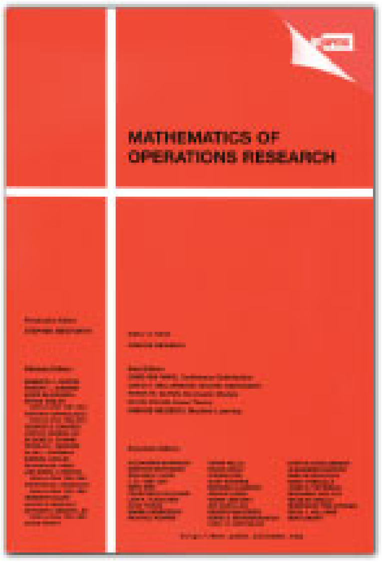

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Singular Control and Optimal Stopping of SPDEs, and Backward SPDEs with Reflection

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We consider general singular control problems for random fields given by a stochastic partial differential equation (SPDE). We show that under some conditions the optimal singular control can be identified with the solution of a coupled system of SPDE and a *reflected backward* SPDE (RBSPDE). As an illustration we apply the result to a singular optimal harvesting problem from a population whose density is modeled as a stochastic reaction-diffusion equation. Existence and uniqueness of solutions of RBSPDEs are established, as well as comparison theorems. We then establish a relation between RBSPDEs and optimal stopping of SPDEs, and apply the result to a *risk-minimizing stopping problem*.

Key words: stochastic partial differential equations (SPDEs); singular control of SPDEs; maximum principles; comparison theorem for SPDEs; reflected SPDEs; optimal stopping of SPDEs
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1. Introduction. As a motivation for the problem studied here we consider a problem of optimal harvesting from a fish population in a lake, D. Suppose the density $Y(t,x)$ of the population at time $t\in[0,T]$ and at the point x is given by a stochastic reaction-diffusion equation of the form

$$\begin{aligned} dY(t,x) &= [\Delta Y(t,x)+\alpha Y(t,x)]dt + Y(t,x)\beta dB(t) - \lambda_0\xi(dt,x); \quad (t,x)\in(0,T)\times D,\\ Y(0^-,x) &= y_0(x)>0; \quad x\in D, \\ Y(t,x) &= 0; \quad (t,x)\in(0,T)\times\partial D, \end{aligned} \tag{1}$$

where D is a bounded domain in $\mathbb{R}^d$ and $y_0(x)$ is a given bounded deterministic function. Here $B(t)=B_t$, $t\ge 0$, is an m-dimensional Brownian motion on a filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$; α, $\lambda_0>0$ are given constants; β is a given m-dimensional vector; and $\Delta := \sum_{i=1}^d \partial^2/\partial x_i^2$ is the Laplacian differential operator. We may regard $\xi(dt,x)$ as the harvesting effort rate and $\lambda_0>0$ as the harvesting efficiency coefficient. The performance coefficient is assumed to be

$$J(\xi)=E\left[\int_D\int_0^T h_0(t,x)Y(t,x)\xi(dt,x)\,dx + \int_D h_0(T,x)Y(T,x)\,dx\right], \tag{2}$$

where $h_0(t,x)>0$ is the net unit price of the fish and $T>0$ is a fixed terminal time. Thus $J(\xi)$ represents the expected total net income from the harvesting. The problem is to maximize $J(\xi)$ over all admissible harvesting strategies $\xi(t,x)$. We say that ξ is admissible and write $\xi\in\mathcal{A}$ if $\xi(t,x)$ is $\mathcal{F}_t$-adapted, nondecreasing in t and $\xi(0,x)=0$ for each x. In this example we also require that $Y(t,x)\ge 0$ for all $(t,x)\in[0,T]\times D$.

This optimal harvesting problem is a special case of a general singular control problem of stochastic partial differential equations (SPDE) driven by a multiplicative noise of finite dimension. The aim of this paper is to study these problems. In particular, we want to establish stochastic maximum principles providing optimality conditions, and to study relations with some associated reflected backward SPDEs.

It is wellknown that the stochastic maximum principle method for solving a stochastic control problem for SPDEs involves a backward SPDE for the adjoint processes $p(t,x)$, $q(t,x)$ (see Øksendal et al. [15]). We will show that in the case of a *singular* control problem for SPDE we arrive at a BSPDE *with reflection* for the adjoint processes.

Several papers are devoted to the study of backward SPDEs (without reflection) and maximum principles of SPDEs; see, e.g., Bensoussan [5], Hu and Peng [10], Hu and Peng [9], Hu et al. [11], Guatteri and Masiero [7]. In a finite-dimensional setup, maximum principles for singular stochastic control problems have been studied in Andersson [1], Bahlali and Mezerdi [2], Bahlali et al. [3], Bahlali et al. [4], and in the recent paper by Øksendal and Sulem [14], where connections between singular stochastic control, reflected BSDEs under partial information are also established. For the study of SPDEs with reflection, we refer to Donati-Martin and Pardoux [6], Haussmann and Pardoux [8], Nualart and Pardoux [13], Zhang [19].

The paper is organized as follows: In §2, we study a class of *singular* control problems for SPDEs and prove a maximum principle for the solution of such problems. This maximum principle leads to an adjoint equation which is a *reflected* backward stochastic partial differential equation. Both the necessary and sufficient properties of the maximum principle are discussed and, similar to the finite-dimensional case, the sufficient condition is established under suitable concavity properties of the coefficients.

As an illustration, at the end of §2 we apply the result to the singular optimal harvesting problem above. In §3, we study existence and uniqueness of solutions of backward stochastic partial differential equations (BSPDEs) with reflection, and in §4 we establish comparison theorems for BSPDEs and reflected BSPDEs. In §5, we establish connections between reflected BSPDEs and optimal stopping of SPDEs, and in §6 we consider an application to a *risk-minimizing stopping* problem in a market with mean-field interactions.

2. Singular control of SPDEs. Let D be a regular domain in $\mathbb{R}^d$. Denote by $a(x)=(a_{ij}(x))$ a matrix-valued function on $\mathbf{R}^d$ satisfying the uniform ellipticity condition:

$$\frac{1}{c}I_d \le a(x) \le cI_d \quad \text{for some constant } c\in(0,\infty).$$

Let $b(x)$ be a vector field on D with $b\in L^p(D)$ for some $p>d$, and let $q(x)$ be a measurable real-valued function on D such that $q\in L^{p_1}(D)$ for some $p_1>d/2$. Introduce the following second-order partial differential operator:

$$Au(x) = -\operatorname{div}(a(x)\nabla u(x)) + b(x)\cdot\nabla u(x) + q(x)u(x).$$

Suppose the state equation is an SPDE of the form

$$\begin{aligned} dY(t,x) &= \{AY(t,x)+b(t,x,Y(t,x))\}dt + \sigma(t,x,Y(t,x))dB(t)\\ &\quad + \lambda(t,x,Y(t,x))\xi(dt,x); \quad (t,x)\in[0,T]\times D, \qquad (3)\\ Y(0^-,x) &= y_0(x); \quad x\in D,\\ Y(t,x) &= 0; \quad (t,x)\in(0,T)\times\partial D. \qquad (4)\end{aligned}$$

Here $y_0\in K:=L^2(D)$ and $y_1\in L^2([0,T]\times D)$ are given functions. We assume that b, σ, and λ are C^1 with respect to y. Let $V:=W_0^{1,2}(D)$ be the Sobolev space of order one with zero boundary condition. Then Y is understood as a weak (variational) solution to (3), in the sense that $Y\in C([0,T];K)\cap L^2([0,T];V)$ and for $\phi\in C_0^\infty(D)$,

$$\begin{aligned}\langle Y(t,\cdot),\phi\rangle_K &= \langle y_0(\cdot),\phi\rangle_K + \int_0^t\langle Y(s,\cdot),A^*\phi\rangle\,ds + \int_0^t\langle b(s,\cdot,Y(s,\cdot)),\phi\rangle_K\,ds\\ &\quad + \int_0^t\langle\sigma(s,\cdot,Y(s,\cdot)),\phi\rangle_K\,dB(s), \qquad (5)\end{aligned}$$

where A^* is the adjoint operator of A, and $\langle,\rangle$ denotes the dual pairing between the space V and its dual V^*. Under this framework the Itô formula can be applied to such SPDEs. See Pardoux [16], Prévôt and Röckner [17]. The *performance functional* is given by

$$\begin{aligned}J(\xi) := E\bigg[&\int_D\int_0^T f(t,x,Y(t,x))\,dt\,dx + \int_D g(x,Y(T,x))\,dx\\ &+ \int_D\int_0^T h(t,x,Y(t,x))\xi(dt,x)\,dx\bigg], \qquad (6)\end{aligned}$$

where $f(t,x,y)$, $g(x,y)$, and $h(t,x,y)$ are bounded measurable functions that are differentiable in the argument y and continuous w.r.t. t.

We want to maximize $J(\xi)$ over all $\xi \in \mathcal{A}$, where $\mathcal{A}$ is the set of all adapted processes $\xi(t,x)$, that are nondecreasing and left-continuous w.r.t. t for all x, $\xi(0,x)=0$, $\xi(T,x)<\infty$ and such that the performance functional is finite. We call $\mathcal{A}$ the set of admissible singular controls. Thus we want to find $\xi^* \in \mathcal{A}$ (called an optimal control) such that

$$\sup_{\xi\in\mathcal{A}} J(\xi) = J(\xi^*).$$

We study this problem by using an extension to SPDEs of the maximum principle in Øksendal and Sulem [14]: Define the *Hamiltonian* H by

$$H(t,x,y,p,q)(dt,\xi(dt,x)) := \{f(t,x,y)+b(t,x,y)p+\sigma(t,x,y)q\}dt + \{\lambda(t,x,y)p+h(t,x,y)\}\xi(dt,x). \tag{7}$$

To this Hamiltonian we associate the following *backward* SPDE (BSPDE) in the unknown process $(p(t,x),q(t,x))$:

$$dp(t,x) = -\left\{A^*p(t,x)dt + \frac{\partial H}{\partial y}(t,x,Y(t,x),p(t,x),q(t,x))(dt,\xi(dt,x))\right\} + q(t,x)dB(t); \quad (t,x)\in(0,T)\times D, \tag{8}$$

$$p(T,x) = \frac{\partial g}{\partial y}(x,Y(T,x)); \quad x\in D, \tag{9}$$

$$p(t,x)=0; \quad (t,x)\in(0,T)\times\partial D. \tag{10}$$

Here A^* denotes the adjoint of the operator A. We assume that a unique solution $p(t,x)$, $q(t,x)$ of (8)–(10) exists for each $\xi\in\mathcal{A}$.

Theorem 1 (Sufficient Maximum Principle for Singular Control of SPDE). *Let $\hat{\xi}\in\mathcal{A}$ with corresponding solutions $\hat{Y}(t,x)$, $\hat{p}(t,x)$, $\hat{q}(t,x)$. Assume that*

$$y\to h(x,y) \text{ is concave,} \tag{11}$$

$$(y,\xi)\to H(t,x,y,\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \text{ is concave,} \tag{12}$$

$$E\left[\int_D\left(\int_0^T\{(Y^\xi(t,x)-\hat{Y}(t,x))^2\hat{q}^2(t,x)+\hat{p}^2(t,x)(\sigma(t,x,Y^\xi(t,x))-\sigma(t,x,\hat{Y}(t,x)))^2\}\,dt\right)dx\right]<\infty, \quad \text{for all } \xi\in\mathcal{A}. \tag{13}$$

Moreover, assume that the following maximum condition holds:

$$\hat{\xi}(dt,x)\in \arg\max_{\xi\in\mathcal{A}} H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)); \tag{14}$$

i.e.,

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\xi(dt,x) \le \{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \quad \text{for all } \xi\in\mathcal{A}. \tag{15}$$

Then $\hat{\xi}$ is an optimal singular control.

Remark. By saying that $(y,\xi)\to H(y,\xi)$ is concave, we mean that for all $a\in[0,1]$ and all (y,ξ), $(\bar{y},\bar{\xi})$, we have $H(a(y,\xi)+(1-a)(\bar{y},\bar{\xi}))\ge aH(y,\xi)+(1-a)H(\bar{y},\bar{\xi})$. Since H is $\mathcal{C}^1$, this is equivalent to

$$H(y,\xi)-H(\bar{y},\bar{\xi}) \le \frac{\partial H}{\partial y}(\bar{y},\bar{\xi})(y-\bar{y}) + \nabla_\xi H(\bar{y},\bar{\xi})(\xi-\bar{\xi}),$$

where $\nabla_\xi H(\bar{y},\bar{\xi})$ is the Fréchet derivative of H with respect to ξ and $\nabla_\xi H(\bar{y},\bar{\xi})(\xi-\bar{\xi})$ is the result of applying this linear operator to $\xi-\bar{\xi}$.

Proof of Theorem 1. Choose $\xi \in \mathcal{A}$ and put $Y = Y^{\xi}$. Then by (6) we can write

$$J(\xi) - J(\hat{\xi}) = I_1 + I_2 + I_3, \tag{16}$$

where

$$I_1 = E\left[\int_0^T \int_D \{f(t,x,Y(t,x)) - f(t,x,\hat{Y}(t,x))\}\,dx\,dt\right], \tag{17}$$

$$I_2 = E\left[\int_D \{g(x,Y(T,x)) - g(x,\hat{Y}(T,x))\}\,dx\right], \tag{18}$$

$$I_3 = E\left[\int_0^T \int_D \{h(t,x,Y(t,x))\xi(dt,x) - h(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}\right]. \tag{19}$$

By our definition of H we have

$$\begin{aligned} I_1 = E\bigg[&\int_0^T \int_D \{H(t,x,Y(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \\ &- H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\hat{\xi}(dt,x))\} \\ &- \int_0^T \int_D \{b(t,x,Y(t,x)) - b(t,x,\hat{Y}(t,x))\}\hat{p}(t,x)\,dx\,dt \\ &- \int_0^T \int_D \{\sigma(t,x,Y(t,x)) - \sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)\,dx\,dt \\ &- \int_0^T \int_D \hat{p}(t,x)\{\lambda(t,x,Y(t,x))\xi(dt,x) - \lambda(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}\,dx \\ &- \int_0^T \int_D \{h(t,x,Y(t,x))\xi(dt,x) - h(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}\,dx\bigg]. \end{aligned} \tag{20}$$

By (13) and concavity of g we have, with $\tilde{Y} = Y - \hat{Y}$,

$$\begin{aligned} I_2 &\le E\left[\int_D \frac{\partial g}{\partial y}(x,\hat{Y}(T,x))(Y(T,x) - \hat{Y}(T,x))\,dx\right] = E\left[\int_D \hat{p}(T,x)\tilde{Y}(T,x)\,dx\right] \\ &= E\bigg[\int_D \int_0^T \tilde{Y}(t,x)\,d\hat{p}(t,x)\,dx + \int_D \int_0^T \hat{p}(t,x)\,d\tilde{Y}(t,x)\,dx \\ &\qquad + \int_D \int_0^T \{\sigma(t,x,Y(t,x)) - \sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)\,dt\,dx\bigg] \\ &= E\bigg[\int_D \int_0^T \tilde{Y}(t,x)\left\{-A^*\hat{p}(t,x)\,dt - \frac{\partial H}{\partial y}(t,x,\hat{Y},\hat{p},\hat{q})(dt,\hat{\xi}(dt,x))\right\}dx \\ &\qquad + \int_D \int_0^T \hat{p}(t,x)\{A\tilde{Y}(t,x) + b(t,x,Y(t,x)) - b(t,x,\hat{Y}(t,x))\}\,dt\,dx \\ &\qquad + \int_D \int_0^T \hat{p}(t,x)\{\lambda(t,x,Y(t,x))\xi(dt,x) - \lambda(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}\,dx \\ &\qquad + \int_D \int_0^T \{\sigma(t,x,Y(t,x)) - \sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)\,dt\,dx\bigg]. \end{aligned} \tag{21}$$

The rigorous meaning of the expressions $\int_D \int_0^T \tilde{Y}(t,x)A^*\hat{p}(t,x)\,dt\,dx$, and $\int_D \int_0^T \hat{p}(t,x)A\tilde{Y}(t,x)\,dt\,dx$ are

$$\int_D \int_0^T \tilde{Y}(t,x)A^*\hat{p}(t,x)\,dt\,dx = \int_0^T \langle \tilde{Y}(t,\cdot), A^*\hat{p}(t,\cdot)\rangle\,dt,$$

$$\int_D \int_0^T \hat{p}(t,x)A\tilde{Y}(t,x)\,dt\,dx = \int_0^T \langle \hat{p}(t,\cdot), A\tilde{Y}(t,\cdot)\rangle\,dt,$$

where $\langle , \rangle$ stands for the dual pairing between the space $V = H_0^{1,2}(D)$ and its dual V^*.

In view of $\langle \tilde{Y}(t,\cdot), A^*\hat{p}(t,\cdot)\rangle = \langle \hat{p}(t,\cdot), A\tilde{Y}(t,\cdot)\rangle$, combining (16)–(21) and concavity of H, we have

$$\begin{aligned} J(\xi)-J(\hat{\xi}) &\le E\bigg[\int_D\int_0^T \Big\{H(t,x,Y(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \\ &\qquad - H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\hat{\xi}(dt,x)) - \tilde{Y}(t,x)\frac{\partial H}{\partial y}(t,x,\hat{Y},\hat{p},\hat{q})(dt,\hat{\xi}(dt,x))\Big\}\,dx\bigg] \\ &\le \bigg[\int_D\int_0^T \nabla_\xi H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(\xi(dt,x)-\hat{\xi}(dt,x))\,dx\bigg] \\ &= E\bigg[\int_D\int_0^T \{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}(\xi(dt,x)-\hat{\xi}(dt,x))\,dx\bigg] \\ &\le 0 \text{ by (15)}. \end{aligned}$$

This proves that $\hat{\xi}$ is optimal. □

For $\xi \in \mathcal{A}$ we let $\mathcal{V}(\xi)$ denote the set of adapted processes $\zeta(t,x)$ of finite variation w.r.t. t such that there exists $\delta = \delta(\xi) > 0$ (possibly depending on ζ) such that $\xi + y\zeta \in \mathcal{A}$ for all $y \in [0,\delta]$.

Proceeding as in Øksendal and Sulem [13] we prove the following useful result:

Lemma 1. *The inequality (15) is equivalent to the following two variational inequalities:*

$$\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x)) \le 0 \quad \textit{for all } t,\ x, \tag{22}$$

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) = 0 \quad \textit{for all } t,\ x. \tag{23}$$

Proof. (i). Suppose (15) holds. Choosing $\xi = \hat{\xi} + y\zeta$ with $\zeta \in \mathcal{V}(\hat{\xi})$ and $y \in (0,\delta(\hat{\xi}))$ we deduce that

$$\{\lambda(s,x,\hat{Y}(s,x))\hat{p}(s,x)+h(s,x,\hat{Y}(s,x))\}\zeta(ds,x) \le 0; \quad (s,x)\in(0,T)\times D \tag{24}$$

for all $\zeta \in \mathcal{V}(\hat{\xi})$. In particular, this holds if we fix $t \in (0,T)$ and put

$$\zeta(ds,x) = a(\omega)\delta_t(ds)\phi(x); \qquad (s,x,\omega)\in(0,T)\times D\times\Omega,$$

where $a(\omega) \ge 0$ is $\mathcal{F}_t$-measurable and bounded, $\phi(x) \ge 0$ is bounded, deterministic, and $\delta_t(ds)$ denotes the Dirac measure at t. Note that $\zeta \in \mathcal{V}(\xi)$. Then we get

$$\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x)) \le 0 \quad \text{for all } t,\ x \tag{25}$$

which is (22).

On the other hand, clearly $\zeta(dt,x) := \hat{\xi}(dt,x) \in \mathcal{V}(\hat{\xi})$ and this choice of ζ in (24) gives

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \le 0; \quad (t,x)\in(0,T)\times D. \tag{26}$$

Similarly, we can choose $\zeta(dt,x) = -\hat{\xi}(dt,x) \in \mathcal{V}(\hat{\xi})$, and this gives

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \ge 0; \quad (t,x)\in(0,T)\times D. \tag{27}$$

Combining (26) and (27) we get

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) = 0$$

which is (23). Together with (25) this proves (i).

(ii). Conversely, suppose (22) and (23) hold. Since $\xi(dt,x) \ge 0$ for all $\xi \in \mathcal{A}$ we see that (15) follows. □

We may formulate what we have proved as follows:

Theorem 2 (Sufficient Maximum Principle II). *Suppose the conditions of Theorem 1 hold. Suppose $\xi \in \mathcal{A}$, and that ξ together with its corresponding processes $Y^\xi(t,x)$, $p^\xi(t,x)$, $q^\xi(t,x)$ solve the coupled system consisting of the SPDE (3)–(4) together with the reflected backward SPDE (RBSPDE) given by*

$$dp^\xi(t,x) = -\left\{A^* p^\xi(t,x) + \frac{\partial f}{\partial y}(t,x,Y^\xi(t,x)) + \frac{\partial b}{\partial y}(t,x,Y^\xi(t,x))p^\xi(t,x) + \frac{\partial \sigma}{\partial y}(t,x,Y^\xi(t,x))q^\xi(t,x)\right\}dt$$
$$- \left\{\frac{\partial \lambda}{\partial y}(t,x,Y^\xi(t,x))p^\xi(t,x) + \frac{\partial h}{\partial y}(t,x,Y^\xi(t,x))\right\}\xi(dt,x) + q(t,x)dB(t); \quad (t,x) \in [0,T]\times D,$$
$$\lambda(t,x,Y^\xi(t,x))p^\xi(t,x) + h(t,x,Y^\xi(t,x)) \le 0; \quad \textit{for all } t,\ x,\ \textit{a.s.},$$
$$\{\lambda(t,x,Y^\xi(t,x))p^\xi(t,x) + h(t,x,Y^\xi(t,x))\}\xi(dt,x) = 0; \quad \textit{for all } t,\ x,\ \textit{a.s.},$$
$$p(T,x) = \frac{\partial g}{\partial y}(x,Y^\xi(T,x)); \quad x \in D,$$
$$p(t,x) = 0; \quad (t,x) \in (0,T)\times \partial D.$$

Then ξ maximizes the performance functional $J(\xi)$.

It is also of interest to have a maximum principle of "necessary type". To this end, we first prove some auxiliary results.

Lemma 2. *Let $\xi(dt,x) \in \mathcal{A}$ and choose $\zeta(dt,x) \in \mathcal{V}(\xi)$. Suppose that the derivative process*

$$\mathcal{Y}(t,x) = \lim_{y\to 0^+} \frac{1}{y}(Y^{\xi+y\zeta}(t,x) - Y^\xi(t,x)) \tag{28}$$

exists. Then $\mathcal{Y}$ satisfies the SPDE

$$\begin{aligned} d\mathcal{Y}(t,x) &= A\mathcal{Y}(t,x)dt + \mathcal{Y}(t,x)\bigg[\frac{\partial b}{\partial y}(t,x,Y(t,x))dt \\ &\quad + \frac{\partial \sigma}{\partial y}(t,x,Y(t,x))dB(t) + \frac{\partial \lambda}{\partial y}(t,x,Y(t,x))\xi(dt,x)\bigg] \\ &\quad + \lambda(t,x,Y(t,x))\zeta(dt,x); \quad (t,x)\in[0,T]\times D, \\ \mathcal{Y}(t,x) &= 0; \quad (t,x)\in(0,T)\times\partial D, \\ \mathcal{Y}(0,x) &= 0; \quad x \in D. \end{aligned} \tag{29}$$

Proof. By (3), we have:

$$\begin{aligned} &\lim_{y\to 0^+}\frac{1}{y}(Y^{\xi+y\zeta}(t,x) - Y^\xi(t,x)) \\ &= \lim_{y\to 0^+}\frac{1}{y}\bigg[\int_0^t A(Y^{\xi+y\zeta} - Y^\xi)(s,x)\,ds + \int_0^t (b(s,x,Y^{\xi+y\zeta}(s,x)) - b(s,x,Y^\xi(s,x)))\,ds \\ &\qquad + \int_0^t (\sigma(s,x,Y^{\xi+y\zeta}(s,x)) - \sigma(s,x,Y^\xi(s,x)))\,dB(s) \\ &\qquad + \int_0^t (\lambda(s,x,Y^{\xi+y\zeta}(s,x))(\xi(ds,x) + y\zeta(ds,x)) - \lambda(s,x,Y^\xi(s,x))\xi(ds,x))\bigg] \\ &= \int_0^t A\mathcal{Y}(s,x)\,ds + \int_0^t \frac{\partial b}{\partial y}(s,x,Y^\xi(s,x))\mathcal{Y}(s,x)\,ds + \int_0^t \frac{\partial \sigma}{\partial y}(s,x,Y^\xi(s,x))\mathcal{Y}(s,x)\,dB(s) \\ &\quad + \int_0^t \frac{\partial \lambda}{\partial y}(s,x,Y^\xi(s,x))\mathcal{Y}(s,x)\xi(ds,x) + \int_0^t \lambda(s,x,Y^\xi(s,x))\zeta(ds,x). \end{aligned}$$

By (4), we have

$$\mathcal{Y}(t,x) = 0; \quad (t,x)\in(0,T)\times\partial D,$$
$$\mathcal{Y}(0,x) = 0; \quad x\in D. \qquad \square$$

Remark. The existence of the limit in (28) is a nontrivial issue, and we do not discuss conditions for this in this paper. Here we simply assume that the limit exists. We refer to Prévôt and Röckner [17] for a study about this issue in a related setting.

Lemma 3. *Let $\xi(dt,x)\in\mathcal{A}$ and let $\zeta(dt,x)\in\mathcal{V}(\xi)$. Put $\eta=\xi+y\zeta$; $y\in[0,\delta(\xi)]$. Assume that*

$$E\bigg[\int_D\bigg(\int_0^T\{(Y^\eta(t,x)-Y^\xi(t,x))^2q^2(t,x) + p^2(t,x)(\sigma(t,x,Y^\eta(t,x))-\sigma(t,x,Y^\xi(t,x)))^2\}\,dt\bigg)\,dx\bigg]<\infty \quad \text{for all } y\in[0,\delta(\xi)], \tag{30}$$

where $(p(t,x),q(t,x))$ is the solution of (7)–(9) corresponding to $Y^\xi(t,x)$. Then

$$\lim_{y\to0^+}\frac{1}{y}(J(\xi+y\zeta)-J(\xi)) = E\bigg[\int_D\bigg(\int_0^T\{\lambda(t,x,Y(t,x))p(t,x)+h(t,x,Y(t,x))\}\zeta(dt,x)\bigg)\,dx\bigg]. \tag{31}$$

Proof. By (6) and (28), we have

$$\begin{aligned}\lim_{y\to0^+}\frac{1}{y}(J(\xi+y\zeta)-J(\xi)) &= E\bigg[\int_D\bigg\{\int_0^T\frac{\partial f}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)\,dt+\frac{\partial g}{\partial y}(x,Y(T,x))\mathcal{Y}(T,x)\bigg\}\,dx\\ &\quad+\int_D\int_0^T\frac{\partial h}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)\xi(dt,x)\,dx+\int_D\int_0^T h(t,x,Y(t,x))\zeta(dt,x)\,dx\bigg].\end{aligned} \tag{32}$$

By (7) we obtain

$$\begin{aligned}E\bigg[\int_D\int_0^T\frac{\partial f}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)\,dt\,dx\bigg] &= E\bigg[\int_D\bigg(\int_0^T\mathcal{Y}(t,x)\bigg\{\frac{\partial H}{\partial y}(dt,\xi(dt,x))-p(t,x)\frac{\partial b}{\partial y}(t,x)\,dt\\ &\quad-q(t,x)\frac{\partial\sigma}{\partial y}(t,x)dt-\bigg(p(t,x)\frac{\partial\lambda}{\partial y}(t,x)+\frac{\partial h}{\partial y}(t,x)\bigg)\xi(dt,x)\bigg\}\bigg)dx\bigg],\end{aligned} \tag{33}$$

where we have used the abbreviated notation

$$\frac{\partial H}{\partial y}(dt,\xi(dt,x))=\frac{\partial H}{\partial y}(t,x,Y(t,x),p(t,x),q(t,x))(dt,\xi(dt,x)),$$

etc.

By the Itô formula and (29) we see that

$$\begin{aligned}E\bigg[\int_D\frac{\partial g}{\partial y}(x)\mathcal{Y}(T,x)\,dx\bigg] &= E\bigg[\int_D p(T,x)\mathcal{Y}(T,x)\,dx\bigg]\\ &= E\bigg[\int_D\bigg(\int_0^T\{p(t,x)d\mathcal{Y}(t,x)+\mathcal{Y}(t,x)dp(t,x)\}+[p(\cdot,x),\mathcal{Y}(\cdot,x)](T)\bigg)\,dx\bigg]\\ &= E\bigg[\int_D\bigg(\int_0^T\bigg[p(t,x)\bigg\{A\mathcal{Y}(t,x)dt+\mathcal{Y}(t,x)\frac{\partial b}{\partial y}(t,x)dt\\ &\qquad+\mathcal{Y}(t,x)\frac{\partial\lambda}{\partial y}(t,x)\xi(dt,x)+\lambda(t,x)\zeta(dt,x)\bigg\}\\ &\qquad+\mathcal{Y}(t,x)\bigg\{-A^*p(t,x)dt-\frac{\partial H}{\partial y}(dt,\xi(dt,x))\bigg\}\\ &\qquad+\mathcal{Y}(t,x)\frac{\partial\sigma}{\partial y}(t,x)q(t,x)\bigg]\,dt\bigg)\,dx\bigg],\end{aligned} \tag{34}$$

where $[p(\cdot,x),\mathcal{Y}(\cdot,x)](t)$ denotes the covariation process of $p(\cdot,x)$ and $\mathcal{Y}(\cdot,x)$.

Since $p(t,x)=\mathcal{Y}(t,x)=0$ for $x\in\partial D$, we deduce that

$$\int_D p(t,x)A\mathcal{Y}(t,x)\,dx=\int_D A^*p(t,x)\mathcal{Y}(t,x)\,dx. \tag{35}$$

Therefore, substituting (33) and (34) into (32), we get

$$\lim_{y\to0^+}\frac{1}{y}(J(\xi+y\zeta)-J(\xi))$$
$$=E\bigg[\int_D\bigg(\int_0^T\{\lambda(t,x)p(t,x)+h(t,s)\}\zeta(dt,x)\bigg)\,dx\bigg]. \qquad \square$$

We can now state our necessary maximum principle:

Theorem 3 (Necessary Maximum Principle). (i) *Suppose $\xi^*\in\mathcal{A}$ is optimal, i.e., $\max_{\xi\in\mathcal{A}}J(\xi)=J(\xi^*)$. Let Y^*, (p^*,q^*) be the corresponding solution of (3)–(4) and (8)–(10), respectively, and assume that (30) holds with $\xi=\xi^*$. Then*

$$\lambda(t,x,Y^*(t,x))p^*(t,x)+h(t,x,Y^*(t,x))\le0\quad\textit{for all }(t,x)\in[0,T]\times D,\ \textit{a.s.}, \tag{36}$$

and

$$\{\lambda(t,x,Y^*(t,x))p^*(t,x)+h(t,x,Y^*(t,x))\}\xi^*(dt,x)=0\quad\textit{for all }(t,x)\in[0,T]\times D,\ \textit{a.s.} \tag{37}$$

(ii) *Conversely, suppose that there exists $\hat\xi\in\mathcal{A}$ such that the corresponding solutions $\hat Y(t,x)$, $(\hat p(t,x),\hat q(t,x))$ of (3)–(4) and (8)–(10), respectively, satisfy*

$$\lambda(t,x,\hat Y(t,x))\hat p(t,x)+h(t,x,\hat Y(t,x))\le0\quad\textit{for all }(t,x)\in[0,T]\times D,\ \textit{a.s.}, \tag{38}$$

and

$$\{\lambda(t,x,\hat Y(t,x))\hat p(t,x)+h(t,x,\hat Y(t,x))\}\hat\xi(dt,x)=0\quad\textit{for all }(t,x)\in[0,T]\times D,\ \textit{a.s.} \tag{39}$$

Then $\hat\xi$ is a directional sub-critical point for $J(\cdot)$, in the sense that

$$\lim_{y\to0^+}\frac{1}{y}(J(\hat\xi+y\zeta)-J(\hat\xi))\le0\quad\textit{for all }\zeta\in\mathcal{V}(\hat\xi). \tag{40}$$

Proof. This is proved in a similar way as in Theorem 2.4 in Øksendal and Sulem [14]. For completeness we give the details:

(i) If $\xi\in\mathcal{A}$ is optimal, we get by Lemma 3

$$0\ge\lim_{y\to0^+}\frac{1}{y}(J(\xi+y\zeta)-J(\xi))$$
$$=E\bigg[\int_D\int_0^T\{\lambda(t,x)p(t,x)+h(t,x)\}\zeta(dt,x)\,dx\bigg],\quad\text{for all }\zeta\in\mathcal{V}(\xi). \tag{41}$$

In particular, this holds if we choose ζ such that

$$\zeta(ds,x)=a(\omega)\delta_t(s)\phi(x) \tag{42}$$

for some fixed $t\in[0,T]$ and some bounded $\mathcal{F}_t$-measurable random variable $a(\omega)\ge0$, and some bounded, deterministic $\phi(x)\ge0$, where $\delta_t(s)$ is a Dirac measure at t. Then (41) gets the form

$$E\bigg[\int_D\{\lambda(t,x)p(t,x)+h(t,x)\}a(\omega)\phi(x)\,dx\bigg]\le0.$$

Since this holds for all such $a(\omega)$, $\phi(x)$ we deduce that

$$\lambda(t,x)p(t,x)+h(t,x)\le0\quad\text{for all }t,\ x,\ a.s. \tag{43}$$

Next, if we choose $\zeta(dt,x)=\xi(dt,x)\in\mathcal{V}(\xi)$, we get from (41)

$$E\bigg[\int_D\int_0^T\{\lambda(t,x)p(t,x)+h(t,x)\}\xi(dt,x)\,dx\bigg]\le0. \tag{44}$$

On the other hand, we can also choose $\zeta(dt,x) = -\xi(dt,x) \in \mathcal{V}(\xi)$, and this gives

$$E\left[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x)\,dx\right] \geq 0. \tag{45}$$

Combining (44) and (45) we get

$$E\left[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x)\,dx\right] = 0. \tag{46}$$

Combining (43) and (46) we see that

$$\{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x) = 0 \quad \text{for all } t,\ x, \text{ a.s.}, \tag{47}$$

as claimed. This proves (i).

(ii) Conversely, suppose $\hat{\xi} \in \mathcal{A}$ is as in (ii). Then (40) follows from Lemma 3. □

2.1. Application to optimal harvesting. We now return to the problem of optimal harvesting from a fish population in a lake D stated in the introduction. Thus we suppose the density $Y(t,x)$ of the population at time $t \in [0,T]$ and at the point $x \in D$ is given by the stochastic reaction-diffusion Equation (1), and the performance criterion is assumed to be as in (2). In this case the Hamiltonian in (7) is

$$\begin{aligned} &H(t,x,y,p,q)(dt,\xi(dt,x)) \\ &\quad = (\alpha y p + \beta y q)dt + [-\lambda_0 p + h_0(t,x)y]\xi(dt,x), \end{aligned} \tag{48}$$

and the adjoint Equation (8)–(10) is

$$\begin{aligned} dp(t,x) &= -[\Delta p(t,x) + \alpha p(t,x) + \beta q(t,x)]dt \\ &\quad - h_0(t,x)\xi(dt,x) + q(t,x)dB(t); \quad (t,x) \in (0,T) \times D, \\ p(T,x) &= h_0(T,x); \quad x \in D, \\ p(t,x) &= 0; \quad (t,x) \in (0,T) \times \partial D. \end{aligned} \tag{49}$$

The variational inequalities (36)–(37) for an optimal control $\xi(dt,x)$ are:

$$-\lambda_0 p(t,x) + h_0(t,x)Y(t,x) \leq 0; \quad (t,x) \in [0,T] \times D, \tag{50}$$

$$[-\lambda_0 p(t,x) + h_0(t,x)Y(t,x)]\xi(dt,x) = 0; \quad (t,x) \in [0,T] \times D. \tag{51}$$

We can rewrite the variational inequalities above as follows:

$$\begin{aligned} &p(t,x) \geq \frac{h_0(t,x)Y(t,x)}{\lambda_0}; \quad (t,x) \in [0,T] \times D, \\ &\left[p(t,x) - \frac{h_0(t,x)Y(t,x)}{\lambda_0}\right]\xi(dt,x) = 0; \quad (t,x) \in [0,T] \times D. \end{aligned} \tag{52}$$

We summarize the above in the following:

Theorem 4. (a) *Suppose $\xi(dt,x) \in \mathcal{A}$ is an optimal singular control for the harvesting problem*

$$\sup_{\xi \in \mathcal{A}} E\left[\int_D \int_0^T h_0(t,x)Y(t,x)\xi(dt,x)\,dx + \int_D h_0(T,x)Y(T,x)\,dx\right], \tag{53}$$

where $Y(t,x)$ is given by the SPDE (1). Then $\xi(dt,x)$ solves the reflected BSPDE (49), (52).

(b) *Conversely, suppose $\xi(dt,x)$ is a solution of the reflected BSPDE (49), (52). Then $\xi(dt,x)$ is a directional sub-critical point for the performance $J(\cdot)$ given by (2).*

Heuristically we can interpret the optimal harvesting strategy as follows:

- As long as $p(t,x) > (h_0(t,x)Y(t,x))/\lambda_0$, we do nothing.
- If $p(t,x) = (h_0(t,x)Y(t,x))/\lambda_0$, we harvest immediately from $Y(t,x)$ at a rate $\xi(dt,x)$ which is exactly enough to prevent $p(t,x)$ from dropping below $(h_0(t,x)Y(t,x))/\lambda_0$ in the next moment.
- If $p(0,x) < (h_0(0,x)Y(0^-,x))/\lambda_0$, we harvest immediately what is necessary to bring $(h_0(0,x)Y(0,x))/\lambda_0$ down to the level of $p(0,x)$.

3. Existence and uniqueness results of reflected backward SPDEs. In this section we prove existence and uniqueness results for reflected backward stochastic partial differential equations. For notational simplicity, we choose the operator A to be the Laplacian operator Δ. However, our methods work equally well for general second-order differential operators such as

$$A = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right),$$

where $a = (a_{ij}(x))\colon D \to \mathbb{R}^{d\times d}$ $(d \ge 2)$ is a measurable, symmetric matrix-valued function which satisfies the uniform ellipticity condition

$$\lambda|z|^2 \le \sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \le \Lambda |z|^2, \quad \forall z \in \mathbb{R}^d \quad \text{and} \quad x \in D,$$

for some constants λ, $\Lambda > 0$.

Let $V := W_0^{1,2}(D)$ be the Sobolev space of order one with the usual norm $\|\cdot\|$. As before, let $K := L^2(D)$. Consider the reflected backward stochastic partial differential equation:

$$\begin{aligned} du(t,x) &= -\Delta u(t,x)dt - b(t,u(t,x),Z(t,x))dt + Z(t,x)dB_t - h_0(t,x)\eta(dt,x), \quad t\in(0,T), \quad x\in D,\\ u(t,x) &\ge L(t,x), \quad t\in(0,T), \quad x\in D,\\ \int_0^T &\int_D (u(t,x)-L(t,x))\eta(dt,x)\,dx = 0,\\ u(t,x) &= 0; \quad (t,x)\in(0,T)\times\partial D,\\ u(T,x) &= \phi(x) \quad \text{a.s.} \end{aligned} \tag{54}$$

The optimality Equations (49)–(52) for the optimal harvesting problem above are typically of this form.

Theorem 5. *Assume that $E[|\phi|_K^2] < \infty$ and that*

$$|b(s,u_1,z_1) - b(s,u_2,z_2)| \le C(|u_1-u_2| + |z_1-z_2|).$$

Let $L(t,x)$ be a measurable function which is differentiable in t and twice differentiable in x such that

$$\int_0^T \int_D L'(t,x)^2\,dx\,dt < \infty, \qquad \int_0^T \int_D |\Delta L(t,x)|^2\,dx\,dt < \infty.$$

Let $h_0(t,x) > 0$ be a given bounded predictable process. Then there exists a unique $K \times L^2(D,\mathbb{R}^m)\times K$-valued progressively measurable process $(u(t,x), Z(t,x), \eta(t,x))$ such that

(i) $$E\left[\int_0^T \|u(t)\|_V^2\,dt\right] < \infty, \qquad E\left[\int_0^T |Z(t)|^2_{L^2(D,R^m)}\,dt\right] < \infty.$$

(ii) *η is a K-valued continuous process, nonnegative, nondecreasing in t, and $\eta(0,x) = 0$.*

(iii) $$\begin{aligned} u(t,x) &= \phi(x) + \int_t^T \Delta u(t,x)\,ds + \int_t^T b(s,u(s,x),Z(s,x))\,ds - \int_t^T Z(s,x)\,dB_s\\ &\quad + \int_t^T h_0(s,x)\eta(ds,x); \quad 0\le t\le T, \quad x\in D. \end{aligned} \tag{55}$$

(iv) $$u(t,x) \ge L(t,x) \quad a.e. \quad x\in D, \quad \forall t\in[0,T].$$

(v) $$\int_0^T \int_D (u(t,x)-L(t,x))\eta(dt,x)\,dx = 0.$$

(vi) $$u(t,x) = 0; \quad (t,x)\in(0,T)\times\partial D,$$

where $u(t)$ stands for the K-valued continuous process $u(t,\cdot)$ and (iii) is understood as an equation in the dual space V^* of V.

For the proof of the theorem, without loss of generality we assume $h_0(t,x)\equiv 1$ and introduce the penalized BSPDEs:

$$\begin{aligned} du^n(t,x) &= -\Delta u^n(t,x)dt - b(t,u^n(t,x),Z^n(t,x))dt + Z^n(t,x)dB_t\\ &\quad - n(u^n(t,x)-L(t,x))^- dt, \quad t\in(0,T),\\ u^n(T,x) &= \phi(x) \quad \text{a.s.} \end{aligned} \tag{56}$$

According to Øksendal et al. [14], the solution (u^n, Z^n) of the above equation exists and is unique. We are going to show that the sequence (u^n, Z^n) has a limit, which will be a solution of the Equation (55). First we need some a priori estimates:

LEMMA 4. *Let* (u^n, Z^n) *be the solution of Equation* (56). *We have*

$$\sup_n E\left[\sup_t |u^n(t)|_K^2\right] < \infty, \tag{57}$$

$$\sup_n E\left[\int_0^T \|u^n(t)\|_V^2\right] < \infty, \tag{58}$$

$$\sup_n E\left[\int_0^T |Z^n(t)|^2_{L^2(D,R^m)}\right] < \infty. \tag{59}$$

PROOF. Take a function $f(t,x) \in C_0^{2,2}([-1, T+1] \times D)$ satisfying $f(t,x) \geq L(t,x)$. Applying Itô's formula, it follows that

$$\begin{aligned}
|u^n(t) - f(t)|_K^2 &= |\phi - f(T)|_K^2 + 2\int_t^T \langle u^n(s) - f(s), \Delta u^n(s)\rangle\, ds \\
&\quad + 2\int_t^T \langle u^n(s) - f(s), b(s, u^n(s), Z^n(s))\rangle\, ds \\
&\quad - 2\int_t^T \langle u^n(s) - f(s), Z^n(s)\rangle\, dB_s \\
&\quad + 2n\int_t^T \langle u^n(s) - f(s), (u^n(s) - L(s))^-\rangle\, ds - \int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds \\
&\quad + 2\int_t^T \langle u^n(s) - f(s), f'(s)\rangle\, ds, \quad \text{a.s.,}
\end{aligned} \tag{60}$$

where $\langle , \rangle$ denotes the inner product in K. Now we estimate each of the terms on the right-hand side:

$$\begin{aligned}
2\int_t^T \langle u^n(s) - f(s), \Delta u^n(s)\rangle\, ds &= -2\int_t^T \|u^n(s)\|_V^2\, ds + 2\int_t^T \left\langle \frac{\partial f(s)}{\partial x}, \frac{\partial u^n(s)}{\partial x}\right\rangle ds \\
&\leq -\int_t^T \|u^n(s)\|_V^2\, ds + \int_t^T \|f(s)\|_V^2\, ds,
\end{aligned} \tag{61}$$

$$\begin{aligned}
&2\int_t^T \langle u^n(s) - f(s), b(s, u^n(s), Z^n(s))\rangle\, ds \\
&\quad = 2\int_t^T \langle u^n(s) - f(s), b(s, u^n(s), Z^n(s)) - b(s, f(s), Z^n(s))\rangle\, ds \\
&\qquad + 2\int_t^T \langle u^n(s) - f(s), b(s, f(s), Z^n(s)) - b(s, f(s), 0)\rangle\, ds \\
&\qquad + 2\int_t^T \langle u^n(s) - f(s), b(s, f(s), 0)\rangle\, ds \\
&\quad \leq C\int_t^T |u^n(s) - f(s)|_H^2\, ds + \frac{1}{2}\int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds + C\int_t^T |b(s, f(s), 0)|_H^2\, ds
\end{aligned} \tag{62}$$

$$\begin{aligned}
&2n\int_t^T \langle u^n(s) - f(s), (u^n(s) - L(s))^-\rangle\, ds, \\
&\quad = 2n\int_t^T \int_D (u^n(s,x) - f(s,x))\chi_{\{u^n(s,x) \leq L(s,x)\}}(L(s,x) - u^n(s,x))\, ds\, dx \leq 0.
\end{aligned} \tag{63}$$

Substituting (61), (3), and (63) into (60) we obtain

$$\begin{aligned}
&|u^n(t) - f(t)|_K^2 + \int_t^T \|u^n(s)\|_V^2\, ds + \frac{1}{2}\int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds \\
&\quad \leq |\phi - f(T)|_K^2 + C\int_t^T |u^n(s) - f(s)|_K^2\, ds + C\int_t^T |b(s, f(s), 0)|_K^2\, ds \\
&\qquad + \int_t^T \|f(s)\|_V^2\, ds - 2\int_t^T \langle u^n(s) - f(s), Z^n(s)\rangle\, dB_s.
\end{aligned} \tag{64}$$

We take expectation and use the Gronwall inequality to obtain

$$\sup_n \sup_t E[|u^n(t)|_K^2] < \infty, \tag{65}$$

$$\sup_n E\left[\int_0^T \|u^n(t)\|_V^2\right] < \infty, \tag{66}$$

$$\sup_n E\left[\int_0^T |Z^n(t)|^2_{L^2(D,\mathbb{R}^m)}\, dt\right] < \infty. \tag{67}$$

By virtue of (67), (65) can be further strengthend to (57). Indeed, by the Burkholder inequality,

$$\begin{aligned} &E\left[2\sup_{v\le t\le T}\left|\int_t^T \langle u^n(s)-f(s), Z^n(s)\rangle dB_s\right|\right]\\ &\le CE\left[\left(\int_v^T |u^n(s)-f(s)|_K^2 |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds\right)^{1/2}\right]\\ &\le CE\left[\sup_{v\le s\le T}(|u^n(s)-f(s)|_K)\left(\int_v^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds\right)^{1/2}\right]\\ &\le \frac{1}{2}E\left[\sup_{v\le s\le T}(|u^n(s)-f(s)|_K^2)\right] + CE\left[\int_v^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds\right]. \end{aligned} \tag{68}$$

With (68), taking supremum over $t\in[v,T]$ on both sides of (60) we obtain (57). □

We need the following estimates:

Lemma 5. *Suppose the conditions in Theorem 5 hold. Then there is a constant C such that*

$$E\left[\int_0^T \int_D ((u^n(t,x)-L(t,x))^-)^2\, dx\, dt\right] \le \frac{C}{n^2}. \tag{69}$$

Proof. For $m\ge 1$, define the functions $\psi_m(z)$, $f_m(x)$ as follows (see Donati-Martin and Pardoux [6]).

$$\psi_m(z) = \begin{cases} 0 & \text{if } z\le 0,\\ 2mz & \text{if } 0\le z\le \dfrac{1}{m},\\ 2 & \text{if } z > \dfrac{1}{m}, \end{cases} \tag{70}$$

$$f_m(x) = \begin{cases} 0 & \text{if } x\le 0,\\ \displaystyle\int_0^x dy \int_0^y \psi_m(z)\, dz & \text{if } x>0. \end{cases} \tag{71}$$

We have

$$f'_m(x) = \begin{cases} 0 & \text{if } x\le 0,\\ mx^2 & \text{if } x\le \dfrac{1}{m},\\ 2x-\dfrac{1}{n} & \text{if } x > \dfrac{1}{m}. \end{cases} \tag{72}$$

Then $f_m(x)\uparrow (x^+)^2$ and $f'_m(x)\uparrow 2x^+$ as $m\to\infty$. For $h\in K$, set

$$G_m(h) = \int_D f_m(-h(x))\, dx.$$

It is easy to see that for $h_1, h_2\in K$,

$$G'_m(h)(h_1) = -\int_D f'_m(-h(x))h_1(x)\, dx, \tag{73}$$

$$G''_n(h)(h_1,h_2) = \int_D f''_m(-h(x))h_1(x)h_2(x)\, dx. \tag{74}$$

Applying Itô's formula we get

$$\begin{aligned} G_m(u^n(t)-L(t)) \;=\; & G_m(\phi-L(T))+\int_t^T G'_m(u^n(s)-L(s))(\Delta u^n(s))\,ds \\ & +\int_t^T G'_m(u^n(s)-L(s))(b(s,u^n(s),Z^n(s)))\,ds \\ & +n\int_t^T G'_m(u^n(s)-L(s))((u^n(s)-L(s))^-)\,ds \\ & +\int_t^T G'_m(u^n(s)-L(s))(L'(s))\,ds \\ & -\int_t^T G'_m(u^n(s)-L(s))(Z^n(s))\,dB_s \\ & -\frac{1}{2}\int_t^T G''_m(Z^n(s),Z^n(s))\,ds \\ =:\; & I^1_m+I^2_m+I^3_m+I^4_m+I^5_m+I^6_m+I^7_m. \end{aligned} \tag{75}$$

Now,

$$\begin{aligned} I^2_m &= \int_t^T G'_m(u^n(s)-L(s))(\Delta u^n(s))\,ds \\ &= -\int_t^T\int_D f'_m(L(s,x)-u^n(s,x))(\Delta(u^n(s,x)-L(s,x)))\,dx\,ds \\ &\quad -\int_t^T\int_D f'_m(L(s,x)-u^n(s,x))(\Delta L(s,x))\,dx\,ds \\ &\le -\int_t^T\int_D f''_m(L(s,x)-u^n(s,x))|\nabla(u^n(s,x)-L(s,x))|^2\,dx\,ds \\ &\quad +\frac{1}{4}n\int_t^T\int_D f'_m(L(s,x)-u^n(s,x))^2 x\,ds \\ &\quad +\frac{C}{n}\int_t^T\int_D(\Delta L(s,x))^2\,dx\,ds, \end{aligned} \tag{76}$$

$$\begin{aligned} I^3_m &= -\int_t^T\int_D f'_m(L(s,x)-u^n(s,x))b(s,u^n(s,x),Z^n(s,x))\,dx\,ds \\ &\le \frac{1}{4}n\int_t^T\int_D f'_m(L(s,x)-u^n(s,x))^2\,ds \\ &\quad +\frac{C}{n}\int_t^T\int_D(b(s,u^n(s,x),Z^n(s,x)))^2\,dx\,ds, \end{aligned} \tag{77}$$

$$\begin{aligned} I^5_m &= -\int_t^T\int_D f'_m(L(s,x)-u^n(s,x))(L'(s,x))\,dx\,ds \\ &\le \frac{1}{4}n\int_t^T\int_D f'_m(L(s,x)-u^n(s,x))^2\,ds \\ &\quad +\frac{C}{n}\int_t^T\int_D(L'(s,x))^2\,dx\,ds. \end{aligned} \tag{78}$$

Combining (75)–(78) and taking expectation we obtain

$$\begin{aligned} & E[G_m(u^n(t)-L(t))] \\ &\quad \le E[G_m(\phi-L(T))]+\frac{3}{4}n\int_t^T\int_D f'_m(L(s,x)-u^n(s,x))^2\,ds \\ &\qquad +\frac{C}{n}E\left[\int_t^T\int_D(L'(s,x))^2\,dx\,ds\right]+\frac{C}{n}E\left[\int_t^T\int_D(\Delta L(s,x))^2\,dx\,ds\right] \end{aligned}$$

$$+\frac{C}{n}E\left[\int_t^T\int_D (b(s,u^n(s,x),Z^n(s,x)))^2\,dx\,ds\right]$$
$$-nE\left[\int_t^T\int_D f_m'(L(s,x)-u^n(s,x))((u^n(s,x)-L(s,x))^-)\,ds\right]. \tag{79}$$

Letting $m\to\infty$ we conclude that

$$E\left[\int_D((u^n(t,x)-L(t,x))^-)^2\,dx\right]$$
$$\le \frac{3}{4}nE\left[\int_t^T\int_D((u^n(s,x)-L(s,x))^-)^2\,dx\,ds\right]$$
$$-nE\left[\int_t^T\int_D((u^n(s,x)-L(s,x))^-)^2\,dx\,ds\right]+\frac{C'}{n}, \tag{80}$$

where the Lipschitz condition of b and Lemma 4 have been used. In particular we have

$$E\left[\int_t^T\int_D((u^n(s,x)-L(s,x))^-)^2\,dx\,ds\right]\le\frac{C'}{n^2}. \quad\square \tag{81}$$

Lemma 6. *Let (u^n, Z^n) be the solution of Equation (56). We have*

$$\lim_{n,m\to\infty}E[\sup_{0\le t\le T}|u^n(t)-u^m(t)|_K^2]=0, \tag{82}$$
$$\lim_{n,m\to\infty}E\left[\int_0^T\|u^n(t)-u^m(t)\|_V^2\,dt\right]=0, \tag{83}$$
$$\lim_{n,m\to\infty}E\left[\int_0^T|Z^n(t)-Z^m(t)|^2_{L^2(D,\mathbb{R}^m)}\,dt\right]=0. \tag{84}$$

Proof. Applying Itô's formula, it follows that

$$|u^n(t)-u^m(t)|_K^2$$
$$=2\int_t^T\langle u^n(s)-u^m(s),\Delta(u^n(s)-u^m(s))\rangle\,ds$$
$$+2\int_t^T\langle u^n(s)-u^m(s),b(s,u^n(s),Z^n(s))-b(s,u^m(s),Z^m(s))\rangle\,ds$$
$$-2\int_t^T\langle u^n(s)-u^m(s),Z^n(s)-Z^m(s)\rangle\,dB_s$$
$$+2\int_t^T\langle u^n(s)-u^m(s),n(u^n(s)-L(s))^- - m(u^m(s)-L(s))^-\rangle\,ds.$$
$$-\int_t^T|Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)}\,ds \tag{85}$$

Now we estimate each of the terms on the right side:

$$2\int_t^T\langle u^n(s)-u^m(s),\Delta(u^n(s)-u^m(s))\rangle\,ds$$
$$=-2\int_t^T\|u^n(s)-u^m(s)\|_V^2\,ds. \tag{86}$$

By the Lipschitz continuity of b and the inequality $ab\le\varepsilon a^2+C_\varepsilon b^2$, one has

$$2\int_t^T\langle u^n(s)-u^m(s),b(s,u^n(s),Z^n(s))-b(s,u^m(s),Z^m(s))\rangle\,ds$$
$$\le C\int_t^T|u^n(s)-u^m(s)|_K^2ds+\frac{1}{2}\int_t^T|Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)}\,ds. \tag{87}$$

In view of (81),

$$
\begin{aligned}
&2E\left[\int_t^T \langle u^n(s)-u^m(s), n(u^n(s)-L(s))^- - m(u^m(s)-L(s))^-\rangle\, ds\right]\\
&= 2nE\left[\int_t^T \langle u^n(s)-L(s), (u^n(s)-L(s))^-\rangle\, ds\right]\\
&\quad + 2mE\left[\int_t^T \langle L(s)-u^n(s), (u^m(s)-L(s))^-\rangle\, ds\right]\\
&\quad + 2mE\left[\int_t^T \langle u^m(s)-L(s), (u^m(s)-L(s))^-\rangle\, ds\right]\\
&\quad + 2nE\left[\int_t^T \langle L(s)-u^m(s), (u^n(s)-L(s))^-\rangle\, ds\right]\\
&\le 2mE\left[\int_t^T \langle L(s)-u^n(s), (u^m(s)-L(s))^-\rangle\, ds\right]\\
&\quad + 2nE\left[\int_t^T \langle L(s)-u^m(s), (u^n(s)-L(s))^-\rangle\, ds\right]\\
&\le 2mE\left[\int_t^T \int_D (u^n(s,x)-L(s,x))^-(u^m(s,x)-L(s,x))^-\, dx\, ds\right]\\
&\quad + 2nE\left[\int_t^T \int_D (u^m(s,x)-L(s,x))^-(u^n(s,x)-L(s,x))^-\, dx\, ds\right]\\
&\le 2m\left(E\left[\int_t^T \int_D ((u^n(s,x)-L(s,x))^-)^2\, dx\, ds\right]\right)^{1/2}\left(E\left[\int_t^T \int_D ((u^m(s,x)-L(s,x))^-)^2\, dx\, ds\right]\right)^{1/2}\\
&\quad + 2n\left(E\left[\int_t^T \int_D ((u^n(s,x)-L(s,x))^-)^2\, dx\, ds\right]\right)^{1/2}\left(E\left[\int_t^T \int_D ((u^m(s,x)-L(s,x))^-)^2\, dx\, ds\right]\right)^{1/2}\\
&\le C'\left(\frac{1}{n}+\frac{1}{m}\right).
\end{aligned}
\tag{88}
$$

It follows from (85) and (86) that

$$
\begin{aligned}
&E[|u^n(t)-u^m(t)|_K^2] + \frac{1}{2}E\left[\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds\right]\\
&\quad + E\left[\int_t^T \|u^n(s)-u^m(s)\|_V^2\, ds\right]\\
&\le C\int_t^T E[|u^n(s)-u^m(s)|_K^2]\, ds + C'\left(\frac{1}{n}+\frac{1}{m}\right).
\end{aligned}
\tag{89}
$$

Application of the Gronwall inequality yields

$$
\lim_{n,m\to\infty}\left\{E[|u^n(t)-u^m(t)|_K^2] + \frac{1}{2}E\left[\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds\right]\right\} = 0, \tag{90}
$$

$$
\lim_{n,m\to\infty} E\left[\int_t^T \|u^n(s)-u^m(s)\|_V^2\, ds\right] = 0. \tag{91}
$$

By (90) and the Burkholder inequality we can further show that

$$
\lim_{n,m\to\infty} E\left[\sup_{0\le t\le T} |u^n(t)-u^m(t)|_K^2\right] = 0. \tag{92}
$$

The proof is complete. □

Proof of Theorem 5. From Lemma 6 we know that (u^n, Z^n), $n \geq 1$, forms a Cauchy sequence. Denote by $u(t,x)$, $Z(t,x)$ the limit of u^n and Z^n. Put

$$\bar{\eta}^n(t,x) = n(u^n(t,x) - L(t,x))^-.$$

Lemma 5 implies that $\bar{\eta}^n(t,x)$ admits a nonnegative weak limit, denoted by $\bar{\eta}(t,x)$, in the following Hilbert space:

$$\bar{K} = \left\{ h;\ h \text{ is a } K\text{-valued adapted process such that } E\left[\int_0^T |h(s)|_K^2\, ds\right] < \infty \right\}$$

with inner product

$$\langle h_1, h_2 \rangle_{\bar{K}} = E\left[\int_0^T \int_D h_1(t,x) h_2(t,x)\, dt\, dx\right].$$

Set $\eta(t,x) = \int_0^t \bar{\eta}(s,x)\, ds$. Then η is a continuous K-valued process which is increasing in t. Keeping Lemma 6 in mind and letting $n \to \infty$ in (56) we obtain

$$\begin{aligned} u(t,x) &= \phi(x) + \int_t^T \Delta u(t,x)\, ds + \int_t^T b(s, u(s,x), Z(s,x))\, ds - \int_t^T Z(s,x)\, dB_s \\ &\quad + \eta(T,x) - \eta(t,x); \quad 0 \leq t \leq T. \end{aligned} \tag{93}$$

Recall from Lemma 5 that

$$E\left[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2\, dx\, ds\right] \leq C' \frac{1}{n^2}.$$

By the Fatou Lemma, this implies that $E[\int_t^T \int_D ((u(s,x) - L(s,x))^-)^2\, dx\, ds] = 0$. In view of the continuity of u in t, we conclude that $u(t,x) \geq L(t,x)$ a.e. in x, for every $t \geq 0$. Combining the strong convergence of u^n and the weak convergence of $\bar{\eta}^n$, we also have

$$\begin{aligned} &E\left[\int_0^T \int_D (u(s,x) - L(s,x)) \eta(dt,x)\, dx\right] \\ &\quad = E\left[\int_0^T \int_D (u(s,x) - L(s,x)) \bar{\eta}(t,x)\, dt\, dx\right] \\ &\quad \leq \lim_{n\to\infty} E\left[\int_0^T \int_D (u^n(s,x) - L(s,x)) \bar{\eta}^n(t,x)\, dt\, dx\right] \leq 0. \end{aligned} \tag{94}$$

Hence,

$$\int_0^T \int_D (u(s,x) - L(s,x)) \eta(dt,x)\, dx = 0, \quad \text{a.s.}$$

We have shown that (u, Z, η) is a solution to the reflected BSPDE (54).

It remains to prove that the solution is unique. Let (u_1, Z_1, η_1), (u_2, Z_2, η_2) be two solutions of Equation (55). By Itô's formula, we have

$$\begin{aligned} &|u_1(t) - u_2(t)|_K^2 \\ &\quad = 2\int_t^T \langle u_1(s) - u_2(s), \Delta(u_1(s) - u_2(s)) \rangle\, ds \\ &\qquad + 2\int_t^T \langle u_1(s) - u_2(s), b(s, u_1(s), Z_1(s)) - b(s, u_2(s), Z_2(s)) \rangle\, ds \\ &\qquad - 2\int_t^T \langle u_1(s) - u_2(s), Z_1(s) - Z_2(s) \rangle\, dB_s \\ &\qquad + 2\int_t^T \langle u_1(s) - u_2(s), \eta_1(ds) - \eta_2(ds) \rangle \\ &\qquad - \int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D,\mathbb{R}^m)}\, ds. \end{aligned} \tag{95}$$

Similar to the proof of Lemma 6, we have

$$2\int_t^T \langle u_1(s)-u_2(s), \Delta(u_1(s)-u_2(s))\rangle\, ds \le 0, \tag{96}$$

and

$$\begin{aligned}&2\int_t^T \langle u_1(s)-u_2(s), b(s,u_1(s),Z_1(s))-b(s,u_2(s),Z_2(s))\rangle\, ds\\ &\quad\le C\int_t^T |u_1(s)-u_2(s)|_K^2\,ds+\frac{1}{2}\int_t^T |Z_1(s)-Z_2(s)|^2_{L^2(D,\mathbb{R}^m)}\,ds.\end{aligned}\tag{97}$$

On the other hand,

$$\begin{aligned}&2E\bigg[\int_t^T \langle u_1(s)-u_2(s), \eta_1(ds)-\eta_2(ds)\rangle\bigg]\\ &=2E\bigg[\int_t^T\int_D (u_1(s,x)-L(s,x))\eta_1(ds,x)\,dx\bigg]\\ &\quad-2E\bigg[\int_t^T\int_D (u_1(s,x)-L(s,x))\eta_2(ds,x)\,dx\bigg]\\ &\quad+2E\bigg[\int_t^T\int_D (u_2(s,x)-L(s,x))\eta_2(ds,x)\,dx\bigg]\\ &\quad-2E\bigg[\int_t^T\int_D (u_2(s,x)-L(s,x))\eta_1(ds,x)\,dx\bigg]\\ &\le 0.\end{aligned}\tag{98}$$

Combining (95)—(98) we arrive at

$$\begin{aligned}&E[|u_1(t)-u_2(t)|_K^2]+\frac{1}{2}E\bigg[\int_t^T |Z_1(s)-Z_2(s)|^2_{L^2(D,\mathbb{R}^m)}\,ds\bigg]\\ &\le C\int_t^T E[|u_1(s)-u_2(s)|_K^2]\,ds.\end{aligned}\tag{99}$$

Appealing to the Gronwall inequality, this implies

$$u_1=u_2,\qquad Z_1=Z_2,$$

which further gives $\eta_1=\eta_2$ from the equation they satisfy. This completes the proof of Theorem 5 . □

4. Comparison theorems for BSPDEs and reflected BSPDEs. We now establish some comparison theorems for backward stochastic partial differential equations and BSPDEs without and with reflection, which are useful in the application to risk measures below, and which are also of independent interest. Consider two backward SPDEs:

$$\begin{aligned}du_1(t,x)&=-\Delta u_1(t,x)dt-b_1(t,u_1(t,x),Z_1(t,x))dt+Z_1(t,x)dB_t,\quad t\in(0,T),\\ u_1(T,x)&=\phi_1(x)\quad\text{a.s.},\end{aligned}\tag{100}$$

$$\begin{aligned}du_2(t,x)&=-\Delta u_2(t,x)dt-b_2(t,u_2(t,x),Z_2(t,x))dt+Z_2(t,x)dB_t,\quad t\in(0,T),\\ u_2(T,x)&=\phi_2(x)\quad\text{a.s.}\end{aligned}\tag{101}$$

From now on, if $u(t,x)$ is a function of (t,x), we write $u(t)$ for the function $u(t,\cdot)$.

Theorem 6 (Comparison Theorem for BSPDEs). *Assume that b_i, ϕ_i, $i=1,2$, satisfy the conditions in Theorem 5. Suppose $\phi_1(x)\le\phi_2(x)$ and $b_1(t,u,z)\le b_2(t,u,z)$. Then we have $u_1(t,x)\le u_2(t,x)$, $x\in D$, a.e. for every $t\in[0,T]$.*

Proof. For $h \in K := L^2(D)$, set

$$F_n(h) = \int_D f_n(h(x))\,dx,$$

where f_n is defined in (71). Applying Itô's formula we get

$$\begin{aligned}
&F_n(u_1(t)-u_2(t))\\
&\quad = F_n(\phi_1-\phi_2) + \int_t^T F_n'(u_1(s)-u_2(s))(\Delta(u_1(s)-u_2(s)))\,ds\\
&\qquad + \int_t^T F_n'(u_1(s)-u_2(s))(b_1(s,u_1(s),Z_1(s)) - b_2(s,u_2(s),Z_2(s)))\,ds\\
&\qquad - \int_t^T F_n'(u_1(s)-u_2(s))(Z_1(s)-Z_2(s))\,dB_s\\
&\qquad - \frac{1}{2}\int_t^T F_n''(u_1(s)-u_2(s))(Z_1(s)-Z_2(s), Z_1(s)-Z_2(s))\,ds\\
&\quad =: I_n^1 + I_n^2 + I_n^3 + I_n^4 + I_n^5,
\end{aligned} \tag{102}$$

where

$$\begin{aligned}
I_n^2 &= \int_t^T F_n'(u_1(s)-u_2(s))(\Delta(u_1(s)-u_2(s)))\,ds\\
&= \int_t^T \int_D f_n'(u_1(s,x)-u_2(s,x))(\Delta(u_1(s,x)-u_2(s,x)))\,dx\,ds\\
&= -\int_t^T f_n''(u_1(s,x)-u_2(s,x))|\nabla(u_1(s,x)-u_2(s,x))|^2\,dx\,ds \le 0,
\end{aligned} \tag{103}$$

$$\begin{aligned}
I_n^5 &= -n\int_t^T\int_D \chi_{\{0\le u_1(s,x)-u_2(s,x)\le 1/n\}}(u_1(s,x)-u_2(s,x))|Z_1(s,x)-Z_2(s,x)|^2\,dx\,ds\\
&\quad - \int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>1/n\}}|Z_1(s,x)-Z_2(s,x)|^2\,dx\,ds.
\end{aligned} \tag{104}$$

For I_n^3, we have

$$\begin{aligned}
I_n^3 &= \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_1(s,u_1(s,x),Z_1(s,x)) - b_2(s,u_2(s,x),Z_2(s,x)))\,dx\,ds\\
&= \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_1(s,u_1(s,x),Z_1(s,x)) - b_2(s,u_1(s,x),Z_1(s,x)))\,dx\,ds\\
&\quad + \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_1(s,x),Z_1(s,x)) - b_2(s,u_2(s,x),Z_1(s,x)))\,dx\,ds\\
&\quad + \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_2(s,x),Z_1(s,x)) - b_2(s,u_2(s,x),Z_2(s,x)))\,dx\,ds\\
&\le \int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_2(s,x),Z_1(s,x)) - b_2(s,u_2(s,x),Z_2(s,x)))\,dx\,ds\\
&\quad + C\int_t^T\int_D ((u_1(s,x)-u_2(s,x)^+)^2\,dx\,ds := I_{n,1}^3 + I_{n,2}^3,
\end{aligned} \tag{105}$$

where the Lipschitz condition of b and the assumption $b_1 \le b_2$ have been used. $I_{n,1}^3$ can be estimated as follows:

$$\begin{aligned}
I_{n,1}^3 &\le C\int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))|Z_1(s,x)-Z_2(s,x)|\,dx\,ds\\
&= C\int_t^T\int_D \chi_{\{0\le u_1(s,x)-u_2(s,x)\le 1/n\}} n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|\,dx\,ds\\
&\quad + C\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>1/n\}}\left[2(u_1(s,x)-u_2(s,x)) - \frac{1}{n}\right]|Z_1(s,x)-Z_2(s,x)|\,dx\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>1/n\}}\left(2(u_1(s,x)-u_2(s,x))-\frac{1}{n}\right)^2 dx\,ds\\
&\quad+\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>1/n\}}|Z_1(s,x)-Z_2(s,x)|^2\,dx\,ds\\
&\quad+\frac{1}{4}C^2\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq 1/n\}}n(u_1(s,x)-u_2(s,x))^3\,dx\,ds\\
&\quad+\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq 1/n\}}n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|^2\,dx\,ds\\
&\leq C'\int_t^T\int_D ((u_1(s,x)-u_2(s,x))^+)^2\,dx\,ds\\
&\quad+\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>1/n\}}|Z_1(s,x)-Z_2(s,x)|^2\,dx\,ds\\
&\quad+\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq 1/n\}}n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|^2\,dx\,ds. \qquad (106)
\end{aligned}
$$

Equations (104), (105), and (106) imply that

$$
I_n^3+I_n^5\leq C\int_t^T\int_D((u_1(s,x)-u_2(s,x))^+)^2\,dx\,ds. \qquad (107)
$$

Thus it follows from (102), (103), and (107) that

$$
\begin{aligned}
&F_n(u_1(t)-u_2(t))\\
&\quad\leq F_n(\phi_1-\phi_2)+C\int_t^T\int_D((u_1(s,x)-u_2(s,x))^+)^2\,dx\,ds\\
&\qquad-\int_t^T F_n'(u_1(s)-u_2(s))(Z_1(s)-Z_2(s))\,dB_s. \qquad (108)
\end{aligned}
$$

Take expectation and let $n\to\infty$ to get

$$
E\left[\int_D((u_1(t,x)-u_2(t,x))^+)^2\,dx\right]\leq\int_t^T ds\,E\left[\int_D((u_1(s,x)-u_2(s,x))^+)^2\,dx\right]. \qquad (109)
$$

Gronwall's inequality yields that

$$
E\left[\int_D((u_1(t,x)-u_2(t,x))^+)^2\,dx\right]=0, \qquad (110)
$$

which completes the proof of the theorem. □

Remark 1. Comparison theorems for BSPDEs were also proved in Ma et al. [12] and Hu et al. [11]. However, the results in these articles could not cover our theorem and the proofs are quite different.

We now state the comparison theorem for BSPDEs with reflection. For $i=1,2$, consider the reflected backward stochastic partial differential equation:

$$
\begin{cases}
du_i(t,x)=-\Delta u_i(t,x)dt-b_i(t,u_i(t,x),Z_i(t,x))dt\\
\qquad\qquad+Z_i(t,x)dB_t-h_0(t,x)\eta_i(dt,x), \quad t\in(0,T),\ x\in D,\\
u_i(t,x)\geq L_i(t,x), \quad t\in(0,T),\ x\in D,\\
\int_0^T\int_D(u_i(t,x)-L_i(t,x))\eta(dt,x)\,dx=0,\\
u_i(t,x)=0; \quad (t,x)\in(0,T)\times\partial D,\\
u_i(T,x)=\phi_i(x) \quad \text{a.s.}
\end{cases} \qquad (111)
$$

Let $u_i(t,x)$, $i=1,2$, be solutions of the above equations.

Theorem 7 (Comparison Theorem for Reflected BSPDEs). *Assume that b_i, ϕ_i, L_i, $i=1,2$, satisfy the conditions in Theorem 5. Suppose $\phi_1(x)\le\phi_2(x)$, $L_1(t,x)\le L_2(t,x)$, and $b_1(t,u,z)\le b_2(t,u,z)$. Then we have $u_1(t,x)\le u_2(t,x)$, $x\in D$, a.e. for every $t\in[0,T]$.*

Proof. Let $u_i^n(t,x)$, $i=1,2$, $n\ge 1$, be the solutions of the penalized backward SPDEs:

$$\begin{aligned} du_i^n(t,x) &= -\Delta u_i^n(t,x)dt - b(t,u_i^n(t,x),Z_i^n(t,x))dt + Z_i^n(t,x)dB_t \\ &\quad - n(u_i^n(t,x)-L_i(t,x))^- h_0(t,x)dt, \quad t\in(0,T), \\ u_i^n(T,x) &= \phi_i(x) \quad \text{a.s.} \end{aligned} \tag{112}$$

From the proof of Theorem 5 we know that $u_i^n(t,x)\to u_i(t,x)$ as $n\to\infty$. By Theorem 6, we have $u_1^n(t,x)\le u_2^n(t,x)$, $x\in D$, a.e. for every $t\in[0,T]$ and $n\ge 1$. Hence, $u_1(t,x)\le u_2(t,x)$, $x\in D$, a.e. for every $t\in[0,T]$. □

5. Link to optimal stopping. In this section, we provide a link between the solution of a reflected backward stochastic partial differential equation and an optimal stopping problem.

Let $\mathcal{S}_{t,T}$ be the set of all stopping times τ satisfying $t\le\tau\le T$ a.s. For $\tau\in\mathcal{S}_{t,T}$, let $(Y,k)=(Y^\tau,k^\tau)$ be the solution of the BSPDE:

$$\begin{aligned} dY(t,x) &= -\Delta Y(t,x)dt - g(t,x,Y(t,x),k(t,x))dt + k(t,x)dB(t), \quad (t,x)\in(0,\tau)\times\mathbb{R}^d, \\ Y(\tau,x) &= L(\tau,x)\chi_{\tau<T} + \phi(x)\chi_{\tau=T}L(\tau,x); \quad x\in\mathbb{R}^d, \end{aligned} \tag{113}$$

which gives in integral form

$$Y(t,x)=\int_t^\tau P_{s-t}g(s,x,Y(s,x),k(s,x))\,ds+P_{\tau-t}L(\tau,x)\chi_{\{\tau<T\}}+P_{\tau-t}\phi(x)\chi_{\{\tau=T\}}-\int_t^\tau P_{s-t}k(s,x)\,dB_s, \tag{114}$$

where P_t denotes the semigroup generated by the Laplacian operator Δ, i.e.,

$$P_tf(x)=(4\pi t)^{-d/2}\int_{\mathbb{R}^d} f(y)\exp\left(-\frac{|y-x|^2}{4t}\right)dy; \quad f\in L^1(\mathbb{R}^d).$$

Now let $(u(t,x), Z(t,x), \eta(t,x))$ be the solution of the following reflected BSPDE:

$$\begin{aligned} u(t,x) &= \phi(x)+\int_t^T \Delta u(s,x)\,ds+\int_t^T g(s,x,u(s,x),Z(s,x))\,ds-\int_t^T Z(s,x)\,dB_s \\ &\quad + \eta(T,x)-\eta(t,x); \quad 0\le t\le T, \quad x\in\mathbb{R}^d, \\ & u(t,x)\ge L(t,x), \quad (t,x)\in[0,T]\times\mathbb{R}^d, \\ & \int_0^T\int_D (u(s,x)-L(s,x))\eta(ds,x)\,dx=0 \quad \text{a.s.} \end{aligned} \tag{115}$$

We have the following result:

Theorem 8. *$u(t,x)$ is the value function of the optimal stopping problem associated with $Y^\tau(t,x)$, i.e.,*

$$u(t,x)=ess\sup_{\tau\in\mathcal{S}_{t,T}} Y^\tau(t,x). \tag{116}$$

Moreover,

$$\hat\tau := \hat\tau(t,x) := \inf\{s\in[t,T) \mid u(s,x)=L(s,x)\}\wedge T \tag{117}$$

is an optimal stopping time.

Proof. Observe that u admits the following mild representation:

$$\begin{aligned} u(t,x) &= P_{T-t}\phi(x)+\int_t^T P_{s-t}(g(s,u(s,x),Z(s,x)))\,ds-\int_t^T P_{s-t}(Z(s,x))\,dB_s \\ &\quad +\int_t^T P_{s-t}\eta(ds,x); \quad 0\le t\le T. \end{aligned} \tag{118}$$

More generally, for any stopping time τ with $t \le \tau \le T$, we have

$$u(t,x) = P_{\tau-t}(u(\tau,x)) + \int_t^\tau P_{s-t}(g(s,x,u(s,x),Z(s,x)))\,ds - \int_t^\tau P_{s-t}(Z(s,x))\,dB(s) + \int_t^\tau P_{s-t}\eta(ds,x); \quad 0 \le t \le \tau. \tag{119}$$

Since $\eta(s,x)$ is increasing in s and $u(s,x) \ge L(s,x)$ for $s \le T$, it follows that

$$u(t,x) \ge Y^\tau(t,x) - \int_t^\tau P_{s-t}(Z(s,x))\,dB(s) + \int_t^\tau P_{s-t}(k(s,x))\,dB_s.$$

Taking conditional expectation with respect to $\mathcal{F}_t$ on both sides we get

$$u(t,x) \ge E[Y^\tau(t,x) \mid \mathcal{F}_t] = Y^\tau(t,x). \tag{120}$$

As τ is arbitrary, we obtain

$$u(t,x) \ge ess \sup_{\tau \in \mathcal{S}_{t,T}} Y^\tau(t,x). \tag{121}$$

Now, define

$$\hat{\tau} = \hat{\tau}(t,x) = \inf\{s \in [t,T) \mid u(s,x) = L(s,x)\} \wedge T.$$

From the property of η, it is not increasing on the interval $[t,\hat{\tau}]$. Thus, $\int_t^{\hat{\tau}} P_{s-t}\eta(ds,x) = 0$. So we have from (119) that

$$\begin{aligned} u(t,x) &= P_{\tau-t}(u(\tau,x))|_{\tau=\hat{\tau}} + \int_t^{\hat{\tau}} P_{s-t}(g(s,x,u(s,x),Z(s,x)))\,ds \\ &\quad - \int_t^{\hat{\tau}} P_{s-t}(Z(s,x))dB_s + \int_t^{\hat{\tau}} P_{s-t}(k(s,x))\,dB_s \\ &= Y^\tau(t,x)|_{\tau=\hat{\tau}} - \int_t^{\hat{\tau}} P_{s-t}(Z(s,x))dB_s + \int_t^{\hat{\tau}} P_{s-t}(k(s,x))\,dB_s \\ &= Y^{\hat{\tau}}(t,x) - \int_t^{\hat{\tau}} P_{s-t}(Z(s,x))\,dB_s + \int_t^{\hat{\tau}} P_{s-t}(k(s,x))\,dB_s. \end{aligned} \tag{122}$$

Taking conditional expectation yields that

$$u(t,x) = E[Y^{\hat{\tau}}(t,x) \mid \mathcal{F}_t] = Y^{\hat{\tau}}(t,x).$$

Combining this with (120) we obtain the theorem. □

6. Application to risk-minimizing stopping. Let $\tau \in \mathcal{S}_{0,T}$, the set of stopping times with values between 0 and T. Suppose that $g(t,x,y,k)$ is concave with respect to (y,k) for all t,x and g satisfies the conditions on b in Theorem 5. Let $F(t,x)$ be a given square integrable adapted càdlàg process satisfying the conditions on L in Theorem 5. In analogy with the definition of a convex risk measure in finance in terms of (ordinary) backward stochastic differential equations, we may consider $F^\tau(x) = F(\tau,x)$ as the financial standing at (τ,x), and we define the *risk* $\rho(F^\tau)(t,x)$ of $F^\tau(x)$ at time $t \le \tau$ and at the point x by

$$\rho(F^\tau)(t,x) = -Y_{F^\tau}(t,x), \tag{123}$$

where $(Y(t,x) = Y_{F^\tau}(t,x),\ k(t,x))$ is the solution of the BSPDE

$$\begin{aligned} dY(t,x) &= -\Delta Y(t,x)dt - g(t,x,Y(t,x),k(t,x))dt + k(t,x)dB(t), \quad (t,x) \in (0,\tau) \times \mathbb{R}^d, \\ Y(\tau,x) &= F^\tau(x); \quad x \in \mathbb{R}^d. \end{aligned} \tag{124}$$

Note that the monotonicity of such "risk measures" is ensured by the comparison in Theorem 6. We consider the *risk-minimizing stopping problem* to find $\tau^* \in \mathcal{S}_{0,T}$ and $\rho(F^{\tau^*})(t,x)$ such that

$$\rho(F^{\tau^*})(t,x) = ess \inf_{\tau \in \mathcal{S}_{t,T}} \rho(F^\tau)(t,x). \tag{125}$$

We may consider the space diffusion effect stemming from the Laplacian operator, as a representation of a mean-field effect in a market with many agents with interacting notions of risk.

Note that the solution of the BSPDE for $Y_{F^\tau}(t,x)$ is

$$Y_{F^\tau}(t,x) = \int_t^\tau P_{s-t}g(s,x)\,ds + P_{\tau-t}F(\tau,x) - \int_t^\tau P_{s-t}(k(s,x))\,dB(s). \tag{126}$$

Therefore, comparing with the Equation (119) above for $Y^\tau(t,x)$, we see that $Y_F^\tau(t,x)$ coincides with $Y^\tau(t,x)$ if we choose $L(t,x)$ and $\phi(x)$ such that

$$F(t,x) = L(t,x)\chi_{t<T} + \phi(x)\chi_{t=T}. \tag{127}$$

Applying Theorem 6 above to this choice of $L(t,x)$ and $\phi(x)$ we get the following result, which is a space-time version of a known result (see Quenez and Sulem [18]):

Theorem 9 (Risk-Minimizing Stopping Theorem).

$$ess \inf_{\tau\in\mathcal{S}_{t,T}} \rho(F^\tau)(t,x) = -u(t,x), \tag{128}$$

where $u(t,x)$, $Z(t,x)$, $\eta(t,x)$ is the solution of the reflected BSPDE

$$\begin{aligned} u(t,x) &= F(T,x) + \int_t^T \Delta u(s,x)\,ds + \int_t^T g(s,x,u(s,x),Z(s,x))\,ds \\ &\quad - \int_t^T Z(s,x)\,dB(s) + \eta(T,x) - \eta(t,x); \quad (t,x)\in(0,T)\times\mathbb{R}^d, \\ &u(t,x) \geq F(t,x); \quad (t,x)\in(0,T)\times\mathbb{R}^d, \\ &\int_0^T \int_{\mathbb{R}^d} (u(s,x)-F(s,x))\eta(ds,x)\,dx = 0 \quad a.s. \end{aligned} \tag{129}$$

Moreover, the stopping time $\hat\tau = \hat\tau(t,x)$ defined by

$$\hat\tau(t,x) = \inf\{s\in[t,T) \mid u(s,x) = F(s,x)\} \wedge T$$

is optimal.

7. Conclusion. In this paper we study singular control problems of stochastic partial differential equations (SPDEs). We prove a sufficient and a necessary maximum principle for the solution of such problems and show that the solution is linked to a reflected backward SPDE (RBSPDE). The existence and uniqueness of such equations is established, and we also prove comparison theorems for BSPDEs and reflected BSPDEs, which is of independent interest.

We give two applications of our general results:

(i) In the first application we solve a problem of optimal harvesting from a population whose density dynamics is modeled by a stochastic reaction diffusion equation,

(ii) In the second we show that the solution of an optimal stopping problem for a BSPDE can be expressed in terms of an associated reflected BSPDE. The result is applied to the problem of risk-minimizing stopping in a financial market with mean-field type of interactions between the agents.

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