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A game theoretic approach to martingale measures in incomplete markets

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Abstract

We consider a stochastic differential game in a financial jump diffusion market, where the agent chooses a portfolio which maximizes the utility of her terminal wealth, while the market chooses a scenario (represented by a probability measure) which minimizes this maximal utility. We show that the optimal strategy for the market is to choose an equivalent martingale measure.

1 Introduction

When pricing derivatives in a financial market (not necessarily complete), it is common to apply no-arbitrage arguments to show that the price has to be given by an expectation of the discounted payoff of the derivative, the expectation taken with respect to some equivalent martingale (or *risk free*) measure P_0. Such a measure P_0 can then be found by using the Girsanov theorem. However, if the market is incomplete the measure P_0 is not unique, and the no-arbitrage argument gives no information about which measure to use.

The purpose of this paper is to put the pricing question into the framework of a stochastic differential game:

We represent the traders by a representative agent with a given utility function U. This agent is player number 1. Player number 2 is the market itself. Player number 1 chooses a portfolio which *maximizes* her expected

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discounted utility, the expectation being taken with respect to a probability measure Q. The measure Q represents the "scenario", which is chosen by the market. And the market chooses the scenario which *minimizes* the maximal expected utility of the representative agent. This leads to a min-max problem. We put this problem into a stochastic differential game framework by representing the market prices by a jump diffusion $S(t)$ and the scenarios by a parametrized family $\{Q_\theta\}_{\theta\in\Theta}$ of the probability measures. Then we show that the solution of this game is for the market to choose a risk free measure $Q_{\hat{\theta}}$ and for the agent to put all the money in the risk free asset. Thus the use of a risk free measure by the market appears as an equilibrium point in this game.

In the next section we explain this in more detail.

The problem studied in this paper is related to some "worst case scenario" problems studied in the literature. See e.g. [BMS], [ES], [G], [KM] and [S]. For more information about differential games we refer to [FS], [FlSo], [I] and [KS]. Our Theorem 2.1 may be regarded as a generalization of Example 3.1 in [MØ], which again is an extension of a result in [PS].

2 The stochastic differential game model

Consider the following jump diffusion market

(2.1) (risk free asset) $$dS_0(t) = r(t)S_0(t)dt; \quad S_0(0) = 1$$

(2.2) (risky asset) $$dS_1(t) = S_1(t^-)\Big[\alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\Big]; \quad S_1(0) > 0,$$

where $B(t)$ and $\tilde{N}(dt,dz)$ is a Brownian motion and a compensated Poisson random measure, respectively, on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$. Here $r(t), \beta(t)$ and $\gamma(t,z)$ are given $\mathcal{F}_t$-adapted processes, satisfying the following integrability condition:

$$E\Big[\int_0^T \{|r(s)| + |\alpha(s)] + \tfrac{1}{2}\beta^2(s) + \int_{\mathbb{R}} |\log(1+\gamma(s,z)) - \gamma(s,z)|\nu(dz)\}ds\Big] < \infty \tag{2.3}$$

where $T > 0$ is a fixed given constant. We also assume that

$$\gamma(s,z) \geq -1 \qquad \text{for a.a. } s,z \in [0,T]\times\mathbb{R}_0, \tag{2.4}$$

where $\mathbb{R}_0 = \mathbb{R}\setminus\{0\}$.

Then it is well-known (see e.g. [ØS, Ch. 1]) that the solution $S_1(t)$ of (2.2) is

$$S_1(t)=S_1(0)\exp\Big[\int_0^t\{\alpha(s)-\tfrac{1}{2}\beta^2(s)+\int_{\mathbb{R}}(\log(1+\gamma(s,z))-\gamma(s,z))\nu(dz)\}ds$$

$$+\int_0^t\beta(s)dB(s)+\int_0^t\int_{\mathbb{R}}\gamma(s,z)\tilde{N}(ds,dz)\Big];\quad t\in[0,T]. \tag{2.5}$$

We now introduce a family $\mathcal{M}$ of measures Q_θ parametrized by *processes* $\theta=(\theta_0(t),\theta_1(t,z))$ such that

$$dQ_\theta(\omega)=Z_\theta(T)dP(\omega)\quad\text{on } \mathcal{F}_T, \tag{2.6}$$

where

$$\begin{cases} dZ_\theta(t)=Z_\theta(t^-)[-\theta_0(t)dB(t)-\int_{\mathbb{R}}\theta_1(t,z)\tilde{N}(dt,dz)];\quad 0\le t\le T\\ Z_\theta(0)=1.\end{cases} \tag{2.7}$$

We assume that

$$\theta_1(t,z)\le 1\qquad\text{for a.a. } t,z,\omega \text{ and} \tag{2.8}$$

$$\int_0^T\{\theta_0^2(s)+\int_{\mathbb{R}}\theta_1^2(s,z)\nu(dz\}ds<\infty\ \text{ a.s.} \tag{2.9}$$

Then the solution of (2.7) is given by

$$Z_\theta(t)=\exp\Big[-\int_0^t\theta_0(s)dB(s)-\tfrac{1}{2}\int_0^t\theta_0^2(s)ds$$

$$+\int_0^t\int_{\mathbb{R}}\log(1-\theta_1(s,z))\tilde{N}(ds,dz)$$

$$+\int_0^t\int_{\mathbb{R}}\{\log(1-\theta_1(s,z))+\theta_1(s,z)\}\nu(dz)ds\Big];\quad 0\le t\le T \tag{2.10}$$

If

$$E[Z_\theta(T)]=1 \tag{2.11}$$

then $Q_\theta(\Omega)=1$, i.e. Q_θ is a probability measure.

If, in addition, $\theta_0(t)$ and $\theta_1(t,z)$ satisfy the equation

$$\beta(t)\theta_0(t) + \int_{\mathbb{R}} \gamma(t,z)\theta_1(t,z)\nu(dz) = \alpha(t) - r(t); \qquad t \in [0,T] \tag{2.12}$$

then the measure Q_θ is an *equivalent local martingale measure*. See e.g. [ØS, Ch. 1].

We do *not* assume a priori that (2.12) holds. The set of all $\theta = (\theta_0, \theta_1)$ such that (2.8)–(2.11) hold is denoted by Θ. These are the *admissible controls of the market*.

Next we introduce a *portfolio* in this market, represented by the *fraction* $\pi(t)$ of the wealth invested in the risky asset at time t. We assume that $\pi(t)$ is self-financing, which means that the corresponding wealth process $X^{(\pi)}(t)$ will have the dynamics

$$\begin{aligned} dX^{(\pi)}(t) &= X^{(\pi)}(t^-)\Big[\{r(t) + (\alpha(t) - r(t))\pi(t)\}dt \\ &\quad + \beta(t)\pi(t)dB(t) + \pi(t)\int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\Big]; \quad X^{(\pi)}(0) = x > 0. \end{aligned} \tag{2.13}$$

We assume that $\pi(t)\gamma(t,z) \geq -1$ a.s. and

$$\begin{aligned} &\int_0^T \{|r(s)| + |\alpha(s) - r(s)|\,|\pi(s)| + \beta^2(s)\pi^2(s) \\ &\quad + \pi^2(s)\int_{\mathbb{R}} \gamma^2(s,z)\nu(dz)\}ds < \infty \ \text{ a.s.} \end{aligned} \tag{2.14}$$

Then the solution of (2.13) is

$$\begin{aligned} X^{(\pi)}(t) &= x\exp\Big[\int_0^t \{r(s) + (\alpha(s) - r(s))\pi(s) - \tfrac{1}{2}\beta^2(s)\pi^2(s) \\ &\quad + \int_{\mathbb{R}} (\ln(1 + \pi(s)\gamma(s,z)) - \pi(s)\gamma(s,z))\nu(dz)\}ds \\ &\quad + \int_0^t \pi(s)\beta(s)dB(s) + \int_0^t\int_{\mathbb{R}} \ln(1 + \pi(s)\gamma(s,z))\tilde{N}(ds,dz)\Big]; \quad t \geq 0 \end{aligned} \tag{2.15}$$

The set of portfolios above is denoted by $\mathcal{A}$. Fix a utility function $U : [0,\infty) \to [-\infty,\infty)$, assumed to be increasing, concave and twice continuously differentiable on $(0,\infty)$.

Consider the following stochastic differential game between the *representative agent* and the *market*: Given the *scenario* represented by the measure

Q_θ, the agent chooses the portfolio which maximizes the Q_θ-expected utility of her terminal wealth. The market reacts to this choice by choosing the scenario Q_θ which minimizes this maximal expected utility. This can be expressed as the *zero-sum stochastic differential game* to find $\Phi(s, y_1, y_2)$ and $\theta^* \in \Theta$, $\pi^* \in \mathcal{A}$ such that

$$\Phi(s, y_1, y_2) = \inf_{\theta \in \Theta} \Big(\sup_{\pi \in \mathcal{A}} E_{Q_\theta}[U(X^{(\pi)}(T-s))] \Big) = E_{Q_{\theta^*}}[U(X^{(\pi^*)}(T-s))]. \tag{2.16}$$

Here $s = Y_0(0)$, $y_1 = Y_1(0)$, $y_2 = Y_2(0)$ are the initial values of the process $Y(t) = Y^{\theta,\pi}(t) \in \mathbb{R}^3$ given by

$$\begin{aligned} dY(t) &= \begin{bmatrix} dY_0(t) \\ dY_1(t) \\ dY_2(t) \end{bmatrix} = \begin{bmatrix} dt \\ dZ_\theta(t) \\ dX^{(\pi)}(t) \end{bmatrix} = (dt, dZ_\theta(t), dX^{(\pi)}(t))^T \\ &= (1, 0, X(t)[r(t) + (\alpha(t) - r(t))\pi(t)])^T dt \\ &\quad + (0, -\theta_0(t) Z_\theta(t), X(t)\pi(t)\beta(t))^T dB(t) \\ &\quad + \int_{\mathbb{R}} (0, -Z_\theta(t^-)\theta_1(t,z), X(t^-)\pi(t)\gamma(t,z))^T \tilde{N}(dt, dz). \end{aligned} \tag{2.17}$$

We assume from now on that $r(t)$ is *deterministic* and that $\alpha(t) = \alpha(Y(t))$, $\beta(t) = \beta(Y(t))$, $\gamma(t,z) = \gamma(Y(t), z)$, $\pi(t) = \pi(Y(t))$ and $\theta(t) = (\theta_0(Y(t)),$ $\theta_1(Y(t), z))$ are Markovian. Thus we identify π with a map $\pi : \mathbb{R}^3 \to \mathbb{R}$ and we identify θ with a map $\theta = (\theta_0, \theta_1(\cdot)) : \mathbb{R}^3 \to \mathbb{R} \times \mathbb{R}^{\mathbb{R}}$ (feedback controls).

Then $Y^{\theta,\pi}(t)$ is a Markov process with generator $A^{\theta,\pi}$ given by

$$\begin{aligned} A^{\theta,\pi}\varphi(s, y_1, y_2) &= \tfrac{\partial \varphi}{\partial s} + y_2(r + (\alpha - r)\pi)\tfrac{\partial \varphi}{\partial y_2} \\ &\quad + \tfrac{1}{2}\theta_0^2 y_1^2 \tfrac{\partial^2 \varphi}{\partial y_1^2} + \tfrac{1}{2} y_2^2 \pi^2 \beta^2 \tfrac{\partial^2 \varphi}{\partial y_2^2} - \theta_0 \pi y_1 y_2 \beta \tfrac{\partial^2 \varphi}{\partial y_1 \partial y_2} \\ &\quad + \int_{\mathbb{R}} \{\varphi(s, y_1 - y_1\theta_1(\cdot, z), y_2 + y_2\pi\gamma(\cdot, z)) - \varphi(s, y_1, y_2) \\ &\quad + y_1\theta_1(\cdot, z)\tfrac{\partial \varphi}{\partial y_1} - y_2\pi\gamma(\cdot, z)\tfrac{\partial \varphi}{\partial y_2}\}\nu(dz) \quad \text{for } \varphi \in C^{1,2,2}(\mathbb{R}^3). \end{aligned} \tag{2.18}$$

To solve the problem (2.16) we apply the Hamilton-Jacobi-Bellman (HJB) equation for stochastic differential games given in [MØ]. Applied to our setting this HJB gets the following form:

Theorem 2.1 ([MØ]) *Put $\mathcal{S} = (0,T) \times (0,\infty) \times (0,\infty)$, $y = (y_0, y_1, y_2) = (s, y_1, y_2)$. Suppose there exists a function $\varphi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$ and a Markov control $(\hat{\theta}(y), \hat{\pi}(y))) \in \Theta \times \mathcal{A}$ such that*

(i) $A^{\theta,\hat{\pi}(y)}\varphi(y) \geq 0$ for all $\theta \in \mathbb{R} \times \mathbb{R}^{\mathbb{R}}$, $y \in \mathcal{S}$

(ii) $A^{\hat{\theta}(y),\pi}\varphi(y) \leq 0$ for all $\pi \in \mathbb{R}$, $y \in \mathcal{S}$

(iii) $A^{\hat{\theta}(y),\hat{\pi}(y)}\varphi(y) = 0$ *for all* $y \in \mathcal{S}$

(iv) $\lim_{t\to T^-} \varphi(Y^{\theta,\pi}(t)) = Y_1^{\theta,\pi}(T)U(Y_2^{\theta,\pi}(T))$

(v) the family $\{\ \varphi(Y^{\theta,\pi}(\tau))\}_{\tau\in\mathcal{T}}$ *is uniformly integrable for all* $y \in \mathcal{S}$*, where* $\mathcal{T}$ *is the set of stopping times* $\tau \leq T$.

Then, with $J^{\theta,\pi}(y) = E_{Q_\theta}[U(X^{(\pi)}(T-s))]$,

$$\begin{aligned}\varphi(y) = \Phi(y) &= \inf_{\theta\in\Theta}\Big(\sup_{\pi\in\Pi} J^{\theta,\pi}(y)\Big) = \sup_{\pi\in\Pi}\Big(\inf_{\theta\in\Theta} J^{\theta,\pi}(y)\Big)\\ &= \sup_{\pi\in\Pi} J^{\hat{\theta},\pi}(y) = \inf_{\theta\in\Theta} J^{\theta,\hat{\pi}}(y) = J^{\hat{\theta},\hat{\pi}}(y); \qquad y \in \mathcal{S}\end{aligned}$$

and $(\hat{\theta},\hat{\pi})$ *is an optimal (Markov) control.*

We guess that φ has the form

$$\varphi(s, y_1, y_2) = y_1 U(f(s)y_2) \tag{2.19}$$

for some deterministic function f with $f(T) = 1$ (motivated by (iv)).

Note that conditions (i)–(iii) in Theorem 2.1 can be written

$$\inf_{\theta} A^{\theta,\hat{\pi}}\varphi(y) = A^{\hat{\theta},\hat{\pi}}\varphi(y) = 0$$

and

$$\sup_{\pi} A^{\hat{\theta},\pi}\varphi(y) = A^{\hat{\theta},\hat{\pi}}\varphi(y) = 0.$$

Maximizing $A^{\hat{\theta},\pi}\varphi(s, y_1, y_2)$ over all π gives the following first order condition for a *maximum* point $\hat{\pi}$:

$$\begin{aligned}&y_2(\alpha - r(s))y_1U'(f(s)y_2)f(s) + y_2^2\hat{\pi}\beta^2(y)y_1U''(f(s)y_2)f^2(s)\\ &\quad - \hat{\theta}_0 y_1 y_2\beta(y)U'(f(s)y_2)f(s)\\ &\quad + \int_{\mathbb{R}}\{(y_1 - y_1\hat{\theta}_1(y,z))U'(f(s)(y_2 + y_2\hat{\pi}\gamma(y,z)))f(s)y_2\gamma(y,z)\\ &\quad - y_2\gamma(y,z)y_1U'(f(s)y_2)f(s)\}\nu(dz) = 0,\end{aligned}$$

i.e.

$$\begin{aligned}&(\alpha - r(s))U'(f(s)y_2) + y_2\hat{\pi}\beta^2(y)U''(f(s)y_2)f(s) - \hat{\theta}_0\beta(y)U'(f(s)y_2)\\ &\quad + \int_{\mathbb{R}}\{(1 - \hat{\theta}_1(y,z))U'(f(s)y_2(1 + \hat{\pi}\gamma(y,z)))\\ &\quad - U'(f(s)y_2)\}\gamma(y,z)\nu(dz) = 0.\end{aligned} \tag{2.20}$$

We then minimize $A^{\theta,\hat{\pi}}\varphi(s, y_1, y_2)$ over all $\theta = (\theta_0, \theta_1)$ and get the following first order conditions for a *minimum* point $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1)$:

$$-\hat{\pi} y_1 y_2 \beta(y) U'(f(s)y_2) f(s) = 0 \tag{2.21}$$

and

$$\int_{\mathbb{R}} \{-y_1 U(f(s)y_2(1 + \hat{\pi}\gamma(y,z))) + y_1 U(f(s)y_2)\}\nu(dz) = 0. \tag{2.22}$$

From (2.21) we conclude that

$$\hat{\pi} = 0, \tag{2.23}$$

which substituted into (2.20) gives

$$\begin{aligned}(\alpha - r(s))U'(f(s)y_2) &- \hat{\theta}_0 \beta(y) U'(f(s)y_2) \\ &+ \int_{\mathbb{R}} \{-\hat{\theta}_1(y,z)\gamma(y,z)U'(f(s)y_2)\}\nu(dz) = 0\end{aligned}$$

or

$$\hat{\theta}_0(y)\beta(y) + \int_{\mathbb{R}} \hat{\theta}_1(y,z)\gamma(y,z)\nu(dz) = \alpha(y) - r(s). \tag{2.24}$$

The HJB equation for stochastic differential games states that with these values of $\hat{\pi}$ and $\hat{\theta}$ we should have

$$A^{\hat{\theta},\hat{\pi}}\varphi(s, y_1, y_2) = 0$$

i.e.

$$\begin{aligned}y_1 U'(f(s)y_2) y_2 f'(s) &+ y_2 r(s) y_1 U'(f(s)y_2) f(s) \\ &+ \int_{\mathbb{R}} \{y_1(1 - \theta_1(y,z))U(f(s)y_2) - y_1 U(f(s)y_2) \\ &+ y_1 \hat{\theta}_1 U(f(s)y_2)\}\nu(dz) = 0\end{aligned}$$

or

$$f'(s) + r(s)f(s) = 0$$

i.e.

$$f(s) = \exp\Big(\int_0^{T-s} r(u)du\Big). \tag{2.25}$$

We have proved:

Theorem 2.2 *Let $U \in C([0,\infty)) \cap C^1((0,\infty))$ be concave and increasing. Then the solution of the stochastic differential game (2.16) is for the agent to choose the portfolio*

$$\pi(t) = \hat{\pi}(t) = 0 \tag{2.26}$$

(i.e. to put all the wealth in the risk free asset) and for the market to choose the scenario $Q_{\hat{\theta}}$ where $\hat{\theta} = (\hat{\theta}_0, \hat{\theta}_1)$ satisfies the equation

$$\hat{\theta}_0(Y(t))\beta(Y(t)) + \int_{\mathbb{R}} \hat{\theta}_1(Y(t),z)\gamma(Y(t),z)\nu(dz) = \alpha(Y(t)) - r(t). \tag{2.27}$$

In other words, the market chooses an equivalent martingale measure *(or* risk free *measure) $Q_{\hat{\theta}}$.*

Remark 2.3 Note that there is no no-arbitrage principle used in this paper. In stead, the choice of a scenario represented by an *equivalent martingale measure* is deduced as an equilibrium state of a game between a representative agent and the market.

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