Optimal risk control and dividend policies under excess of loss reinsurance

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Abstract

We study the optimal reinsurance policy and dividend distribution of an insurance company under excess of loss reinsurance. The objective of the insurer is to maximize the expected discounted dividends. We suppose that in the absence of dividend distribution, the reserve process of the insurance company follows a compound Poisson process. We first prove existence and uniqueness results for this optimization problem by using singular stochastic control methods and the theory of viscosity solutions. We then compute the optimal strategy of reinsurance, the optimal dividend strategy and the value function by solving the associated integro-differential Hamilton-Jacobi-Bellman Variational Inequality numerically.

Key words : Stochastic control, jump diffusion, insurance, integro-differential Hamilton-Jacobi-Bellman equation, viscosity solution, Howard algorithm.

MSC Classification (2000) : 93E20, 49L20, 49L25, 65N06.

1 Introduction

A basic problem in insurance is the problem of optimal risk control and/or dividend distribution. In the literature, various criteria are used to formulate this problem such that (i) maximizing expected utility of terminal reserve process, (ii) minimizing the ruin probability of the insurer or (iii) maximizing the cumulative expected discounted dividends.

Touzi (2000) studied the problem of maximizing the expected utility of the terminal reserve in the case of a proportional reinsurance contract. He modelled the reserve process by a Doléans-Dade exponential of jump process and characterized the optimal strategy of reinsurance via a dual formulation. The criterion of maximizing the expected utility of the terminal reserve is usually not relevant in insurance modelling since the insurer who is invited to cover a large risk wants to be risk neutral (see Aase (2002)).

The second criterion is useful for consumers and supervisors and is extremely conservative especially for rich companies. Schmidli (2001) studied the optimal proportional reinsurance policy which minimizes the ruin probability in infinite horizon. He derived the associated Hamilton-Jacobi-Bellman equation, proved the existence of a solution and a verification theorem in the diffusion case. He proved that the ruin probability decreases exponentially and that the optimal proportion to insure is constant. Moreover, he gave some conjecture in the Cramér-Lundberg case.

The third criterion is preferable for shareholders. Jeanblanc-Piqué and Shirayev (1995) studied the problem of optimal dividend distribution policy without optimal risk control. They modelled the evolution of the capital $X = (X_t)_{t\geq 0}$ of a company by $dX_t = \mu dt + \sigma dW_t - dZ_t$ where μ and σ are constants, $W = (W_t)_{t\geq 0}$ is a standard Brownian motion and $Z = (Z_t)_{t\geq 0}$ is a nonnegative, nondecreasing right-continuous and adapted process. The process Z represents the strategy of payment of dividends by the company. They showed that there exists a threshold u_1 such that every excess of the reserve above u_1 is distributed as dividend instantaneously. Højgaard and Taksar (1999) studied the problem of risk control and dividend distribution policies. They modelled the evolution of the process X of the company by $dX_t = a_t(\mu dt + \sigma dW_t) - dZ_t$, where $a = (a_t)_{t\geq 0}$ represents the risk exposure with $0 \leq a_t \leq 1$ for all $t \geq 0$. They found the optimal strategy which maximizes the expected total discounted dividends when there is no restriction on the rate of dividend pay-out. They showed that there exists u_0 and u_1 with $u_0 \leq u_1$ such that every excess of the reserve above u_1 is distributed as dividend and the optimal risk exposure is given by $a(x) = \frac{u_0}{x} \wedge 1$ where x is the current reserve. Asmussen, Højgaard and Taksar (2000) considered the issue of optimal risk control and dividend distribution policies under excess of loss reinsurance which is the most common in the reinsurance industry. Under this contract with dynamic retention level $(\alpha_t)_{t\geq 0}$, the reinsurer covers the excess amount $(y - \alpha_t)_+$ of a claim of size y occuring at time t and receives a certain part of the premium. The authors used a diffusion approximation for the reserve process and reparametrized the problem by considering the drift term as the basic control parameter, which leads to a mixed regular/singular stochastic control problem. They derived an Hamilton Jacobi Bellman variational inequality (HJBVI in short) in the case of unbounded rate of dividends and proved that the value function is a classical solution of the associated

HJBVI. They constructed the solution in the case of unbounded and bounded support of the distribution of the claims. In this paper, we study the same problem but we model the reserve process of the insurer by using a compound Poisson process. Due to the Markovian context, our problem may be studied by a direct dynamic programming approach leading to an integro-differential HJBVI. In general, the value function of control problems is not smooth enough to be a strong solution of the associated HJBVI. The notion of viscosity solution, first introduced by Crandall and Lions (1983), is known to be a powerful tool for this type of problems. We prove here an existence and uniqueness result for the associated HJBVI and then solve it by using an efficient numerical method, the convergence of which is ensured by the uniqueness result. The paper is organized as follows. The problem is formulated in Section 2. In Section 3, we prove that the value function is a viscosity solution of the associated HJBVI. In Section 4 we prove the uniqueness of the viscosity solution. Section 5 is devoted to the numerical analysis of the HJBVI: we perform a finite difference approximation of the HJBVI and then solve the problem by using an algorithm based on a "Howard" or policy iteration algorithm. Numerical results are presented. They provide the optimal policy of reinsurance and the optimal dividend strategy.

2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We assume that the claims are generated by a compound Poisson process. More precisely, we consider an integer-valued random measure $\mu(dt, dy)$ with compensator $\pi(dy)dt$. We denote by $\tilde{\mu}(dt, dy) = \mu(dt, dy) - \pi(dy)dt$ the compensated Poisson random measure. We assume that $\pi(dy) = \beta G(dy)$ where $G(dy)$ is a probability distribution on $B \subseteq I\!R_+$ and β is a positive constant. In this case, the integral with respect to the random measure $\mu(dt, dy)$ is simply a compound Poisson process. We have $\int_0^t \int_B y\mu(du, dy) = \sum_{i=1}^{N_t} Y_i$, where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity β and $\{Y_i, i \in I\!N\}$ is a sequence of random variables with common distribution G which represent the sizes of the claims. We set $\nu := EY_i$ for all $i = 1 \dots N_t$, for all $t \geq 0$.
We denote by $I\!F = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by the random measure $\mu(dt, dy)$.
A retention level process is an $\mathcal{F}_t$-adapted process $\alpha = (\alpha_t, t \geq 0)$ representing an excess of loss treaty specifying that, of any claim of size y at time t, the direct insurer is to cover $y \wedge \alpha_t$ and the reinsurer is to cover the excess amount $(y - \alpha_t)_+$.
Given a retention level α_t at time t, we denote by $p(\alpha_t)$ the difference between the premium rate per unit of time received by the direct insurer and the premium rate per unit of time paid by the direct insurer to the reinsurer at time t.
We consider a premium rate of the same form as in Asmussen, Højgaard and Taksar (2000):

$$p(\alpha_t) \quad = \quad (1+k_1)\beta\nu - (1+k_2)\beta E\left[(Y_i - \alpha_t)_+\right] \quad \text{for all } t \geq 0, \tag{2.1}$$

where k_1 and k_2 are proportional factors satisfying $0 \leq k_1 \leq k_2$. In Equation (2.1), the term $\beta\nu$ represents the expectation of the amount of the claims during a unit of time. The second term of the r.h.s of Equation (2.1) is the premium paid to the reinsurance company to support the difference between the amount of the claims and the retention level α, during a unit of time. Regulations lay down that premia must be nonnegative, which means that

$$p(\alpha_t) \geq 0 \ \text{ for all } t \geq 0. \tag{2.2}$$

Condition (2.2) is equivalent to

$$\alpha_t \geq \underline{\alpha} \quad \text{for all } (t, w) \text{ a.e.} \tag{2.3}$$

where $\underline{\alpha}$, the lowest admissible retention, is the unique solution of $p(\underline{\alpha}) = 0$. We make the following assumption on $\underline{\alpha}$:

$$\textbf{(H)} \quad \underline{\alpha} \leq \sup B.$$

Remark 2.1 *When the mark space is reduced to $B = \{\delta\}$ with $\delta > 0$, then we have an explicit expression of $\underline{\alpha}$ which is $\underline{\alpha} = \frac{\delta(k_2 - k_1)}{1 + k_2}$.*

We denote by $L = (L_t, t \geq 0)$ the $\mathcal{F}_t$-adapted process of the cumulative amount of dividends paid out by the direct insurer. Given an initial reserve x and a policy (α, L), the reserve of the insurance company at time t under this excess of loss contract is then given by :

$$X_t^{x,\alpha,L} \quad = \quad x + \int_0^t p(\alpha_u) du - \int_0^t \int_B (y \wedge \alpha_u)\, \mu(du, dy) - \int_0^t dL_u. \tag{2.4}$$

A strategy (α, L) is said to be admissible if $\alpha = (\alpha_t, t \geq 0)$ satisfies (2.3) and

$$L \text{ is right-continuous, nondecreasing, } L_{0^-} = 0 \text{ and } \triangle L_t \leq X_t^{x,\alpha,L} \text{ for } (t, w) \text{ a.e.} \tag{2.5}$$

where $\triangle L_s := L_s - L_{s^-}$. The last inequality expresses the fact that the insurer is not allowed to pay out dividends at time t which exceed the level of his reserve at this time. Given an initial reserve x, we denote by $\Pi(x)$ the set of all admissible policies. For $(\alpha, L) \in \Pi(x)$, we define the return function as

$$J(x, \alpha, L) = E_x \int_0^{\bar{\tau}} e^{-rt} dL_t,$$

where $r > 0$ is a discount factor and $\bar{\tau}$ is the ruin time defined by

$$\bar{\tau} = \inf\{t \geq 0, X_t^{x,\alpha,L} \leq 0\}.$$

The objective is to find the value function which is defined as

$$v(x) = \sup_{(\alpha,L) \in \Pi(x)} J(x, \alpha, L). \tag{2.6}$$

For $x \in \mathbb{R}_+$ and $(\alpha, L) \in \Pi(x)$, we have for all $t \in [0, \bar{\tau})$, $X_t^{x,\alpha,L} \geq 0$. Sending $t \longrightarrow 0^+$, we get $x - \alpha_0 \wedge y - L_0 \geq 0$ for all $y \in B$ and so

$$x - \alpha_0 \wedge y \geq 0 \text{ for all } y \in B. \tag{2.7}$$

The constraint (2.2), the inequality (2.7) and the assumption (H) imply that it is optimal to distribute all the current reserve as dividend when $x \leq \underline{\alpha}$ and so $v(x) = x$ for all $x \in [0, \underline{\alpha}]$.

3 Characterization of the value function as a viscosity solution of a HJBVI

In this section, we prove that the value function defined in (2.6) is a viscosity solution of the integro-differential Hamilton Jacobi Bellman variational inequality

$$\max\left\{H(x,v,v^{'}),1-v^{'}(x)\right\}=0 \quad \text{in } (\underline{\alpha},\infty) \tag{3.1}$$

with Dirichlet boundary conditions

$$v(\underline{\alpha})=\underline{\alpha} \tag{3.2}$$

where

$$H(x,v,v^{'}):=\sup_{\alpha\in\Lambda(x)}\left\{-rv(x)+p(\alpha)v^{'}(x)+\int_B (v(x-y\wedge\alpha)-v(x))\pi(dy)\right\},$$

and

$$\Lambda(x)=\{\alpha\geq\underline{\alpha} \text{ s.t. } x\geq\alpha\wedge y \text{ for all } y\in B\}.$$

We begin by giving a heuristic derivation of (3.1) by following the technique used e.g. by Davis and Norman (1990) and He and Pagès (1993). Assume that L is absolutely continuous with respect to t, i.e there exists $\lambda=(\lambda_t)_{t\geq 0}$ such that $dL_t=\lambda_t dt$ and $\lambda_t\geq 0$. The evolution of the reserve process X is then

$$X_t^{x,\alpha,L} \;=\; x+\int_0^t p(\alpha_u)du-\int_0^t\int_B (y\wedge\,\alpha_u)\,\mu(du,dy)-\int_0^t\lambda_u du.$$

and the associated Hamilton Jacobi Bellman equation becomes

$$\sup_{\lambda\geq 0}\left\{H(x,v,v')+\lambda(1-v^{'}(x))\right\}=0 \quad \text{in } (\underline{\alpha},\infty) \tag{3.3}$$

with Dirichlet boundary conditions $v(\underline{\alpha})=\underline{\alpha}$. This is only valid if $1-v'(x)\leq 0$. We obtain the following charaterization for λ:

$$\begin{aligned} \lambda\in[0,+\infty] &\quad \text{if} \quad 1-v^{'}(x)=0,\\ \lambda=0 &\quad \text{if} \quad 1-v^{'}(x)<0. \end{aligned}$$

Since $\lambda(1-v^{'}(x))=0$ and $1-v^{'}(x)\leq 0$, equation (3.3) can be written as (3.1).

We can also apply a verification theorem for integro-differential HJBVIs which states that a solution of (3.1)-(3.2) in $\mathcal{C}^2((\underline{\alpha},\infty))\cap\mathcal{C}([\underline{\alpha},\infty))$ which satisfies some technical conditions actually coincides with the value function (2.6) (see Theorem 5.2 in Øksendal and Sulem (2005) and Ishikawa (2004)).

However, our value function is not $\mathcal{C}^1$ in general and not even a priori continuous everywhere. Indeed the value function is not concave and one cannot derive as usual the continuity of the value function as a consequence of the concavity property. Nevertheless, we will show that the value function does satisfy equation (3.1)-(3.2) if we interpret these equations in the appropriate sense of viscosity solutions.

We now state some useful properties for the value function.

Lemma 3.1 *The value function v is nondecreasing in $I\!R_+$ and satisfies*

$$v(x) \leq x + K$$

where K is a positive constant.

Proof. Let x, x' in $I\!R_+$ such that $x \leq x'$. Then clearly $\Pi(x) \subset \Pi(x')$ and consequently $v(x) \leq v(x')$.
Let $(\alpha, L) \in \Pi(x)$. Using generalized Itô's formula for $e^{-rt}L_t$, we have

$$d(e^{-rt}L_t) = e^{-rt}dL_t - rL_{t^-}e^{-rt}dt. \tag{3.4}$$

From (2.4) and since $p(\alpha_t) \leq (1+k_1)\beta\nu t$ for all $t \in [0, \bar{\tau}]$ and L is nondecreasing we deduce that for all $t \in [0, \bar{\tau}]$

$$L_{t^-} \leq L_t \leq L_0 + x + (1+k_1)\beta\nu t. \tag{3.5}$$

Setting by convention $dL_t = 0$ for all $t \geq \bar{\tau}$, we obtain, by combining (3.4) and (3.5)

$$\lim_{t\to\infty} e^{-rt}L_t - L_0 \geq \int_0^\infty e^{-rt}dL_t - r\int_0^\infty e^{-rt}\left(L_0 + x + (1+k_1)\beta\nu t\right)dt.$$

Consequently

$$\int_0^\infty e^{-rt}dL_t \leq x + K$$

where K is a positive constant independent of (α, L). Taking the supremum over all admissible strategies, we get

$$v(x) \leq x + K.$$

□

We define now the upper and the lower semicontinuous envelope of the function v.

Definition 3.1 *(i) The upper semi-continuous envelope of the function v is defined as*

$$\begin{aligned} v^*(x) &:= \limsup_{x'\to x} v(x') \\ &= \lim_{\varrho\searrow 0} \sup\{v(y), y \in [\underline{\alpha}, \infty) \textit{ and } |y-x| \leq \varrho\}, \quad \textit{for all } x \in [\underline{\alpha}, \infty). \end{aligned} \tag{3.6}$$

(ii) The lower semi-continuous envelope of the function v is defined as

$$\begin{aligned} v_*(x) &:= \liminf_{x'\to x} v(x') \\ &= \lim_{\varrho\searrow 0} \inf\{v(y), y \in [\underline{\alpha}, \infty) \textit{ and } |y-x| \leq \varrho\}, \quad \textit{for all } x \in [\underline{\alpha}, \infty). \end{aligned} \tag{3.7}$$

Since the Hamiltonian H may not be continuous w.r.t. its arguments, we define the upper and the lower semi-continuous envelope of H by $H^*(x, v, v^{'}) = \limsup_{x'\to x} H(x', v, v^{'})$ and $H_*(x, v, v^{'}) = \liminf_{x'\to x} H(x', v, v^{'})$.
Extending the definition of viscosity solutions introduced by Crandall and Lions (1983) and then by Soner (1986) and Sayah (1991) to first integro-differential operators, we define the viscosity solution as follows:

Definition 3.2 *(i) A function v is a viscosity super-solution of* (3.1) *in* $(\underline{\alpha}, \infty)$ *if*

$$\max\left\{H_*(x, \psi, \psi^{'}), 1-\psi^{'}(x)\right\} \leq 0 \tag{3.8}$$

whenever $\psi \in C^1(N_x)$, N_x is a neighbourhood of x and $v_* - \psi$ has a global strict minimum at $x \in (\underline{\alpha}, \infty)$.
(ii)A function v is a viscosity sub-solution of (3.1) *in* $(\underline{\alpha}, \infty)$ *if*

$$\max\left\{H^*(x, \psi, \psi^{'}), 1-\psi^{'}(x)\right\} \geq 0 \tag{3.9}$$

whenever $\psi \in C^1(N_x)$, N_x is a neighbourhood of x and $v^* - \psi$ has a global strict maximum at $x \in (\underline{\alpha}, \infty)$.
(iii) A function v is a viscosity solution of (3.1) *in* $(\underline{\alpha}, \infty)$ *if it is both a super and a sub-solution in* $(\underline{\alpha}, \infty)$.

We define

$$\mathcal{S}_1(I\!R_+) = \{f : I\!R_+ \longrightarrow I\!R, f \text{ is nondecreasing and } \sup_{x \in I\!R_+} \frac{f(x)}{1+x} < \infty\}.$$

Remark 3.1 *It is easy to check that v^* and v_* are in $\mathcal{S}_1(I\!R_+)$.*

We need the following *dynamic programming principle*: For any stopping time $\tau \in [0, \bar{\tau}]$ and any $0 \leq t \leq T$,

$$v(x) = \sup_{(\alpha,L)\in\Pi(x)} E\left[e^{-r(t\wedge\tau)} v\left(X^{x,\alpha,L}_{t\wedge\tau}\right) + \int_0^{t\wedge\tau} e^{-rs} dL_s\right], \tag{3.10}$$

where $a \wedge b = \min(a, b)$. This principle is well known in the diffusion case (see Krylov (1980) (Theorem 9 and Theorem 11, p. 134) and Fleming-Soner (1993)(Theorem 2.1 p. 219). In the case of jump diffusions the proof can be found in Ishikawa (2004).

Theorem 3.1 *The value function v is a viscosity solution of (3.1) in $(\underline{\alpha}, \infty)$.*

Proof. We first prove that v is a viscosity super-solution of (3.1) in $(\underline{\alpha}, \infty)$. Let $x_0 \in (\underline{\alpha}, \infty)$ and $\psi \in C^1(I\!R_+)$ such that without loss of generality

$$0 = (v_* - \psi)(x_0) = \min_{(\underline{\alpha}, \infty)} (v_* - \psi).$$

From the definition of v_*, there exists a sequence $(x_n)_{n\geq 1} \in (\underline{\alpha}, \infty)$ such that $x_n \longrightarrow x_0$ and $v(x_n) \longrightarrow v_*(x_0)$ when $n \longrightarrow \infty$.
For $\alpha \geq \underline{\alpha}$ and $\delta > 0$, we set $L_s = \delta$ and $\alpha_s = \alpha$ for all $s \geq 0$. Then $X^{x_n,\alpha,L}_{0^+} = x_n - \delta$. The dynamic programming principle (3.10) yields

$$\psi(x_n) + \gamma_n \geq \psi(x_n - \delta) + \delta, \tag{3.11}$$

where the sequence

$$\gamma_n := v(x_n) - \psi(x_n)$$

is deterministic and converges to zero when n tends to infinity. Sending $n \longrightarrow \infty$ in (3.11), we get

$$\psi(x_0) \geq \psi(x_0 - \delta) + \delta.$$

Sending now $\delta \longrightarrow 0^+$, we obtain

$$1 - \psi^{'}(x_0) \leq 0. \tag{3.12}$$

It remains to prove

$$H_*(x_0, \psi, \psi^{'}) \leq 0. \tag{3.13}$$

We choose $L_s = 0$ and $\alpha_s = \alpha$ for all $s \geq 0$. We set

$$\theta_n = \inf\{t \geq 0, X_t^{x_n,\alpha,L} \notin B(x_n, \eta)\},$$

where η is a positive constant and $B(x_n, \eta) = \{x, |x - x_n| \leq \eta\}$. Applying Itô's formula to $e^{-r(t\wedge\theta_n)}\psi(X_{t\wedge\theta_n}^{x_n,\alpha,L})$, using (3.11) and the martingale property of

$$\int_0^{t\wedge\theta_n} \int_B e^{-rs} \left(\psi(X_{s^-}^{x_n,\alpha,L} - y \wedge \alpha_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right) \tilde{\mu}(ds, dy),$$

we get for all $t \in [0, T]$

$$\begin{aligned} & E\left[\frac{1}{t}\int_0^{t\wedge\theta_n} -re^{-rs}\psi(X_s^{x_n,\alpha,L}) + e^{-rs}p(\alpha_s)\psi^{'}(X_s^{x_n,\alpha,L})ds\right] \\ + \quad & E\left[\frac{1}{t}\int_0^{t\wedge\theta_n} \int_B e^{-rs} \left(\psi(X_{s^-}^{x_n,\alpha,L} - y \wedge \alpha_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right) \pi(dy)ds\right] \leq \frac{\gamma_n}{t}. \end{aligned} \tag{3.14}$$

Two cases are now to be considered:
case 1: the set $\{n \geq 0 : \gamma_n = 0\}$ is finite. Then there exists a subsequence renamed $(\gamma_n)_{n\geq 0}$ such that $\gamma_n \neq 0$ for all n and we take $t = \sqrt{\gamma_n}$.
case 2: the set $\{n \geq 0 : \gamma_n = 0\}$ is not finite. Then there exists a subsequence renamed $(\gamma_n)_{n\geq 0}$ such that $\gamma_n = 0$ for all n.
In both cases $\dfrac{\gamma_n}{t} \longrightarrow 0$ as n tends to ∞ . Sending n to infinity, using dominated convergence theorem and mean value theorem, (3.14) implies

$$-r\psi(x_0) + p(\alpha)\psi^{'}(x_0) + \int_B \left(\psi(x_0 - y \wedge \alpha) - \psi(x_0)\right) \pi(dy) \leq 0$$

and so (3.13) is proved. Combining (3.13) and (3.12), we conclude that v is a viscosity super-solution.
For sub-solution inequality (3.9), let $\psi \in C^1(\mathbb{R}_+)$, and let $x_0 \in (\underline{\alpha}, \infty)$ be a strict global maximizer of $v^* - \psi$ such that $(v^* - \psi)(x_0) = \max\limits_{(\underline{\alpha},\infty)}(v^* - \psi) = 0$. We have to show that

$$\max\left\{H^*\left(x_0, \psi, \psi^{'}\right), 1 - \psi^{'}(x_0)\right\} \geq 0. \tag{3.15}$$

Suppose that (3.15) does not hold. Hence the left-hand side of (3.15) is negative. By smoothness of ψ and since H^* is upper semi-continuous, there exists δ and ξ satisfying:

$$\max\left\{H^*\left(x,\psi,\psi^{'}\right), 1-\psi^{'}(x)\right\} < -r\xi \tag{3.16}$$

for all $x \in B(x_0,\delta)$ as well as

$$v^*(x') \le -\xi + \psi(x') \tag{3.17}$$

where $x' = x_0 \pm \delta$. By changing δ, we may assume that $B(x_0,\delta) \subset (\underline{\alpha},\infty)$.
From the definition of v^*, there exists a sequence $(x_n)_{n\ge1} \in (\underline{\alpha},\infty)$ such that $x_n \longrightarrow x_0$ and $v(x_n) \longrightarrow v^*(x_0)$ when $n \longrightarrow \infty$. We suppose that $x_n \in B(x_0,\delta)$ for all $n \in I\!N$. Let $(\alpha, L) \in \Pi(x_n)$ be given and define the stopping time τ_n as

$$\tau_n = \inf\{t \ge 0, X_t^{x_n,\alpha,L} \notin B(x_0,\delta)\}.$$

We truncate τ_n by a constant T in order to make it finite and set $\tau^* = \tau_n \wedge T$. On the set $\{\tau^* = \tau_n\}$, using that v is nondecreasing, we get from (3.17)

$$v(X_{\tau^*}) \le v(x') \le \psi(x') - \xi \le \psi(X_{\tau^*-}) - \xi.$$

On the set $\{\tau^* = T\}$, we have

$$v(X_{\tau^*}) \le \psi(X_{\tau^*-}).$$

Applying Itô's formula to $e^{-r\tau^*}\psi(X_{\tau^{*-}}^{x_n,\alpha,L})$, we get (with L_t^c denoting the continuous part of L_t)

$$\begin{aligned}
e^{-r\tau^*}v(X_{\tau^*}^{x_n,\alpha,L}) &\le e^{-r\tau^{*-}}\psi(X_{\tau^{*-}}^{x_n,\alpha,L}) - \xi e^{-r\tau^*}\mathbf{1}_{\{\tau^*=\tau_n\}} \\
&\le \psi(x_n) + \int_0^{\tau^*}(-re^{-rs}\psi(X_s^{x_n,\alpha,L}) + e^{-rs}p(\alpha_s)\psi^{'}(X_s^{x_n,\alpha,L}))ds \\
&+ \int_0^{\tau^{*-}}\int_B e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - y\wedge\alpha_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right)\mu(ds,dy) \\
&- \int_0^{\tau^{*-}} e^{-rs}\psi^{'}(X_s^{x_n,\alpha,L})dL_s^c \\
&+ \sum_{s=0}^{\tau^{*-}} e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - \triangle L_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right) - \xi e^{-r\tau^*}\mathbf{1}_{\{\tau^*=\tau_n\}}
\end{aligned} \tag{3.18}$$

where $\triangle L_s = L_s - L_{s^-}$. For $0 \le s \le \tau^{*-}$, (3.16) implies

$$\begin{aligned}
&- r\psi(X_s^{x_n,\alpha,L}) + p(\alpha_s)\psi^{'}(X_s^{x_n,\alpha,L}) \\
&+ \int_B (\psi(X_s^{x_n,\alpha,L} - y\wedge\alpha_s) - \psi(X_s^{x_n,\alpha,L}))\pi(dy) < -r\xi
\end{aligned} \tag{3.19}$$

and

$$1 - \psi^{'}(X_s^{x_n,\alpha,L}) < 0. \tag{3.20}$$

Integrating (3.20), we get

$$
\begin{aligned}
- & \int_0^{\tau^*-} e^{-rs}\psi'(X_s^{x_n,\alpha,L})dL_s^c + \sum_{s=0}^{\tau^*-} e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - \triangle L_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right) \\
\leq & \ -\int_0^{\tau^*} e^{-rs}dL_s. \qquad (3.21)
\end{aligned}
$$

Substituting (3.19) and (3.21) into (3.18) and using the martingale property of

$$\int_0^{\tau^*-}\int_B e^{-rs}\left(\psi(X_{s^-}^{x_n,\alpha,L} - y\wedge\alpha_s) - \psi(X_{s^-}^{x_n,\alpha,L})\right)\tilde{\mu}(ds,dy),$$

we obtain

$$\psi(x_n) \geq E\left[e^{-r\tau^*}v(X_{\tau^*}^{x_n,\alpha,L}) + \int_0^{\tau^*} e^{-rs}dL_s\right] + \xi(1-e^{-rT}). \qquad (3.22)$$

Inequality (3.22) implies

$$v(x_n) + \delta_n \geq E\left[e^{-r\tau^*}v(X_{\tau^*}^{x_n,\alpha,L}) + \int_0^{\tau^*} e^{-rs}dL_s\right] + \xi(1-e^{-rT}), \qquad (3.23)$$

where $\delta_n := \psi(x_n) - v(x_n)$. Since $\delta_n = \psi(x_n) - \psi(x_0) + v^*(x_0) - v(x_n)$, there exists $n_0 \in I\!N$ such that for all $n \geq n_0$, $\delta_n \leq \frac{\xi}{2}(1-e^{-rT})$ and inequality (3.23) implies

$$v(x_n) \geq \sup_{(\alpha,L)\in\Pi(x_n)} E\left[e^{-r\tau^*}v(X_{\tau^*}^{x_n,\alpha,L}) + \int_0^{\tau^*} e^{-rs}dL_s\right] + \frac{\xi}{2}(1-e^{-rT}),$$

which is a contradiction with the dynamic programming principle. □

Remark 3.2 *The proof of Theorem 3.1 remains valid in the case of a general Poisson random measure μ.*

We need now to specify the boundary conditions for the *usc* and *lsc* envelopes of v. Since v may be discontinuous, we need to characterize $v^*(\underline{\alpha})$ and $v_*(\underline{\alpha})$.

Theorem 3.2 *The upper and the lower semi-continuous envelope of v satisfy*

$$v^*(\underline{\alpha}) = v_*(\underline{\alpha}) = \underline{\alpha}. \qquad (3.24)$$

Proof. Since $v(x) \geq \underline{\alpha}$ for all $x \in [\underline{\alpha}, \infty)$, we also have $v_*(x) \geq \underline{\alpha}$ and so $v_*(\underline{\alpha}) \geq \underline{\alpha}$. The opposite inequality holds since $v_*(\underline{\alpha}) \leq v(\underline{\alpha}) = \underline{\alpha}$.
It remains to prove $v^*(\underline{\alpha}) = \underline{\alpha}$. Obviously we have $v^*(\underline{\alpha}) \geq v(\underline{\alpha}) = \underline{\alpha}$. We need to show $v^*(\underline{\alpha}) \leq \underline{\alpha}$. Suppose it is not true: there exists $\eta > 0$ such that $v^*(\underline{\alpha}) \geq 2\eta + \underline{\alpha}$. From the definition of v^*, there exists a sequence $(x_n)_n$ such that $x_n \longrightarrow \underline{\alpha}$ and $v(x_n) \longrightarrow v^*(\underline{\alpha})$ when $n \longrightarrow \infty$, which implies that there exists $n_0 \in I\!N$ such that for all $n \geq n_0$, we have $v(x_n) \geq \eta + \underline{\alpha}$. Let $(\alpha, L) \in \Pi(x_n)$ be given and define the stopping time τ_n as

$$\tau_n = \inf\{t \geq 0, X_t^{x_n,\alpha,L} \leq 0\}.$$

Since $x_n \longrightarrow \underline{\alpha}$ when $n \longrightarrow \infty$, we have $\tau_n \longrightarrow 0^+$. From hypothesis (2.5), we have $\triangle L_0 \leq X_0^{x_n,\alpha,L} = x_n - L_0$ and so $\triangle L_0 \leq x_n$. Let $\epsilon > 0$, there exists $n_1 \in I\!N$ such that for all $n \geq n_1$, we have $\int_0^{\tau_n} e^{-rs} dL_s \leq \underline{\alpha} + \epsilon$. Taking the expectation and then the supremum over all admissible strategies we obtain $v(x_n) \leq \underline{\alpha} + \epsilon$. Sending ϵ to 0^+, we obtain a contradiction. □

4 Uniqueness of the viscosity solution

Some uniqueness proofs for viscosity solutions of first-order integro-differential operators are given in Soner (1986) for bounded viscosity solutions and in Sayah (1991) and in Pham (1998) for unbounded viscosity solutions. As in Soner (1986) Lemma 2.1 or in Sayah (1991) Proposition 2.1, we give an equivalent formulation for viscosity solutions which is needed to prove a comparison theorem.

Proposition 4.1 *Let v be a function defined on $I\!R_+$, then*
i) v is a viscosity super-solution of (3.1) *in* $(\underline{\alpha}, \infty)$ *if and only if*

$$\max\left\{H_*(x_0, v_*, \psi^{'}), 1 - \psi^{'}(x_0)\right\} \leq 0 \tag{4.1}$$

whenever $\psi \in C^1(N_{x_0})$, $v_* - \psi$ has a global strict minimum at $x_0 \in (\underline{\alpha}, \infty)$, N_{x_0} is a neighbourhood of x_0 and

$$H_*(x_0, v_*, \psi^{'}) = \liminf_{x \longrightarrow x_0} H(x, v_*, \psi^{'}).$$

ii) v is a viscosity sub-solution of (3.1) *in* $(\underline{\alpha}, \infty)$ *if and only if*

$$\max\left\{H^*(x_0, v^*, \psi^{'}), 1 - \psi^{'}(x_0)\right\} \geq 0 \tag{4.2}$$

whenever $\psi \in C^1(N_{x_0})$, $v^* - \psi$ has a global strict maximum at $x_0 \in (\underline{\alpha}, \infty)$, N_{x_0} is a neighbourhood of x_0 and

$$H^*(x_0, v^*, \psi^{'}) = \limsup_{x \longrightarrow x_0} H(x, v^*, \psi^{'}).$$

Proof. We prove the statement for sub-solutions only, the other statement is proved similarly. Let v be such that

$$\max\left\{H^*(x_0, v^*, \psi^{'}), 1 - \psi^{'}(x_0)\right\} \geq 0,$$

whenever ψ and x_0 are as above. Since $v^*(x) - v^*(x_0) \leq \psi(x) - \psi(x_0)$ for all $x \in (\underline{\alpha}, \infty)$, then

$$H^*(x_0, v^*, \psi^{'}) \leq H^*(x_0, \psi, \psi^{'}).$$

Hence v is a viscosity sub-solution of (3.1) in $(\underline{\alpha}, \infty)$.
Conversely, let $\psi \in C^1(N_{x_0})$ and $x_0 \in (\underline{\alpha}, \infty)$ such that

$$(v^* - \psi)(x_0) = \max_{(\underline{\alpha}, \infty)} (v^* - \psi)(x) = 0.$$

For each ϵ, $\delta > 0$, we define

$$\Phi_{\epsilon,\delta}(x) = \begin{cases} \psi(x) & \text{if } x \in B(x_0,\epsilon) \\ v^*(x)+\delta & \text{if } x \notin B(x_0,\epsilon), \end{cases}$$

where $B(x_0,\epsilon)$ is the open ball centred in x_0 with radius ϵ. We have $v^*(x_0) = \Phi_{\epsilon,\delta}(x_0)$ and $v^*(x) - \Phi_{\epsilon,\delta}(x) < 0$ for all $x \in (\underline{\alpha},\infty) - \{x_0\}$. Hence

$$(v^* - \Phi_{\epsilon,\delta})(x_0) = \max_{x\in(\underline{\alpha},\infty)} (v^* - \Phi_{\epsilon,\delta})(x).$$

Thus the hypothesis of the Proposition yields

$$\max\left\{H^*(x_0, \Phi_{\epsilon,\delta}, \Phi'_{\epsilon,\delta}), 1 - \psi'(x_0)\right\} \geq 0.$$

From the definition of H, we have the following estimate

$$H^*(x_0, \Phi_{\epsilon,\delta}, \psi') - H^*(x_0, v^*, \psi') \leq G^*(x_0), \tag{4.3}$$

where $G^*(x_0) := \limsup_{x\longrightarrow x_0} G(x)$ and

$$\begin{aligned} G(x) &:= \sup_{\alpha\in\Lambda(x)} \Big\{ -r\left(\Phi_{\epsilon,\delta}(x) - v^*(x)\right) + p(\alpha)\left(\Phi'_{\epsilon,\delta}(x) - \psi'(x)\right) \\ &+ \int_B \left(\Phi_{\epsilon,\delta}(x - y\wedge\alpha) - v^*(x-y\wedge\alpha)\right)\pi(dy) - \int_B \left(\Phi_{\epsilon,\delta}(x) - v^*(x)\right)\pi(dy) \Big\}. \end{aligned}$$

From the definition of G^*, there exists a sequence $(x_n)_n \in (\underline{\alpha},\infty)$ such that $x_n \longrightarrow x_0$ and $G(x_n) \longrightarrow G^*(x_0)$ when $n \longrightarrow \infty$. We suppose that $x_n \in B(x_0,\epsilon)$ for all $n \in I\!N$. From the definition of $\Phi_{\epsilon,\delta}$ we have $\Phi'_{\epsilon,\delta}(x)(x_n) = \psi'(x_n)$ and $v^*(x_n) - \Phi_{\epsilon,\delta}(x_n) \leq 0$ and so

$$G(x_n) \leq \sup_{\alpha\in\Lambda(x_n)} \left\{ \int_B \left(\Phi_{\epsilon,\delta}(x_n - y\wedge\alpha) - v^*(x_n - y\wedge\alpha)\right)\pi(dy) - \int_B \left(\Phi_{\epsilon,\delta}(x_n) - v^*(x_n)\right)\pi(dy) \right\}.$$

We choose $\alpha \in \Lambda(x_n)$, and consider the two cases: $\underline{\alpha} > 0$ and $\underline{\alpha} = 0$.

(i) If $\underline{\alpha} > 0$, then $\alpha > 0$. We set $B_1^\alpha := \{y \in B, x_n - y\wedge\alpha \in B(x_0,\epsilon)\}$ and $B_2^\alpha := \{y \in B, x_n - y\wedge\alpha \notin B(x_0,\epsilon)\}$. Observe that for $(x_n - y\wedge\alpha) \notin B(x_0,\epsilon)$, we have $\Phi_{\epsilon,\delta}(x_n - y\wedge\alpha) - v^*(x_n - y\wedge\alpha) = \delta$ and for $(x_n - y\wedge\alpha) \in B(x_0,\epsilon)$, we have $\Phi^{\epsilon,\delta}(x_0 - y\wedge\alpha) - v^*(x_0 - y\wedge\alpha) = \psi(x_n - y\wedge\alpha) - v^*(x_n - y\wedge\alpha)$. Since $v^*(x_n) - \Phi_{\epsilon,\delta}(x_n) \leq 0$ we get

$$\begin{aligned} &\int_B \left(\Phi^{\epsilon,\delta}(x_n - y\wedge\alpha) - v^*(x_n - y\wedge\alpha)\right)\pi(dy) - \int_B \left(\Phi_{\epsilon,\delta}(x_n) - v^*(x_n)\right)\pi(dy) \\ \leq\ & \delta\int_{B_2^\alpha} \pi(dy) + K\pi(B_1^\alpha) \\ \leq\ & \delta\int_B \pi(dy) + K\pi([0,\epsilon]) < +\infty \end{aligned} \tag{4.4}$$

where K is a constant independent of α and the last inequality is derived for ϵ sufficiently small ($\epsilon \leq \dfrac{\alpha}{2}$) and n sufficiently large .

(ii) If $\underline{\alpha} = 0$, then $\alpha = 0$ or $\alpha > 0$. If $\alpha = 0$, we have

$$\int_B (\Phi_{\epsilon,\delta}(x_n - y \wedge \alpha) - v^*(x_n - y \wedge \alpha))\, \pi(dy) - \int_B (\Phi_{\epsilon,\delta}(x_n) - v^*(x_n))\, \pi(dy) = 0. \quad (4.5)$$

The case $\alpha > \epsilon$ is similar to case (i). If $0 < \alpha \leq \epsilon$, there exists $\epsilon^{'}$ such that $\alpha > \epsilon^{'}$. As in (4.4), we obtain

$$\begin{aligned} & \int_B \Big(\Phi^{\epsilon,\delta}(x_n - y \wedge \alpha) - v^*(x_n - y \wedge \alpha)\Big)\, \pi(dy) - \int_B (\Phi_{\epsilon,\delta}(x_n) - v^*(x_n))\, \pi(dy) \\ \leq \;& \delta \int_B \pi(dy) + K\pi([0, \epsilon^{'}]) \\ \leq \;& \delta \int_B \pi(dy) + K\pi([0, \epsilon]). \end{aligned} \quad (4.6)$$

From (4.4), (4.5) and (4.6) we deduce that

$$\sup_{\alpha \in \Lambda(x_n)} \left\{ \int_B (\Phi_{\epsilon,\delta}(x_n - y \wedge \alpha) - v^*(x_n - y \wedge \alpha))\pi(dy) \right\} \longrightarrow 0$$

when ϵ and δ tend to 0^+. Sending n to infinity, inequality (4.3) implies

$$H^*(x_0, v^*, \psi^{'}) \geq 0,$$

and so (4.2) is proved. $\square$

Uniqueness of the solution of the HJBVI (3.1) with boundary conditions (3.2) is a consequence of the following theorem.

Theorem 4.1 (Comparison theorem) *Let v_1 and v_2 in $\mathcal{S}_1(I\!R_+)$ be a viscosity sub-solution and a super-solution respectively of* (3.1) *in $(\underline{\alpha}, \infty)$ such that $v_1^*(\underline{\alpha}) = (v_2)_*(\underline{\alpha}) = \underline{\alpha}$. Then*

$$v_1^*(x) \leq v_{2*}(x) \;\; \textit{for all } x \in (\underline{\alpha}, \infty). \quad (4.7)$$

Proof. Due to the linear growth of the viscosity sub-solution v_1 (resp. super-solution v_2), the function u_1 (resp u_2) defined by $u_1(x) = v_1(x)e^{-\lambda x}$ (resp $u_2(x) = v_2(x)e^{-\lambda x}$) for $\lambda \in I\!R_+^*$ and $x \in [\underline{\alpha}, \infty)$ is bounded. For $\epsilon > 0$, we define $\Phi : [\underline{\alpha}, \infty) \times [\underline{\alpha}, \infty) \longrightarrow I\!R \cup \{-\infty\}$ as

$$\Phi(x, z) \;\; := \;\; u_1^*(x) - u_{2*}(z) - \frac{1}{\epsilon}(x - z)^2.$$

Since u_1 and u_2 are bounded and Φ is upper semi-continuous, Φ has a global maximum at point $(x^*, z^*) \in [\underline{\alpha}, \infty) \times [\underline{\alpha}, \infty)$. Using $\Phi(\underline{\alpha}, \underline{\alpha}) \leq \Phi(x^*, z^*)$ and that u_1 and u_2 are bounded, it follows that

$$\begin{aligned} |x^* - z^*|^2 \;\; & \leq \;\; \epsilon(u_1^*(x^*) - u_{2*}(z^*)) \\ & \leq \;\; C_\lambda \epsilon, \end{aligned} \quad (4.8)$$

where C_λ is a constant depending only on λ.
If $x^* = \underline{\alpha}$, then using $\Phi(x,x) \le \Phi(\underline{\alpha}, z^*)$ for all $x \in [\underline{\alpha}, \infty)$ and $u_1^*(\underline{\alpha}) = \underline{\alpha} e^{-\lambda \underline{\alpha}}$, we get

$$u_1^*(x) - u_{2*}(x) \quad \le \quad \underline{\alpha} e^{-\lambda \underline{\alpha}} - u_{2*}(z^*). \tag{4.9}$$

From inequality (4.8) and since $x^* = \underline{\alpha}$, we deduce that $z^* \longrightarrow \underline{\alpha}$ when $\epsilon \longrightarrow 0$.
Since u_{2*} is lower semi-continuous, it follows that $\liminf\limits_{\epsilon \to 0} u_{2*}(z^*) \ge u_{2*}(\underline{\alpha})$.
Taking the limit when $\epsilon \longrightarrow 0^+$ in (4.9), we obtain $u_1^*(x) \le u_{2*}(x)$ and so

$$v_1^*(x) \le v_{2*}(x).$$

If $z^* = \underline{\alpha}$, then for all $x \in [\underline{\alpha}, \infty)$, we have

$$u_1^*(x) - u_{2*}(x) \le u_1^*(x^*) - \underline{\alpha} e^{-\lambda \underline{\alpha}}. \tag{4.10}$$

From inequality (4.8) and since $z^* = \underline{\alpha}$, we deduce that $x^* \longrightarrow \underline{\alpha}$ when $\epsilon \longrightarrow 0$.
Since u_1^* is upper semi-continuous, it follows that $\limsup\limits_{\epsilon \to 0} u_1^*(x^*) \le u_1^*(\underline{\alpha})$.
Taking the limit when $\epsilon \longrightarrow 0^+$ in (4.10), we obtain $u_1^*(x) \le u_{2*}(x)$ and so

$$v_1^*(x) \le v_{2*}(x).$$

It remains to study the case when $z^* \ne \underline{\alpha}$ and $x^* \ne \underline{\alpha}$. The functions u_1^* and u_{2*} are respectively sub-solution and super-solution of the variational inequality

$$\max\left\{H^{'}(x,u,u^{'}), 1 - e^{\lambda x}(u^{'}(x) + \lambda u(x))\right\} = 0 \text{ in } (\underline{\alpha}, \infty), \tag{4.11}$$

where

$$H^{'}(x,u,u^{'}) := \sup_{\alpha \in \Lambda(x)} \left\{-ru(x) + p(\alpha)\left(u^{'}(x) + \lambda u(x)\right) + \int_B (u(x - y \wedge \alpha) e^{\lambda y \wedge \alpha} - u(x))\pi(dy)\right\}.$$

The function $(u_1^* - \psi_1)(x)$ reaches its maximum in x^* where

$$\psi_1(x) \equiv u_{2*}(z^*) + \frac{1}{\epsilon}(x - z^*)^2.$$

Consequently, from Proposition (4.1), we get

$$\max\left\{H^{'*}(x^*, u_1^*, \psi_1'), 1 - e^{\lambda x^*}(\psi_1'(x^*) + u_1^*(x^*))\right\} \ge 0,$$

which implies

$$\max\left\{H^{'}(x^*, u_1^*, \psi_1'), 1 - e^{\lambda x^*}(\psi_1'(x^*) + u_1^*(x^*))\right\} \ge 0.$$

Similarly $(u_{2*} - \psi_2)(z)$ reaches its minimum in z^* where

$$\psi_2(z) \equiv u_1^*(x^*) - \frac{1}{\epsilon}(x^* - z)^2.$$

Since u_{2*} is a super-solution of (4.11), we have

$$\max\left\{H_*^{'}(z^*, u_{2*}, \psi_2^{'}), 1 - e^{\lambda z^*}(\psi_2^{'}(z^*) + u_{2*}(z^*))\right\} \le 0,$$

which implies

$$\max\left\{H^{'}(z^*,u_{2*},\psi_2^{'}),1-e^{\lambda z^*}(\psi_2^{'}(z^*)+u_{2*}(z^*))\right\}\le 0.$$

Observing that $\max\{a,b\}-\max\{d,e\}\le 0$ implies either $a\le d$ or $b\le e$, we consider two cases:

(i) the case

$$H^{'}(z^*,u_{2*},\psi_2^{'})-H^{'}(x^*,u_1^*,\psi_1^{'})\le 0,$$

which implies

$$\begin{aligned}0 \quad\le\quad & \sup_{\alpha\in\Lambda(x^*)\cap\Lambda(z^*)}\Big\{-r\left(u_1^*(x^*)-u_{2*}(z^*)\right) \\ +\quad & p(\alpha)\left(\psi_1'(x^*)-\psi_2'(z^*)+\lambda\left(u_1^*(x^*)-u_{2*}(z^*)\right)\right) \\ +\quad & \int_B\Big(u_1^*(x^*-y\wedge\alpha)e^{-\lambda(y\wedge\alpha)}-u_{2*}(z^*-y\wedge\alpha)e^{-\lambda(y\wedge\alpha)}-u_1^*(x^*)+u_{2*}(z^*)\Big)\,\pi(dy)\Big\}.\end{aligned}\tag{4.12}$$

Since (x^*,z^*) is a maximum point of Φ in $[\underline{\alpha},\infty)\times[\underline{\alpha},\infty)$ and $\Phi(x^*,z^*)\ge\Phi(\underline{\alpha},\underline{\alpha})=0$, we have

$$\Phi(x^*,z^*)\ge\Phi(x^*-y\wedge\alpha,z^*-y\wedge\alpha)e^{-\lambda(y\wedge\alpha)}\quad\text{for all } y\in B,$$

which implies

$$\left(u_1^*(x^*-y\wedge\alpha)-u_{2*}(z^*-y\wedge\alpha)\right)e^{-\lambda(y\wedge\alpha)}-u_1^*(x^*)+u_{2*}(z^*)\le 0 \text{ for all } y\in B.$$

From inequality (4.12) and using the fact that $\psi_1'(x^*)=\psi_2'(z^*)=\frac{2}{\epsilon}(x^*-z^*)$, we have

$$\sup_{\alpha\in\Lambda(x^*)\cap\Lambda(z^*)}\left\{\left(-r+\lambda p(\alpha)\right)\left(u_1^*(x^*)-u_{2*}(z^*)\right)\right\}\ge 0.$$

Since $p(\alpha)$ is bounded, choosing λ sufficiently small, we obtain

$$u_1^*(x^*)-u_{2*}(z^*)\le 0.$$

Using that $\Phi(x,x)\le\Phi(x^*,z^*)$, we conclude that $u_1^*(x)\le u_{2*}(x)$ and

$$v_1^*(x)\le v_{2*}(x).$$

(ii) the second case occurs if

$$e^{\lambda x^*}(\frac{2}{\epsilon}(x^*-z^*)+u_1^*(x^*))-e^{\lambda z^*}(\frac{2}{\epsilon}(x^*-z^*)+u_{2*}(z^*))\le 0,$$

which implies

$$e^{\lambda x^*}u_1^*(x^*)-e^{\lambda z^*}u_{2*}(z^*)\le\frac{2}{\epsilon}(e^{\lambda z^*}-e^{\lambda x^*})(x^*-z^*)\le 0,$$

and so we obtain

$$v_1^*(x)\le v_{2*}(x).$$

□

5 Numerical study

Given an initial reserve x and a policy (α, L), the reserve of the insurance company at time t is then given by (2.4) where $\alpha \in \Pi(x)$, L satisfies (2.5) and $p(\alpha_t)$ is given by (2.1). Our purpose is to solve the following equation

$$\begin{cases} \max\left\{ \sup_{\alpha\in\Lambda(x)} \left\{ A^{\alpha}(x, v, v^{'}) \right\}, 1 - v^{'}(x) \right\} = 0 & \text{in} \quad (\underline{\alpha}, \infty) \\ v(x) = x & \text{in} \quad [0, \underline{\alpha}], \end{cases} \tag{5.1}$$

where

$$A^{\alpha}(x, v, v^{'}) = -rv(x) + p(\alpha)v^{'}(x) + \int_B (v(x - \alpha \wedge y) - v(x))\, \pi(dy).$$

We proceed with a technical change of variable which brings $I\!R_+$ into $[0, 1)$, namely

$$\begin{cases} z = \frac{x}{1+x} \\ \psi(z) = v(x). \end{cases}$$

The function ψ is defined in $[0, 1)$ and satisfies

$$\begin{cases} \max\left\{ \sup_{\alpha\in\Lambda(z)} \left\{ \bar{A}^{\alpha}(z, \psi, \psi^{'}) \right\}, 1 - (1-z)^2\psi^{'}(z) \right\} = 0 & \text{in} \quad (\frac{\underline{\alpha}}{1+\underline{\alpha}}, 1) \\ \psi(z) = \frac{z}{1-z} & \text{in} \quad [0, \frac{\underline{\alpha}}{1+\underline{\alpha}}], \end{cases} \tag{5.2}$$

where

$$\bar{A}^{\alpha}(z, \psi, \psi^{'}) = -r\psi(z) + p(\alpha)(1-z)^2\psi^{'}(z) + \int_B \left(\psi(\frac{z - (1-z)\alpha \wedge y}{1 - (1-z)\alpha \wedge y}) - \psi(z) \right) \pi(dy),$$

and $\Lambda(z) = \{\alpha \geq \underline{\alpha} \text{ s.t. } \frac{z}{1-z} \geq \alpha \wedge y \text{ for all } y \in B\}$.
In Sections 3 and 4, we have proved that the value function (2.6), within a change of variables, is the unique viscosity solution of HJBVI (5.2). This solution is approximated by performing the following numerical method:
(i) approximate HJBVI (5.2) by using a finite difference approximation
(ii) solve the approximating equation by means of the Howard or policy iteration algorithm.
Finally a reverse change of variables is performed in order to display the solution of Equation (5.1).

5.1 Finite difference approximation

First, we compute $\underline{\alpha}$ solution of the equation

$$p(\underline{\alpha}) \quad = \quad (1 + k_1)\beta\nu - (1 + k_2)\beta E\left[(Y_i - \underline{\alpha})_+\right] = 0.$$

We take $B = [b_{min}, b_{max}]$ with $0 \leq b_{min} \leq b_{max} < \infty$. We suppose that the claims are uniformly distribued on $[b_{min}, b_{max}]$.

To approximate the integral term, we use the following approximation with uniform step $\triangle b$:

$$\begin{aligned} & \int_{b_{min}}^{b_{max}} \left(\psi(\frac{z-(1-z)\alpha\wedge y}{1-(1-z)\alpha\wedge y}) - \psi(z) \right) \pi(dy) \\ \approx \quad & \frac{\beta \triangle b}{b_{max}-b_{min}} \sum_{i=0}^{k-1} \left(\psi(\frac{z-(1-z)\alpha\wedge y_i}{1-(1-z)\alpha\wedge y_i}) - \psi(\frac{z}{1-z}) \right), \end{aligned}$$

where $y_i = b_{min} + i\,\Delta b;\ i = 0 \dots k-1$. Let $h = \frac{1}{M}$, $(M \in I\!N^*)$ denote the finite difference step in the state coordinate. Define the grids $\Omega_{1h} = \{z_i \equiv ih, z_i > \frac{\underline{\alpha}}{1+\underline{\alpha}}, 0 \le i \le M-1\}$ and $\Omega_{2h} = \{z_i \equiv ih, z_i \le \frac{\underline{\alpha}}{1+\underline{\alpha}}, 0 \le i \le M-1\}$. We define the finite-difference operators

$$\begin{aligned} \partial^+\psi(z) \quad &:= \tfrac{1}{h}(\psi(z+h) - \psi(z)) \\ \partial^-\psi(z) \quad &:= \tfrac{1}{h}(\psi(z) - \psi(z-h)) \\ \partial_2\psi(z) \quad &:= \tfrac{1}{h^2}(\psi(z+h) - 2\psi(z) + \psi(z-h)). \end{aligned}$$

We want to find a monotone, stable and consistent scheme : indeed we know by a result of Barles and Souganidis (1991) that such a scheme will converge to the viscosity solution of HJBVI (5.2) since a comparison theorem holds for the limiting equation (Theorem 4.1). This is achieved by using the one sided difference approximation

$$p(\alpha)\psi'(x) \sim p(\alpha)\partial^+\psi(z) \tag{5.3}$$

since the operator $\bar{A}^\alpha$ is degenerate and $p(\alpha) > 0$. (see Kushner and Dupuis (1992) and Lapeyre, Sulem and Talay). However, to increase the numerical stability of our method, we split $p(\alpha)\psi'(x)$ into a negative and a positive part and use the following scheme:

$$p(\alpha)\psi'(x) \quad \sim -(1+k_2)\beta E\left[(Y_i - \alpha)_+\right]\partial^-\psi(z) + (1+k_1)\beta\nu\partial^+\psi(z). \tag{5.4}$$

The approximation (5.4) is equivalent to adding a viscosity term to the operator. This scheme satisfies all the requirements of stability, consistency and monotonicity.

This finite difference approximation leads to a system of inequalities with unknowns $\{\psi(z_i)\ , z_i \in \Omega_{1h}\}$:

$$\begin{cases} \max\left\{ \sup_{\alpha\in[\underline{\alpha},\delta]} \left\{ \bar{A}_h^\alpha \psi_h \right\}, \mathbf{1} - \bar{B}\psi_h \right\} = 0 \ \text{ in } \Omega_{1h} \\ \psi(z_i) = \frac{z_i}{1-z_i} \ \text{ in } \Omega_{2h} \end{cases} \tag{5.5}$$

where ψ_h is the vector $(\psi(z_i))_{z_i\in\Omega_{1h}}$, $\bar{A}_h^\alpha$ is the matrix associated to the approximation of the operator $\bar{A}^\alpha$, $\bar{B}$ is the matrix associated to the second term of our variational inequality, defined as

$$\begin{cases} \bar{B}(i,i) = -\frac{(1-z_i)^2}{h} \text{ for all } z_i \in \Omega_{1h} \\ \bar{B}(i,i-1) = \frac{(1-z_i)^2}{h} \text{ for all } z_i \in \Omega_{1h} \\ \bar{B}(i,j) = 0 \text{ if } j \notin \{i, i-1\}, \end{cases}$$

and $\mathbf{1}$ is the vector which all entries equal to 1.

5.2 The Howard algorithm

To solve Equation (5.5), we use the Howard algorithm (see Howard (1960) and Lapeyre, Sulem and Talay), also named policy iteration. It consists of computing two sequences $\{\alpha^n, D^n\}_{n\geq 1}$ and $(\psi_h^n)_{n\geq 1}$, (starting from ψ_h^1) defined by:

- Step $2n-1$. Given ψ_h^n, compute a feedback strategy α^n defined as

$$\alpha^n \in \text{argmax}_{\alpha\in[\underline{\alpha},\delta]} \left\{\bar{A}_h^\alpha \psi_h^n\right\}$$

  and define the subset D^n of the grid Ω_h as

$$\bar{A}^{\alpha^n}\psi_h^n \geq \mathbf{1} - \bar{B}\psi_h^n \text{ in } D^n,$$

$$\bar{A}^{\alpha^n}\psi_h^n < \mathbf{1} - \bar{B}\psi_h^n \text{ in } \Omega_h \backslash D^n.$$

- Step $2n$. Define ψ_h^{n+1} as the solution of the linear systems:

$$\bar{A}^{\alpha^n}\psi_h^{n+1} = 0 \text{ in } D^n,$$

  and

$$\mathbf{1} - \bar{B}\psi_h^{n+1} = 0 \text{ in } \Omega_h \backslash D^n.$$

- If $|\psi_h^{n+1} - \psi_h^n| \leq \epsilon$ stop, otherwise, go to step $2n+1$.

Assume that that there exists states $z \in \Omega_{1h}$ such that

$$\bar{A}_h^\alpha \psi_h(z) > \mathbf{1} - \bar{B}\psi_h(z).$$

Then the Howard algorithm converges to (5.5) since the matrices $\bar{A}^{\alpha^n}$ are diagonally dominant (see Chancelier-Messaoud-Sulem (2004)).

5.3 Numerical results

Equation (5.5) has been solved by using the Howard algorithm. This algorithm is very efficient and converges in twenty iterations. Three tests have been performed with parameter values given in Table 1.

	k_1	k_2	r	β	b_{min}	b_{max}
Test 1	0.1	0.15	0.07	0.5	1.1	1.5
Test 2	0.1	0.2	0.07	0.5	1.1	1.5
Test 3	0.1	0.2	0.07	0.6	1.1	1.5

Table 1: The parameters of the numerical tests

The optimal policy has the following form: every excess of the reserve above some critical threshold u is distributed as dividend. When the reserve process is below a level l, it is optimal to distribute all the current reserve as dividends because of the constraint (2.3). When the reserve process is in (l, u), then the insurer doesn't distribute any dividend.

When the cost of reinsurance k_2 increases, the optimal critical level decreases (Compare Tests 1 and 2 in Table 2). When the intensity of the claims β increases, the level u increases (Compare Tests 2 and 3 in Table 2). These phenomena have the following economic explanation: When k_2 increases, the premium $p(\alpha)$ decreases and the potential dividends also decrease. Consequently the dividend payments are started at a low level. On the other hand, when β increases, the premium $p(\alpha)$ increases and the potential reserve is higher. The dividends are thus distributed at a higher level.

Test 1	$l = \underline{\alpha}$=0.42	u=2.57
Test 2	$l = \underline{\alpha}$=0.64	u=2.12
Test 3	$l = \underline{\alpha}$=0.64	u=3.16

Table 2: Lower and upper critical thresholds

The optimal retention level α is displayed in Figures 1 and 2 as a function of the reserve level x. It has the following form: $\alpha(x) = x$ for $l \leq x \leq b_{max}$ and $\alpha(x) = b_{max}$ for $b_{max} \leq x \leq u$. Similar results were obtained by Højgaard and Taksar (1999) in the case of a diffusion model and proportional reinsurance. Figures 3 and 5 display the value function v in terms of the reserve level x. The value function v is nondecreasing. It is linear in $[0, l]$ and $[u, \infty)$. Figures 4 and 6 enlarge the region of the level l.

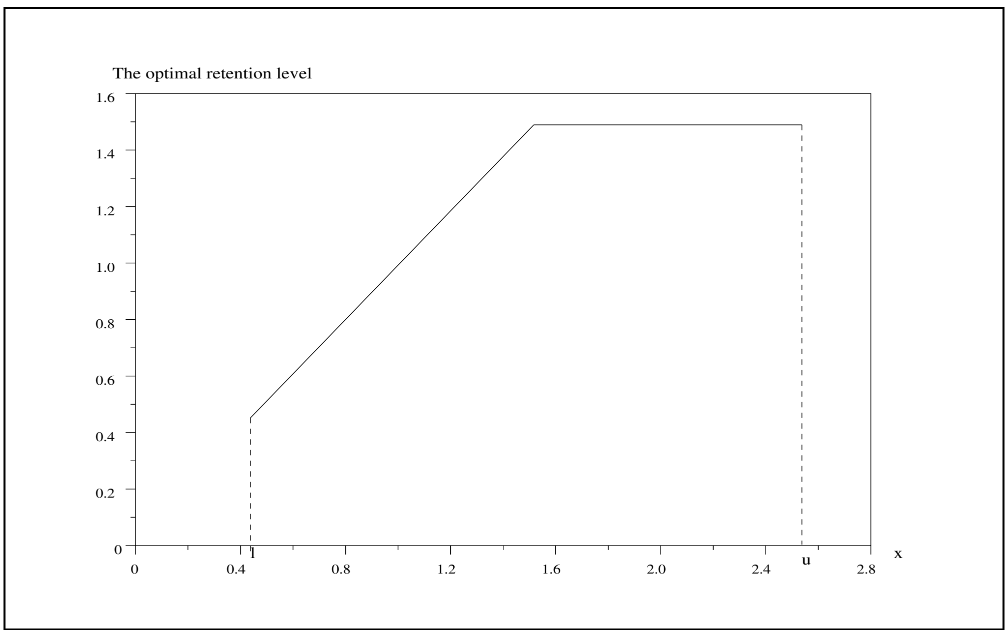

Figure 1: The optimal retention level $\alpha(x)$ for Test 1

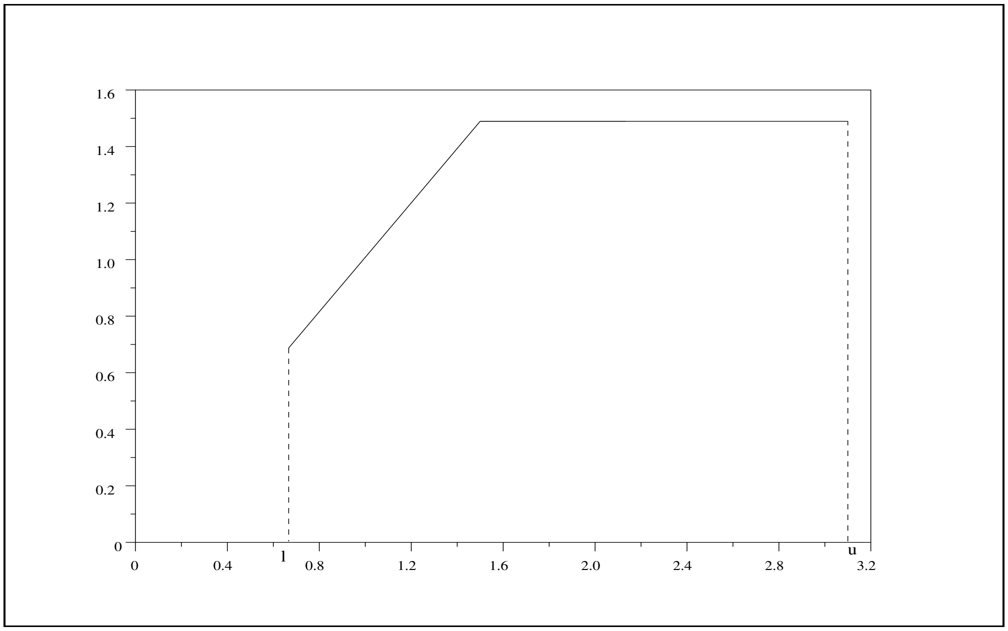

Figure 2: The optimal retention level $\alpha(x)$ for Test 3

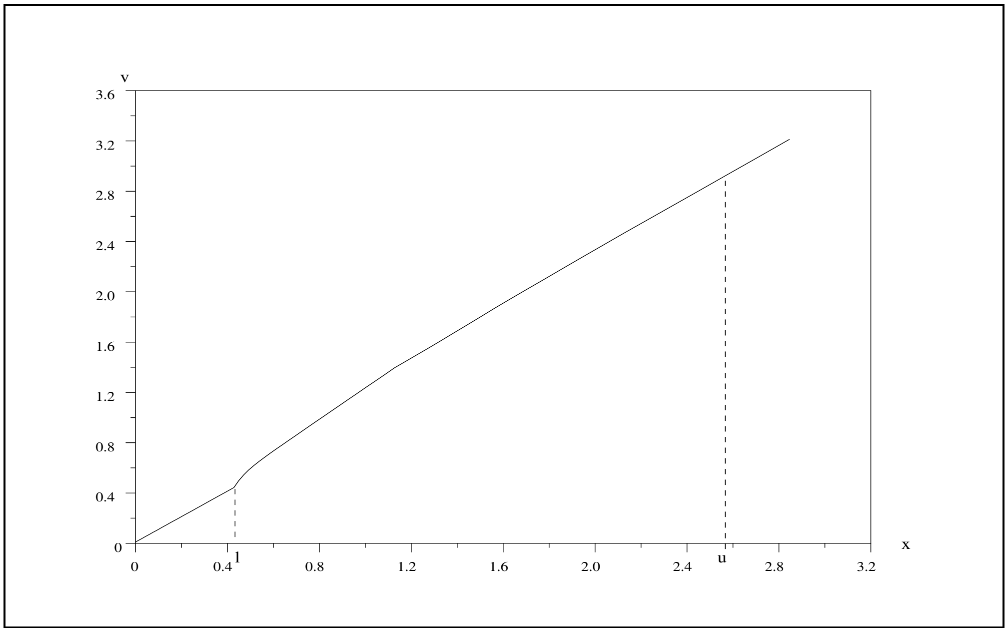

Figure 3: The value function for Test 1

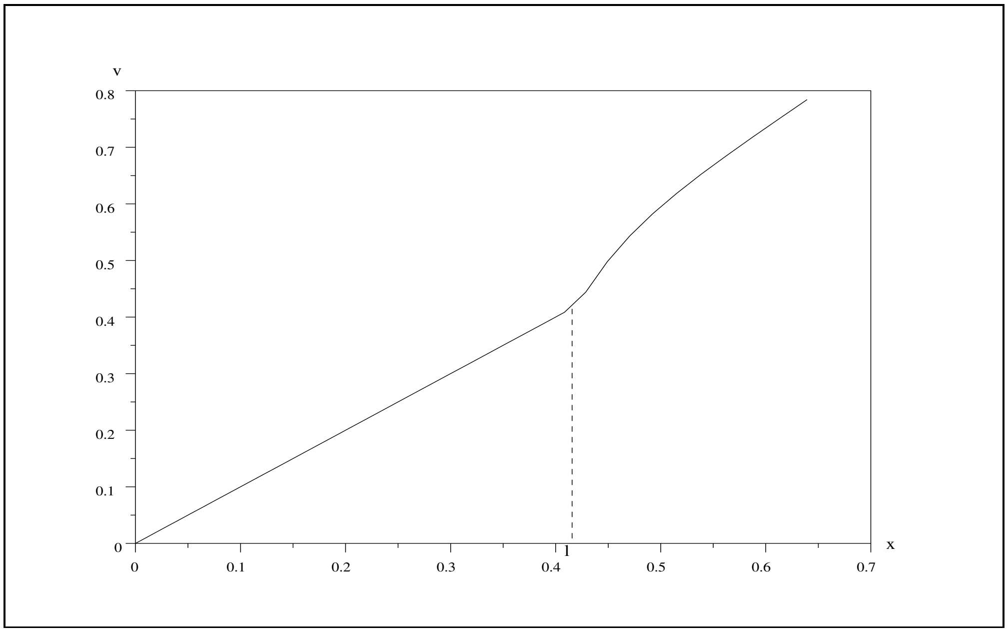

Figure 4: Enlargement of Figure 3 in the neighbourhood of l

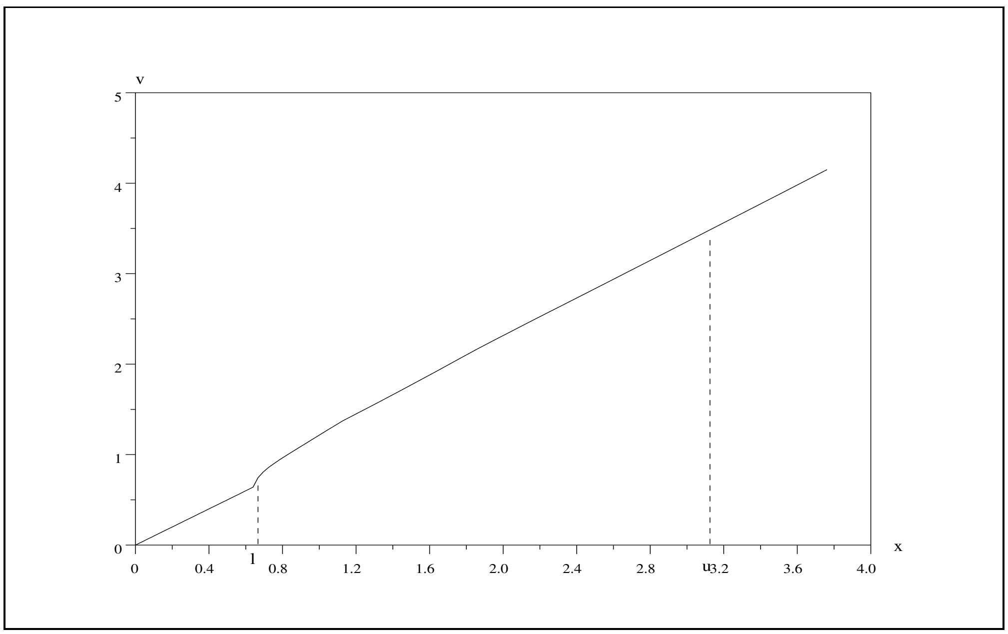

Figure 5: The value function for Test 3

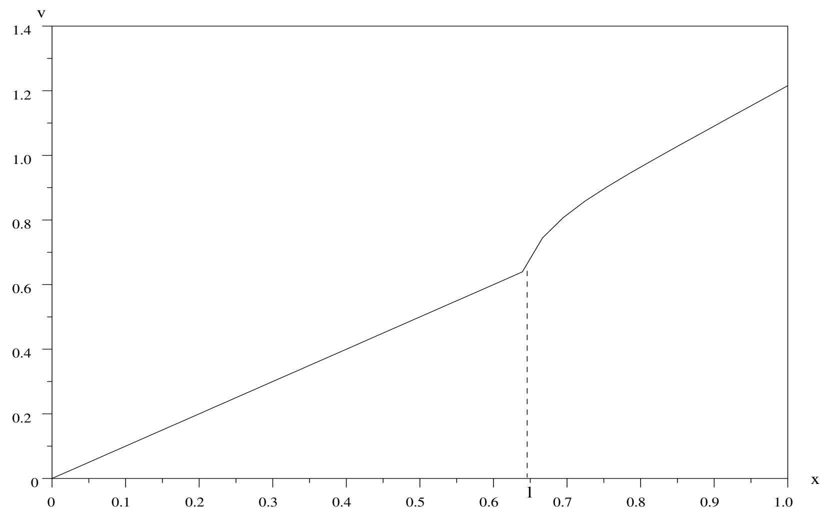

Figure 6: Enlargement of Figure 5 in the neighbourhood of l

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