof. By estimates (4.1) and (4.2), we derive that

$$\begin{aligned}|u^{\alpha^1}(s,x^1) - u^{\alpha^2}(s,x^2)|^2 &= |Y_s^{\alpha^1,s,x^1}[g(X_T^{\alpha^1,t,x^1})] - Y_s^{\alpha^2,s,x^2}[g(X_T^{\alpha^2,t,x^2})]|^2 \\ &\le C'(\|\alpha^1-\alpha^2\|^2_{\mathbb{H}^2_s} + |x^1-x^2|^2),\end{aligned}$$

where C' is a constant depending only on T and the Lipschitz constant C of the map f. It follows that the map $(\alpha, x) \mapsto u^\alpha(s,x)$ is Lipschitz-continuous.
Hence, since $u(s,x) = \sup_\alpha u^\alpha(s,x)$, the map $x \mapsto u(s,x)$ is Lipschitz-continuous. □

In order to show the *strong* dynamic programming principle, we first prove that the value function u is continuous with respect to (t,x). We have already shown that u is continuous with respect to x uniformly in t (see Lemma 4.5). It is thus sufficient to show the continuity of u with respect to t. To this purpose, we first prove a *strong* dynamic programming principle at deterministic times.

Lemma 4.6. *Suppose that g, h_1 and h_2 satisfy the continuity Assumption 4.4. Let $t \in [0,T]$. For all $s \ge t$, the value function u defined by (2.5) satisfies the equality*

$$u(t,x) = \sup_{\alpha \in \mathcal{A}_t^t} Y_{t,s}^{\alpha,t,x}\left[u(s, X_s^{\alpha,t,x})\right]. \tag{4.7}$$

Proof. We first show that:

$$u(t,x) \le \sup_{\alpha \in \mathcal{A}_t^t} Y_{t,s}^{\alpha,t,x}\left[u(s, X_s^{\alpha,t,x})\right]. \tag{4.8}$$

By the flow property for doubly reflected BSDEs (see [10]), we have that:

$$Y_{t,T}^{\alpha,t,x} = Y_{t,s}^{\alpha,t,x}[Y_{s,T}^{\alpha,t,x}].$$

Note that for almost-every ω, at fixed ${}^s\omega$, the process $\alpha({}^s\omega, T^s)$ (denoted also by $\alpha({}^s\omega,\cdot)$ belongs to $\mathcal{A}_s^s$ (see Section A.2 in the Appendix). Moreover, by Proposition A.2, for almost-every ω, at fixed ${}^s\omega$, $Y_{s,T}^{\alpha,t,x}({}^s\omega)$ coincides with $Y_{s,T}^{\alpha({}^s\omega,\cdot),s,X_s^{\alpha,t,x}({}^s\omega)}$, the solution at time s of the doubly reflected BSDE associated with control $\alpha({}^s\omega,\cdot)$, initial conditions $s, X_s^{\alpha,t,x}({}^s\omega)$, with respect to the filtration $\mathbb{F}^s$ and driven by the s-translated Brownian motion and s-translated Poisson measure. Now, by using the definition of u^α, we get that for almost-every ω,

$$Y_{s,T}^{\alpha,t,x}({}^s\omega) = Y_{s,T}^{\alpha({}^s\omega,\cdot),s,X_s^{\alpha,t,x}({}^s\omega)} = u^{\alpha({}^s\omega,\cdot)}(s, X_s^{\alpha,t,x}({}^s\omega)) \le u(s, X_s^{\alpha,t,x}({}^s\omega)).$$

Finally, the comparison theorem for doubly reflected BSDEs (see Th. 5.1 in [16]) leads to:

$$Y_{t,T}^{\alpha,t,x} = Y_{t,s}^{\alpha,t,x}[Y_{s,T}^{\alpha,t,x}] \le Y_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})].$$

Taking the supremum over $\alpha \in \mathcal{A}_t^t$ in this inequality, we get inequality (4.8).

It remains to show the inequality:

$$\sup_{\alpha \in \mathcal{A}_t^t} Y_{t,s}^{\alpha,t,x}\left[u(s, X_s^{\alpha,t,x})\right] \leq u(t,x). \tag{4.9}$$

Fix $s \in [t,T]$ and $\alpha \in \mathcal{A}_t^t$. By Corollary A.4, there exists an "optimizing" sequence of measurable controls for $u(s, X_s^{\alpha,t,x})$, satisfying appropriate measurable properties. More precisely, there exists a sequence $(\alpha^n)_{n\in\mathbb{N}}$ of controls belonging to $\mathcal{A}_s^t$ such that, for P-almost every ω, we have

$$u(s, X_s^{\alpha,t,x}({}^s\omega)) = \lim_{n\to\infty} u^{\alpha^n({}^s\omega,\cdot)}(s, X_s^{\alpha,t,x}({}^s\omega)) = \lim_{n\to\infty} Y_{s,T}^{\alpha^n({}^s\omega,\cdot),s,X_s^{\alpha,t,x}({}^s\omega)}. \tag{4.10}$$

For each $n \in \mathbb{N}$, we set:

$$\tilde{\alpha}_u^n := \alpha_u \mathbf{1}_{u<s} + \alpha_u^n \mathbf{1}_{s\leq u\leq T}.$$

Note that $\tilde{\alpha}^n \in \mathcal{A}_t^t$. By the splitting property for doubly reflected BSDEs (see Proposition A.2), for P-almost every ω, we have

$$Y_{s,T}^{\alpha^n({}^s\omega,\cdot),s,X_s^{\alpha,t,x}({}^s\omega)} = Y_{s,T}^{\tilde{\alpha}^n,t,x}(\omega).$$

Hence, by (4.10), applying the continuity property of doubly reflected BSDEs with respect to terminal condition (see Proposition 3.2), we obtain

$$Y_{t,s}^{\alpha,t,x}\left[u(s, X_s^{\alpha,t,x})\right] = Y_{t,s}^{\alpha,t,x}\left[\lim_{n\to\infty} Y_{s,T}^{\tilde{\alpha}^n,t,x}\right] = \lim_{n\to\infty} Y_{t,s}^{\alpha,t,x}\left[Y_{s,T}^{\tilde{\alpha}^n,t,x}\right]. \tag{4.11}$$

Now, by the flow property of doubly reflected BSDEs, for each n, we have $Y_{t,s}^{\alpha,t,x}\left[Y_{s,T}^{\tilde{\alpha}^n,t,x}\right] = Y_{t,T}^{\tilde{\alpha}^n,t,x}$. We thus get

$$Y_{t,s}^{\alpha,t,x}\left[u(s, X_s^{\alpha,t,x})\right] = \lim_{n\to\infty} Y_{t,T}^{\tilde{\alpha}^n,t,x} \leq u(t,x).$$

Taking the supremum on $\alpha \in \mathcal{A}_t^t$ in this inequality, we get (4.9), which ends the proof. □

Using this *strong* dynamic programming principle at deterministic times, we derive the continuity property of u with respect to time t.

Theorem 4.7. *Suppose that g, h_1 and h_2 satisfy the continuity Assumption 4.4. The value function u is then continuous with respect to t, uniformly in x.*

Proof. Since (4.6) holds, we have $h_1(t,x) \leq u(t,x) \leq h_2(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$.
Let $0 \leq t < s \leq T$. We have

$$|u(t,x) - u(s,x)| \leq |u(t,x) - \sup_{\alpha \in \mathcal{A}_t^t} \mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})]| + |\sup_{\alpha \in \mathcal{A}_t^t} \mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})] - u(s,x)|.$$

We start by estimating $|\sup_{\alpha \in \mathcal{A}_t^t} \mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})] - u(s,x)|$.

$$\begin{aligned}
&|\sup_{\alpha \in \mathcal{A}_t^t} \mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})] - u(s,x)| \leq \sup_{\alpha \in \mathcal{A}_t^t} |\mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})] - u(s,x)| \\
&\leq \sup_{\alpha \in \mathcal{A}_t^t} |\mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})] - \mathcal{E}_{t,s}^{0}[u(s,x)]| \leq C\mathbb{E}[\sup_{t\leq r\leq s} (X_r^{\alpha,t,x} - x)^2]^{\frac{1}{2}} \leq C|s-t|(1+x^{2p})^{\frac{1}{2}}. \quad (4.12)
\end{aligned}$$

Here, $\mathcal{E}^0$ denotes the conditional expectation associated with the driver equal to 0. In order to obtain the above relation, we have used BSDEs estimates (see [31]), the Lipschitz property of u with respect to x (see Lemma 4.1) and the polynomial growth of u (see Lemma 4.2).
We now estimate $|u(t,x) - \sup_{\alpha\in\mathcal{A}_t^t} \mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})]|$. Using the strong dynamic programming principle for deterministic times (see Lemma 4.6), we derive that:

$$u(t,x) = \sup_{\alpha\in\mathcal{A}_t^t} Y_{t,s}^{\alpha,t,x}\left[u(s, X_s^{\alpha,t,x})\right].$$

Since the solution Y of the doubly reflected BSDE is smaller than the solution of the reflected BSDE with the same lower barrier, denoted by $\tilde{Y}$, we have

$$Y_{t,s}^{\alpha,t,x}\left[u(s, X_s^{\alpha,t,x})\right] \leq \tilde{Y}_{t,s}^{\alpha,t,x}\left[h_1(r, X_r^{\alpha,t,x})\mathbf{1}_{t\leq r<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{r=s}\right].$$

By the characterization of the solution of a reflected BSDE, we derive that

$$\begin{aligned} u(t,x) - \sup_{\alpha\in\mathcal{A}_t^t} \mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})] &\leq \sup_{\alpha\in\mathcal{A}_t^t}\sup_{\tau\in\mathcal{T}_t^t}(\mathcal{E}_{t,\tau\wedge s}^{\alpha,t,x}[h_1(\tau, X_\tau^{\alpha,t,x})\mathbf{1}_{\tau<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{\tau\geq s}] \\ &\quad -\mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})\mathbf{1}_{\tau<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{\tau\geq s}]) \\ &\leq A, \end{aligned} \tag{4.13}$$

where

$$A := \sup_{\alpha\in\mathcal{A}_t^t}\sup_{\tau\in\mathcal{T}_t^t}|\mathcal{E}_{t,s}^{\tilde{f}^\alpha}[h_1(\tau, X_\tau^{\alpha,t,x})\mathbf{1}_{\tau<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{\tau\geq s}] - \mathcal{E}_{t,s}^{f^\alpha}[h_1(s, X_s^{\alpha,t,x})\mathbf{1}_{\tau<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{\tau\geq s}]|$$

with $\tilde{f}^\alpha(s,\cdot) := f^{\alpha,t,x}(s,\cdot)\mathbf{1}_{s\leq\tau}$, because $u \geq h_1$. By the Lipschitz property in x of h_1, the polynomial growth of h_1 and f in x, and the standard estimates for BSDEs and SDEs, we have

$$\begin{aligned} A^2 &\leq C\sup_{\alpha\in\mathcal{A}_t^t}\sup_{\tau\in\mathcal{T}_t^t}\mathbb{E}[(h_1(\tau, X_\tau^{\alpha,t,x})\mathbf{1}_{\tau<s} - h_1(s, X_s^{\alpha,t,x})\mathbf{1}_{\tau<s})^2] + \mathbb{E}[\int_{\tau\wedge s}^s f^2(\alpha_r, r, h_1(\tau, X_\tau^{\alpha,t,x}), 0, 0)dr] \\ &\leq C[\sup_{t\leq r<s}\sup_x |h_1(r,x) - h_1(s,x)|^2 + (1+|x|^p)^2|s-t| + |s-t|(1+|x|^q)]. \end{aligned} \tag{4.14}$$

Also, the solution $Y^{\alpha,t,x}$ of the doubly reflected BSDE is greater than the solution of the reflected BSDE with the same upper barrier, denoted by $\bar{Y}^{\alpha,t,x}$. Hence,

$$Y_{t,s}^{\alpha,t,x}\left[u(s, X_s^{\alpha,t,x})\right] \geq \bar{Y}_{t,s}^{\alpha,t,x}\left[h_2(r, X_r^{\alpha,t,x})\mathbf{1}_{t\leq r<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{r=s}\right].$$

By the characterization of the solution of a reflected BSDE with upper barrier , we derive that

$$\bar{Y}_{t,s}^{\alpha,t,x}\left[h_2(r, X_r^{\alpha,t,x})\mathbf{1}_{t\leq r<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{r=s}\right] = \inf_{\sigma\in\mathcal{T}_t^t}\mathcal{E}_{t,\sigma\wedge s}^{\alpha,t,x}[h_2(\sigma, X_\sigma^{\alpha,t,x})\mathbf{1}_{\sigma<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{\sigma\geq s}]$$

It follows that

$$\begin{aligned} u(t,x) - \sup_{\alpha\in\mathcal{A}_t^t} \mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})] &\geq \sup_{\alpha\in\mathcal{A}_t^t}\inf_{\sigma\in\mathcal{T}_t^t}(\mathcal{E}_{t,\sigma\wedge s}^{\alpha,t,x}[h_2(\sigma, X_\sigma^{\alpha,t,x})\mathbf{1}_{\sigma<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{\sigma\geq s}] \\ &\quad -\mathcal{E}_{t,s}^{\alpha,t,x}[u(s, X_s^{\alpha,t,x})\mathbf{1}_{\sigma<s} + u(s, X_s^{\alpha,t,x})\mathbf{1}_{\sigma\geq s}]) \\ &\geq B, \end{aligned} \tag{4.15}$$

where

$$B := \sup_{\alpha\in\mathcal{A}_t^t}\inf_{\sigma\in\mathcal{T}_t^t}|\mathcal{E}^{\bar{f}^\alpha}_{t,s}[h_2(\sigma,X^{\alpha,t,x}_\sigma)\mathbf{1}_{\sigma<s}+u(s,X^{\alpha,t,x}_s)\mathbf{1}_{\sigma\geq s}]-\mathcal{E}^{f^\alpha}_{t,s}[h_2(s,X^{\alpha,t,x}_s)\mathbf{1}_{\sigma<s}+u(s,X^{\alpha,t,x}_s)\mathbf{1}_{\sigma\geq s}]|$$

with $\bar{f}^\alpha(s,\cdot) := f^{\alpha,t,x}(s,\cdot)\mathbf{1}_{s\leq\sigma}$, because $u \leq h_2$. Now, by the Lipschitz property in x of h_2, the polynomial growth of h_2 and f in x, we have

$$\begin{aligned} B^2 &\leq C\sup_{\alpha\in\mathcal{A}_t^t}\inf_{\sigma\in\mathcal{T}_t^t}\mathbb{E}[(h_2(\sigma,X^{\alpha,t,x}_\sigma)\mathbf{1}_{\sigma<s}-h_2(s,X^{\alpha,t,x}_s)\mathbf{1}_{\sigma<s})^2]+\mathbb{E}[\int_{\sigma\wedge s}^s f^2(\alpha_r,r,h_2(\sigma,X^{\alpha,t,x}_\sigma),0,0)dr] \\ &\leq C[\sup_{t\leq r<s}\sup_x|h_2(r,x)-h_2(s,x)|^2+(1+|x|^p)^2|s-t|+|s-t|(1+|x|^q)]. \end{aligned} \tag{4.16}$$

By (4.13) and (4.15), we get

$$|u(t,x)-\sup_{\alpha\in\mathcal{A}_t^t}\mathcal{E}^{\alpha,t,x}_{t,s}[u(s,X^{\alpha,t,x}_s)]|^2 \leq A^2+B^2.$$

This, together with inequalities (4.14), (4.16) and inequality (4.12) implies that

$$\begin{aligned} |u(t,x)-u(s,x)|^2 \leq & C[\sup_{t\leq r<s}\sup_x(|h_1(r,x)-h_1(s,x)|^2+\sup_{t\leq r<s}\sup_x(|h_2(r,x)-h_2(s,x)|^2] \\ &+C[(s-t)^2+|s-t|((1+|x|^p)^2+1+|x|^q). \end{aligned}$$

Now, since h_1 and h_2 are continuous with respect to t on $[0,T]$ uniformly in x, we derive that u is continuous with respect to time t, uniformly in x. The proof is thus ended. □

From this theorem, we derive the continuity property of u with respect to (t,x).

Corollary 4.8. *Suppose that g, h_1 and h_2 satisfy the continuity Assumption 4.4. The value function u is then continuous in $[0,T]\times\mathbb{R}$.*

Proof. Since Assumption 4.4 is satisfied, u is continuous with respect to x (see Lemma 4.1). This property together with the previous theorem implies that u is continuous with respect to (t,x). □

This result yields that under Assumption 4.4, we have $u^* = u_* = u$. Hence, by the *weak* dynamic programming principle (which still holds even if the set $\mathbf{A}$ is not compact because of the continuity Assumption 4.4) it follows that the value function u satisfies the following *strong* dynamic programming principle at stopping times.

Theorem 4.9 (Strong dynamic programming principle)**.** *Suppose that g, h_1 and h_2 satisfy the continuity Assumption 4.4. For each $t\in[0,T]$ and for each stopping time $\theta\in\mathcal{T}_t^t$, we have*

$$u(t,x) = \sup_{\alpha\in\mathcal{A}_t^t}Y^{\alpha,t,x}_{t,\theta}\left[u(\theta,X^{\alpha,t,x}_\theta)\right]. \tag{4.17}$$

This result still holds with θ replaced by θ^α, given a family of stopping times indexed by controls $\{\theta^\alpha,\alpha\in\mathcal{A}_t^t\}$.

5 Generalized HJB variational inequalities

In this section, we do not suppose that Assumption 4.4 holds.
We introduce the following Hamilton Jacobi Bellman variational inequalities (HJBVIs):

$$\begin{cases} h_1(t,x) \le u(t,x) \le h_2(t,x), \quad t<T \\ \text{if } u(t,x) < h_2(t,x) \text{ then } \inf_{\alpha\in A} \mathcal{H}^\alpha u(t,x) \ge 0 \\ \text{if } h_1(t,x) < u(t,x) \text{ then } \inf_{\alpha\in A} \mathcal{H}^\alpha u(t,x) \le 0, \end{cases} \tag{5.1}$$

with terminal condition $u(T,x) = g(x), x \in \mathbb{R}$. Here, $L^\alpha := A^\alpha + K^\alpha$, and for $\phi \in C^2(\mathbb{R})$,

- $A^\alpha\phi(x) := \frac{1}{2}\sigma^2(x,\alpha)\frac{\partial^2\phi}{\partial x^2}(x) + b(x,\alpha)\frac{\partial\phi}{\partial x}(x)$
- $K^\alpha\phi(x) := \int_{\mathbf{E}} \left(\phi(x+\beta(x,\alpha,e)) - \phi(x) - \frac{\partial\phi}{\partial x}(x)\beta(x,\alpha,e)\right)\nu(de)$
- $B^\alpha\phi(x) := \phi(x+\beta(x,\alpha,\cdot)) - \phi(x)$.
- $\mathcal{H}^\alpha\phi(x) := -\frac{\partial u}{\partial t}(t,x) - L^\alpha u(t,x) - f(\alpha,t,x,u(t,x),(\sigma\frac{\partial u}{\partial x})(t,x), B^\alpha u(t,x))$.

Definition 5.1. • An upper semicontinuous function u is said to be a *viscosity subsolution* of (5.1) if for any point $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and for any $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its minimum at (t_0,x_0), if $h_1(t_0,x_0) < u(t_0,x_0)$ then $\inf_{\alpha\in A}\mathcal{H}^\alpha\phi(t,x) \le 0$.

• A lower semicontinuous function u is said to be a *viscosity supersolution* of (5.1) if for any point $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and for any $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ such that $\phi(t_0,x_0) = u(t_0,x_0)$ and $\phi - u$ attains its maximum at (t_0,x_0), if $u(t_0,x_0) < h_2(t_0,x_0)$ then $\inf_{\alpha\in A}\mathcal{H}^\alpha\phi(t,x) \ge 0$.

5.1 The irregular case

Using the weak dynamic programming principle (Theorem 4.3), we now prove that the value function of our problem is a viscosity solution of the above HJBVI in a weak sense.

Theorem 5.2. *Suppose that* $\mathbf{A}$ *is compact. The map* u *is a* weak viscosity solution *of* (5.1) *in the sense that* u^* *is a viscosity subsolution of* (5.1) *and* u_* *is a viscosity supersolution of* (5.1).

Remark 5.3. *Using this theorem, when* $h_1(T,x) \le g(x) \le h_2(T,x)$ *for all* $x\in\mathbb{R}$*, we show in the next section that* u_* *is a viscosity supersolution of* (5.1) *with terminal value greater than* g_* *(see Corollary 6.2). Moreover, when* g *is l.s.c, we show that the value function* u *is the minimal viscosity supersolution of* (5.1) *with terminal value greater than* g *(see Theorem 6.5).*

Proof. We first prove that u_* is a viscosity supersolution of (5.1).
Let $(t_0,x_0) \in [0,T[\times\mathbb{R}$ and $\phi \in C^{1,2}([0,T]\times\mathbb{R})$ be such that $\phi(t_0,x_0) = u_*(t_0,x_0)$ and $\phi(t,x) \le u_*(t,x)$, for each $(t,x) \in [0,T]\times\mathbb{R}$. Without loss of generality, we can suppose that the maximum is strict in (t_0,x_0).
Let us assume that $u_*(t_0,x_0) < h_2(t_0,x_0)$ and that

$$\inf_{\alpha\in A}\left(-\frac{\partial}{\partial t}\phi(t_0,x_0) - L^\alpha\phi(t_0,x_0) - f\left(\alpha,t_0,x_0,\phi(t_0,x_0),(\sigma\frac{\partial\phi}{\partial x})(t_0,x_0),B^\alpha\phi(t_0,x_0)\right)\right) < 0.$$

By continuity, we can suppose that there exists $\alpha \in A$, $\varepsilon > 0$ and $\eta_\varepsilon > 0$ such that:
$\forall (t,x)$ such that $t_0 \leq t \leq t_0 + \eta_\varepsilon < T$ and $|x - x_0| \leq \eta_\varepsilon$, we have $\phi(t,x) \leq h_2(t,x) - \varepsilon$ and

$$-\frac{\partial}{\partial t}\phi(t,x) - L^\alpha \phi(t,x) - f\left(\alpha, t, x, \phi(t,x), (\sigma \frac{\partial \phi}{\partial x})(t,x), B^\alpha \phi(t,x)\right) \leq -\varepsilon. \tag{5.2}$$

Let $B_{\eta_\varepsilon}(t_0, x_0)$ be the ball of radius η_ε and center (t_0, x_0). Let $(t_n, x_n)_{n\in\mathbb{N}}$ be a sequence in $B_{\eta_\varepsilon}(t_0, x_0)$ with $t_n \geq t_0$ for each n, such that the sequence $(t_n, x_n, u(t_n, x_n))_{n\in\mathbb{N}}$ tends to $(t_0, x_0, u_*(t_0, x_0))$. We introduce the state process X^{α, t_n, x_n} associated with the above constant control α.
Let $\theta^{\alpha,n}$ be the stopping time defined by

$$\theta^{\alpha,n} := (t_0 + \eta_\varepsilon) \wedge \inf\{s \geq t_n \, , \, |X_s^{\alpha,t_n,x_n} - x_0| \geq \eta_\varepsilon\}.$$

By applying Itô's lemma to $\phi(s, X_s^{\alpha,t_n,x_n})$, one can derive that

$$\left(\phi(s, X_s^{\alpha,t_n,x_n}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{\alpha,t_n,x_n}), B^\alpha \phi(s, X_{s^-}^{\alpha,t_n,x_n}); s \in [t_n, \theta^{\alpha,n}]\right) \tag{5.3}$$

is the solution of the BSDE associated with terminal time $\theta^{\alpha,n}$, terminal value $\phi(\theta^{\alpha,n}, X_{\theta^{\alpha,n}}^{\alpha,t_n,x_n})$ and driver $-\psi^\alpha(s, X_s^{\alpha,t_n,x_n})$, where $\psi^\alpha(s,x) := \dfrac{\partial}{\partial s}\phi(s,x) + L^\alpha\phi(s,x)$. By the definition of the stopping time $\theta^{\alpha,n}$ together with inequality (5.2), we obtain:

$$-\psi^\alpha(s, X_s^{\alpha,t_n,x_n}) \leq f(\alpha, s, X_s^{\alpha,t_n,x_n}, \phi(s, X_s^{\alpha,t_n,x_n}), (\sigma \frac{\partial \phi}{\partial x})(s, X_s^{\alpha,t_n,x_n}), B^\alpha \phi(s, X_s^{\alpha,t_n,x_n})) - \varepsilon, \tag{5.4}$$

for $t_n \leq s \leq \theta^{\alpha,n}$ $ds \otimes dP$-a.s. Now, since the maximum (t_0, x_0) is strict, there exists γ_ε, which depends on η_ε, such that

$$u_*(t,x) \geq \phi(t,x) + \gamma_\varepsilon \text{ on } [0,T] \times \mathbb{R} \setminus B_{\eta_\varepsilon}(t_0, x_0). \tag{5.5}$$

Note now that

$$\phi(\theta^{\alpha,n} \wedge t, X_{\theta^{\alpha,n}\wedge t}^{\alpha,t_n,x_n}) = \phi(t, X_t^{\alpha,t_n,x_n})\mathbf{1}_{t<\theta^{\alpha,n}} + \phi(\theta^{\alpha,n}, X_{\theta^{\alpha,n}}^{\alpha,t_n,x_n})\mathbf{1}_{t\geq\theta^{\alpha,n}}, \quad t_n \leq t \leq T.$$

Using the inequalities $\phi(t,x) \leq h_2(t,x) - \varepsilon$ and (5.5) together with the definition of $\theta^{\alpha,n}$, we derive that for each $t \in [t_n, \theta^{\alpha,n}]$,

$$\phi(t, X_t^{\alpha,t_n,x_n}) \leq (h_2(t, X_t^{\alpha,t_n,x_n}) - \delta_\varepsilon)\mathbf{1}_{t<\theta^{\alpha,n}} + (u_*(\theta^{\alpha,n}, X_{\theta^{\alpha,n}}^{\alpha,t_n,x_n}) - \delta_\varepsilon)\mathbf{1}_{t=\theta^{\alpha,n}} \quad \text{a.s.},$$

where $\delta_\varepsilon := \min(\varepsilon, \gamma_\varepsilon)$. Hence, by the inequality (5.4) between the driver process $-\psi^\alpha(s, X_s^{\alpha,t_n,x_n})$ and the driver $f(\alpha, s, X_s^{\alpha,t_n,x_n}, .)$ computed along the solution (5.3), applying a comparison theorem between a BSDE and a reflected BSDE (see [17, Proposition 20]), we obtain

$$\phi(t_n, x_n) \leq \bar{Y}_{t_n,\theta^{\alpha,n}}^{\alpha,t_n,x_n}[h_2(t, X_t^{\alpha,t_n,x_n})\mathbf{1}_{t<\theta^{\alpha,n}} + u_*(\theta^{\alpha,n}, X_{\theta^{\alpha,n}}^{\alpha,t_n,x_n})\mathbf{1}_{t=\theta^{\alpha,n}}] - \delta_\varepsilon K, \tag{5.6}$$

where $\bar{Y}$ is the solution of the reflected BSDE associated with upper barrier
$h_2(t, X_t^{\alpha,t_n,x_n})\mathbf{1}_{t<\theta^{\alpha,n}} + u_*(\theta^{\alpha,n}, X_{\theta^{\alpha,n}}^{\alpha,t_n,x_n})\mathbf{1}_{t=\theta^{\alpha,n}}$ and K is a positive constant which only depends on T and the Lipschitz constant of f.

Moreover, since the solution of the reflected BSDE with upper barrier is smaller than the solution of a doubly reflected BSDE with the same upper barrier, we have

$$\bar{Y}^{\alpha,t_n,x_n}_{t_0,\theta^{\alpha,n}}[h_2(t,X^{\alpha,t_n,x_n}_t)\mathbf{1}_{t<\theta^{\alpha,n}}+u_*(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})\mathbf{1}_{t=\theta^{\alpha,n}}] \quad\leq\quad Y^{\alpha,t_n,x_n}_{t_n,\theta^{\alpha,n}}[u_*(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})].$$

Hence, using inequality (5.6), we get

$$\phi(t_n,x_n)\leq Y^{\alpha,t_n,x_n}_{t_n,\theta^{\alpha,n}}[u_*(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})]-\delta_\varepsilon K. \tag{5.7}$$

Now, ϕ is continuous with $\phi(t_0,x_0)=u_*(t_0,x_0)$, and the sequence $(t_n,x_n,u(t_n,x_n))$ converges to $(t_0,x_0,u_*(t_0,x_0))$ as n tends to $+\infty$. We can thus assume that n is sufficiently large so that $|\phi(t_n,x_n)-u(t_n,x_n)|\leq\dfrac{\delta_\varepsilon K}{2}$. By inequality (5.7), we thus get

$$u(t_n,x_n)\leq Y^{\alpha,t_n,x_n}_{t_n,\theta^{\alpha,n}}[u_*(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{\alpha,n}})]-\frac{\delta_\varepsilon K}{2}.$$

Now, by the super-optimality dynamic programming principle (4.4), since $\bar{u}_*\geq u_*$, we have $u(t_n,x_n)\geq Y^{\alpha,t_n,x_n}_{t_n,\theta^{\alpha,n}}[u_*(\theta^{\alpha,n},X^{\alpha,t_n,x_n}_{\theta^{n,\alpha}})]$. We thus obtain a contradiction. Hence, u_* is a viscosity supersolution of (5.1). It remains to prove that u^* is a viscosity subsolution of (5.1). The proof, which is based on quite similar arguments, is omitted. □

Remark 5.4. *In a Brownian setting with a continuous terminal reward, this property corresponds to a result shown in [10] by a penalization approach.*

5.2 The continuous case

Theorem 5.5. *Suppose that the set* $\mathbf{A}$ *is a closed subset of* $\mathbb{R}^p$ *and that the continuity Assumption 4.4 holds. Then the value function* u *is a viscosity solution in the classical sense, that is both a viscosity sub- and super-solution of the generalized HJBVIs* (5.1).

Proof. Since Assumption 4.4 holds, by Corollary 4.8, the value function u is continuous with respect to (t,x), which implies that $u^*=u_*=u$. Moreover, by Theorems 4.3 and 5.2 (which do not require the compactness of $\mathbf{A}$ in the continuous case), u is a *weak* viscosity solution of the generalized HJBVIs (5.1). The result thus follows. □

An uniqueness result. Suppose Assumption 4.4 holds and $\mathbf{A}$ is compact. Assume moreover that $\mathbf{E}=\mathbb{R}^*$, $\mathcal{K}=\mathcal{B}(\mathbb{R}^*)$ and $\int_{\mathbf{E}}(1\wedge e^2)\nu(de)<+\infty$, and the following

Assumption 5.6. *1.* $f(\alpha,x,y,z,k):=\overline{f}(\alpha,x,y,z,\int_{\mathbf{E}}k(e)\gamma(x,e)\nu(de))$ *where* $\overline{f}:A\times[0,T]\times\mathbb{R}^4\to\mathbb{R}$ *is Borelian and satisfies:*

(i) $|\overline{f}(\alpha,t,x,0,0,0)|\leq C$, *for any* $x\in\mathbb{R}$, $t\in[0,T]$, $\alpha\in A$.

(ii) $|\overline{f}(\alpha,t,x,y,z,k)-\overline{f}(\alpha',t,x',y',z',k')|\leq C(|\alpha-\alpha'|+|y-y'|+|z-z'|+|k-k'|)$, *for any* $x,x'\in\mathbb{R}$, $t\in[0,T]$, $y,y'\in\mathbb{R}$, $z,z'\in\mathbb{R}$, $k,k'\in\mathbb{R}$, $\alpha,\alpha'\in A$.

(iii) $k\to\overline{f}(\alpha,t,x,y,z,k)$ *is non-decreasing, for any* $(\alpha,t,x,y,z,k)\in A\times[0,T]\times\mathbb{R}^4$.

2. *For each $R > 0$, there exists a continuous function $m_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that $m_R(0) = 0$ and*

$$|\overline{f}(\alpha, t, x, v, p, q) - \overline{f}(\alpha, t, y, v, p, q)| \leq m_R(|x - y|(1 + |p|)),$$

for any $t \in [0, T], |x|, |y|, |v| \leq R, p, q \in \mathbb{R}, \alpha \in A$.

3. *$\gamma : \mathbb{R} \times \mathbf{E} \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbf{E})$-measurable,*
$|\gamma(x, e) - \gamma(x', e)| < C|x - x'|(1 \wedge e^2), x, x' \in \mathbb{R}, e \in \mathbf{E}$
$|\gamma(x, e)| \leq C(1 \wedge |e|)$ and $\gamma(x, e) \geq 0$, $e \in \mathbf{E}$

4. *There exists $r > 0$ such that for any $x \in \mathbb{R}$, $t \in [0, T]$, $u, v \in \mathbb{R}$, $p \in \mathbb{R}$, $l \in \mathbb{R}$, $\alpha \in A$:*

$$\overline{f}(\alpha, t, x, v, p, l) - \overline{f}(\alpha, t, x, u, p, l) \geq r(u - v) \text{ when } u \geq v.$$

5. *$|g(x)| + |h(t, x)| \leq C$, for any $x \in \mathbb{R}$, $t \in [0, T]$.*

Lemma 5.7 (Comparison principle)**.** *Suppose that the above assumptions hold. If U is a bounded viscosity subsolution and V is a bounded viscosity supersolution of the HJBVIs (5.1) with $U(T, x) \leq g(x) \leq V(T, x)$, $x \in \mathbb{R}$. Then, $U(t, x) \leq V(t, x)$, for each $(t, x) \in [0, T] \times \mathbb{R}$.*

The proof is similar to the case studied in [16] without controls and is thus omitted.

Moreover, using this comparison principle together with Theorem 5.5, we derive the characterization of u as the unique solution of the HJBVIs.

Theorem 5.8 (Characterization of the value function)**.** *Suppose that the above assumptions hold. The value function u is then the unique viscosity solution of the HJBVIs (5.1) in the class of bounded continuous functions, in the sense that u is a viscosity sub and super solution of (5.1) with terminal condition $u(T, x) = g(x)$.*

Using the comparison principle (Lemma 5.7), we show below some new results in the irregular case.

6 Complementary results in the discontinuous case

Using the previous results both in the discontinuous case and the continuous case, we provide some additional properties of the value function u when the terminal reward map is discontinuous. In particular, we prove some complementary results on the l.s.c. envelop u_* of u. We also show that, when g is l.s.c. , the value function u is l.s.c., and that it is characterized as the minimal viscosity supersolution of the HJBVIs (5.1) with terminal value greater or equal to g.

Let g be a Borelian function such that there exists $K > 0$ and $p \in \mathbb{N}$ with $|g(x)| \leq K(1 + |x|^p)$ and satisfying $h_1(T, x) \leq g(x) \leq h_2(T, x)$ for all $x \in \mathbb{R}$. Let g_* be the l.s.c. envelope of g.
We denote by $u^g(t, x)$ the value function of our problem associated with terminal reward g.

Proposition 6.1. *Let $(g_n)_{n \in \mathbb{N}}$ be a non decreasing sequence of Lipschitz continuous maps satisfying for each n, $h_1(T, x) \leq g_n(x) \leq h_2(T, x)$ and $|g_n(x)| \leq K(1 + |x|^p)$, $x \in \mathbb{R}$, and such that $g_* = lim_{n \to \infty} \uparrow g_n$ (which exists by analysis results). Then, we have*

$$u^{g_*}(t, x) = lim_{n \to \infty} \uparrow u^{g_n}(t, x).$$

Moreover, for each n, u^{g_n} is continuous and u^{g_*} is l.s.c.

Proof. By definition of the value function,

$$u^{g_*}(t,x) = \sup_{\alpha \in \mathcal{A}_t^t} Y_{t,T}^{\alpha,t,x}[g_*(X_T^{\alpha,t,x})].$$

By the continuity property of doubly reflected BSDEs with respect to the terminal condition (see Proposition 3.2) and by the comparison theorem, we have:

$$Y_{t,T}^{\alpha,t,x}[g_*(X_T^{\alpha,t,x})] = \lim_{n\to\infty} \uparrow Y_{t,T}^{\alpha,t,x}[g_n(X_T^{\alpha,t,x})].$$

From the two above equalities, we derive that

$$u^{g_*}(t,x) = \sup_{\alpha \in \mathcal{A}_t^t} \sup_{n \in \mathbb{N}} Y_{t,T}^{\alpha,t,x}[g_n(X_T^{\alpha,t,x})] = \sup_{n \in \mathbb{N}} \sup_{\alpha \in \mathcal{A}_t^t} Y_{t,T}^{\alpha,t,x}[g_n(X_T^{\alpha,t,x})] = \lim_{n\to\infty} \uparrow u^{g_n}(t,x).$$

By the continuity property of the value function in the continuous case (see Corollary 4.8), u^{g_n} is continuous because g_n is continuous. Hence, u^{g_*} is l.s.c. as supremum of continuous functions. □

Corollary 6.2. *We have:*
1. If g is l.s.c., then u^g is l.s.c.
2. $u_*^g \geq u^{g_*}$.

Proof. 1. Since g is l.s.c., $g_* = g$. Hence, Point 1. directly follows from the above Proposition.
2. As $g \geq g_*$, we have $u^g \geq u^{g_*}$. By Point 1, u^{g_*} is l.s.c., and we thus obtain $u_*^g \geq u^{g_*}$. □

Corollary 6.3. *Suppose that* $\mathbf{A}$ *is compact.*
1. The l.s.c. envelop u_*^g of the value function u^g is a viscosity supersolution of the HJBVIs (5.1), with terminal condition $u_*^g(T,x) \geq g_*(x),\ x \in \mathbb{R}$.
2. If g is l.s.c., then u^g is l.s.c., and is a viscosity supersolution of the HJBVIs (5.1) with terminal condition $u^g(T,x) = g(x),\ x \in \mathbb{R}$.

Proof. 1. By Point 2. of Corollary 6.2, $u_*^g(T,x) \geq u^{g_*}(T,x) = g_*(x),\ x \in \mathbb{R}$. Using Theorem 5.2, we obtain that u_*^g is a viscosity supersolution of the HJBVIs (5.1), with terminal condition $u_*^g(T,x) \geq g_*(x),\ x \in \mathbb{R}$.
2. By Point 1. of Corollary 6.2, since g is l.s.c., u^g is l.s.c. Hence, by Theorem 5.2, u^g is a viscosity supersolution of (5.1). □

Proposition 6.4. *Suppose that* $\mathbf{A}$ *is compact and Assumption 5.6 holds. The function u^{g_*} is the minimal viscosity supersolution of the HJBVIs (5.1) with terminal value greater or equal to g_*.*

Proof. By point 2. of Corollary 6.3 applied to the l.s.c. function g_*, the mapping u^{g_*} is l.s.c. and is a viscosity supersolution of the HJBVIs (5.1) with terminal value $u^{g_*}(T,x) = g_*(x),\ x \in \mathbb{R}$. Let v be a viscosity supersolution of the HJBVIs (5.1) with $v(T,x) \geq g_*(x),\ x \in \mathbb{R}$. Let us show that $v \geq u^{g_*}$. Let $(g_n)_{n \in \mathbb{N}}$ be a non decreasing sequence of continuous maps satisfying the assumptions of Proposition 6.1. For each n, we have $v(T,x) \geq g_n(x)$, $x \in \mathbb{R}$. As u^{g_n} is a continous viscosity solution, then it is a subsolution. We also have $u^{g_n}(T,x) = g_n(x)$, $x \in \mathbb{R}$. By using the comparison principle (see Lemma 5.7), we have $v \geq u^{g_n}$, for all n. Hence, we get

$$v \geq \lim_n \uparrow u^{g_n} = u^{g_*},$$

where the last equality follows from Proposition 6.1. □

By the above result, we derive the following characterization of the value function when g is l.s.c.

Theorem 6.5. *Suppose* $\mathbf{A}$ *is compact and Assumption 5.6 holds. If* g *is l.s.c., then the value function* u^g *of our mixed problem is l.s.c. and it is the minimal viscosity supersolution of the HJBVIs (5.1) with terminal value greater or equal to* g*.*

Proof. Since g is l.s.c. , we have $g_* = g$. The result then follows from Proposition 6.4.

A Appendix

A.1 A decomposition property

We show a property which ensures that under our assumptions (in particular h_1 or h_2 is $\mathcal{C}^{1,2}$), then, for each $t \in [0,T]$ and for each $\alpha \in \mathcal{A}_t^t$, the barriers $h_1(s, X_s^{\alpha,t,x})\mathbf{1}_{s<T} + g(X_T^{\alpha,t,x})\mathbf{1}_{s=T}$ and $h_2(s, X_s^{\alpha,t,x})\mathbf{1}_{s<T} + g(X_T^{\alpha,t,x})\mathbf{1}_{s=T}$ satisfy Mokobodzki's condition (see (2.7)).

Proposition A.1. *Let* $h : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be* $\mathcal{C}^{1,2}$ *with bounded derivatives, and let* $g : \mathbb{R} \to \mathbb{R}$ *be a Borelian map. Suppose that* $|h(t,x)| + |g(x)| \leq C(1 + |x|^p)$*. Let* $t \in [0,T]$ *and* $\alpha \in \mathcal{A}_t^t$*. There exist two nonnegative* $\mathbb{F}^t$*-supermartingales* $H^{\alpha,t,x}$ *and* $H'^{\alpha,t,x}$ *in* $\mathcal{S}_t^2$ *such that*

$$h(s, X_s^{\alpha,t,x})\mathbf{1}_{s<T} + g(X_T^{\alpha,t,x})\mathbf{1}_{s=T} = H_s^{\alpha,t,x} - H_s'^{\alpha,t,x}, \;\; t \leq s \leq T \quad \text{a.s.}$$

Proof. Set $\xi_s := h(s, X_s^{\alpha,t,x})\mathbf{1}_{s<T} + g(X_T^{\alpha,t,x})\mathbf{1}_{s=T}$. By applying Itô's formula to $h(s, X_s^{\alpha,t,x})$, one can show that there exist three $\mathbb{F}^t$-predictable processes (f_r), (φ_r^1) and $(\varphi^2(r,.))$ such that $dh(r, X_r^{\alpha,t,x}) = f_r dr + \varphi_r^1 dW_r^t + \int_E \varphi^2(r,e) d\tilde{N}^t(dr,de)$, with $|f_r| + |\varphi_r^1| + ||\varphi^2(r,.)||_\nu \leq K(1 + |X_{r^-}^{\alpha,t,x}|^p)$. Integrating this equation between $s \geq t$ and T, and then taking the conditional expectation with respect to $\mathcal{F}_s^t$, we derive that

$$h(s, X_s^{\alpha,t,x}) = E[h(T, X_T^{\alpha,t,x}) + \int_s^T f_r dr \,|\, \mathcal{F}_s^t] = I_s - I_s',$$

where (I_s) and (I_s') are defined for each $s \in [t,T]$ by $I_s := E[h_1^+(T, X_T^{\alpha,t,x}) + \int_s^T f_r^+ dr \,|\, \mathcal{F}_s^t]$ and $I_s' := E[h_1^-(T, X_T^{\alpha,t,x}) + \int_s^T f_r^- dr \,|\, \mathcal{F}_s^t]$. They are both nonnegative supermartingales belonging to $\mathcal{S}_t^2$. For each $s \in [t,T]$, we have $h(s, X_s^{\alpha,t,x}) = I_s - I_s'$. For each $s \in [t,T]$, set

$$\tilde{\xi}_s := \xi_s - E[g(X_T^{\alpha,t,x})|\mathcal{F}_s^t]. \tag{A.1}$$

Let us now show that there exist two nonnegative supermartingales $\tilde{H}$ and $\tilde{H}'$ such that

$$\tilde{H}_T = \tilde{H}'_T = 0 \quad \text{and} \quad \tilde{\xi}_s = \tilde{H}_s - \tilde{H}_s', \; t \leq s \leq T. \tag{A.2}$$

By (A.1), we have $\tilde{\xi}_T = 0$. We thus have that for each $s \in [t,T]$,

$$\tilde{\xi}_s = (h(s, X_s^{\alpha,t,x}) - E[g(X_T^{\alpha,t,x})|\mathcal{F}_s^t])\mathbf{1}_{s<T} = (I_s - I_s' - E[g(X_T^{\alpha,t,x})|\mathcal{F}_s^t])\mathbf{1}_{s<T}.$$

Now, the processes $(I_s + E[g^-(X_T^{\alpha,t,x})|\mathcal{F}_s^t])_{t \leq s \leq T}$ and $(I_s' + E[g^+(X_T^{\alpha,t,x})|\mathcal{F}_s^t])_{t \leq s \leq T}$ are nonnegative supermartingales as the sum of two nonnegative supermartingales.

It follows that the processes $(\tilde{H}_s)_{t\leq s\leq T}$ and $(\tilde{H}'_s)_{t\leq s\leq T}$ defined by
$\tilde{H}_s := (I_s + E[g^-(X_T^{\alpha,t,x})|\mathcal{F}_s^t])\mathbf{1}_{s<T}$ and $\tilde{H}'_s := (I'_s + E[g^+(X_T^{\alpha,t,x})|\mathcal{F}_s^t])\mathbf{1}_{s<T}$, $t \leq s \leq T$, are also nonnegative supermartingales, which are moreover equal to 0 at time T. They satisfy the equality $\tilde{\xi}_s = \tilde{H}_s - \tilde{H}'_s$ for each $s \in [t,T]$. Hence, $\tilde{H}$ and $\tilde{H}'$ satisfy equalities (A.2).
Using equality (A.1), we derive that the processes H and H' defined by $H_s := \tilde{H}_s + E[g^+(X_T^{\alpha,t,x})|\mathcal{F}_s^t]$ and $H'_s := \tilde{H}'_s + E[g^-(X_T^{\alpha,t,x})|\mathcal{F}_s^t]$ are nonnegative supermartingales satisfying $\xi_s = H_s - H'_s$, $t \leq s \leq T$, which ends the proof. □

Note that we cannot apply Itô's formula to $\bar{h}(s, X_s^{\alpha,t,x})$ with $\bar{h}(s,x) := h(s,x)\mathbf{1}_{s<T} + g(x)\mathbf{1}_{s=T}$ since g is irregular. The change of variable (A.1) allows us to deal more easily with the irregularity of g.

A.2 Measurability and splitting properties

For each $\omega \in \Omega$, let ${}^s\omega := (\omega_{r\wedge s})_{0\leq r\leq T}$. Let S^s (resp. T^s) be the operator defined on Ω by $S^s(\omega) := {}^s\omega$ (resp. $T^s(\omega) := \omega^s$). Note that S^s and T^s are independent and for each $\omega \in \Omega$ we have $\omega = S^s(\omega) + T^s(\omega)\mathbf{1}_{]s,T]} = {}^s\omega + \omega^s\mathbf{1}_{]s,T]}$. When there is no ambiguity, we identify ω with $({}^s\omega, \omega^s)$.

Let $t \in [0,T]$, $\alpha \in \mathcal{A}_t^t$ and $s \geq t$. For P-almost every ω, the process $\alpha({}^s\omega, T^s)$ (denoted also by $\alpha({}^s\omega, .)$) defined by

$$\alpha({}^s\omega, T^s) : \Omega \times [s,T] \to \mathbb{R}\,;\, (\omega', r) \mapsto \alpha_r({}^s\omega, T^s(\omega'))$$

belongs to $\mathcal{A}_s^s$ (see [17, Lemma 4]).

Recall that the forward process satisfies the splitting property (see [17]): for all $t \in [0,T]$, $\alpha \in \mathcal{A}_t^t$ and $s \geq t$, for almost every $\omega \in \Omega$, setting $\tilde{\omega} = {}^s\omega$, the process $X^{\alpha,t,x}(\tilde{\omega}, T^s)$ (denoted also by $X^{\alpha,t,x}(\tilde{\omega}, .)$) coincides with the solution of the forward SDE on $\Omega \times [s,T]$, driven by W^s and $\tilde{N}^s$, associated with control $(\alpha_r(\tilde{\omega}, \cdot))_{r\geq s}$ and initial condition $X_s^{\alpha(\tilde{\omega}),t,x}(\tilde{\omega})$ at time s.

By similar arguments as in [17], we have an analogous property for doubly reflected BSDEs:

Proposition A.2. *(Splitting property for doubly reflected BSDEs)*
Let $t \in [0,T]$, $\alpha \in \mathcal{A}_t^t$ and $s \in [t,T]$. For almost every $\omega \in \Omega$, setting $\tilde{\omega} = {}^s\omega$, the process $(Y_r^{\alpha,t,x}(\tilde{\omega}, .))_{s\leq r\leq T}$ coincides with the solution of the doubly reflected BSDE on $\Omega \times [s,T]$, associated with driver $f^{\alpha(\tilde{\omega},.),s,X_s^{\alpha(\tilde{\omega}),t,x}(\tilde{\omega})}$, barriers $h_i(r, X_r^{\alpha(\tilde{\omega},T^s),s,X_s^{\alpha(\tilde{\omega}),t,x}(\tilde{\omega})})_{r<T}$, $i = 1,2$, terminal condition $g(X_T^{\alpha(\tilde{\omega},T^s),s,X_s^{\alpha(\tilde{\omega}),t,x}(\tilde{\omega})})$, filtration $\mathbb{F}^s$, and driven by W^s and $\tilde{N}^s$. In particular, we have

$$Y_s^{\alpha,t,x}(\tilde{\omega}) = Y_s^{\alpha(\tilde{\omega},\cdot),s,X_s^{\alpha(\tilde{\omega}),t,x}(\tilde{\omega})} = u^{\alpha(\tilde{\omega},\cdot)}(s, X_s^{\alpha(\tilde{\omega}),t,x}(\tilde{\omega})). \tag{A.3}$$

Let $\eta \in L^2(\mathcal{F}_s^t)$. Since η is $\mathcal{F}_s$-measurable, up to a P-null set, it can be written as a measurable map, still denoted by η, of the past trajectory ${}^s\omega$. For each $\omega \in \Omega$, by using the definition of the function u, we have:

$$u(s, \eta({}^s\omega)) = \sup_{\alpha\in\mathcal{A}_s^s} u^\alpha(s, \eta({}^s\omega)). \tag{A.4}$$

For each (t,s) with $s \geq t$, we introduce the set $\mathcal{A}_s^t$ of restrictions to $[s,T]$ of the controls in $\mathcal{A}_t^t$. They can also be identified to the controls α in $\mathcal{A}_t^t$ which are equal to 0 on $[t,s]$.
Using the measurability and continuity results of Section 2.1, we show the following result.

Proposition A.3. *Let $t \in [0,T]$, $s \in [t,T]$ and $\eta \in L^2(\mathcal{F}^t_s)$. Let $\varepsilon > 0$. Suppose that **A** is compact. There exists $\alpha^\varepsilon \in \mathcal{A}^t_s$ such that, for almost every $\omega \in \Omega$,*

$$u(s, \eta(^s\omega)) \leq u^{\alpha^\varepsilon(^s\omega,.)}(s, \eta(^s\omega)) + \varepsilon. \tag{A.5}$$

Moreover, the ε-optimal control α^ε can be constructed so that it depends on the past trajectory $^s\omega$ only through $\eta(^s\omega)$.
*If Assumption 4.4 holds, this result still holds when **A** is a nonempty closed subset of $\mathbb{R}$, not necessarily compact.*

Proof. To simplify notation, we suppose that $t = 0$. We introduce Ω^s the set of the restrictions to $[s,T]$ of the paths $\omega \in \Omega$. By Lemma 4.1, the map $(x, \alpha) \mapsto u^\alpha(s,x)$ is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathcal{A}^s_s)$-measurable.

By Proposition 7.50 in [5] together with a result of measure theory (see e.g. Lemma 26 in [17]), we derive that there exists a Borelian map $\hat{\alpha}^\varepsilon : \mathbb{R} \to \mathcal{A}^s_s\,;\; x \mapsto \hat{\alpha}^\varepsilon(x, \cdot)$ such that

$$u(s,x) \leq u^{\hat{\alpha}^\varepsilon(x,\cdot)}(s,x) + \varepsilon \quad \text{for} \quad \mathrm{Q}-\text{almost every } x \in \mathbb{R}, \tag{A.6}$$

where Q is the law of η under P. Let $\{e^i, i \in \mathbb{N}\}$ be a countable orthonormal basis of the separable Hilbert space $\mathbb{H}^2_s$. For each x, the map $\hat{\alpha}^\varepsilon_u(x,\cdot)$, which belongs to $\mathbb{H}^2_s$, admits the following decomposition: $\hat{\alpha}^\varepsilon_u(x,\cdot) = \sum_i \beta^{i,\varepsilon}(x) e^i_u(\cdot)\ dP \otimes du$-a.s. and in $\mathbb{H}^2_s$, where $\beta^{i,\varepsilon} : \mathbb{R} \mapsto \mathbb{R}$ is a Borelian map defined by $\beta^{i,\varepsilon}(x) =< \hat{\alpha}^\varepsilon(x,\cdot), e^i(\cdot) >_{\mathbb{H}^2_s}$. Consider the predictable process α^ε defined for each $(r,\omega) \in [s,T] \times \Omega$ by $\alpha^\varepsilon_r(\omega) := \sum_i \beta^{i,\varepsilon}(\eta(^s\omega)) e^i_r(\omega^s)$. Since **A** is bounded, α^ε is bounded, and hence belongs to $\mathbb{H}^2_s$ and thus to $\mathcal{A}_s$, which gives the desired result.

When **A** is only a nonempty closed subset of $\mathbb{R}$, not necessarily bounded, the control $\hat{\alpha}^\varepsilon$ can also be supposed to be bounded because the set of bounded controls in $\mathcal{A}^s_s$, denoted by $\mathcal{A}^{s,b}_s$, is dense in $\mathcal{A}^s_s$. Indeed, let $a \in \mathbf{A}$. If $\alpha \in \mathcal{A}^s_s$, the bounded controls $\alpha_r \mathbf{1}_{|\alpha_r| \leq n} + a \mathbf{1}_{|\alpha_r| > n}$ belong to $\mathcal{A}^s_s$ and converge to α in $\mathbb{H}^2_s$ as n tends to $+\infty$.
Now, by the continuity Assumption 4.4, u^α is continuous with respect to α (see Lemma 4.5). Hence,

$$u(s,x) = \sup_{\alpha \in \mathcal{A}^s_s} u^\alpha(s,x) = \sup_{\alpha \in \mathcal{A}^{s,b}_s} u^\alpha(s,x).$$

It follows that there exists a bounded Borelian map $\hat{\alpha}^\varepsilon : \mathbb{R} \to \mathcal{A}^s_s\,;\; x \mapsto \hat{\alpha}^\varepsilon(x,\cdot)$ which satisfies inequality (A.6). The proof is thus complete. □

By applying this property to $\varepsilon = \frac{1}{n}$ for each $n \in \mathbb{N}$, we derive that under Assumption 4.4, there exists an "optimizing sequence" for the value function $u(s, \eta(\cdot))$.

Corollary A.4. *Suppose Assumption 4.4 holds. Let $t \in [0,T]$, $s \in [t,T]$ and $\eta \in L^2(\mathcal{F}^t_s)$. There exists a sequence $(\alpha^n)_{n \in \mathbb{N}} \in \mathcal{A}^t_s$ such that for almost every $\omega \in \Omega$,*

$$u(s, \eta(^s\omega)) = \lim_{n \to \infty} u^{\alpha^n(^s\omega,\cdot)}(s, \eta(^s\omega)).$$

Moreover, the processes α^n, $n \in \mathbb{N}$, can be chosen so that they depend on the past trajectory $^s\omega$ only through $\eta(^s\omega)$.

A.3 Complementary results on BSDEs and reflected BSDEs with jumps

We first state a version of Fatou lemma for $\mathcal{E}^f$-conditional expectations (or equivalently for BSDEs) where the limit involves both terminal condition and terminal time.

Lemma A.5 (A Fatou lemma for BSDEs). *Let $T > 0$. Let f be a given Lipschitz driver satisfying Assumption 3.1. Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}$, converging a.s. to $\theta \in \mathcal{T}$ as n tends to ∞. Let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $\mathbb{E}[\sup_n(\xi^n)^2] < +\infty$, and for each n, ξ^n is $\mathcal{F}_{\theta^n}$-measurable. Then, for each stopping time τ with $\tau \leq \theta$ a.s., we have*

$$\mathcal{E}^f_{\tau,\theta}(\liminf_{n\to+\infty} \xi^n) \leq \liminf_{n\to+\infty} \mathcal{E}^f_{\tau,\theta^n}(\xi^n) \quad \text{and} \quad \mathcal{E}^f_{\tau,\theta}(\limsup_{n\to+\infty} \xi^n) \geq \limsup_{n\to+\infty} \mathcal{E}^f_{\tau,\theta^n}(\xi^n) \quad a.s.$$

Proof. Let us show the first inequality. For each n, by the nondecreasing property of $\mathcal{E}^f$, we have $\mathcal{E}^f_{\tau,\theta^n}(\inf_{p\geq n} \xi^p) \leq \mathcal{E}^f_{\tau,\theta^n}(\xi^n)$ a.s. Thus, $\liminf_{n\to+\infty} \mathcal{E}^f_{\tau,\theta^n}(\inf_{p\geq n} \xi^p) \leq \liminf_{n\to+\infty} \mathcal{E}^f_{\tau,\theta^n}(\xi^n)$ a.s. Since $\lim_{n\to+\infty} \inf_{p\geq n} \xi^p = \liminf_{n\to+\infty} \xi^n$ a.s., by a continuity property the $\mathcal{E}^f$-conditional expectation (see Proposition A.6 in [31]), we get $\mathcal{E}^f_{\tau,\theta}(\liminf_{n\to+\infty} \xi^n) = \lim_{n\to+\infty} \mathcal{E}^f_{\tau,\theta^n}(\inf_{p\geq n} \xi^p)$ a.s., from which the desired inequality follows. The proof of the second one is similar. $\square$

Using this lemma, we now show a continuity property of the solutions of reflected BSDEs with respect to both terminal time and terminal condition, which extends the result established in [17] at time 0 (see Proposition 13) to any time $\tau \in \mathcal{T}$.

Lemma A.6 (A continuity property for reflected BSDEs). *Let $T > 0$. Let (η_t) be an RCLL process in $\mathcal{S}^2$. Let f be a given Lipschitz driver satisfying Assumption 3.1. Let $(\theta^n)_{n\in\mathbb{N}}$ be a non increasing sequence of stopping times in $\mathcal{T}$, converging a.s. to $\theta \in \mathcal{T}$ as n tends to ∞. Let $(\xi^n)_{n\in\mathbb{N}}$ be a sequence of random variables such that $\mathbb{E}[\operatorname{ess\,sup}_n(\xi^n)^2] < +\infty$, and for each n, ξ^n is $\mathcal{F}_{\theta^n}$-measurable. Suppose that ξ^n converges a.s. to an $\mathcal{F}_\theta$-measurable random variable ξ as n tends to ∞. Suppose that*

$$\eta_\theta \leq \xi \quad \text{a.s.} \tag{A.7}$$

Let $Y_{\cdot,\theta^n}(\xi^n)$; $Y_{\cdot,\theta}(\xi)$ be the solutions of the reflected BSDEs associated with driver f, obstacle $(\eta_s)_{s<\theta^n}$ (resp. $(\eta_s)_{s<\theta}$), terminal time θ^n (resp. θ), terminal condition ξ^n (resp. ξ). Then, for each stopping time τ with $\tau \leq \theta$ a.s.,

$$Y_{\tau,\theta}(\xi) = \lim_{n\to+\infty} Y_{\tau,\theta^n}(\xi^n) \quad a.s.$$

When for each n, $\theta_n = \theta$ a.s., the result still holds without Assumption (A.7).

Proof. Note that the case where $\tau = 0$ has been solved in [17] by using the classical a priori estimates on reflected BSDEs. The case where τ is any stopping time requires some additional arguments. It could be shown by using again Itô's calculus. We adopt here a less classical approach which requires less computations.
Step 1: Let us first consider the simpler case when for each n, $\theta_n = \theta$ a.s.
Using the a priori estimates on reflected BSDEs provided [15] (see Proposition A.1 in [15]) and the convergence of ξ^n to ξ, one can show that $Y_{\tau,\theta}(\xi) = \lim_{n\to+\infty} Y_{\tau,\theta}(\xi^n)$ a.s.
Step 2: Let us now consider the general case. The difficulty is here to deal with the variation of

the terminal time together with the presence of the obstacle (η_t). In particular, Proposition A.1 in [15] is not appropriate to this case. By the flow property for reflected BSDEs, we have:

$$Y_{\tau,\theta_n}(\xi^n) = Y_{\tau,\theta}(Y_{\theta,\theta_n}(\xi^n)).$$

By step 1, it is thus sufficient to show that $\lim_{n\to} Y_{\theta,\theta_n}(\xi^n) = \xi$ a.s.
Since the solution of the reflected BSDE associated with terminal condition ξ^n and terminal time θ_n is greater than the solution of the nonreflected BSDE associated with terminal condition ξ^n and terminal time θ_n, we have:

$$Y_{\theta,\theta_n}(\xi^n) \geq \mathcal{E}_{\theta,\theta_n}(\xi^n) \text{ a.s.} \tag{A.8}$$

Now, by the continuity property of BSDEs with respect to both terminal time and terminal condition (Proposition A.6 in [31]), we have $\lim_{n\to\infty} \mathcal{E}_{\theta,\theta_n}(\xi^n) = \mathcal{E}_{\theta,\theta}(\xi) = \xi$ a.s. Taking the lim inf in (A.8), we obtain:

$$\liminf_{n\to\infty} Y_{\theta,\theta_n}(\xi^n) \geq \lim_{n\to\infty} \mathcal{E}_{\theta,\theta_n}(\xi^n) = \xi, \quad \text{a.s.}$$

It remains to show that:

$$\limsup_{n\to\infty} Y_{\theta,\theta_n}(\xi^n) \leq \xi, \quad \text{a.s.} \tag{A.9}$$

By the characterization of the solution of a reflected BSDE, we obtain:

$$Y_{\theta,\theta^n} = ess \sup_{\tau\in\mathcal{T}_\theta} \mathcal{E}_{\theta,\tau\wedge\theta_n}(\eta_\tau \mathbf{1}_{\tau<\theta_n} + \xi^n \mathbf{1}_{\tau\geq\theta_n}).$$

Fix $\varepsilon > 0$. By the second assertion of Theorem 3.3 in [32], there exists an ε-optimal stopping time $\tau_n^\varepsilon \in \mathcal{T}_\theta$, that is such that

$$Y_{\theta,\theta_n}(\xi^n) \leq \mathcal{E}_{\theta,\tau_n^\varepsilon\wedge\theta_n}(\eta_{\tau_n^\varepsilon} \mathbf{1}_{\tau_n^\varepsilon<\theta_n} + \xi^n \mathbf{1}_{\tau_n^\varepsilon\geq\theta_n}) + \varepsilon \quad \text{a.s.} \tag{A.10}$$

Note that we have the following property: let X, X', X_n, X'_n, $n \in \mathbb{N}$, be real valued random variables with $X \leq X'$, and let A_n, $n \in \mathbb{N}$ be measurable sets of $\mathcal{F}_T$.

$$\text{If } X_n \to X \text{ and } X'_n \to X' \text{ a.s., then } \limsup_{n\to\infty} (X_n \mathbf{1}_{A_n} + X'_n \mathbf{1}_{A_n^c}) \leq X' \text{ a.s.} \tag{A.11}$$

Now, for each n, $\tau_n^\varepsilon \wedge \theta_n \geq \theta$ a.s. and $\tau_n^\varepsilon \wedge \theta_n$ tends to θ a.s. as $n \to +\infty$. Hence, by the right-continuity property of the obstacle (η_t), we get $\eta_{\tau_n^\varepsilon\wedge\theta_n} \to \eta_\theta \leq \xi$ a.s., where the last inequality holds by Assumption A.7. By applying Property (A.11) and since $\xi^n \to \xi$ a.s., we thus obtain

$$\limsup_{n\to\infty} (\eta_{\tau_n^\varepsilon} \mathbf{1}_{\tau_n^\varepsilon<\theta_n} + \xi^n \mathbf{1}_{\tau_n^\varepsilon\geq\theta_n}) \leq \xi \text{ a.s.}$$

Now, the Fatou property for BSDEs (see Lemma A.5) together with (A.10) implies that
$\limsup_{n\to\infty} Y_{\theta,\theta_n}(\xi^n) \leq \xi + \varepsilon$ a.s. Since this inequality holds for each $\varepsilon > 0$, we derive inequality (A.9). The proof is thus complete. $\square$

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