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Forward–Backward Stochastic Differential Games and Stochastic Control under Model Uncertainty

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Abstract We study optimal stochastic control problems with jumps under model uncertainty. We rewrite such problems as stochastic differential games of forward–backward stochastic differential equations. We prove general stochastic maximum principles for such games, both in the zero-sum case (finding conditions for saddle points) and for the nonzero sum games (finding conditions for Nash equilibria). We then apply these results to study robust optimal portfolio-consumption problems with penalty. We establish a connection between market viability under model uncertainty and equivalent martingale measures. In the case with entropic penalty, we prove a general reduction theorem, stating that a optimal portfolio-consumption problem under model uncertainty can be reduced to a classical portfolio-consumption problem under model *certainty*, with a change in the utility function, and we relate this to risk sensitive control. In particular, this result shows that model uncertainty increases the Arrow–Pratt risk aversion index.

Keywords Forward–backward SDEs · Stochastic differential games · Maximum principle · Model uncertainty · Robust control · Viability · Optimal portfolio · Optimal consumption · Jump diffusions

1 Introduction

One of the aftereffects of the financial crisis is the increased awareness of the need for more advanced modeling in mathematical finance, and a focus of attention is on

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the problem of *model uncertainty*. This paper is motivated by a topic of this type. We consider a stochastic system described by a general Itô–Lévy process controlled by an agent. The performance functional is expressed as the Q-expectation of an integrated profit rate plus a terminal payoff, where Q is a probability measure equivalent to the original probability measure P, which is often called a reference measure. We may regard Q as a *scenario measure* controlled by the market or the environment. If $Q = P$, the problem becomes a classical stochastic control problem of the type studied in [1]. If Q is uncertain, however, the agent might seek the strategy which maximizes the performance in the worst possible choice of Q. This leads to a *stochastic differential game* between the agent and the market. Our approach is the following: We write the performance functional as the value at time $t = 0$ of the solution of an associated controlled *backward* stochastic differential equation (BSDE). Thus, we arrive at a (zero-sum) stochastic differential game of a system of *forward–backward* SDEs (FBSDEs) that we study by the maximum principle approach.

There are several papers of related content. Stochastic control of forward–backward SDEs (FBSDEs) has been studied in [2, 3], and in [4] a maximum principle for stochastic differential g-expectation games of SDEs is developed. The recent papers [5–9] also study optimal portfolio under model uncertainty by means of BSDEs. The approaches in the three latter papers are strongly linked to the exponential utility case. A key feature of the current paper is that it applies to general utility functions and also general dynamics for the state process.

Our paper is organized as follows. In Sect. 2, we state general stochastic maximum principles for stochastic differential games with partial information, both in the zero-sum case (finding conditions for saddle points) and for the nonzero sum games (finding conditions for Nash equilibria). The proofs are given in Appendix A. In Sect. 3, we consider stochastic control problems under model uncertainty, also called robust control problems. We formulate these problems as (zero-sum) stochastic differential games of *forward–backward* SDEs (FBSDEs), and we study them by the maximum principle approach of Sect. 2. In Sect. 4, we apply these techniques to study a robust optimal portfolio-consumption problem with penalty. We establish a connection between market viability under model uncertainty and equivalent martingale measures. Finally, we study the case with entropic penalty, and we prove a general reduction theorem, stating that any optimal portfolio-consumption problem under model uncertainty can be reduced to a classical portfolio-consumption problem under model certainty, with a change in the utility function. In particular, we obtain a connection to risk-sensitive control, and we show that model uncertainty increases the Arrow–Pratt risk aversion index.

2 Maximum Principles for Stochastic Differential Games of Forward–Backward Stochastic Differential Equations

2.1 The Case of General (Nonzero) Stochastic Differential Games

In this section, we formulate and prove a sufficient and necessary maximum principle for general stochastic differential games (not necessarily zero-sum games) of forward–backward SDEs. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space, where P is a reference probability measure. Consider a controlled *forward* SDE of the form

$$\begin{aligned} dX(t) &= dX^u(t) \\ &= b\big(t, X(t), u(t), \omega\big)\, dt + \sigma\big(t, X(t), u(t), \omega\big)\, dB(t) \\ &\quad + \int_{\mathbb{R}} \gamma\big(t, X\big(t^-\big), u(t), \zeta, \omega\big)\tilde{N}(dt, d\zeta), \quad 0 \leq t \leq T, \\ X(0) &= x \in \mathbb{R}, \end{aligned} \tag{1}$$

where B is a Brownian motion, and $\tilde{N}(dt, d\zeta) = N(dt, d\zeta) - \nu(d\zeta)\, dt$ is an independent compensated Poisson random measure, where ν is the Lévy measure of N such that $\int_{\mathbb{R}} \zeta^2 \nu(d\zeta) < \infty$. We assume that $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ is the P-augmentation of the natural filtration associated with B and N. Here $u = (u_1, u_2)$, where $u_i(t)$ is the control of player i, $i = 1, 2$. We assume that we are given two subfiltrations

$$\mathcal{E}_t^{(i)} \subseteq \mathcal{F}_t; \quad t \in [0, T], \tag{2}$$

representing the information available to player i at time t; $i = 1, 2$. We let $\mathcal{A}_i$ denote a given set of admissible control processes for player i, contained in the set of $\mathcal{E}_t^{(i)}$-predictable processes; $i = 1, 2$, with values in $A_i \subset \mathbf{R}^d$, $d \geq 1$. Denote $\mathbb{U} := A_1 \times A_2$. We assume that $b(\cdot, x, u, \omega)$, $\sigma(\cdot, , x, u, \omega)$, $\gamma(\cdot, , x, u, \zeta, \omega)$ are given predictable processes for each x in $\mathbb{R}$, u in $\mathbb{U}$, and ζ in $\mathbb{R}_0 := \mathbb{R}\backslash\{0\}$ such that (1) has a unique solution for each u in $\mathbb{U}$.

We consider the associated controlled *backward* SDEs (i.e., BSDEs) in the unknowns $(Y_i^u(t), Z_i^u(t), K_i^u(t, \zeta)) = (Y_i(t), Z_i(t), K_i(t, \zeta))$ of the form

$$\begin{aligned} dY_i(t) &= -g_i\big(t, X(t), Y_i(t), Z_i(t), K_i(t, \cdot), u(t), \omega\big)\, dt \\ &\quad + Z_i(t)\, dB(t) + \int_{\mathbb{R}} K_i(t, \zeta)\tilde{N}(dt, d\zeta), \quad 0 \leq t \leq T, \\ Y_i(T) &= h_i\big(X(T), \omega\big), \quad i = 1, 2. \end{aligned} \tag{3}$$

Here $g_i(\cdot, x, y, z, k, u, \omega)$ are given predictable processes for each x in $\mathbb{R}$, y in $\mathbb{R}$, z in $\mathbb{R}$, k in $\mathbb{R}^{\mathbb{R}_0}$, u in $\mathbb{U}$, ζ, and $h_i(x, \omega)$ is $\mathcal{F}_T$-measurable for each given x in $\mathbb{R}$, such that the BSDEs (3) have unique solutions for each u in $\mathbb{U}$.

Let $f_i(t, x, u) : [0, T] \times \mathbb{R} \times \mathbb{U} \to \mathbb{R}$, $\varphi_i(x) : \mathbb{R} \to \mathbb{R}$, and $\psi_i(x) : \mathbb{R} \to \mathbb{R}$ be given profit rates, bequest functions, and "risk evaluations" respectively, of player i, $i = 1, 2$. Define

$$J_i(u) := E\bigg[\int_0^T f_i\big(t, X^u(t), u(t), \omega\big)\, dt + \varphi_i\big(X^u(T), \omega\big) + \psi_i\big(Y_i^u(0)\big)\bigg], \quad i = 1, 2, \tag{4}$$

provided that the integrals and expectations exist. We call $J_i(u)$ the *performance functional* of player i, $i = 1, 2$. We assume that $b, \sigma, \gamma, g_i, h_i, f_i, \varphi_i$, and ψ_i are C^1 with respect to x, y, z, u and that

$$\psi_i'(x) \geq 0 \quad \text{for all } x, i = 1, 2. \tag{5}$$

A *Nash equilibrium* for the FBSDE game (1)–(4) is a pair $(\hat{u}_1, \hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$J_1(u_1, \hat{u}_2) \le J_1(\hat{u}_1, \hat{u}_2) \quad \text{for all } u_1 \in \mathcal{A}_1 \tag{6}$$

and

$$J_2(\hat{u}_1, u_2) \le J_2(\hat{u}_1, \hat{u}_2) \quad \text{for all } u_2 \in \mathcal{A}_2. \tag{7}$$

Heuristically, this means that player i has no incentive to deviate from the control $\hat{u}_i$, as long as player j $(j \ne i)$ does not deviate from $\hat{u}_j$, $i = 1, 2$. Therefore a Nash equilibrium is in some cases a likely outcome of a game. Suppose that there exists a Nash equilibrium $(\hat{u}_1, \hat{u}_2)$. We now present a method to find it, based on the maximum principle for stochastic control. Our result may be regarded as an extension of the maximum principles for FBSDEs in [2] and for (forward) SDE games in [4].

Define the *Hamiltonians*

$$H_i(t, x, y, z, k, u_1, u_2, \lambda, p, q, r) : [0, T] \times \mathbb{R}^3 \times \mathcal{R} \times \mathbb{U} \times \mathbb{R}^3 \times \mathcal{R} \to \mathbb{R}$$

of this game by

$$\begin{aligned} &H_i(t, x, y, z, k, u_1, u_2, \lambda, p, q, r) \\ &\quad := f_i(t, x, u_1, u_2) + \lambda g_i(t, x, y, z, k, u_1, u_2) + p b(t, x, u_1, u_2) \\ &\qquad + q\sigma(t, x, u_1, u_2) + \int_{\mathbb{R}} r(\zeta)\gamma(t, x, u_1, u_2, \zeta)\nu(d\zeta), \quad i = 1, 2, \end{aligned} \tag{8}$$

where $\mathcal{R}$ is the set of functions from $\mathbb{R}_0$ into $\mathbb{R}$ such that the integral in (8) converges.

We assume that H_i is Fréchet differentiable ($\mathcal{C}^1$) in the variables x, y, z, k, u, $i = 1, 2$, and that $\nabla_k H_i(t, \zeta)$ as a random measure is absolutely continuous with respect to ν, $i = 1, 2$. We also assume that H_i and its derivatives with respect to u_1 and u_2 are integrable with respect to P, $i = 1, 2$.

In the following, we are using the shorthand notation

$$\frac{\partial H_i}{\partial y}(t) = \frac{\partial H_i}{\partial y}\big(t, X(t), Y_i(t), Z_i(t), K_i(t, \cdot), u_1(t), u_2(t), \lambda_i(t), p_i(t), q_i(t), r_i(t, \cdot)\big)$$

and similarly for the other partial derivatives of H_i.

To these Hamiltonians we associate a system of FBSDEs in the adjoint processes $\lambda_i(t)$, $p_i(t)$, $q_i(t)$, and $r_i(t, \zeta)$ as follows:

1. Forward SDE in $\lambda_i(t)$:

$$\begin{cases} d\lambda_i(t) = \dfrac{\partial H_i}{\partial y}(t)\,dt + \dfrac{\partial H_i}{\partial z}(t)\,dB(t) + \displaystyle\int_{\mathbb{R}} \frac{d}{d\nu}\nabla_k H_i(t, \zeta)\tilde{N}(dt, d\zeta) \\ \qquad = \lambda_i(t)\left[\dfrac{\partial g_i}{\partial y}(t)\,dt + \dfrac{\partial g_i}{\partial z}(t)\,dB(t) + \displaystyle\int_{\mathbb{R}} \frac{d}{d\nu}\nabla_k g_i(t, \zeta)\tilde{N}(dt, d\zeta)\right], \\ \quad 0 \le t \le T, \\ \lambda_i(0) = \psi_i'\big(Y_i(0)\big)\left(= \dfrac{d\psi_i}{dy}\big(Y_i(0)\big)\right) \ge 0, \end{cases} \tag{9}$$

where $\frac{d}{d\nu}\nabla_k g_i(t,\zeta)$ is the Radon–Nikodym derivative of $\nabla_k g_i(t,\zeta)$ with respect to $\nu(\zeta)$.

2. Backward SDE in $p_i(t), q_i(t), r_i(t,\zeta)$:

$$\begin{cases} dp_i(t) = -\dfrac{\partial H_i}{\partial x}(t)\,dt + q_i(t)\,dB(t) + \displaystyle\int_{\mathbb{R}} r_i(t,\zeta)\tilde{N}(dt,d\zeta), \quad 0 \le t \le T, \\ p_i(T) = \varphi_i'\big(X(T)\big) + h_i'\big(X(T)\big)\lambda_i(T). \end{cases} \tag{10}$$

See Appendix A for an explanation of the gradient operator $\nabla_k H_i(t,\zeta) = \nabla_k H_i(t,\zeta)(\cdot)$.

Theorem 2.1 (Sufficient maximum principle for FBSDE games) *Let $(\hat{u}_1,\hat{u}_2) \in \mathcal{A}_1 \times \mathcal{A}_2$ with corresponding solutions $\hat{X}(t)$, $\hat{Y}_i(t)$, $\hat{Z}_i(t)$, $\hat{K}_i(t)$, $\hat{\lambda}_i(t)$, $\hat{p}_i(t)$, $\hat{q}_i(t)$, $\hat{r}_i(t,\zeta)$ of Eqs.* (1), (3), (9), *and* (10) *for $i = 1, 2$. Suppose that the following holds*:

- (*Concavity I*) *The functions $x \to h_i(x), x \to \varphi_i(x), x \to \psi_i(x)$ are concave, $i = 1, 2$.*
- (*The conditional maximum principle*)

$$\begin{aligned} &\operatorname{ess\,sup}_{v\in A_1} E\big[H_1\big(t,\hat{X}(t),\hat{Y}_1(t),\hat{Z}_1(t),\hat{K}_1(t,\cdot),\\ &\qquad v,\hat{u}_2(t),\hat{\lambda}_1(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,\cdot)\big) \mid \mathcal{E}_t^{(1)}\big]\\ &\quad = E\big[H_1\big(t,\hat{X}(t),\hat{Y}_1(t),\hat{Z}_1(t),\hat{K}_1(t,\cdot),\\ &\qquad \hat{u}_1(t),\hat{u}_2(t),\hat{\lambda}_1(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,\cdot)\big) \mid \mathcal{E}_t^{(1)}\big] \end{aligned} \tag{11}$$

and similarly

$$\begin{aligned} &\operatorname{ess\,sup}_{v\in A_2} E\big[H_2\big(t,\hat{X}(t),\hat{Y}_2(t),\hat{Z}_2(t),\hat{K}_2(t,\cdot),\\ &\qquad u_1(t),v,\hat{\lambda}_2(t),\hat{p}_2(t),\hat{q}_2(t),\hat{r}_2(t,\cdot)\big) \mid \mathcal{E}_t^{(2)}\big]\\ &\quad = E\big[H_2\big(t,\hat{X}(t),\hat{Y}_2(t),\hat{Z}_2(t),\hat{K}_2(t,\cdot),\\ &\qquad \hat{u}_1(t),\hat{u}_2(t),\hat{\lambda}_2(t),\hat{p}_2(t),\hat{q}_2(t),\hat{r}_2(t,\cdot)\big) \mid \mathcal{E}_t^{(2)}\big]. \end{aligned} \tag{12}$$

- (*Concavity II*) (*The Arrow conditions*) *The functions*

$$\begin{aligned} &\hat{\mathcal{H}}_1(x,y,z,k)\\ &\quad := \operatorname{ess\,sup}_{v_1\in A_1} E\big[H_1\big(t,x,y,z,k,v_1,\hat{u}_2(t),\hat{\lambda}_1(t),\hat{p}_1(t),\hat{q}_1(t),\hat{r}_1(t,\cdot)\big) \mid \mathcal{E}_t^{(1)}\big] \end{aligned}$$

and

$$\begin{aligned} &\hat{\mathcal{H}}_2(x,y,z,k)\\ &\quad := \operatorname{ess\,sup}_{v_2\in A_2} E\big[H_2\big(t,x,y,z,k,\hat{u}_1(t),v_2,\hat{\lambda}_2(t),\hat{p}_2(t),\hat{q}_2(t),\hat{r}_2(t,\cdot)\big) \mid \mathcal{E}_t^{(2)}\big] \end{aligned}$$

are concave for all t, a.s.

- *Assume that*

$$\frac{d}{d\nu}\nabla_k \hat{g}_i(t,\zeta) > -1$$

for $i=1,2$.
- *Moreover, assume that the following growth conditions hold*:

$$\begin{aligned}&E\biggl[\int_0^T \biggl\{\hat{p}_i^2(t)\biggl[\bigl(\sigma(t)-\hat{\sigma}(t)\bigr)^2+\int_{\mathbb{R}}\bigl(r_i(t,\zeta)-\hat{r}_i(t,\zeta)\bigr)^2\nu(d\zeta)\biggr]\\&\quad+\bigl(X(t)-\hat{X}(t)\bigr)^2\biggl[\hat{q}_i^2(t)+\int_{\mathbb{R}}\hat{r}_i^2(t,\zeta)\nu(d\zeta)\biggr]\\&\quad+\bigl(Y_i(t)-\hat{Y}_i(t)\bigr)^2\biggl[\biggl(\frac{\partial \hat{H}_i}{\partial z}\biggr)^2(t)+\int_{\mathbb{R}}\bigl\|\nabla_k \hat{H}_i(t,\zeta)\bigr\|^2\nu(d\zeta)\biggr]\\&\quad+\hat{\lambda}_i^2(t)\biggl[\bigl(Z_i(t)-\hat{Z}_i(t)\bigr)^2+\int_{\mathbb{R}}\bigl(K_i(t,\zeta)-\hat{K}_i(t,\zeta)\bigr)^2\nu(d\zeta)\biggr]\biggr\}dt\biggr]<\infty\\&\quad\textit{for } i=1,2.\end{aligned}\tag{13}$$

Then $\hat{u}(t)=(\hat{u}_1(t),\hat{u}_2(t))$ *is a Nash equilibrium for* (1)–(4).

Above and in the proof in Appendix A, we have used the following shorthand notation:

If $i=1$, then $X(t)=X^{(u_1,\hat{u}_2)}(t)$ and $Y_1(t)=Y_1^{(u_1,\hat{u}_2)}(t)$ are the processes corresponding to the control $u(t)=(u_1(t),\hat{u}_2(t))$, while $\hat{X}(t)=X^{\hat{u}}(t)$ and $\hat{Y}_1(t)=Y_1^{\hat{u}}(t)$ are those corresponding to the control $\hat{u}(t)=(\hat{u}_1(t),\hat{u}_2(t))$. An analogue notation is used for $i=2$. Moreover, we put

$$\frac{\partial \hat{H}_i}{\partial x}(t)=\frac{\partial H_i}{\partial x}\bigl(t,\hat{X}(t),\hat{Y}_i(t),\hat{Z}_i(t),\hat{K}_i(t,\cdot),\hat{u}(t),\hat{\lambda}_i(t),\hat{p}_i(t),\hat{q}_i(t),\hat{r}_i(t,\cdot)\bigr)$$

and similarly with $\frac{\partial \hat{H}_i}{\partial z}(t)$ and $\nabla_k \hat{H}_i(t,\zeta)$, $i=1,2$.

Proof See Appendix A. □

It is also of interest to prove a version of the maximum principle which does not require the concavity conditions. One such version is the following *necessary maximum principle* (Theorem 2.2) which requires the following assumptions:

- For all $t_0\in[0,T]$ and all bounded, $\mathcal{E}_t^{(i)}$-measurable random variables $\alpha_i(\omega)$, the control

$$\beta_i(t):=\chi_{(t_0,T)}(t)\alpha_i(\omega)\quad \text{belongs to } \mathcal{A}_i,\ i=1,2.\tag{14}$$

- For all $u_i,\beta_i\in\mathcal{A}_i$ with β_i bounded, there exists $\delta_i>0$ such that the control

$$\begin{aligned}&\tilde{u}_i(t):=u_i(t)+s\beta_i(t),\quad t\in[0,T],\\&\quad\text{belongs to } \mathcal{A}_i \text{ for all } s\in(-\delta_i,\delta_i),\ i=1,2.\end{aligned}\tag{15}$$

- The following derivative processes exist and belong to $L^2([0,T]\times\Omega)$:

$$\begin{aligned}
x_1(t) &= \frac{d}{ds}X^{(u_1+s\beta_1,u_2)}(t)\mid_{s=0},\\
y_1(t) &= \frac{d}{ds}Y_1^{(u_1+s\beta_1,u_2)}(t)\mid_{s=0},\\
z_1(t) &= \frac{d}{ds}Z_1^{(u_1+s\beta_1,u_2)}(t)\mid_{s=0},\\
k_1(t,\zeta) &= \frac{d}{ds}K_1^{(u_1+s\beta_1,u_2)}(t)\mid_{s=0},
\end{aligned} \tag{16}$$

  and, similarly,

$$x_2(t) = \frac{d}{ds}X^{(u_1,u_2+s\beta_2)}(t)\mid_{s=0}, \qquad \text{etc.}$$

Note that since $X^u(0)=x$ for all u, we have $x_i(0)=0$ for $i=1,2$.

In the following we write

$$\frac{\partial b}{\partial x}(t) \quad \text{for } \frac{\partial b}{\partial x}\big(t,X(t),u(t)\big), \qquad \text{etc.}$$

By (1) and (3) we have, using the estimates in [4],

$$\begin{aligned}
dx_1(t) &= \left\{\frac{\partial b}{\partial x}(t)x_1(t) + \frac{\partial b}{\partial u_1}(t)\beta_1(t)\right\}dt + \left\{\frac{\partial \sigma}{\partial x}(t)x_1(t) + \frac{\partial \sigma}{\partial u_1}(t)\beta_1(t)\right\}dB(t)\\
&\quad + \int_{\mathbb{R}}\left\{\frac{\partial \gamma}{\partial x}(t,\zeta)x_1(t) + \frac{\partial \gamma}{\partial u_1}(t,\zeta)\beta_1(t)\right\}\tilde{N}(dt,d\zeta),
\end{aligned} \tag{17}$$

$$\begin{aligned}
dy_1(t) &= -\left\{\frac{\partial g_1}{\partial x}(t)x_1(t) + \frac{\partial g_1}{\partial y}(t)y_1(t) + \frac{\partial g_1}{\partial z}(t)z_1(t)\right.\\
&\quad \left. + \int_{\mathbb{R}}\nabla_k g_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta) + \frac{\partial g_1}{\partial u_1}(t)\beta_1(t)\right\}dt\\
&\quad + z_i(t)\,dB(t) + \int_{\mathbb{R}}k_1(t,\zeta)\tilde{N}(dt,d\zeta), \quad 0\le t\le T,\\
y_1(T) &= h_1'\big(X^{(u_1,u_2)}(T)\big)x_1(T),
\end{aligned} \tag{18}$$

and similarly for $dx_2(t), dy_2(t)$.

We are now ready to state a necessary maximum principle, which is an extension of Theorem 3.1 in [4] and Theorem 3.1 in [2]. In the sequel, $\frac{\partial H}{\partial v}$ means $\nabla_v H$.

Theorem 2.2 (Necessary maximum principle) *Suppose that $u\in\mathcal{A}$ with corresponding solutions $X(t), Y_i(t), Z_i(t), K_i(t,\zeta), \lambda_i(t), p_i(t), q_i(t), r_i(t,\zeta)$ of Eqs.* (1), (3), (9) *and* (10). *Suppose that* (14), (15), *and* (16) *hold.*

Moreover, assume that

$$
\begin{aligned}
&E\bigg[\int_0^T \bigg\{p_i^2(t)\bigg[\bigg(\frac{\partial\sigma}{\partial x}(t)x_i(t)+\frac{\partial\sigma}{\partial u_i}(t)\beta_i(t)\bigg)^2 \\
&\quad+\int_{\mathbb{R}}\bigg(\frac{\partial\gamma}{\partial x}(t,\zeta)x_i(t)+\frac{\partial\gamma}{\partial u_i}(t,\zeta)\beta_i(t)\bigg)^2\nu(d\zeta)\bigg] \\
&\quad+x_i^2(t)\bigg(q_i^2(t)+\int_{\mathbb{R}}r_i^2(t,\zeta)\nu(d\zeta)\bigg)+\lambda_i^2(t)\bigg(z_i^2(t)+\int_{\mathbb{R}}k_i^2(t,\zeta)\nu(d\zeta)\bigg) \\
&\quad+y_i^2(t)\bigg(\bigg(\frac{\partial H_i}{\partial z}\bigg)^2(t)+\int_{\mathbb{R}}\big\|\nabla_k H_i(t,\zeta)\big\|^2\nu(d\zeta)\bigg)\bigg\}\bigg]dt<\infty \quad \text{for } i=1,2.
\end{aligned}
\tag{19}
$$

Then the following are equivalent:

(1) $\dfrac{d}{ds}J_1(u_1+s\beta_1,u_2)\mid_{s=0}=\dfrac{d}{ds}J_2(u_1,u_2+s\beta_2)\mid_{s=0}=0$

for all bounded $\beta_1\in\mathcal{A}_1$, $\beta_2\in\mathcal{A}_2$.

(2)
$$
\begin{aligned}
&E\bigg[\frac{\partial}{\partial v_1}H_1\big(t,X(t),Y_1(t),Z_1(t),K_1(t,\cdot), \\
&\qquad v_1,u_2(t),\lambda_1(t),p_1(t)q_1(t),r_1(t,\cdot)\big)\mid\mathcal{E}_t^{(1)}\bigg]_{v_1=u_1(t)} \\
&=E\bigg[\frac{\partial}{\partial v_2}H_2\big(t,X(t),Y_2(t),Z_2(t),K_2(t,\cdot), \\
&\qquad u_1(t),v_2,\lambda_2(t),p_2(t),q_2(t),r_2(t,\cdot)\big)\mid\mathcal{E}_t^{(2)}\bigg]_{v_2=u_2(t)} \\
&=0.
\end{aligned}
$$

Proof See Appendix A. □

2.2 The Zero-Sum Game Case

In the *zero-sum case* we have

$$J_1(u_1,u_2)+J_2(u_1,u_2)=0. \tag{20}$$

Then the Nash equilibrium $(\hat{u}_1,\hat{u}_2)\in\mathcal{A}_1\times\mathcal{A}_2$ satisfying (6)–(7) becomes a *saddle point* for $J(u_1,u_2):=J_1(u_1,u_2)$. To see this, note that (6)–(7) imply that

$$J_1(u_1,\hat{u}_2)\le J_1(\hat{u}_1,\hat{u}_2)=-J_2(\hat{u}_1,\hat{u}_2)\le -J_2(\hat{u}_1,u_2)$$

and hence

$$J(u_1,\hat{u}_2)\le J(\hat{u}_1,\hat{u}_2)\le J(\hat{u}_1,u_2) \quad \text{for all } u_1,u_2.$$

From this we deduce that

$$\begin{aligned}\inf_{u_2\in\mathcal{A}_2}\sup_{u_1\in\mathcal{A}_1} J(u_1,u_2) &\le \sup_{u_1\in\mathcal{A}_1} J(u_1,\hat{u}_2) \le J(\hat{u}_1,\hat{u}_2)\\ &\le \inf_{u_2\in\mathcal{A}_2} J(\hat{u}_1,u_2) \le \sup_{u_1\in\mathcal{A}_1}\inf_{u_2\in\mathcal{A}_2} J(u_1,u_2).\end{aligned} \tag{21}$$

Since we always have $\inf\sup \ge \sup\inf$, we conclude that

$$\begin{aligned}\inf_{u_2\in\mathcal{A}_2}\sup_{u_1\in\mathcal{A}_1} J(u_1,u_2) &= \sup_{u_1\in\mathcal{A}_1} J(u_1,\hat{u}_2) = J(\hat{u}_1,\hat{u}_2)\\ &= \inf_{u_2\in\mathcal{A}_2} J(\hat{u}_1,u_2) = \sup_{u_1\in\mathcal{A}_1}\inf_{u_2\in\mathcal{A}_2} J(u_1,u_2),\end{aligned} \tag{22}$$

i.e., $(\hat{u}_1,\hat{u}_2)\in\mathcal{A}_1\times\mathcal{A}_2$ is a *saddle point* for $J(u_1,u_2)$.

In this case, only one Hamiltonian is needed, and only one set of adjoint equations. Indeed, let $g_1=g_2=:g$, $h_1=h_2=:h$, $f_1=-f_2:=f$, $\varphi_1=-\varphi_2:=\varphi$, and $\psi_1=-\psi_2:=\psi$. Then

$$\begin{aligned}&H_1(t,x,y,z,k,u_1,u_2,\lambda,p,q,r)\\ &\quad= f(t,x,u_1,u_2)+\lambda g(t,x,y,z,k,u_1,u_2)+pb(t,x,u_1,u_2)\\ &\qquad+q\sigma(t,x,u_1,u_2)+\int_{\mathbb{R}} r(\zeta)\gamma(t,x,u_1,u_2,\zeta)\nu(d\zeta)\end{aligned} \tag{23}$$

and

$$\begin{aligned}&H_2(t,x,y,z,k,u_1,u_2,\lambda,p,q,r)\\ &\quad= -f(t,x,u_1,u_2)+\lambda g(t,x,y,z,k,u_1,u_2)+pb(t,x,u_1,u_2)\\ &\qquad+q\sigma(t,x,u_1,u_2)+\int_{\mathbb{R}} r(\zeta)\gamma(t,x,u_1,u_2,\zeta)\nu(d\zeta).\end{aligned} \tag{24}$$

For $u=(u_1,u_2)\in\mathcal{A}_1\times\mathcal{A}_2$, let $X^u(t)$ be defined by (1), and $(Y^u(t),Z^u(t),K^u(t,\zeta))$ be defined by (3) with $g_i=g$ and $h_i=h$. The adjoint processes λ_i, $i=1,2$, satisfy

$$\begin{cases} d\lambda_i(t)=\lambda_i(t)\Big[\dfrac{\partial g}{\partial y}\big(t,X(t),Y(t),Z(t),K(t,\cdot),u_1(t),u_2(t)\big)\,dt\\ \qquad\qquad+\dfrac{\partial g}{\partial z}\big(t,X(t),Y(t),Z(t),K(t,\cdot),u_1(t),u_2(t)\big)\,dB(t)\\ \qquad\qquad+\displaystyle\int_{\mathbb{R}}\frac{d\nabla_k g(t,\zeta)}{d\nu(\zeta)}\tilde{N}(dt,d\zeta)\Big],\quad 0\le t\le T,\\ \lambda_i(0)=\psi_i'(Y(0)).\end{cases} \tag{25}$$

It follows that

$$\lambda_2(t)=-\lambda_1(t),\quad t\in[0,T]. \tag{26}$$

The adjoint processes for p_i, q_i, and $r_i,\ i = 1, 2$, become by (10)

$$\begin{cases} dp_1(t) = -\Big[\frac{\partial f}{\partial x}(t) + \lambda_1(t)\frac{\partial g}{\partial x}(t) + p_1(t)\frac{\partial b}{\partial x}(t) + q_1(t)\frac{\partial \sigma}{\partial x}(t) \\ \qquad + \int_{\mathbb{R}} r_1(t,\zeta)\frac{\partial \gamma}{\partial x}(t,\zeta)\nu(d\zeta)\Big]dt \\ \qquad + q_1(t)\,dB(t) + \int_{\mathbb{R}} r_1(t,\zeta)\tilde{N}(dt,d\zeta), \quad 0 \le t \le T, \\ p_1(T) = \varphi_1'\big(X(T)\big) + h_1'\big(X(T)\big)\lambda_1(T), \end{cases} \tag{27}$$

and, by (10) and (26),

$$\begin{cases} dp_2(t) = -\Big[-\frac{\partial f}{\partial x}(t) - \lambda_1(t)\frac{\partial g}{\partial x}(t) + p_2(t)\frac{\partial b}{\partial x}(t) + q_2(t)\frac{\partial \sigma}{\partial x}(t) \\ \qquad + \int_{\mathbb{R}} r_2(t,\zeta)\frac{\partial \gamma}{\partial x}(t,\zeta)\nu(d\zeta)\Big]dt \\ \qquad + q_2(t)\,dB(t) + \int_{\mathbb{R}} r_2(t,\zeta)\tilde{N}(dt,d\zeta), \quad 0 \le t \le T, \\ p_2(T) = -\varphi_1'\big(X(T)\big) - h_1'\big(X(T)\big)\lambda_1(T). \end{cases} \tag{28}$$

Thus, we see that

$$\big(p_2(t), q_2(t), r_2(t)\big) = -\big(p_1(t), q_1(t), r_1(t)\big), \quad t \in [0, T]. \tag{29}$$

Consequently,

$$\begin{aligned} &- H_2\big(t, X(t), Y(t), Z(t), K(t,\cdot), u_1(t), u_2(t), \lambda_2(t), p_2(t), q_2(t), r_2(t,\cdot)\big) \\ &\quad = H_1\big(t, X(t), Y(t), Z(t), K(t,\cdot), u_1(t), u_2(t), \lambda_1(t), p_1(t), q_1(t), r_1(t,\cdot)\big). \end{aligned}$$

We thus conclude that in the zero-sum game case, we only need one Hamiltonian and one quadruple of controlled adjoint processes. In the following, we set:

$$\begin{aligned} &J(u_1, u_2) := E\Big[\int_0^T f\big(t, X^u(t), u(t)\big)\,dt + \varphi\big(X^{(u)}(T)\big) + \psi\big(Y^u(0)\big)\Big], \\ &H := H_1 \quad \text{as defined in (23), and} \\ &\big(\lambda(t), p(t), q(t), r(t,\cdot)\big) := \big(\lambda_1(t), p_1(t), q_1(t), r_1(t,\cdot)\big) \quad \text{as defined in (25), (27).} \end{aligned} \tag{30}$$

We can now state the necessary and sufficient maximum principles for the zero-sum game:

Theorem 2.3 (Necessary maximum principle for zero-sum forward–backward games) *Assume that the conditions of Theorem 2.2 hold. Then the following are equivalent*:

(1) $\frac{d}{ds}J(u_1 + s\beta_1, u_2)\mid_{s=0} = \frac{d}{ds}J(u_1, u_2 + s\beta_2)\mid_{s=0} = 0$ (31)

for all bounded $\beta_1 \in \mathcal{A}_1, \beta_2 \in \mathcal{A}_2$.

$$
\begin{aligned}
(2)\quad & E\bigg[\frac{\partial}{\partial v_1} H\big(t, X(t), Y(t), Z(t), K(t,\cdot), \\
&\quad v_1, u_2(t), \lambda(t), p(t), q(t), r(t,\cdot)\big) \mid \mathcal{E}_t^{(1)}\bigg]_{v_1=u_1(t)} \\
&= E\bigg[\frac{\partial}{\partial v_2} H\big(t, X(t), Y(t), Z(t), K(t,\cdot), \\
&\quad u_1(t), v_2, \lambda(t), p(t), q(t), r(t,\cdot)\big) \mid \mathcal{E}_t^{(2)}\bigg]_{v_2=u_2(t)} \\
&= 0.
\end{aligned} \tag{32}
$$

Proof The proof is similar to that of Theorem 2.2 and is omitted. □

Corollary 2.1 *Let $u=(u_1,u_2)\in\mathcal{A}_1\times\mathcal{A}_2$ be a Nash equilibrium (saddle point) for the zero-sum game in Theorem 2.3. Then (32) holds.*

Proof This follows from Theorem 2.3 by noting that if $u=(u_1,u_2)$ is a Nash equilibrium, then (31) holds by (22). □

Similarly, we get

Theorem 2.4 (Sufficient maximum principle for zero-sum forward–backward games) *Let $(\hat{u}_1,\hat{u}_2)\in\mathcal{A}_1\times\mathcal{A}_2$, with corresponding solutions $\hat{X}(t)$, $\hat{Y}(t)$, $\hat{Z}(t)$, $\hat{K}(t)$, $\hat{\lambda}(t)$, $\hat{p}(t)$, $\hat{q}(t)$, $\hat{r}(t,\zeta)$. Suppose that the following hold:*

- *The functions $x\to\varphi(x)$ and $x\to\psi(x)$ are affine, and the function $x\to h(x)$ is concave.*
- *(The conditional maximum principle)*

$$
\begin{aligned}
&\operatorname{ess\,sup}_{v_1\in A_1} E\big[H\big(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \\
&\qquad v_1, \hat{u}_2(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\big) \mid \mathcal{E}_t^{(1)}\big] \\
&\quad = E\big[H\big(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \\
&\qquad \hat{u}_1(t), \hat{u}_2(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\big) \mid \mathcal{E}_t^{(1)}\big]
\end{aligned} \tag{33}
$$

and

$$
\begin{aligned}
&\operatorname{ess\,inf}_{v_2\in A_2} E\big[H\big(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \\
&\qquad \hat{u}_1(t), v_2, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\big) \mid \mathcal{E}_t^{(2)}\big] \\
&\quad = E\big[H\big(t, \hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \\
&\qquad \hat{u}_1(t), \hat{u}_2(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\big) \mid \mathcal{E}_t^{(2)}\big].
\end{aligned} \tag{34}
$$

- (*The Arrow conditions*) *The function*

$$\hat{\mathcal{H}}(x,y,z,k) := \operatorname{ess\,sup}_{v_1\in A_1} E\big[H\big(t,x,y,z,k,v_1,\hat{u}_2(t),\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)\big) \mid \mathcal{E}_t^{(1)}\big]$$

is concave, *and the function*

$$\overset{\vee}{\mathcal{H}}(x,y,z,k) := \operatorname{ess\,inf}_{v_2\in A_2} E\big[H\big(t,x,y,z,k,\hat{u}_1(t),v_2,\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)\big) \mid \mathcal{E}_t^{(2)}\big]$$

is convex for all $t\in[0,T]$, *a.s.*
- *Assume that*

$$\frac{d\nabla_k g(t,\zeta)}{d\nu(\zeta)} \geq -1.$$

- *The growth condition* (13) *holds with* $\hat{p}_i=\hat{p}$, *etc.*

Then $\hat{u}(t)=(\hat{u}_1(t),\hat{u}_2(t))$ *is a saddle point for* $J(u_1,u_2)$.

Proof The proof is similar to that of Theorem 2.1 and is omitted. □

3 Stochastic Control under Model Uncertainty

Let $X(t)=X_x^v(t)$ be a controlled Itô–Lévy process of the form

$$\begin{aligned} dX(t) &= b\big(t,X(t),v(t),\omega\big)\,dt+\sigma\big(t,X(t),v(t),\omega\big)\,dB(t) \\ &\quad + \int_{\mathbb{R}} \gamma\big(t,X\big(t^-\big),v(t),\zeta,\omega\big)\tilde{N}(dt,d\zeta), \quad 0\leq t\leq T, \\ X(0) &= x\in\mathbb{R}, \end{aligned} \tag{35}$$

where $v(\cdot)$ is the control process, and b,σ, and γ are as in Sect. 2.1.

We consider a *model uncertainty* setup, represented by a probability measure $Q=Q^\theta$ which is equivalent to P, with the Radon–Nikodym derivative on $\mathcal{F}_t$ given by

$$\frac{d(Q\mid\mathcal{F}_t)}{d(P\mid\mathcal{F}_t)} = G^\theta(t), \tag{36}$$

where, for $0\leq t\leq T$, $G^\theta(t)$ is a martingale of the form

$$dG^\theta(t) = G^\theta\big(t^-\big)\bigg[\theta_0(t)\,dB(t)+\int_{\mathbb{R}}\theta_1(t,\zeta)\tilde{N}(dt,d\zeta)\bigg]; \quad G^\theta(0)=1. \tag{37}$$

Here $\theta=(\theta_0,\theta_1)$ may be regarded as a *scenario control*. Let $\mathcal{A}_1$ denote a given family of admissible controls v, and $\mathcal{A}_2$ denote a given set of admissible scenario controls θ such that $E[\int_0^T\{|\theta_0^2(t)|+\int_{\mathbb{R}}\theta_1^2(t,\zeta)\nu(d\zeta)\}dt]<\infty$ and $\theta_1(t,\zeta)\geq -1+\epsilon$ for some $\epsilon > 0$. Let $\mathcal{E}^{(1)}_{0\le t\le T}$ and $\mathcal{E}^{(2)}_{0\le t\le T}$ be given subfiltrations of $\mathcal{F}_{0\le t\le T}$, representing the information available to the controllers at time t. It is required that $v \in \mathcal{A}_1$ be $\mathcal{E}^1_t$-predictable, and $\theta \in \mathcal{A}_2$ be $\mathcal{E}^2_t$-predictable. We set $u = (v, \theta)$.

We consider the stochastic differential game to find $(\hat{v}, \hat{\theta}) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\sup_{v\in\mathcal{A}_1} \inf_{\theta\in\mathcal{A}_2} E_{Q^\theta}\big[W(v,\theta)\big] = E_{Q^{\hat{\theta}}}\big[W(\hat{v},\hat{\theta})\big] = \inf_{\theta\in\mathcal{A}_2} \sup_{v\in\mathcal{A}_1} E_{Q^\theta}\big[W(v,\theta)\big], \tag{38}$$

where

$$W(v,\theta) = U\big(X^v(T)\big) + \int_0^T F\big(s, X^v(s), v(s), \theta(s)\big)\, ds, \tag{39}$$

where U and F are given functions.

For example, U is a given utility function, and $F(t,x,v,\theta) = U_1(t,x,v) + \rho(\theta)$, with U_1 a utility function and ρ a convex function. The term $E_{Q^\theta}[\int_0^T \rho(\theta(t))\, dt]$ can then be seen as a penalty term, penalizing the difference between Q^θ and the original probability measure P.

We have

$$E_{Q^\theta}\big[W(v,\theta)\big] = E\bigg[G^\theta(T) U\big(X^v(T)\big) + \int_0^T G^\theta(s) F\big(s, X^v(s), u(s)\big)\, ds\bigg]. \tag{40}$$

We now define $Y(t) = Y^{v,\theta}(t)$ by

$$Y(t) = E\bigg[\frac{G^\theta(T)}{G^\theta(t)} U\big(X^v(T)\big) + \int_t^T \frac{G^\theta(s)}{G^\theta(t)} F\big(s, X^v(s), u(s)\big)\, ds \mid \mathcal{F}_t\bigg], \quad t \in [0,T]. \tag{41}$$

Then we recognize $Y(t)$ as the solution of the linear BSDE (see Lemma B.1)

$$\begin{aligned} dY(t) = &-\bigg[F\big(t, X^v(t), u(t)\big) + \theta_0(t) Z(t) + \int_{\mathbb{R}} \theta_1(t,\zeta) K(t,\zeta) \nu(d\zeta)\bigg] dt \\ &+ Z(t)\, dB(t) + \int_{\mathbb{R}} K(t,\zeta) \tilde{N}(dt, d\zeta), \quad 0 \le t \le T, \\ Y(T) = &\; U\big(X^v(T)\big). \end{aligned} \tag{42}$$

Note that

$$Y(0) = Y^{v,\theta}(0) = E_{Q^\theta}\big[W(v,\theta)\big]. \tag{43}$$

Therefore, problem (38) can be written as

$$\sup_{v\in\mathcal{A}_1} \inf_{\theta\in\mathcal{A}_2} Y^{v,\theta}(0) = Y^{\hat{v},\hat{\theta}}(0) = \inf_{\theta\in\mathcal{A}_2} \sup_{v\in\mathcal{A}_1} Y^{v,\theta}(0), \tag{44}$$

where $Y^{v,\theta}(t)$ is given by the forward–backward system (35) and (42). This is a zero-sum *stochastic differential game (SDG) of forward–backward SDEs* of the form (30) with $f = \varphi = 0$, $\psi = Id$, and $h(x) = U(x)$.

Proceeding as in Sect. 2, define the Hamiltonian

$$H:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}_0\times\mathcal{R}\times A_1\times A_2\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathcal{R}\to\mathbb{R}$$

by

$$\begin{aligned}
&H(t,x,y,z,k,v,\theta,\lambda,p,q,r)\\
&\quad:=\left[F(t,x,u)+\theta_0 z+\int_{\mathbb{R}}\theta_1(t,\zeta)k(t,\zeta)\nu(d\zeta)\right]\lambda\\
&\qquad+b(t,x,v)p+\sigma(t,x,v)q+\int_{\mathbb{R}}\gamma(t,x,v,\zeta)r(\zeta)\nu(d\zeta).
\end{aligned}\tag{45}$$

Define a pair of FBSDEs in the adjoint processes $\lambda(t), p(t), q(t), r(t,\zeta)$ as follows.

Forward SDE for $\lambda(t)$:

$$\begin{aligned}
d\lambda(t)&=\frac{\partial H}{\partial y}(t)\,dt+\frac{\partial H}{\partial z}(t)\,dB(t)+\int_{\mathbb{R}}\frac{d\nabla_k H_i(t,\zeta)}{d\nu(\zeta)}\tilde{N}(dt,d\zeta)\\
&=\lambda(t)\theta_0(t)\,dB(t)+\lambda(t)\int_{\mathbb{R}}\theta_1(t,\zeta)(\cdot)\tilde{N}(dt,d\zeta),\quad t\in[0,T],\\
\lambda(0)&=1.
\end{aligned}\tag{46}$$

Backward SDE for $(p(t),q(t),r(t,\zeta))$:

$$\begin{aligned}
dp(t)&=-\frac{\partial H}{\partial x}(t)\,dt+q(t)\,dB(t)+\int_{\mathbb{R}}r(t,\zeta)\tilde{N}(dt,d\zeta)\\
&=-\left\{\frac{\partial F}{\partial x}(t)\lambda(t)+p(t)\frac{\partial b}{\partial x}(t)+q(t)\frac{\partial\sigma}{\partial x}(t)+\int_{\mathbb{R}}r(t,\zeta)\frac{\partial\gamma}{\partial x}(t,\zeta)\nu(d\zeta)\right\}dt\\
&\quad+q(t)\,dB(t)+\int_{\mathbb{R}}r(t,\zeta)\tilde{N}(dt,d\zeta),\quad t\in[0,T],\\
p(T)&=\lambda(T)U'\big(X(T)\big).
\end{aligned}\tag{47}$$

Here we have used the abbreviated notation

$$\frac{\partial H}{\partial y}(t)=\frac{\partial H}{\partial y}\big(t,X(t),Y(t),Z(t),K(t,\cdot),v(t),\theta(t),\lambda(t),p(t),q(t),r(t,\cdot)\big)$$

and similarly for the other partial derivatives. We now present a necessary maximum principle for the forward–backward stochastic differential game (35), (42), (44) by adapting Theorem 2.3 to this case.

Theorem 3.1 *Suppose that the conditions of Theorem 2.2 hold. Let $\hat{u}=(\hat{v},\hat{\theta})\in\mathcal{A}_1\times\mathcal{A}_2$, with corresponding solutions $\hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)$*

of Eqs. (35), (42), (47), *and* (46). *Suppose that* (44) *holds, together with* (13). *Then the following holds*:

$$E\left[\hat{\lambda}(t)\frac{\partial F}{\partial v}\big(t,\hat{X}(t),\hat{u}(t)\big)+\hat{p}(t)\frac{\partial b}{\partial v}\big(t,\hat{X}(t),\hat{v}(t)\big)\right.$$

$$\left.+\hat{q}(t)\frac{\partial \sigma}{\partial v}\big(t,\hat{X}(t),\hat{v}(t)\big)+\int_{\mathbb{R}}\hat{r}(t,\zeta)\frac{\partial \gamma}{\partial v}\big(t,\hat{X}(t),\hat{v}(t),\zeta\big)\nu(d\zeta)\mid \mathcal{E}_t^{(1)}\right]=0,$$

$$E\left[\hat{\lambda}(t)\left(\frac{\partial F}{\partial \theta_0}\big(t,\hat{X}(t),\hat{u}(t)\big)+\hat{Z}(t)\right)\mid \mathcal{E}_t^{(2)}\right]=0,$$

$$E\left[\hat{\lambda}(t)\left(\nabla_{\theta_1}F\big(t,\hat{X}(t),\hat{u}(t)\big)+\int_{\mathbb{R}}(\cdot)\hat{K}(t,\zeta)\nu(d\zeta)\right)\mid \mathcal{E}_t^{(2)}\right]=0.$$

Note that both $\nabla_{\theta_1}F$ and $\int_{\mathbb{R}}(\cdot)\hat{K}(t,\zeta)\nu(d\zeta)$ are linear functionals, the latter being defined by the action

$$\varphi\to\int_{\mathbb{R}}\varphi(\zeta)\hat{K}(t,\zeta)\nu(d\zeta)$$

for all bounded continuous functions $\varphi:\mathbb{R}_0\mapsto\mathbb{R}$.

4 Robust Optimal Portfolio and Consumption with Penalty

We now apply this to the following portfolio problem under model uncertainty. We restrict here ourselves to the case $\mathcal{E}_t^{(1)}=\mathcal{E}_t^{(2)}=\mathcal{F}_t$, $t\in[0,T]$.

Consider a financial market consisting of a bond with unit price $S_0(t)=1$, $0\le t\le T$, and a stock, with unit price $S(t)$ given by

$$dS(t)=S\big(t^-\big)\left[b_0(t)\,dt+\sigma_0(t)\,dB(t)+\int_{\mathbb{R}}\gamma_0(t,\zeta)\tilde{N}(dt,d\zeta)\right],\tag{48}$$

where $b_0(t)=b_0(t,\omega)$, $\sigma_0(t)=\sigma_0(t,\omega)$, and $\gamma_0(t,\zeta)=\gamma_0(t,\zeta,\omega)$ are given $\{\mathcal{F}_t\}$-predictable processes such that $\gamma_0\ge -1+\epsilon$ for some $\epsilon>0$, and

$$E\left[\int_0^T\left\{\big|b_0(t)\big|+\sigma_0^2(t)+\int_{\mathbb{R}}\gamma_0^2(t,\zeta)\nu(d\zeta)\right\}dt\right]<\infty.$$

Note that this system is non-Markovian since the coefficients are random processes.

Let $X(t)=X^{\pi,c}(t)$ be the wealth process corresponding to a portfolio $\pi(t)$ and a consumption rate $c(t)$, i.e.,

$$\begin{cases}dX(t)=\pi(t)\left[b_0(t)\,dt+\sigma_0(t)\,dB(t)+\int_{\mathbb{R}}\gamma_0(t,\zeta)\tilde{N}(dt,d\zeta)\right]-c(t)\,dt,\\ \quad t\in[0,T],\\ X(0)=x\in\mathbb{R}.\end{cases}\tag{49}$$

For π and c to be admissible, we require that $X^{\pi,c}\ge 0$ for all $t\in[0,T]$.

We consider the stochastic differential game to find $(\hat{\pi}, \hat{c}, \hat{\theta}) \in \mathcal{A}_1 \times \mathcal{A}_2$ such that

$$\sup_{\pi,c\in\mathcal{A}_1} \inf_{\theta\in\mathcal{A}_2} E_{Q^\theta}\big[W(\pi,c,\theta)\big] = E_{Q^{\hat{\theta}}}\big[W(\hat{\pi},\hat{c},\hat{\theta})\big] = \inf_{\theta\in\mathcal{A}_2} \sup_{\pi,c\in\mathcal{A}_1} E_{Q^\theta}\big[W(\pi,c,\theta)\big] \tag{50}$$

with

$$W(\pi,c,\theta) = U\big(X^{\pi,c}(T)\big) + \int_0^T U_1\big(c(s)\big)\,ds + \int_0^T \rho\big(\theta(s)\big)\,ds, \tag{51}$$

where U and U_1 are utility functions, and ρ is a convex function. We have seen in Sect. 3 that this problem can be written as

$$\sup_{\pi,c\in\mathcal{A}_1} \inf_{\theta\in\mathcal{A}_2} Y^{\pi,c,\theta}(0) = Y^{\hat{\pi},\hat{c},\hat{\theta}}(0) = \inf_{\theta\in\mathcal{A}_2} \sup_{\pi,c\in\mathcal{A}_1} Y^{\pi,c,\theta}(0), \tag{52}$$

where $Y(t) = Y^{\pi,c,\theta}(t)$ is given by

$$dY(t) = -\bigg[U_1\big(c(t)\big) + \rho\big(\theta(t)\big) + \theta_0(t)Z(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)K(t,\zeta)\nu(d\zeta)\bigg]dt$$

$$+ Z(t)\,dB(t) + \int_{\mathbb{R}} K(t,\zeta)\tilde{N}(dt,d\zeta), \quad t\in[0,T], \tag{53}$$

$$Y(T) = U\big(X(T)\big). \tag{54}$$

In particular,

$$Y^{\pi,c,\theta}(0) = E\bigg[G^\theta(T)U\big(X(T)\big) + \int_0^T G^\theta(s)\big[U_1\big(c(s)\big) + \rho\big(\theta(s)\big)\big]\,ds\bigg]. \tag{55}$$

The Hamiltonian for the problem (52) is, by (45),

$$\begin{aligned} &H(t,x,y,z,k,\pi,c,\theta,\lambda,p,q,r) \\ &\quad = \bigg[U_1(c) + \rho(\theta) + \theta_0 z + \int_{\mathbb{R}} \theta_1(\zeta)k(\zeta)\nu(d\zeta)\bigg]\lambda \\ &\qquad + \big(\pi b_0(t)\big) - c)p + \pi\sigma_0(t)q + \pi\int_{\mathbb{R}} \gamma_0(t,\zeta)r(\zeta)\nu(d\zeta). \end{aligned} \tag{56}$$

The forward SDE for $\lambda(t) = \lambda^\theta(t)$ is

$$\begin{aligned} &d\lambda(t) = \lambda(t)\bigg[\theta_0(t)\,dB(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)\tilde{N}(dt,d\zeta)\bigg], \quad t\in[0,T], \\ &\lambda(0) = 1. \end{aligned} \tag{57}$$

By comparing (37) and (57), we see that

$$\lambda(t) = G^\theta(t), \quad t\in[0,T]. \tag{58}$$

The BSDE for $(p^{\pi,c,\theta}(t), q^{\pi,c,\theta}(t), r^{\pi,c,\theta}(t,\zeta)) = (p(t), q(t), r(t,\zeta))$ is (see (46)–(47))

$$\begin{aligned} dp(t) &= q(t)\,dB(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(dt,d\zeta), \quad t \in [0,T], \\ p(T) &= \lambda(T)U'\big(X(T)\big) = G^{\theta}(T)U'\big(X(T)\big). \end{aligned} \tag{59}$$

We get

$$p(t) = E\big[G^{\theta}(T)U'\big(X(T)\big) \mid \mathcal{F}_t\big] > 0, \quad t \in [0,T]. \tag{60}$$

4.1 Viability and Martingale Measures

We now apply the necessary maximum principle given by Theorem 3.1. Maximizing H with respect to π and c and minimizing H with respect to $\theta = (\theta_0, \theta_1)$ gives the following first-order conditions for the optimal portfolio π, the optimal consumption rate c, and the optimal scenario parameter $\theta = (\theta_0, \theta_1)$:

$$b_0(t)p(t) + \sigma_0(t)q(t) + \int_{\mathbb{R}} \gamma_0(t,\zeta)r(t,\zeta)\nu(d\zeta) = 0, \tag{61}$$

$$U_1'\big(c(t)\big)G^{\theta}(t) = p(t) = E\big[G^{\theta}(T)U'\big(X(T)\big) \mid \mathcal{F}_t\big], \tag{62}$$

$$\frac{\partial \rho}{\partial \theta_0}\big(\theta(t)\big) = -Z(t), \tag{63}$$

$$\nabla_{\theta_1}\rho\big(\theta(t)\big)(\cdot) = -\int_{\mathbb{R}} (\cdot)K(t,\zeta)\nu(d\zeta). \tag{64}$$

Equation (61) can be written as

$$b_0(t) + \sigma_0(t)\frac{q(t)}{p(t)} + \int_{\mathbb{R}} \gamma_0(t,\zeta)\frac{r(t,\zeta)}{p(t)}\nu(d\zeta) = 0. \tag{65}$$

By the Girsanov theorem (see, e.g., [1], Chap. 1), this means that the measure $\tilde{Q}$ on $\mathcal{F}_T$, defined by

$$d\tilde{Q}(\omega) := R(T)\,dP(\omega), \tag{66}$$

with $R(t) = R^{\pi,c,\theta}(t)$, $t \in [0,T]$, given by

$$\begin{aligned} dR(t) &= R\big(t^-\big)\left[\frac{q(t)}{p(t^-)}\,dB(t) + \int_{\mathbb{R}} \frac{r(t,\zeta)}{p(t^-)}\tilde{N}(dt,d\zeta)\right], \quad t \in [0,T], \\ R(0) &= 1, \end{aligned} \tag{67}$$

is an equivalent local martingale measure (ELMM) for the market price process $S(t)$ given by (48). By (59) and (67) we get

$$\begin{aligned} R\big(t^-\big)dp(t) &= R\big(t^-\big)\left[q(t)\,dB(t) + \int_{\mathbb{R}} r(t,\zeta)\tilde{N}(dt,d\zeta)\right] \\ &= p\big(t^-\big)dR(t), \end{aligned}$$

i.e.,

$$\frac{dp(t)}{p(t^-)} = \frac{dR(t)}{R(t^-)}.$$

We conclude that

$$p(t) = p(0)R(t) = E\big[G^\theta(T)U'\big(X(T)\big)\big]R(t), \quad 0 \le t \le T.$$

Therefore,

$$d\tilde{Q}(\omega) = \frac{G^\theta(T)U'(X^{\pi,c}(T))}{E[G^\theta(T)U'(X^{\pi,c}(T))]}\, dP(\omega) \quad \text{on } \mathcal{F}_T. \tag{68}$$

This proves the first part of the following result:

Theorem 4.1 (a) *Suppose that there exists an optimal portfolio π, an optimal consumption rate c, and an optimal scenario parameter θ for the model uncertainty portfolio-consumption optimization problem* (52). *Then* (62) *holds, and the measure $\tilde{Q} = Q^{\pi,c,\theta}$ defined by* (68) *is an ELMM for the market* (48).

(b) *Conversely, suppose that there exists a portfolio π, a consumption rate c, and a scenario parameter θ such that* (62) *holds and $\tilde{Q} = Q^{\pi,c,\theta}$ defined by* (68) *is an ELMM for the market* (48). *Suppose that there exists a unique solution $Y(t)$ of the BSDE* (54), *with θ satisfying* (63)–(64).

Define $\hat{\theta} = \hat{\theta}(z,k) = (\hat{\theta}_0(z,k), \hat{\theta}_1(z,k,\cdot))$ as the solution of the equation

$$\nabla\rho(\hat{\theta}) := \left(\frac{\partial\rho}{\partial\theta_0}(\hat{\theta}), \nabla_{\theta_1}\rho(\hat{\theta})\right) = \left(-z, -\int_{\mathbb{R}} (\cdot)k(\zeta)\nu(d\zeta)\right). \tag{69}$$

Suppose that the function

$$\mathcal{H}(z,k) := \rho\big(\hat{\theta}(z,k)\big) + \hat{\theta}_0(z,k)z + \int_{\mathbb{R}} \hat{\theta}_1(z,k,\zeta)k(\zeta)\nu(d\zeta) \tag{70}$$

is concave. Then π is an optimal portfolio, c is an optimal consumption rate, and θ is an optimal scenario parameter for the problem (52).

Proof of (b) If $\tilde{Q}$ is defined by (68), then, by (59),

$$\frac{d\tilde{Q} \mid \mathcal{F}_t}{dP \mid \mathcal{F}_t} = \frac{E[G^\theta(T)U'(X^{\pi,c}(T)) \mid \mathcal{F}_t]}{E[G^\theta(T)U'(X^{\pi,c}(T))]} = \frac{p(t)}{p(0)}, \tag{71}$$

where

$$\begin{aligned} dp(t) &= p\big(t^-\big)\left[\frac{q(t)}{p(t)}\, dB(t) + \int_{\mathbb{R}} \frac{r(t,\zeta)}{p(t^-)} \tilde{N}(dt,d\zeta)\right], \quad t \in [0,T], \\ p(T) &= G^\theta(T)U'\big(X^{\pi,c}(T)\big). \end{aligned} \tag{72}$$

It follows by the Girsanov theorem that if $\tilde{Q}$ is an ELMM for $S(t)$,we must have

$$b_0(t) + \sigma_0(t)\frac{q(t)}{p(t)} + \int_{\mathbb{R}} \gamma_0(t,\zeta)\frac{r(t,\zeta)}{p(t)}\nu(d\zeta) = 0.$$

This implies (61). We conclude that all Eqs. (61)–(64) are satisfied. Moreover, since $\hat{\theta}(z,k)$ defined by (69) is the minimizer w.r.t. θ of $H(t,x,y,z,k,\pi,c,\theta,\lambda,p,q,r)$, it follows by (70) and the Arrow condition in Theorem 2.4 applied to our situation that all conditions of the sufficient maximum principle are satisfied, and we can conclude that π, c, and θ are optimal. □

Remark 4.1 In agreement with the terminology used elsewhere in similar situations, we call the market (48) *model uncertainty viable* if the problem (52) has a solution π, c, θ. Then, under the assumptions of Theorem 4.1, we have proved that the market is model uncertainty viable with optimal π, c, and θ if and only if (62) holds and the measure

$$dQ^{\pi,c,\theta} := \frac{G^\theta(T)U'(X^{\pi,c}(T))}{E[G^\theta(T)U'(X^{\pi,c}(T))]}\,dP$$

is an ELMM. This is an extension to model uncertainty markets of the following result which is well known in classical types of financial markets, mainly the equivalence between (i) the existence of an optimal portfolio (viability) and (ii) the measure $dQ := U'(X(T))/E[U'(X(T))]\,dP$ being an ELMM. See, e.g., [10–13].

Remark 4.2 Using the same method, we can also consider performances of the form (39) with $U(X^{\pi,c}(T))$ replaced by $\int_0^T U(X^{\pi,c}(t))\,dt$.

4.2 The Entropic Penalty Case

We consider now the case where the penalty function has the form

$$\rho_a(\theta) := \frac{1}{a}\rho_1(\theta), \tag{73}$$

where $a > 0$ is a given parameter, and

$$\rho_1(\theta_0,\theta_1)(t) := \frac{1}{2}\theta_0^2(t) + \int_{\mathbb{R}} \big\{\big(1+\theta_1(t,\zeta)\big)\ln\big(1+\theta_1(s,\zeta)\big) - \theta_1(s,\zeta)\big\}\nu(d\zeta). \tag{74}$$

Note that, by (37),

$$G^\theta(t) = \exp\bigg(\int_0^t \theta_0(s)\,dB(s) - \frac{1}{2}\int_0^t \theta_0^2(s)\,ds + \int_0^t\int_{\mathbb{R}} \ln\big(1+\theta_1(s,\zeta)\big)\tilde{N}(ds,d\zeta) + \int_0^t\int_{\mathbb{R}} \big\{\ln\big(1+\theta_1(s,\zeta)\big) - \theta_1(s,\zeta)\big\}\nu(d\zeta)\bigg). \tag{75}$$

Therefore, the relative *entropy* $\mathcal{E}(Q^\theta \mid P)$ of Q^θ with respect to P defined as

$$\mathcal{E}\big(Q^\theta \mid P\big) := E\bigg[\frac{dQ^\theta}{dP}\ln\bigg(\frac{dQ^\theta}{dP}\bigg)\bigg]$$

is seen to be

$$\mathcal{E}\big(Q^\theta \mid P\big) = E\big[G^\theta(T)\ln G^\theta(T)\big] = E\bigg[\int_0^T G^\theta(t)\rho_1\big(\theta(t)\big)\,dt\bigg].$$

We call ρ_a the *entropic penalty* function.

Then, the optimality conditions (63) and (64) for θ_0, θ_1 become

$$\begin{aligned} \frac{1}{a}\theta_0(t) &= -Z(t), \\ \frac{1}{a}\int_{\mathbb{R}} \ln\bigl(1+\theta_1(t,\zeta)\bigr)(\cdot)\nu(d\zeta) &= -\int_{\mathbb{R}} (\cdot)K(t,\zeta)\nu(d\zeta), \end{aligned} \tag{76}$$

i.e.,

$$\frac{1}{a}\ln\bigl(1+\theta_1(t,\zeta)\bigr) = -K(t,\zeta). \tag{77}$$

Substituted into (54), this gives

$$\begin{aligned} dY(t) &= \left[-U_1\bigl(c(t)\bigr) + \frac{1}{2a}\theta_0^2(t) - \frac{1}{a}\int_{\mathbb{R}}\bigl\{\ln\bigl(1+\theta_1(t,\zeta)\bigr) - \theta_1(t,\zeta)\bigr\}\nu(d\zeta)\right]dt \\ &\quad - \frac{1}{a}\theta_0(t)\,dB(t) - \frac{1}{a}\int_{\mathbb{R}} \ln\bigl(1+\theta_1(t,\zeta)\bigr)\tilde{N}(dt,d\zeta); \\ Y(T) &= U\bigl(X(T)\bigr). \end{aligned} \tag{78}$$

It follows that (78) can be written as

$$Y(t) = Y(0) - \frac{1}{a}\ln G^\theta(t) - \int_0^t U_1\bigl(c(s)\bigr)\,ds, \quad t \in [0,T].$$

Taking exponentials gives

$$G^\theta(t) = \exp\left(-aY(t) - a\int_0^t U_1\bigl(c(s)\bigr)\,ds\right)\exp\bigl(aY(0)\bigr), \quad t \in [0,T].$$

In particular, if we put $t = T$, we get

$$G^\theta(T) = \frac{\exp(-aU(X(T)) - a\int_0^T U_1(c(s))\,ds)}{E[\exp(-aU(X(T)) - a\int_0^T U_1(c(s))\,ds)]}. \tag{79}$$

This gives the optimal scenario $G^\theta(T)$ expressed in terms of the optimal terminal wealth $X(T) = X^{\pi,c}(T)$ and the optimal consumption rate c.

Combining this with Theorem 4.1a), we get that

$$d\tilde{Q} := \frac{\exp(-aU(X(T)) - a\int_0^T U_1(c(s))\,ds)U'(X(T))}{E[\exp(-aU(X(T)) - a\int_0^T U_1(c(s))\,ds)U'(X(T))]}\,dP \tag{80}$$

is an ELMM.

With our choice of ρ, we see that the concavity condition for the function $\mathcal{H}$ defined in (70) is satisfied. Therefore, if we combine Theorem 4.1 with the calculations above, we get the following:

Theorem 4.2 *The following*, (a) *and* (b), *are equivalent*:

(a) *There exists an optimal* (π, c, θ) *for the problem* (52)–(54).
(b) *The measure* $\tilde{Q}$ *defined by* (80) *is an ELMM*, *and*

$$U_1'\big(c(t)\big)G^\theta(t) = E\big[G^\theta(T)U'\big(X^{\pi,c}(T)\big) \mid \mathcal{F}_t\big], \quad 0 \le t \le T, \tag{81}$$

with

$$G^\theta(t) = \frac{E[\exp\{-aU(X^{\pi,c}(T)) - a\int_0^T U_1(c(s))\,ds\} \mid \mathcal{F}_t]}{E[\exp\{-aU(X^{\pi,c}(T)) - a\int_0^T U_1(c(s))\,ds\}]}, \quad 0 \le t \le T. \tag{82}$$

An equivalent formulation involving only π and c is as follows:

Theorem 4.3 *The following*, (a) *and* (b), *are equivalent*:

(a) *There exists an optimal* (π, c, θ) *for the problem* (52)–(54).
(b) *The measure* $\tilde{Q}$ *defined by* (80) *is an ELMM*, *and*

$$\begin{aligned} &U_1'\big(c(t)\big)E\left[\exp\left\{-aU\big(X^{\pi,c}(T)\big) - a\int_0^T U_1\big(c(s)\big)\,ds\right\} \mid \mathcal{F}_t\right] \\ &\quad = E\left[\exp\left\{-aU\big(X^{\pi,c}(T)\big) - \int_0^T aU_1\big(c(s)\big)\,ds\right\}U'\big(X^{\pi,c}(T)\big) \mid \mathcal{F}_t\right], \\ &\quad 0 \le t \le T. \end{aligned} \tag{83}$$

Proof (a) $\Rightarrow$ (b): Suppose that a) holds. Then, using Theorem 4.2 and substituting (82) into (81), we get (83).

(b) $\Rightarrow$ (a): Conversely, let (π, c) be such that the measure $\tilde{Q}$ defined by (80) is an ELMM and (83) holds. Define $G(t)$ by

$$G(t) := \frac{E[\exp(-aU(X^{\pi,c}(T)) - a\int_0^T U_1(c(s))\,ds) \mid \mathcal{F}_t]}{E[\exp(-aU(X^{\pi,c}(T)) - a\int_0^T U_1(c(s))\,ds)]} > 0, \quad 0 \le t \le T, \tag{84}$$

and let θ_0, θ_1 be such that

$$dG(t) = G\big(t^-\big)\left[\theta_0(t)\,dB(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)\tilde{N}(dt,d\zeta)\right], \quad 0 \le t \le T, \tag{85}$$

is the Itô representation of the martingale $G(t)$.

Then, with $G^\theta(t) = G(t)$ we see that (π, c, θ) satisfies all the requirements in part (b) of Theorem 4.2, and hence (π, c, θ) is optimal. □

We now compare this result to a *model certainty* problem, as follows.

Let $X(t)$ be as in (49) and choose a utility function V. With U_1, U as above, define the performance functional

$$J_0(\pi, c) := E\left[V\left(U\big(X(T)\big) + \int_0^T U_1\big(c(t)\big)\,dt\right)\right]. \tag{86}$$

We want to find $(\hat{\pi}, \hat{c}) \in \mathcal{A}_1$ such that

$$\sup_{(\pi,c)\in\mathcal{A}_1} J_0(\pi, c) = J_0(\hat{\pi}, \hat{c}). \tag{87}$$

To put this problem into the context of our maximum principle, we define $X_2(t) = X(t)$ and

$$dX_1(t) = U_1\big(c(t)\big)\,dt, \qquad X_1(0) = 0. \tag{88}$$

Then

$$J_0(\pi, c) = E\big[V\big(U\big(X(T)\big) + X_1(T)\big)\big],$$

and the corresponding Hamiltonian becomes

$$H = U_1(c)p_1 + \big(\pi b_0(t) - c\big)p_2 + \pi\sigma_0(t)q_2 + \int_{\mathbb{R}} \pi\ r_2(\zeta)\gamma_0(t,\zeta)\nu(d\zeta). \tag{89}$$

The adjoint equations are

$$\begin{cases} dp_1(t) = q_1(t)\,dB(t) + \int_{\mathbb{R}} r_1(t,\zeta)\tilde{N}(dt, d\zeta), & 0 \le t \le T, \\ p_1(T) = V'\big(U\big(X(T)\big) + X_1(T)\big), \end{cases} \tag{90}$$

$$\begin{cases} dp_2(t) = q_2(t)\,dB(t) + \int_{\mathbb{R}} r_2(t,\zeta)\tilde{N}(dt, d\zeta), & 0 \le t \le T, \\ p_2(T) = V'\big(U\big(X(T)\big) + X_1(T)\big)U'\big(X(T)\big). \end{cases} \tag{91}$$

Arguing as before, we now deduce that (π, c) is optimal for (87) if and only if

$$U_1'\big(c(t)\big) = \frac{E[V'(U(X(T)) + X_1(T))U'(X(T)) \mid \mathcal{F}_t]}{E[V'(U(X(T)) + X_1(T)) \mid \mathcal{F}_t]} \tag{92}$$

and the measure

$$d\tilde{Q}_0 := \frac{V'(U(X(T)) + X_1(T))U'(X(T))}{E[V'(U(X(T)) + X_1(T))U'(X(T))]}\,dP \tag{93}$$

is an ELMM for the market (48).

Therefore, if we choose

$$V(x) := -\frac{1}{a}\exp(-ax) \tag{94}$$

and compare (92) and (93) with (83) and (80), respectively, we obtain the following:

Theorem 4.4 (Model uncertainty reduction theorem) *Suppose that* $(\pi, c) \in \mathcal{A}$ *is optimal for the* (*model certainty*) *portfolio-consumption problem*

$$\sup_{(\pi,c)\in\mathcal{A}_1} -\frac{1}{a}E\left[\exp\left(-aU\left(X^{\pi,c}(T)\right) - a\int_0^T U_1\left(c(t)\right)dt\right)\right]. \tag{95}$$

Then (π, c) *is optimal for the model uncertainty portfolio-consumption problem*

$$\sup_{\pi,c\in\mathcal{A}_1}\inf_{\theta\in\mathcal{A}_2} J(\pi,c,\theta) = \inf_{\theta\in\mathcal{A}_2}\sup_{\pi,c\in\mathcal{A}_1} J(\pi,c,\theta) \tag{96}$$

with

$$J(\pi,c,\theta) = E\left[G^\theta(T)U\left(X^{\pi,c}(T)\right) + \int_0^T G^\theta(s)U_1\left(c(s)\right)ds\right] + \frac{1}{a}\mathcal{E}\left(Q^\theta \mid P\right), \tag{97}$$

where $\mathcal{E}(Q^\theta \mid P)$ *is the relative entropy of* Q^θ *with respect to* P. *Moreover, the Radon–Nikodym derivative of the optimal probability measure* Q^θ *is given by*

$$G^\theta(t) := \frac{E[\exp(-aU(X^{\pi,c}(T)) - a\int_0^T U_1(c(s))\,ds) \mid \mathcal{F}_t]}{E[\exp(-aU(X^{\pi,c}(T)) - a\int_0^T U_1(c(s))\,ds)]}, \quad 0 \le t \le T. \tag{98}$$

Remark 4.3 When the optimal $G^\theta(T)$ is known, we can find the corresponding $\theta_0(t)$ and $\theta_1(t,\zeta)$ in feedback form as follows: By the Clark–Ocone theorem combined with (37), we get

$$\theta_0(t) = \frac{1}{G^\theta(t)}E\left[D_t G^\theta(T) \mid \mathcal{F}_t\right], \quad 0 \le t \le T,$$

$$\theta_1(t,\zeta)(t) = \frac{1}{G^\theta(t)}E\left[D_{t,\zeta} G^\theta(T) \mid \mathcal{F}_t\right], \quad 0 \le t \le T,$$

where D_t and $D_{t,\zeta}$ denote the Malliavin derivatives with respect to $B(\cdot)$ and $N(\cdot,\cdot)$, respectively (see [14]). We refer to [15] for more information on Malliavin calculus for Lévy processes.

Theorem 4.4 shows that the problem of optimal portfolio and consumption under model *uncertainty* with entropic penalty can be reduced to a corresponding model *certainty* problem, but with a different performance functional (different utilities).

Model Uncertainty and Risk Aversion The Arrow–Pratt coefficient of absolute risk aversion at x of a utility function U is defined by

$$\alpha_U(x) := -\frac{U''(x)}{U'(x)}.$$

Define

$$W(x) = -\frac{1}{a}\exp\left(-aU(x)\right), \quad \text{where } a > 0.$$

Then

$$\alpha_W(x) = -\frac{W''(x)}{W'(x)} = -\frac{U''(x) - aU'(x)^2}{U'(x)}.$$

Hence,

$$\alpha_W(x) = \alpha_U(x) + aU'(x).$$

We conclude that the risk aversion of W is bigger that the risk aversion of U. Hence, in view of Theorem 4.4, we can say that, in this sense, *model uncertainty increases the risk aversion*. For more discussion on this topic, see [16] and the references therein.

4.3 Relation of Robust Portfolio-Consumption Problem with Entropic Penalty with Risk-Sensitive Control

Fix π, c and let $X(T)$ and $X_1(T) = \int_0^T U_1(c(t))\,dt$ be the corresponding terminal wealth and total utility from consumption, respectively.

Consider the problem

$$I := \inf_{\theta}\left\{E_{Q^\theta}\left[U\bigl(X(T)\bigr) + \int_0^T U_1\bigl(c(t)\bigr)\,dt + \int_0^T \rho_a\bigl(\theta(t)\bigr)\,dt\right]\right\}, \tag{99}$$

where ρ_a is the entropic penalty defined in (73), so that

$$E\left[\int_0^T G^\theta(t)\rho_a\bigl(\theta(t)\bigr)\,dt\right] = \frac{1}{a}E\bigl[G^\theta(T)\ln G^\theta(T)\bigr]. \tag{100}$$

We have seen that (99) can be written as

$$I = \inf_\theta Y^\theta(0), \tag{101}$$

where $Y^\theta(t)$ solves the BSDE

$$\begin{aligned} dY(t) = {}& -\left[U_1\bigl(c(t)\bigr) + \rho_a\bigl(\theta(t)\bigr) + \theta_0(t)Z(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)K(t,\zeta)\nu(d\zeta)\right]dt \\ & + Z(t)\,dB(t) + \int_{\mathbb{R}} K(t,\zeta)\tilde{N}(dt,d\zeta)], \quad t \in [0,T], \end{aligned} \tag{102}$$

$$Y(T) = U\bigl(X(T)\bigr). \tag{103}$$

By the comparison theorem for BSDEs (see [17] and [18]) , we see that to solve (101), all we need is to minimize

$$\theta \to \rho_a\bigl(\theta(t)\bigr) + \theta_0(t)Z(t) + \int_{\mathbb{R}} \theta_1(t,\zeta)K(t,\zeta)\nu(d\zeta)\,dt.$$

The first-order conditions for optimal $\hat{\theta}$ are

$$\frac{\partial \rho_a}{\partial \theta_0}\bigl(\theta(t)\bigr) = -Z(t), \quad \nabla_{\theta_1}\rho_a\bigl(\theta(t)\bigr)(\cdot) = -\int_{\mathbb{R}} (\cdot)K(t,\zeta)\nu(d\zeta), \tag{104}$$

i.e.,

$$
\begin{aligned}
Z(t) &= -\frac{1}{a}\theta_0(t), \\
K(t,\zeta) &= -\frac{1}{a}\ln\bigl(1+\theta_1(t,\zeta)\bigr).
\end{aligned}
\tag{105}
$$

Substituting this into (103) and arguing as before, we obtain the formula (79), i.e.,

$$
G^{\hat{\theta}}(T) = \frac{\exp(-aU(X(T)) - a\int_0^T U_1(c(s))\,ds)}{E[\exp(-aU(X(T)) - a\int_0^T U_1(c(s))\,ds)]}. \tag{106}
$$

Therefore, since $\hat{\theta}$ is optimal, we get

$$
\begin{aligned}
I &= E\biggl[G^{\hat{\theta}}(T)\biggl(U\bigl(X(T)\bigr) + X_1(T) + \frac{1}{a}\ln G^{\hat{\theta}}(T)\biggr)\biggr] \\
&= E\biggl[G^{\hat{\theta}}(T)\bigl(U\bigl(X(T)\bigr) + X_1(T) - U\bigl(X(T)\bigr) - X_1(T) \\
&\quad - \frac{1}{a}\ln E\bigl[\exp\bigl(-aU\bigl(X(T)\bigr) - aX_1(T)\bigr)\bigr]\bigr)\biggr] \\
&= -\frac{1}{a}E\bigl[G^{\hat{\theta}}(T)\bigr]\ln E\bigl[\exp\bigl(-aU\bigl(X(T)\bigr) - aX_1(T)\bigr)\bigr] \\
&= -\frac{1}{a}\ln E\bigl[\exp\bigl(-aU\bigl(X(T)\bigr) - aX_1(T)\bigr)\bigr].
\end{aligned}
$$

We conclude that our robust portfolio-consumption problem (101) with entropic penalty is related to risk-sensitive control as follows:

$$
\begin{aligned}
&\inf_{\theta}\biggl\{E_{Q^\theta}\biggl[U\bigl(X(T)\bigr) + X_1(T) + \int_0^T \rho_a\bigl(\theta(t)\bigr)\,dt\biggr]\biggr\} \\
&\quad = -\frac{1}{a}\ln E\bigl[\exp\bigl(-aU\bigl(X(T)\bigr) - aX_1(T)\bigr)\bigr].
\end{aligned}
\tag{107}
$$

This is an extension to consumption-portfolio settings of the risk-sensitive control result in [19]. See also the references therein.

Remark 4.4 Equation (107) shows in particular that the sup–inf part of the model uncertainty problem (50) can be reduced to an optimal consumption-portfolio problem with model certainty. Note that the proof of this is relatively easy and does not require the whole machinery that we have set up in the previous sections. However, it seems that the inf–sup part of the same problem cannot be proved so easily; this part requires techniques for optimal control of forward–backward SDE control/games, as developed in this paper.

5 Concluding Remarks

This paper has two main parts:

- The first is a general maximum principle for forward–backward stochastic differential games for Itô–Lévy processes with partial information. This is a result of independent interest, and it has a potential for being useful in many situations.
- The second is an application of the general theory in the first part to optimal portfolio and consumption problems under model uncertainty, in markets modeled by Itô–Lévy processes. The model uncertainty is represented by a family of equivalent probability measures, with a penalty for being "far away" from the original measure P. We obtain a characterization of market viability under model uncertainty in terms of equivalent local martingale measures. If the penalty function is entropic, we prove a reduction theorem saying that the model uncertainty problem can be transformed into a problem without model uncertainty, but with different utility functions.

It is natural to ask if similar results could be obtained with other representations of model uncertainty. For example, one could consider uncertainties in the noise terms or put constraints on the families of probability measures. In particular, can our reduction theorem for the entropic penalty be extended to more general model uncertainty contexts?

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Appendix A: Proofs of the Maximum Principles for FBSDE Games

We first recall some basic concepts and results from Banach space theory. Let V be an open subset of a Banach space $\mathcal{X}$ with norm $\|\cdot\|$, and let $F: V \to \mathbb{R}$.

(i) We say that F has a directional derivative (or Gâteaux derivative) at $x \in X$ in the direction $y \in \mathcal{X}$ if

$$D_y F(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(F(x+\varepsilon y) - F(x)\big)$$

exists.

(ii) We say that F is Fréchet differentiable at $x \in V$ if there exists a linear map

$$L : \mathcal{X} \to \mathbb{R}$$

such that

$$\lim_{\substack{h \to 0 \\ h \in \mathcal{X}}} \frac{1}{\|h\|}\big|F(x+h) - F(x) - L(h)\big| = 0.$$

In this case, we call L the *gradient* (or Fréchet derivative) of F at x, and we write

$$L = \nabla_x F.$$

(iii) If F is Fréchet differentiable, then F has a directional derivative in all directions $y \in \mathcal{X}$, and

$$D_y F(x) = \nabla_x F(y).$$

Proof of Theorem 2.1 (Sufficient maximum principle) We first prove that

$$J_1(u_1, \hat{u}_2) \leq J_1(\hat{u}_1, \hat{u}_2) \quad \text{for all } u_1 \in \mathcal{A}_1.$$

To this end, fix $u_1 \in \mathcal{A}_1$ and consider

$$\Delta := J_1(u_1, \hat{u}_2) - J_1(\hat{u}_1, \hat{u}_2) = I_1 + I_2 + I_3, \tag{A.1}$$

where

$$I_1 = E\left[\int_0^T \left\{f_1\big(t, X(t), u(t)\big) - f_1\big(t, \hat{X}(t), \hat{u}(t)\big)\right\} dt\right], \tag{A.2}$$

$$I_2 = E\big[\varphi_1\big(X(T)\big) - \varphi_1\big(\hat{X}(T)\big)\big], \tag{A.3}$$

$$I_3 = E\big[\psi_1\big(Y_1(0)\big) - \psi_1\big(\hat{Y}_1(0)\big)\big]. \tag{A.4}$$

By (8) we have

$$\begin{aligned} I_1 = E\bigg[\int_0^T \bigg\{ & H_1(t) - \hat{H}_1(t) - \hat{\lambda}_1(t)\big(g_1(t) - \hat{g}_1(t)\big) - \hat{p}_1(t)\big(b(t) - \hat{b}(t)\big) \\ & - \hat{q}_1(t)\big(\sigma(t) - \hat{\sigma}(t)\big) - \int_{\mathbb{R}} \hat{r}_1(t,\zeta)\big(\gamma(t,\zeta) - \hat{\gamma}(t,\zeta)\big)\nu(d\zeta)\bigg\} dt\bigg]. \end{aligned} \tag{A.5}$$

By the concavity of φ_1, (10), and the Itô formula,

$$\begin{aligned} I_2 &\leq E\big[\varphi_1'\big(\hat{X}(T)\big)\big(X(T) - \hat{X}(T)\big)\big] \\ &= E\big[\hat{p}_1(T)\big(X(T) - \hat{X}(T)\big)\big] - E\big[\hat{\lambda}_1(T)h_1'\big(\hat{X}(T)\big)\big(X(T) - \hat{X}(T)\big)\big] \\ &= E\bigg[\int_0^T \hat{p}_1\big(t^-\big)\big(dX(t) - d\hat{X}(t)\big) + \int_0^T \big(X\big(t^-\big) - \hat{X}\big(t^-\big)\big)\, d\hat{p}_1(t) \\ &\quad + \int_0^T \hat{q}_1(t)\big(\sigma(t) - \hat{\sigma}(t)\big)\, dt \\ &\quad + \int_0^T \int_{\mathbb{R}} \hat{r}_1(t,\zeta)\big(\gamma(t,\zeta) - \hat{\gamma}(t,\zeta)\big)\nu(d\zeta)\, dt\bigg] \\ &\quad - E\big[\hat{\lambda}_1(T)h_1'\big(\hat{X}(T)\big)\big(X(T) - \hat{X}(T)\big)\big] \end{aligned}$$

$$
\begin{aligned}
&= E\bigg[\int_0^T \hat{p}_1(t)\big(b(t)-\hat{b}(t)\big)\,dt + \int_0^T \big(X(t)-\hat{X}(t)\big)\bigg(-\frac{\partial \hat{H}_1}{\partial x}(t)\bigg)\,dt \\
&\quad + \int_0^T \hat{q}_1(t)\big(\sigma(t)-\hat{\sigma}(t)\big)\,dt + \int_0^T\int_{\mathbb{R}} \hat{r}_1(t,\zeta)\big(\gamma(t,\zeta)-\hat{\gamma}(t,\zeta)\big)\nu(d\zeta)\,dt\bigg] \\
&\quad - E\big[\hat{\lambda}_1(T)h_1'\big(\hat{X}(T)\big)\big(X(T)-\hat{X}(T)\big)\big].
\end{aligned}
\tag{A.6}
$$

By the concavity of ψ_1, (5), (9), and the concavity of φ,

$$
\begin{aligned}
I_3 &= E\big[\psi_1\big(Y_1(0)\big)-\psi_1\big(\hat{Y}_1(0)\big)\big] \\
&\le E\big[\psi_1'\big(\hat{Y}_1(0)\big)\big(Y_1(0)-\hat{Y}_1(0)\big)\big] \\
&= E\big[\hat{\lambda}_1(0)\big(Y_1(0)-\hat{Y}_1(0)\big)\big] \\
&= E\big[\big(Y_1(T)-\hat{Y}_1(T)\big)\hat{\lambda}_1(T)\big] \\
&\quad - \bigg\{E\bigg[\int_0^T \big(Y_1\big(t^-\big)-\hat{Y}_1\big(t^-\big)\big)d\hat{\lambda}_1(t) + \int_0^T \hat{\lambda}_1\big(t^-\big)\big(dY_1(t)-d\hat{Y}_1(t)\big) \\
&\quad + \int_0^T \frac{\partial \hat{H}_1}{\partial z}(t)\big(Z_1(t)-\hat{Z}_1(t)\big)\,dt \\
&\quad + \int_0^T\int_{\mathbb{R}} \nabla_k \hat{H}_1(t,\zeta)\big(K_1(t,\zeta)-\hat{K}_1(t,\zeta)\big)\nu(d\zeta)\,dt\bigg]\bigg\} \\
&= E\big[\big(h_1\big(X(T)\big)-h_1\big(\hat{X}(T)\big)\big)\hat{\lambda}_1(T)\big] \\
&\quad - \bigg\{E\bigg[\int_0^T \frac{\partial \hat{H}_1}{\partial y}(t)\big(Y_1(t)-\hat{Y}_1(t)\big)\,dt \\
&\quad + \int_0^T \hat{\lambda}_1(t)\big(-g_1(t)+\hat{g}_1(t)\big)\,dt \\
&\quad + \int_0^T \frac{\partial \hat{H}_1}{\partial z}(t)\big(Z_1(t)-\hat{Z}_1(t)\big)\,dt \\
&\quad + \int_0^T\int_{\mathbb{R}} \nabla_k \hat{H}_1(t,\zeta)\big(K_1(t,\zeta)-\hat{K}_1(t,\zeta)\big)\nu(d\zeta)\,dt\bigg]\bigg\} \\
&\le E\big[\hat{\lambda}_1(T)h_1'\big(\hat{X}(T)\big)\big(X(T)-\hat{X}(T)\big)\big] \\
&\quad - \bigg\{E\bigg[\int_0^T \frac{\partial \hat{H}_1}{\partial y}(t)\big(Y_1(t)-\hat{Y}_1(t)\big)\,dt \\
&\quad + \int_0^T \hat{\lambda}_1(t)\big(-g_1(t)+\hat{g}_1(t)\big)\,dt + \int_0^T \frac{\partial \hat{H}_1}{\partial z}(t)\big(Z_1(t)-\hat{Z}_1(t)\big)\,dt \\
&\quad + \int_0^T\int_{\mathbb{R}} \nabla_k \hat{H}_1(t,\zeta)\big(K_1(t,\zeta)-\hat{K}_1(t,\zeta)\big)\nu(d\zeta)\,dt\bigg]\bigg\}.
\end{aligned}
\tag{A.7}
$$

Adding (A.5), (A.6), and (A.7), we get

$$\begin{aligned}\Delta &= I_1 + I_2 + I_3 \\ &\le E\bigg[\int_0^T \bigg\{ H_1(t) - \hat{H}_1(t) - \frac{\partial \hat{H}_1}{\partial x}(t)\big(X(t) - \hat{X}(t)\big) \\ &\quad - \frac{\partial H_1}{\partial y}(t)\big(Y_1(t) - \hat{Y}_1(t)\big) - \frac{\partial \hat{H}_1}{\partial z}(t)\big(Z_1(t) - \hat{Z}_1(t)\big) \\ &\quad - \int_{\mathbb{R}} \nabla_k \hat{H}_1(t,\zeta)\big(K_1(t,\zeta) - \hat{K}_1(t,\zeta)\big)\nu(d\zeta)\bigg\} dt\bigg]. \end{aligned} \tag{A.8}$$

Since $\hat{\mathcal{H}}_1(x,y,z,k)$ is concave, it follows by a standard separating hyperplane argument (see, e.g., [20], Chap. 5, Sect. 23) that there exists a supergradient $a = (a_0, a_1, a_2, a_3(\cdot)) \in \mathbb{R}^3 \times \mathcal{R}$ for $\hat{\mathcal{H}}_1(x,y,z,k)$ at $x = \hat{X}(t),\ y = \hat{Y}_1(t),\ z = \hat{Z}_1(t^-)$, and $k = \hat{K}_1(t^-,\cdot)$ such that if we define

$$\begin{aligned}\varphi_1(x,y,z,k) &:= \hat{\mathcal{H}}_1(x,y,z,k) - \hat{\mathcal{H}}_1\big(\hat{X}\big(t^-\big), \hat{Y}_1\big(t^-\big), \hat{Z}_1\big(t^-\big), \hat{K}_1(t,\cdot)\big) \\ &\quad - \bigg[a_0\big(x - \hat{X}(t)\big) + a_1\big(y - \hat{Y}_1(t)\big) + a_2\big(z - \hat{Z}_1(t)\big) \\ &\quad + \int_{\mathbb{R}} a_3(\zeta)\big(k(\zeta) - \hat{K}(t,\zeta)\big)\nu(d\zeta)\bigg],\end{aligned}$$

then

$$\varphi_1(x,y,z,k) \le 0 \quad \text{for all } x,y,z,k.$$

On the other hand, we clearly have

$$\varphi_1\big(\hat{X}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}_1(t,\cdot)\big) = 0.$$

It follows that

$$\begin{aligned}\frac{\partial \hat{H}_1}{\partial x}(t) &= \frac{\partial \hat{\mathcal{H}}_1}{\partial x}\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big) = a_0, \\ \frac{\partial \hat{H}_1}{\partial y}(t) &= \frac{\partial \hat{\mathcal{H}}_1}{\partial y}\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big) = a_1, \\ \frac{\partial \hat{H}_1}{\partial z}(t) &= \frac{\partial \hat{\mathcal{H}}_1}{\partial z}\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big) = a_2, \\ \nabla_k \hat{H}_1(t,\zeta) &= \nabla_k \hat{\mathcal{H}}_1\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big) = a_3.\end{aligned}$$

Combining this with (A.8), we get

$$\begin{aligned}\Delta &\le \hat{\mathcal{H}}_1\big(X(t), Y_1(t), Z_1(t), K_1(t,\cdot)\big) - \hat{\mathcal{H}}_1\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big) \\ &\quad - \frac{\partial \hat{\mathcal{H}}_1}{\partial x}\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big)\big(X(t) - \hat{X}(t)\big)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\partial \hat{\mathcal{H}}_1}{\partial y}\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big)\big(Y_1(t) - \hat{Y}_1(t)\big) \\
&- \frac{\partial \hat{\mathcal{H}}_1}{\partial z}\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big)\big(Z_1(t) - \hat{Z}_1(t)\big) \\
&- \int_{\mathbb{R}} \nabla_k \hat{\mathcal{H}}_1\big(\hat{X}(t), \hat{Y}_1(t), \hat{Z}_1(t), \hat{K}_1(t,\cdot)\big)\big(K_1(t,\zeta) - \hat{K}_1(t,\zeta)\big)\nu(d\zeta) \\
\le\, & 0 \quad \text{since } \hat{\mathcal{H}}_1 \text{ is concave.}
\end{aligned}
$$

Hence,

$$
J_1(u_1, \hat{u}_2) \le J_1(\hat{u}_1, \hat{u}_2) \quad \text{for all } u_1 \in \mathcal{A}_1.
$$

The inequality

$$
J_2(\hat{u}_1, u_2) \le J_2(\hat{u}_1, \hat{u}_2) \quad \text{for all } u_2 \in \mathcal{A}_2
$$

is proved similarly. This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2 (Necessary maximum principle) Consider

$$
\begin{aligned}
D_1 &:= \frac{d}{ds} J_1(u_1 + s\beta_1, u_2) \,|_{s=0} \\
&= E\bigg[\int_0^T \bigg\{\frac{\partial f_1}{\partial x}(t)x_1(t) + \frac{\partial f_1}{\partial u_1}(t)\beta_1(t)\bigg\}\, dt + \varphi_1'\big(X^{(u_1,u_2)}(T)\big)x_1(T) \\
&\quad + \psi_1'\big(Y_1(0)\big)y_1(0)\bigg].
\end{aligned} \tag{A.9}
$$

By (10), (13), and the Itô formula,

$$
\begin{aligned}
&E\big[\varphi_1'\big(X^{(u_1,u_2)}(T)\big)x_1(T)\big] \\
&\quad = E\big[p_1(T)x_1(T)\big] - E\big[h_1'\big(X^{(u_1,u_2)}(T)\big)\lambda_1(T)\big] \\
&\quad = E\bigg[\int_0^T \bigg\{p_1\big(t^-\big)dx_1(t) + x_1\big(t^-\big)dp_1(t) + q_1(t)\bigg[\frac{\partial \sigma}{\partial x}(t)x_1(t) + \frac{\partial \sigma}{\partial u_1}(t)\beta_1(t)\bigg]dt \\
&\qquad + \int_{\mathbb{R}} r_1(t,\zeta)\bigg[\frac{\partial \gamma}{\partial x}(t,\zeta)x_1(t) + \frac{\partial \gamma}{\partial u_1}(t,\zeta)\beta_1(t,\zeta)\bigg]\nu(d\zeta)\, dt\bigg\}\bigg] \\
&\qquad - E\big[h_1'\big(X^{(u_1,u_2)}(T)\big)\lambda_1(T)\big] \\
&\quad = E\bigg[\int_0^T \bigg\{p_1(t)\bigg[\frac{\partial b}{\partial x}(t)x_1(t) + \frac{\partial b}{\partial u_1}(t)\beta_1(t)\bigg] \\
&\qquad + x_1(t)\bigg(-\frac{\partial H_1}{\partial x}(t)\bigg) + q_1(t)\bigg[\frac{\partial \sigma}{\partial x}(t)x_1(t) + \frac{\partial \sigma}{\partial u_1}(t)\beta_1(t)\bigg] \\
&\qquad + \int_{\mathbb{R}} r_1(t,\zeta)\bigg[\frac{\partial \gamma}{\partial x}(t,\zeta)x_1(t) + \frac{\partial \gamma}{\partial u_1}(t,\zeta)\beta_1(t,\zeta)\bigg]\nu(d\zeta)\bigg\}\, dt\bigg] \\
&\qquad - E\big[h_1'\big(X^{(u_1,u_2)}(T)\big)\lambda_1(T)\big].
\end{aligned} \tag{A.10}
$$

By (9), (13), and the Itô formula,

$$
\begin{aligned}
&E\big[\psi_1'\big(Y_1(0)\big)y_1(0)\big]\\
&\quad= E\big[\lambda_1(0)y_1(0)\big]\\
&\quad= E\big[\lambda_1(T)y_1(T)\big] - E\bigg[\int_0^T \{\lambda_1\big(t^-\big)dy_1(t) + y_1\big(t^-\big)d\lambda_1(t)\\
&\qquad+ \frac{\partial H_1}{\partial z}(t)z_1(t)\,dt + \int_{\mathbb{R}} \nabla_k H_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)\,dt\bigg]\\
&\quad= E\big[\lambda_1(T)h_1'\big(X^{(u_1,u_2)}(T)\big)\big]\\
&\qquad- E\bigg[\int_0^T \bigg\{\lambda_1(t)\bigg[-\frac{\partial g_1}{\partial x}(t)x_1(t) - \frac{\partial g_1}{\partial y}(t)y_1(t) - \frac{\partial g_1}{\partial z}(t)z_1(t).\\
&\qquad- \int_{\mathbb{R}} \nabla_k g_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta) - \frac{\partial g_1}{\partial u_1}(t)\beta_1(t)\bigg]\\
&\qquad+ \frac{\partial H_1}{\partial y}(t)y_1(t) + \frac{\partial H_1}{\partial z}(t)z_1(t) + \int_{\mathbb{R}} \nabla_k H_1(t,\zeta)k_1(t,\zeta)\nu(d\zeta)\bigg\}\,dt\bigg].
\end{aligned}
\tag{A.11}
$$

Adding (A.10) and (A.11), we get, by (A.9),

$$
\begin{aligned}
D_1 &= E\bigg[\int_0^T \bigg\{\bigg[\frac{\partial f_1}{\partial x}(t) + p_1(t)\frac{\partial b}{\partial x}(t) + q_1(t)\frac{\partial \sigma}{\partial x}(t)\\
&\quad+ \int_{\mathbb{R}} r_1(t,\zeta)\frac{\partial \gamma}{\partial x}(t,\zeta)\nu(d\zeta) - \frac{\partial H_1}{\partial x}(t) + \lambda_1(t)\frac{\partial g_1}{\partial x}(t)\bigg]x_1(t)\\
&\quad+ \bigg[-\frac{\partial H_1}{\partial y}(t) + \lambda_1(t)\frac{\partial g_1}{\partial y}(t)\bigg]y_1(t) + \bigg[-\frac{\partial H_1}{\partial z}(t) + \lambda_1(t)\frac{\partial g_1}{\partial z}(t)\bigg]z_1(t)\\
&\quad+ \int_{\mathbb{R}} \big[-\nabla_k H_1(t,\zeta) + \lambda_1(t)\nabla_k g_1(t,\zeta)\big]k_1(t,\zeta)\nu(d\zeta)\\
&\quad+ \bigg[\frac{\partial f_1}{\partial u_1}(t) + p_1(t)\frac{\partial b}{\partial u_1}(t) + q_1(t)\frac{\partial \sigma}{\partial u_1}(t)\\
&\quad+ \int_{\mathbb{R}} r_1(t,\zeta)\frac{\partial \gamma}{\partial u_1}(t,\zeta)\nu(d\zeta) + \frac{\partial g_1}{\partial u_1}(t)\bigg]\beta_1(t)\bigg\}\,dt\bigg]\\
&= E\bigg[\int_0^T \frac{\partial H_1}{\partial u_1}(t)\beta_1(t)\,dt\bigg]\\
&= E\bigg[\int_0^T E\bigg[\frac{\partial H_1}{\partial u_1}(t)\beta_1(t) \mid \mathcal{E}_t^{(1)}\bigg]\,dt\bigg].
\end{aligned}
\tag{A.12}
$$

If $D_1 = 0$ for all bounded $\beta_1 \in \mathcal{A}_1$, then this holds in particular for β_1 of the form in (a1), i.e.,

$$
\beta_1(t) = \chi_{(t_0,T]}(t)\alpha_1(\omega),
$$

where $\alpha_1(\omega)$ is bounded and $\mathcal{E}_{t_0}^{(1)}$-measurable. Hence,

$$E\left[\int_{t_0}^{T} E\left[\frac{\partial H_1}{\partial u_1}(t) \mid \mathcal{E}_t^{(1)}\right]\alpha_1\, dt\right] = 0.$$

Differentiating with respect to t_0, we get

$$E\left[\frac{\partial H_1}{\partial u_1}(t_0)\alpha_1\right] = 0 \quad \text{for a.a. } t_0.$$

Since this holds for all bounded $\mathcal{E}_{t_0}^{(1)}$-measurable random variables α_1, we conclude that

$$E\left[\frac{\partial H_1}{\partial u_1}(t) \mid \mathcal{E}_t^{(1)}\right] = 0 \quad \text{for a.a. } t \in [0, T].$$

A similar argument gives that

$$E\left[\frac{\partial H_2}{\partial u_2}(t) \mid \mathcal{E}_t^{(2)}\right] = 0,$$

provided that

$$D_2 := \frac{d}{ds} J_2(u_1, u_2 + s\beta_2) \mid_{s=0} = 0 \quad \text{for all bounded } \beta_2 \in \mathcal{A}_2.$$

This shows that (i) $\Rightarrow$ (ii). The argument above can be reversed, to give that (ii) $\Rightarrow$ (i). We omit the details. □

Appendix B: Linear BSDEs with Jumps

Lemma B.1 (Linear BSDEs with jumps) *Let Λ be an $\mathcal{F}_T$-measurable and square-integrable random variable. Let β and ξ_0 be bounded predictable processes, and ξ_1 a predictable process such that $\xi_1(t,\zeta) \geq C_1$ with $C_1 > -1$ and $|\xi_1(t,\zeta)| \leq C_2(1 \wedge |\zeta|)$ for a constant $C_2 \geq 0$. Let φ be a predictable process such that $E[\int_0^T \varphi^2(t)\, dt] < \infty$. Then the linear BSDE*

$$\begin{aligned} dY(t) &= -\left[\varphi(t) + \beta(t)Y(t) + \xi_0(t)Z(t) + \int_{\mathbb{R}} \xi_1(t,\zeta)K(t,\zeta)\nu(d\zeta)\right]dt \\ &\quad + Z(t)\, dB(t) + \int_{\mathbb{R}} K(t,\zeta)\tilde{N}(dt,d\zeta), \quad 0 \leq t \leq T, \\ Y(T) &= \Lambda, \end{aligned} \tag{B.1}$$

has the unique solution

$$Y(t) = E\left[\Lambda \Upsilon(t,T) + \int_t^T \Upsilon(t,s)\varphi(s)\, ds \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T, \tag{B.2}$$

where $\Upsilon(t,s),\ 0 \le t \le s \le T$, *is defined by*

$$d\Upsilon(t,s) = \Upsilon\big(t,s^-\big)\bigg[\beta(s)\,ds + \xi_0(s)\,dB(s) + \int_{\mathbb{R}} \xi_1(s,\zeta)\tilde{N}(ds,d\zeta)\bigg], \quad t \le s \le T,$$
$$\Upsilon(t,t) = 1, \tag{B.3}$$

i.e.,

$$\begin{aligned}\Upsilon(t,s) = \exp\bigg(&\int_t^s \Big\{\beta(u) - \frac{1}{2}\xi_0^2(u)\Big\}\,du + \int_t^s \xi_0(u)\,dB(u)\\ &+ \int_t^s\int_{\mathbb{R}} \big\{\ln\big(1+\xi_1(u)\big) - \xi_1(u)\big\}\nu(d\zeta)\,du\\ &+ \int_t^s\int_{\mathbb{R}} \ln\big(1+\xi_1(u)\big)\tilde{N}(du,d\zeta)\bigg).\end{aligned} \tag{B.4}$$

Hence,

$$\Upsilon(t,s) = \frac{\Upsilon(0,s)}{\Upsilon(0,t)}.$$

Proof For completeness, we give the proof, also given in [18]. The existence and uniqueness follow by general theorems for BSDEs with Lipschitz coefficients. See, e.g., [17]. Hence, it only remains to prove that if we define $Y(t)$ to be the solution of (B.1), then (B.2) holds. To this end, define

$$\Upsilon(s) = \Upsilon(0,s).$$

Then by the Itô formula (see, e.g., [1], Chap. 1),

$$\begin{aligned}&d\big(\Upsilon(t)Y(t)\big)\\ &\quad= \Upsilon\big(t^-\big)dY(t) + Y\big(t^-\big)d\Upsilon(t) + d[\Upsilon Y](t)\\ &\quad= \Upsilon\big(t^-\big)\bigg[-\Big\{\varphi(t) + \beta(t)Y(t) + \xi_0(t)Z(t) + \int_{\mathbb{R}} \xi_1(t,\zeta)K(t,\zeta)\nu(d\zeta)\Big\}\,dt\\ &\qquad+ Z(t)\,dB(t) + \int_{\mathbb{R}} K(t,\zeta)\tilde{N}(dt,d\zeta)\bigg]\\ &\qquad+ Y\big(t^-\big)\Upsilon\big(t^-\big)\Big\{\beta(t)\,dt + \xi_0(t)\,dB(t) + \int_{\mathbb{R}} \xi_1(t,\zeta)\tilde{N}(dt,d\zeta)\Big\}\\ &\qquad+ \Upsilon(t)\xi_0(t)Z(t)\,dt + \int_{\mathbb{R}} \Upsilon\big(t^-\big)\xi_1(t,\zeta)K(t,\zeta)\tilde{N}(dt,d\zeta)\\ &\quad= -\Upsilon(t)\varphi(t)\,dt + \big(Z(t) + \xi_0(t)Y(t)\big)\Upsilon(t)\,dB(t)\\ &\qquad+ \int_{\mathbb{R}} \xi_1(t,\zeta)\Upsilon\big(t^-\big)\big(Y\big(t^-\big) + K(t,\zeta)\big)\tilde{N}(dt,d\zeta).\end{aligned}$$

Hence, $\Upsilon(t)Y(t)+\int_0^t \Upsilon(s)\varphi(s)\,ds$ is a martingale, and therefore

$$\Upsilon(t)Y(t)+\int_0^t \Upsilon(s)\varphi(s)\,ds = E\left[\Lambda\Upsilon(T)+\int_0^T \Upsilon(s)\varphi(s)\,ds \mid \mathcal{F}_t\right]$$

or

$$Y(t)=E\left[\Lambda\frac{\Upsilon(T)}{\Upsilon(t)}+\int_t^T \frac{\Upsilon(s)}{\Upsilon(t)}\varphi(s)\,ds \mid \mathcal{F}_t\right],$$

as claimed. □

References

1. Øksendal, B., Sulem, A.: Applied Stochastic Control of Jump Diffusions, 2nd edn. Springer, Berlin (2007)
2. Øksendal, B., Sulem, A.: Maximum principles for optimal control of forward-backward stochastic differential equations with jumps. SIAM J. Control Optim. **48**(5), 2845–2976 (2009)
3. Hamadène, S.: Backward-forward SDE's and stochastic differential games. Stoch. Process. Appl. **77**, 1–15 (1998)
4. An, T.T.K., Øksendal, B.: A maximum principle for stochastic differential games with g-expectation and partial information. Stochastics (2011). doi:10.1080/17442508.2010.532875
5. Bordigoni, G., Matoussi, A., Schweizer, M.: A stochastic control approach to a robust utility maximization problem. In: Benth, F.E., et al. (eds.) Stochastic Analysis and Applications, The Abel Symposium, 2005, pp. 125–152. Springer, Berlin (2007)
6. Jeanblanc, M., Matoussi, A., Ngoupeyou, A.: Robust Utility Maximization in a Discontinuous Filtration (2012)
7. Lim, T., Quenez, M.-C.: Exponential utility maximization and indifference price in an incomplete market with defaults. Electron. J. Probab. **16**, 1434–1464 (2011)
8. Øksendal, B., Sulem, A.: Robust stochastic control and equivalent martingale measures. In: Kohatsu-Higa, A., et al. (eds.) Stochastic Analysis and Applications. Progress in Probability, vol. 65, pp. 179–189 (2011)
9. Øksendal, B., Sulem, A.: Portfolio optimization under model uncertainty and BSDE games. Quant. Finance **11**(11), 1665–1674 (2011)
10. Pliska, S.: Introduction to Mathematical Finance. Blackwell, Oxford (1997)
11. Kreps, D.: Arbitrage and equilibrium in economics with infinitely many commodities. J. Math. Econ. **8**, 15–35 (1981)
12. Loewenstein, M., Willard, G.: Local martingales, arbitrage, and viability. Econ. Theory **16**, 135–161 (2000)
13. Øksendal, B., Sulem, A.: Viability and martingale measures in jump diffusion markets under partial information. Manuscript (2011)
14. Aase, K., Øksendal, B., Privault, N., Ubøe, J.: White noise generalizations of the Clark–Haussmann–Ocone theorem, with application to mathematical finance. Finance Stoch. **4**, 465–496 (2000)
15. Di Nunno, G., Øksendal, B., Proske, F.: Malliavin Calculus for Lévy Processes with Applications to Finance. Springer, Berlin (2009)
16. Maenhout, P.: Robust portfolio rules and asset pricing. Rev. Financ. Stud. **17**, 951–983 (2004)
17. Royer, M.: Backward stochastic differential equations with jumps and related non-linear expectations. Stoch. Process. Appl. **116**, 1358–1376 (2006)
18. Quenez, M.C., Sulem, A.: BSDEs with jumps, optimization and applications to dynamic risk measures. Inria research report rr-7997 (2012)
19. Föllmer, H., Schied, A., Weber, S.: Robust preferences and robust portfolio choice. In: Ciarlet, P., Bensoussan, A., Zhang, Q. (eds.) Mathematical Modelling and Numerical Methods in Finance. Handbook of Numerical Analysis, vol. 15, pp. 29–88 (2009)
20. Rockafellar, R.T.: Convex Analysis. Princeton University Press, Princeton (1970)