Avertissement

Ce document contient l’essentiel d’une présentation effectuée lors de la 11ème Conférence Internationale sur l’Analyse et l’Optimisation des Systèmes, 15–17 juin 1994 à Sophia-Antipolis (actes publiés dans la série Lectures Notes on Control and Information Systems, Vol. 199, G. Cohen et J.-P. Quadrat (Eds.), Springer-Verlag).

Il a été ensuite légèrement remanié et complété à l’occasion d'une nouvelle présentation au groupe de travail “Algèbres tropicales et applications aux systèmes à événements discrets et à la commande optimale” commun au GdR/PRC AMI et Automatique réuni les 6–7 juin 1996 à Paris.

Guy Cohen, 7 juin 1996

Dioids and Discrete Event Systems

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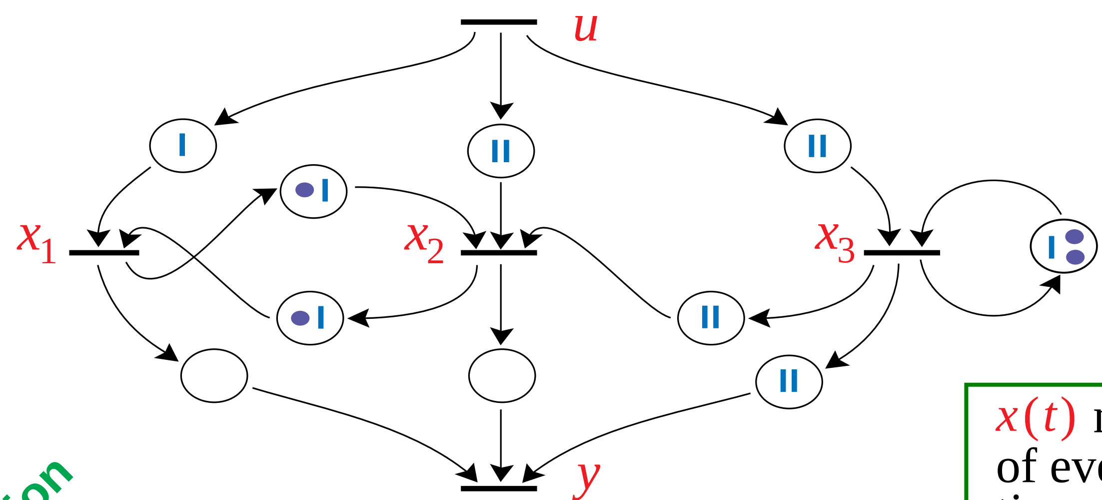
Where the Max-Plus or the Min-Plus Algebra May Pop Up

Dioid Theory

Descriptions of Timed Event Graphs

A Quick Review of Some System-Theoretic Results for Timed Event Graphs

Timed Event Graphs



$x(t)$ number of events at time t at transition x

Counter description

Time domain

$$x_1(t) = \min\left(x_2(t-1)+1, u(t-1)\right)$$

$$x_2(t) = \min\left(x_1(t-1)+1, x_3(t-2), u(t-2)\right)$$

$$x_3(t) = \min\left(x_3(t-1)+2, u(t-2)\right)$$

$$y(t) = \min\left(x_1(t), x_2(t), x_3(t-2)\right)$$

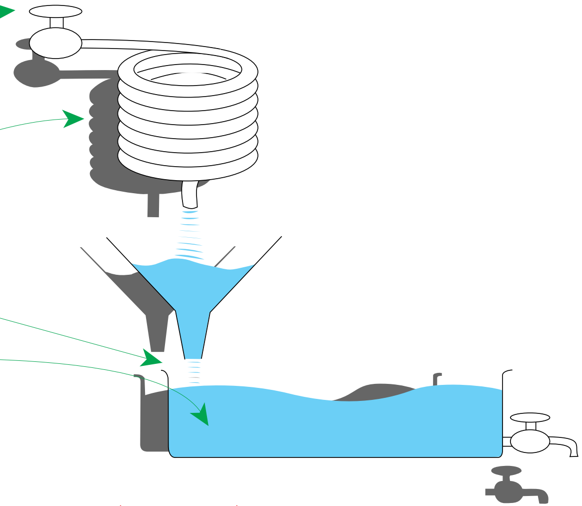
A continuous (linear?) system

$u.t/$ is the cumulated flow from time 0 to t with $u.0/ \quad 0$

the long pipe introduces a delay of 2 s.

the funnel limits the flow to 1 l./s.

$y.t/$ is the volume of the tank at t with $y.0/ \quad 3$ l.

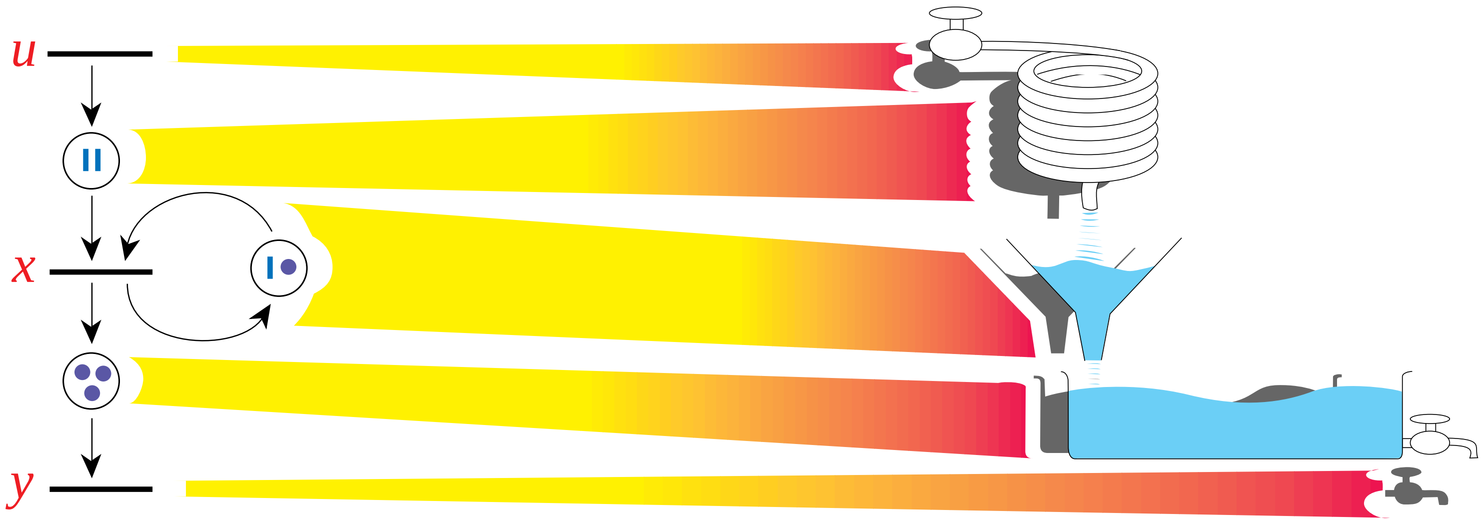


$$y(t) \leq \min(u(t-2)+3, y(t-1)+1)$$

$$y(t) = \inf_{\tau \in \mathbb{R}} \left(h(t-\tau) + u(\tau) \right) \quad \text{with} \quad h(t) = \begin{cases} 3 & \text{for } t \leq 2 \text{ ;} \\ 1+t & \text{otherwise.} \end{cases}$$

$$y(t) = \oint_{\tau \in \mathbb{R}} h(t-\tau) \otimes u(\tau)$$

An event graph analogue



$$y(t) \leq \min(u(t-2)+3; y(t-1)+1)$$

Min-Plus Linearity

if $u(\cdot) \mapsto y(\cdot)$ and $v(\cdot) \mapsto z(\cdot)$,

then $\min(u(\cdot), v(\cdot)) \mapsto \min(y(\cdot), z(\cdot))$ (pointwise min)

and $a + u(\cdot) \mapsto a + y(\cdot)$ for any constant a

Min-plus linear systems can be combined

serially	(inf-convolution)
in parallel	(pointwise min)
in feedback	(‘star’ operation)

Dynamic Programming: a Linear Process

Optimal control problem

$$\min \sum_{t=0}^{T-1} c(u(t)) + \text{`}(x(T)) \quad \text{s.t.} \quad x(t+1) = x(t) - u(t) \; ; \quad x(0) = \text{»}$$

The corresponding dynamic programming equation reads

$$V(x;t) = \inf_y \big(c(x-y) + V(y;t+1)\big) = \oint_y c(x-y)V(y;t+1)$$

$$V(x;T) = \text{`}(x)$$

Hence, the Bellman function V appears as the result of the iterated inf-convolution of the ‘initial’ value ` by the ‘kernel’ c. As such,

V is a min-plus linear function of `.

Asymptotic Exponentials

For $f : \mathbb{R} \to \overline{\mathbb{R}}^{+}$ let $c : f \to \limsup_{x \to +\infty} \log(f(x))/x$

If $f = \sum_{i \in I} \alpha_i \exp(a_i x)$, $\alpha_i \in \mathbb{R}^{+}, a_i \in \overline{\mathbb{R}}$

then $c(f) = \max_{i \in I} a_i$

$$c(f + g) = \max(c(f), c(g))$$

$$c(f \times g) = c(f) + c(g)$$

Relevance

asymptotic Bode plots

large deviations in probability theory

Convexity

Polynomial in one variable x with coefficients in $\mathbb{R}_{\max}$

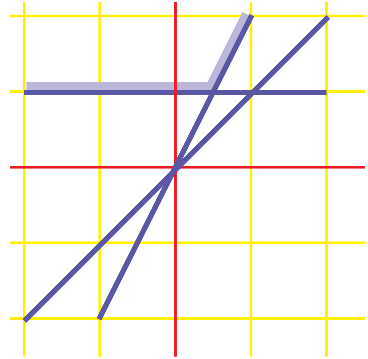
$$p(x) = \bigoplus_i a_i x^i$$

Since $x^i = x \otimes \cdots \otimes x = x + \cdots + x = i \times x$

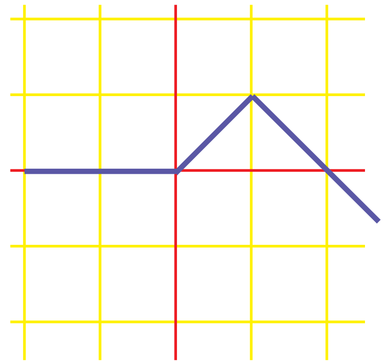
then $p(x) = \max_i (i \times x + a_i)$

As a numerical function, $p(\cdot)$ is piecewise linear convex nondecreasing.

Rational functions are differences of the previous convex functions.



$1 \oplus x \oplus x^2$



$2(x \oplus 0) \oslash (x \oplus 1)^2$

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Dioid: Set $\mathfrak{D}$ endowed with two operations denoted $\oplus$ and $\otimes$

Associativity of addition $$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

Commutativity of addition $$a \oplus b = b \oplus a$$

Associativity of multiplication $$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

Distributivity of multiplication w.r.t. addition

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \qquad c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$$

Existence of a zero element $''$ $$a \oplus '' = a$$

Absorbing zero element $$a \otimes '' = '' \otimes a = ''$$

Existence of an identity element e $$a \otimes e = e \otimes a = a$$

Idempotency of addition $$a \oplus a = a$$

Symmetry?

Suppose that a has an opposite element b such that

$$a \oplus b = ''$$

then $$a \oplus a \oplus b = a$$

hence $$a \oplus b = a$$

thus $$'' = a$$

Commutative dioid

$\mathcal{D}$ is said commutative if $\otimes$ is commutative.

Complete dioid

$\mathcal{D}$ is said complete if it is closed for all infinite sums and if $\otimes$ is distributive with respect to infinite sums.

$\top$ (sum of all elements of $\mathcal{D}$) absorbing for addition

$\top \otimes \text{''} = \text{''}$

Order Structure

Theorem $\{a = a \oplus b\} \Leftrightarrow \{\exists c : a = b \oplus c\}$

($a \succeq b$ if $a = a \oplus b$) defines a (partial) order relation compatible with addition and multiplication,

i.e. if $a \succeq b$, then, for all c, $a \oplus c \succeq b \oplus c$ and $ac \succeq bc$ (and $ca \succeq cb$).

For any two elements a and b in $\mathfrak{D}$, $a \oplus b$ is their least upper bound.

A dioid is a sup-semilattice having a 'bottom' element ε.

If the dioid is complete, this sup-semilattice can be completed to a lattice by the following classical construction of the greatest lower bound $a \wedge b$

$$a \wedge b = \oint_{\substack{x \preceq a \\ x \preceq b}} x$$

Archimedean dioid

$$\forall a \neq '' \; ; \quad \forall b \; ; \quad \exists c \text{ and } d : \quad ac \succeq b \text{ and } da \succeq b$$

Theorem In a complete Archimedean dioid, $\top$ is absorbing for $\otimes$

Distributive dioid

$\mathfrak{D}$ is complete and, for all subsets $\mathfrak{C}$ of $\mathfrak{D}$,

$$\left(\bigwedge_{c \in \mathfrak{C}} c\right) \oplus a = \bigwedge_{c \in \mathfrak{C}} (c \oplus a) \; ; \quad \left(\bigoplus_{c \in \mathfrak{C}} c\right) \wedge a = \bigoplus_{c \in \mathfrak{C}} (c \wedge a)$$

$\oplus$ and $\wedge$ do not play symmetric roles since distributivity of $\otimes$ with respect to $\wedge$ is not always granted

Examples

$\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ ($\mathbb{Z}_{\max}$, $\mathbb{Q}_{\max}$,Ȍ.)

$\mathbb{R}_{\min} = (\mathbb{R} \cup \{+\infty\}, \min, +)$ isomorphic to $\mathbb{R}_{\max}$ by $x \mapsto -x$

$\mathbb{R}_{\max}$ isomorphic to $(\mathbb{R}^{+}, \max, \times)$ by $x \mapsto \exp(x)$

$(\overline{\mathbb{R}}, \max, \min)$

Boole algebra $(\{\varepsilon, e\}, \max, \min)$

$\left(2^{\mathbb{R}^2}, \cup, +\right)$

$(\{(-\infty, x]\}, \cup, +)$ isomorphic to $\mathbb{R}_{\max}$ by the bijection

$$\mathbb{R} \to 2^{\mathbb{R}} : x \mapsto \begin{cases} \varnothing & \text{if } x = \varepsilon \\ (-\infty, x] & \text{otherwise} \end{cases}$$

Properties

$\mathbb{R}_{max}$ not complete, $\overline{\mathbb{R}}_{max}$, $\overline{\mathbb{Z}}_{max}$ complete, $\overline{\mathbb{Q}}_{max}$ not complete!

$\mathbb{R}_{min}$ order reversed with respect to natural order

$(\mathbb{R}^+, \max, \times)$ more convenient than $\mathbb{R}_{max}$ to make drawings

$(\overline{\mathbb{R}}, \max, \min)$ not Archimedian

Boole algebra : you practiced dioids before you knew them!

$\left(2^{\mathbb{R}^2}, \cup, +\right)$ partially ordered

order is given by inclusion

lower bound is given by intersection

multiplication does not distribute with respect to

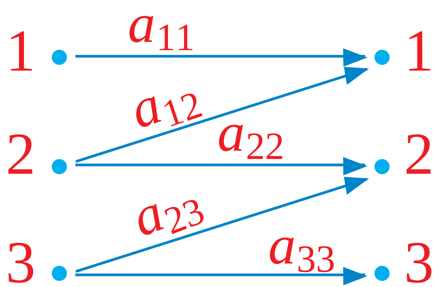
lower bound

Matrix Dioids

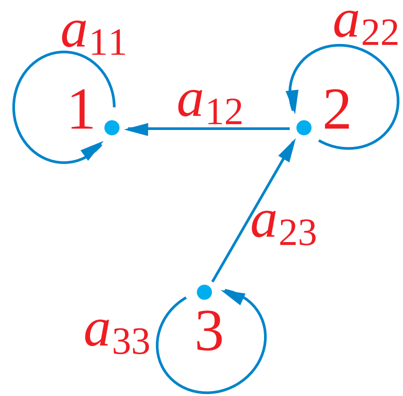
$\mathfrak{D}^{n\times n}$: square $n \times n$ matrices with entries in $\mathfrak{D}$
with sum and product of matrices defined conventionally

$\mathfrak{D}^{n\times n}$ not commutative, partially ordered, complete if $\mathfrak{D}$ is, distributive, not Archimedean

Graph-Theoretic Interpretation



$$\begin{pmatrix} a_{11} & a_{12} & \varepsilon \\ \varepsilon & a_{22} & a_{23} \\ \varepsilon & \varepsilon & a_{33} \end{pmatrix}$$



transition graph

precedence graph

Polynomials and Power Series

$\mathfrak{D}[\![z_1, \ldots, z_m]\!]$ power series in $z_1, \ldots, z_m$ with coefficients in $\mathfrak{D}$ (complete) and with exponents in $\mathbb{N}$ or in $\mathbb{Z}$

$\mathfrak{D}[z_1, \ldots, z_m]$ subdioid of polynomials

Formal polynomials are not isomorphic to their associated numerical functions.

$\mathcal{E}$ associates numerical functions with formal polynomials: homomorphism (for pointwise $\oplus$ and $\otimes$ of numerical functions) but not isomorphism.

$\{a(t)\}_{t\in\mathbb{Z}}$ in $\overline{\mathbb{R}}_{\min}$ $\xrightarrow{z\text{-transform}}$ $A = \bigoplus_{t\in\mathbb{Z}} a(t)z^{-t}$

associated numerical function

$$[\mathcal{E}(A)](x) = \bigoplus_{t\in\mathbb{Z}} a(t)x^{-t} = \inf_{t\in\mathbb{Z}}(a(t) - t \times x) = -\sup_{t\in\mathbb{Z}}(t \times x - a(t))$$

discrete Fenchel transform of the mapping $t \mapsto a(t)$ (up to sign)

An important linear equation

$$x = Ax \oplus b$$

Theorem In a complete dioid, the least solution is given by

$$x = A^* b$$

where

$$A^* = \bigoplus_{p=0}^{+\infty} A^p \,, \qquad A^0 = e$$

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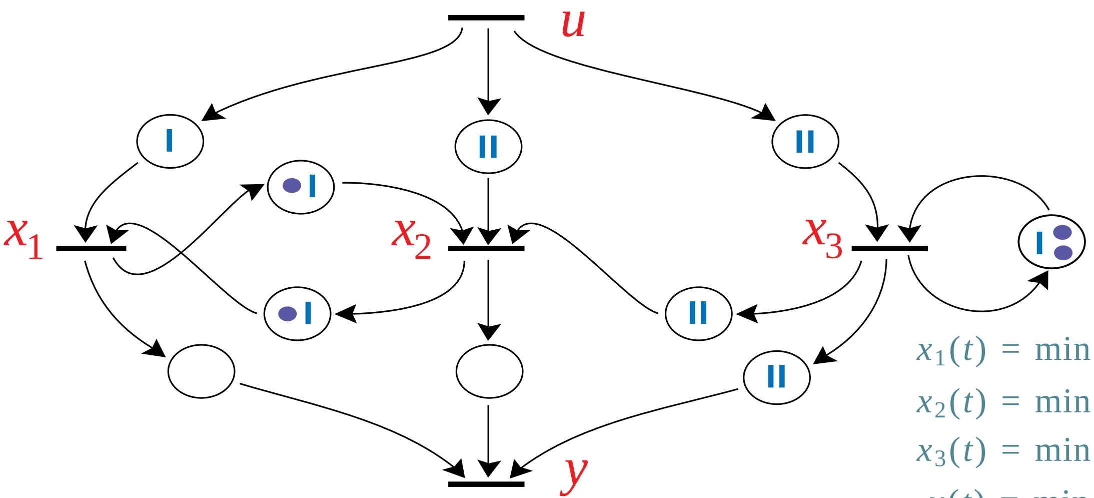

Counter description
Time domain

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$$x_2(t) = \min\left(x_1(t-1)+1,\, x_3(t-2),\, u(t-2)\right)$$
$$x_3(t) = \min\left(x_3(t-1)+2,\, u(t-2)\right)$$
$$y(t) = \min\left(x_1(t),\, x_2(t),\, x_3(t-2)\right)$$

Dater description
Event domain

$$x_1(k) = \max\left(x_2(k-1)+1,\, u(k)+1\right)$$
$$x_2(k) = \max\left(x_1(k-1)+1,\, x_3(k)+2,\, u(k)+2\right)$$
$$x_3(k) = \max\left(x_3(k-2)+1,\, u(k)+2\right)$$
$$y(k) = \max\left(x_1(k),\, x_2(k),\, x_3(k)+2\right)$$

$x(k)$ date of event nr k at transition x

A few words about residuation...

Dater : $k \mapsto d(k) \in \mathbb{Z}$, where $d(k)$ is the date at which the event numbered k occurs.

Daters satisfy max-plus linear dynamic equations in the **event domain**

Counter: inverse mapping $t \mapsto c(t) = k$ such that $d(k) \approx t$. $d(\cdot)$ is monotonic.

Possible definitions:

$$c(t) = \sup\{k \mid d(k) \le t\} \quad \text{or} \quad c(t) = \inf\{k \mid d(k) \ge t\} \; .$$

Counters satisfy min-plus linear dynamic equations in the **time domain**

In lattice-ordered sets, **residuation theory** deals with the problem of finding the least upper bound of the subset $\{x \mid f(x) \preceq y\}$ for a given y and/or the greatest lower bound of the subset $\{x \mid f(x) \succeq y\}$ when f is isotone. Under a condition of 'lower semi-continuity' (or 'upper semi-continuity') of f, the former (or the latter) bound belongs to the corresponding subset and may be called a sub- (or sup-) solution.

γ- and δ-Transforms

γ-transform of daters $x(k)$: $X(\gamma) = \bigoplus_{k \in \mathbb{Z}} x(k)\gamma^k$ in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

δ-transform of counters $x(t)$: $X(\delta) = \bigoplus_{t \in \mathbb{Z}} x(t)\delta^t$ in $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$

In $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, $(m \oplus p)\gamma^n = \max(m,p)\gamma^n$, hence, in $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$, it should be that

$$n(\delta^m \oplus \delta^p) = n\delta^{\max(m,p)}$$

In $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$, $(m \oplus p)\delta^n = \min(m,p)\delta^n$, hence, in $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$, it should be that

$$n(\gamma^m \oplus \gamma^p) = n\gamma^{\min(m,p)}$$

These equalities of operators can be proved true by direct reasoning if applied to transforms of nondecreasing daters or counters.

$\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

In $\mathbb{B}[\![\gamma,\delta]\!]$ the following congruence is considered

$$X(\gamma,\delta) \equiv Y(\gamma,\delta) \iff \gamma^{*}\left(\delta^{-1}\right)^{*} X(\gamma,\delta) = \gamma^{*}\left(\delta^{-1}\right)^{*} Y(\gamma,\delta)$$

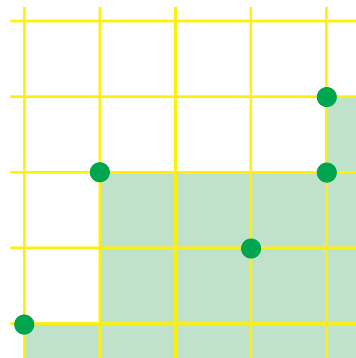
Practical rules

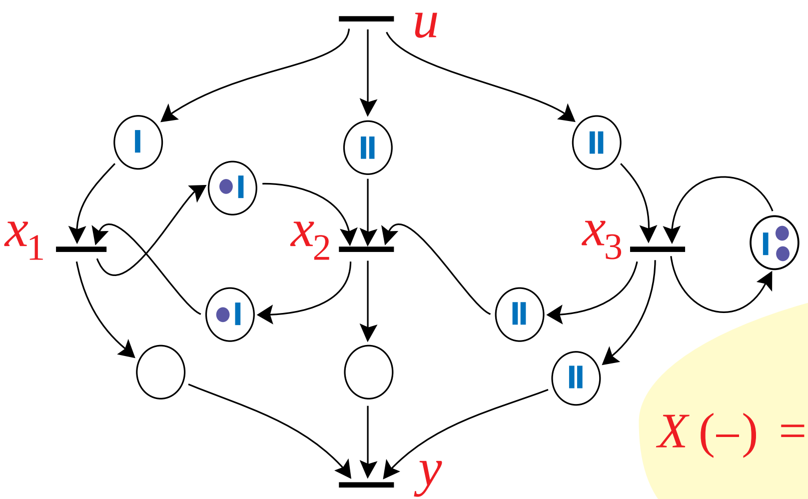
$$\gamma^{m} \oplus \gamma^{p} = \gamma^{\min(m,p)} \qquad \delta^{m} \oplus \delta^{p} = \delta^{\max(m,p)}$$

Geometric interpretation

$\mathbb{B}[\![\gamma,\delta]\!]$ encodes collections of points in $\mathbb{Z}^2$

$\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ encodes collections of South-East cones



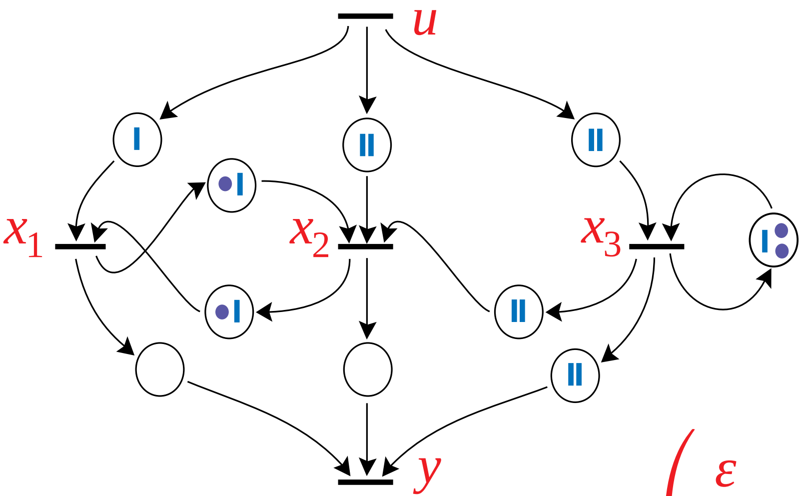


$$X(\text{–}) = \begin{pmatrix} '' & 1\text{–} & '' \\ 1\text{–} & '' & \text{–}^2 \\ '' & '' & 2\text{–} \end{pmatrix} X(\text{–}) \oplus \begin{pmatrix} \text{–} \\ \text{–}^2 \\ \text{–}^2 \end{pmatrix} U(\text{–})$$

$$Y(\text{–}) = \begin{pmatrix} e & e & \text{–}^2 \end{pmatrix} X(\text{–})$$

$$X(\circ) = \begin{pmatrix} '' & 1\circ & '' \\ 1\circ & '' & 2 \\ '' & '' & 1\circ^2 \end{pmatrix} X(\circ) \oplus \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} U(\circ)$$

$$Y(\circ) = \begin{pmatrix} e & e & 2 \end{pmatrix} X(\circ)$$



2D-Domain Description

$$X(\gamma,\delta) = \begin{pmatrix} \varepsilon & \gamma\delta & \varepsilon \\ \gamma\delta & \varepsilon & \delta^2 \\ \varepsilon & \varepsilon & \gamma^2\delta \end{pmatrix} X(\gamma,\delta) \oplus \begin{pmatrix} \delta \\ \delta^2 \\ \delta^2 \end{pmatrix} U(\gamma,\delta)$$

$$Y(\gamma,\delta) = \begin{pmatrix} e & e & \delta^2 \end{pmatrix} X(\gamma,\delta)$$

$$X = AX \oplus BU \qquad Y = CX \oplus DU$$

Transfer Matrices

$$X = AX \oplus BU \qquad Y = CX \oplus DU$$

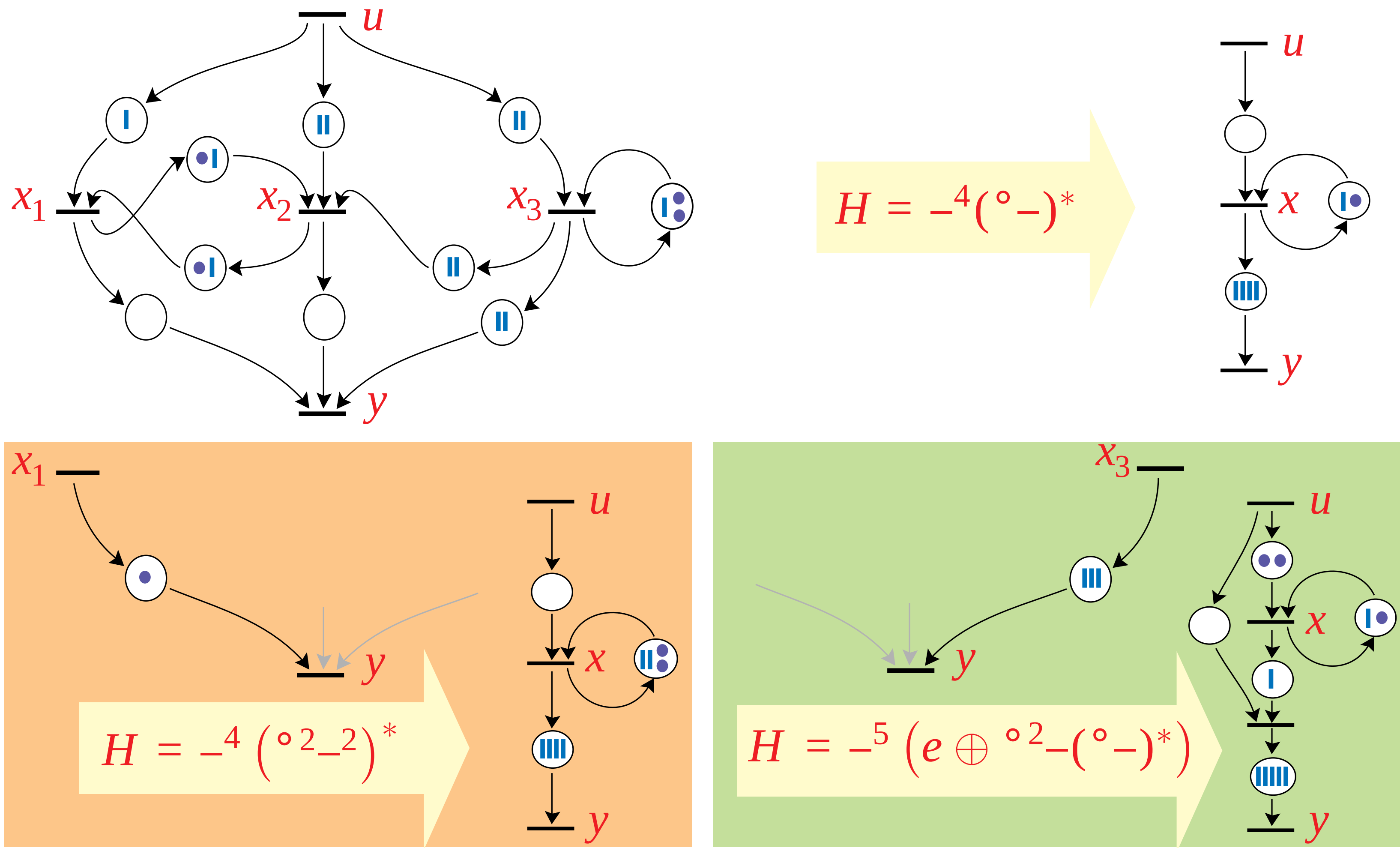
$$X = A^* BU \qquad Y = (CA^* B \oplus D)U$$

$$H = CA^* B \oplus D$$

The meaning of selecting the least solution

- transitions fire as soon as possible
- the most favorable 'initial conditions' take place: all tokens of the initial marking are available since $-\infty$

Any other 'initial conditions' for tokens of the initial marking (sojourn times elapsed prior to initial time) can be enforced by appending auxiliary input controls.


u
x1
x2
x3
y
H = –4(°–)*
u
x
y
x1
y
u
x
y
H = –4 (°2–2)*
x3
u
y
x
y
H = –5 (e ⊕ °2–(°–)*)

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Autonomous System $x(k+1) = Ax(k)$ in $\overline{\mathbb{Z}}^n_{\max}$ (dater equation)

We assume that the precedence graph $\mathcal{G}(A)$ is strongly connected

Asymptotic Behavior

A 'periodic' regime is reached within a finite number of stages:

$$\exists \lambda \in \mathbb{Q}\,,\ \exists c \in \mathbb{N}\,,\ \exists K \in \mathbb{Z} : \forall k > K\,,\quad x(k+c) = \lambda^c x(k)$$

$$\forall i\,,\quad x_i(k+c) = x_i(k) + c \times \lambda$$

$$\lambda = \max_{\text{all circuits of } \mathcal{G}(A)} \frac{\text{total of holding times}}{\text{number of arcs}} = \bigoplus_{j=1}^{n} (\text{trace}(A^j))^{1/j}$$

For systems in AR form $x(k) = \bigoplus_{l=0}^{m} A_l x(k-l)$, 'number of arcs' must be replaced by 'number of tokens'

Eigenvalue λ unique eigenvalue of A: $\exists x$ such that $Ax = \lambda x$

Cyclicity c

Stability

$$\forall i\ , \quad \lim_{k\to\infty} \left\{ \begin{array}{c} x_i(k)^{1/k} \\ \\ x_i(k)/k \end{array} \right\} = \lambda$$

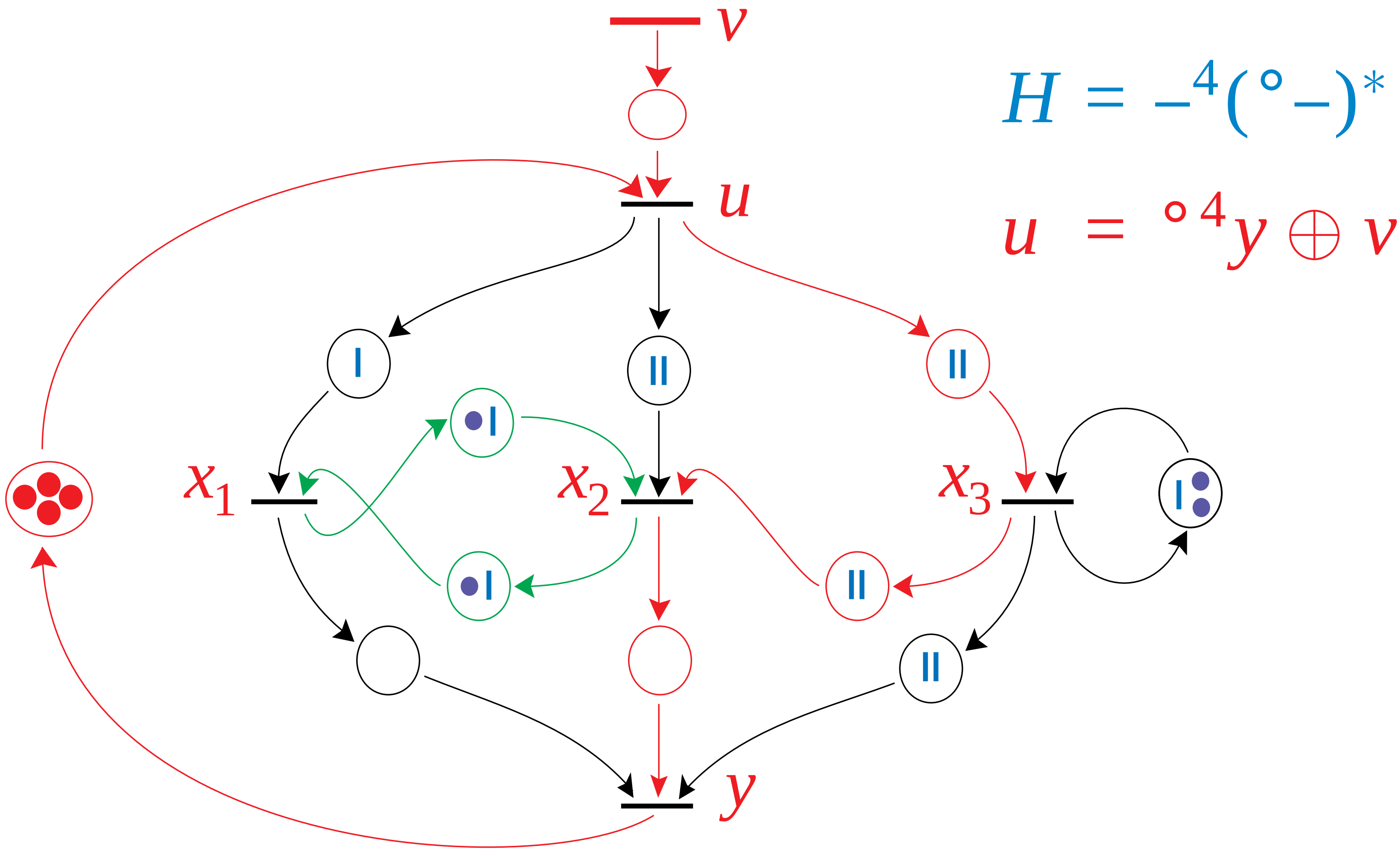
tokens do not accumulate indefinitely inside the graph

Stabilizability by Dynamic Output Feedback

TEG structurally controllable: every internal transition can be reached by a directed path from at least one input transition

TEG structurally observable: every internal transition is the origin of at least one directed path to some output transition

Theorem A TEG which is structurally controllable and observable can be stabilized by dynamic output feedback without altering its open-loop performance



v
u
x1
x2
x3
y
H = –4(°–)*
u = °4y ⊕ v

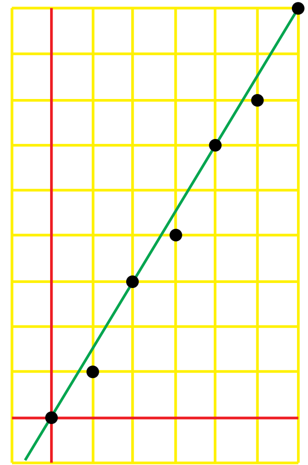
Frequency Responses

Analogues of sine functions

$$s \in \mathbb{Q}, \quad L_s = \bigoplus_{t \leq s \times k} \gamma^k \delta^t$$

average rate of $1/s$ events per unit of time

Example: $L_{3/2} = (e \oplus \gamma\delta)\left(\gamma^2\delta^3\right)^* \left(\gamma^{-2}\delta^{-3}\right)^*$



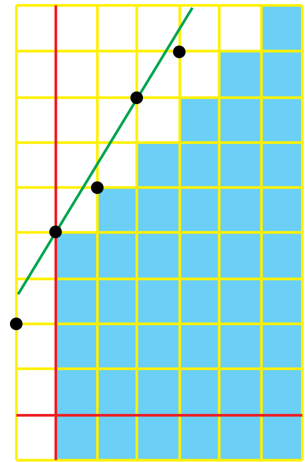
…are indeed eigenfunctions

$$H L_s = \gamma^a \delta^b L_s$$

Rule: pick g and d in $\mathbb{Z}$ such that $s = -g/d$

find a and b in $\mathbb{N}$ such that

$$H(g, d) = a \times g + b \times d$$



Rationality $h \in \mathcal{M}_{in}^{ax}[\![\circ;-]\!]$ is rational (and causal) if it belongs to the rational closure of $\{'';e;\circ;-\}$

Realizability $H \in (\mathcal{M}_{in}^{ax}[\![\circ;-]\!])^{i\times j}$ is realizable if $H = C(\circ A_1 \oplus -A_2)^* B$ where A_1 and A_2 are $n \times n$ matrices, C and B are $n \times j$ and $i \times n$ matrices respectively, and every entry of these matrices is equal to either $''$ or e.

Periodicity $h \in \mathcal{M}_{in}^{ax}[\![\circ;-]\!]$ is periodic if there exist two polynomials p and q and a monomial m (all causal) such that $h = p \oplus qm^*$.

Theorem For $H \in (\mathcal{M}_{in}^{ax}[\![\circ;-]\!])^{i\times j}$, the following three statements are equivalent

(i) H is realizable;

(ii) H is rational;

(iii) H is periodic.

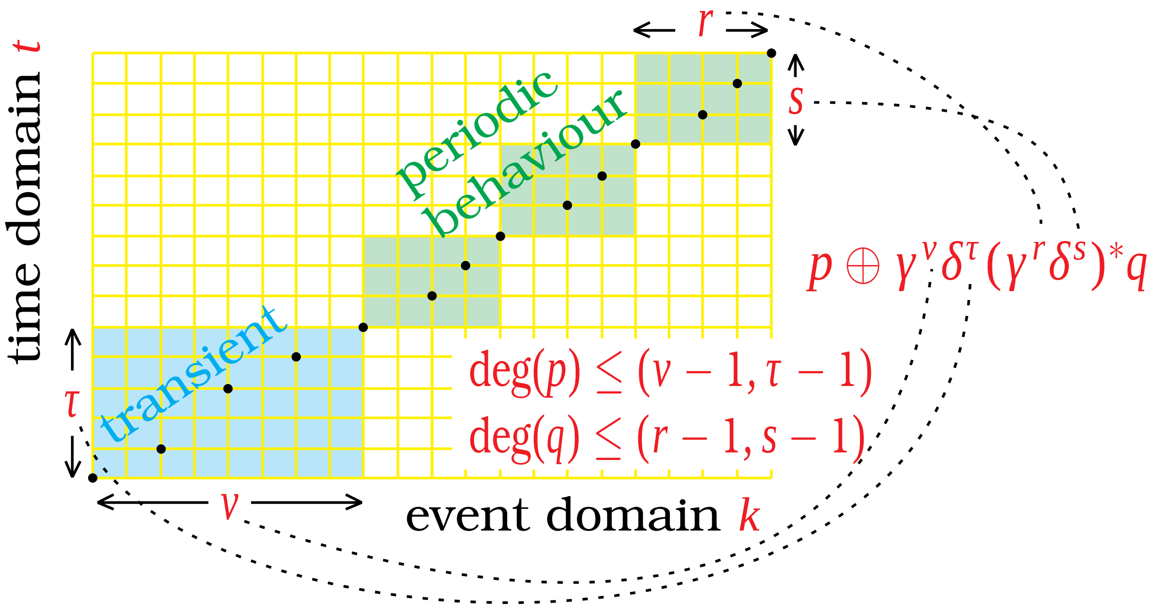
Other definition of Periodicity

$$h = p \oplus (\gamma^{\nu}\delta^{\tau})q(\gamma^{r}\delta^{s})^{*}$$

ν, τ, r, s are nonnegative integers,

p and q are polynomials in (γ, δ) with nonnegative exponents

with a degree in (γ, δ) less than or equal to $(\nu - 1, \tau - 1)$, resp. $(r - 1, s - 1)$



Given an output (dater) trajectory $\{y(k)\}$, find the latest (greatest) input trajectory $\{u(k)\}$ which yields an output trajectory less (earlier) than the given one.

Greatest U such that $HU \preceq Y$: solution $U = H \backslash\!\!\circ Y$

State equations

$$x(k+1) = Ax(k) \oplus Bu(k) \qquad y(k) = Cx(k)$$

Co-State equations

$$\xi(k) = (A \backslash\!\!\circ \xi(k+1)) \wedge (C \backslash\!\!\circ y(k)) \qquad u(k) = B \backslash\!\!\circ \xi(k)$$

Note

$$(A \backslash\!\!\circ b)_i = \min_k (b_k - A_{ki})$$

with conventions

$$+\infty - (+\infty) = +\infty$$
$$(-\infty) - (-\infty) = +\infty$$

Towards second-order theory

$\xi_i(k) - x_i(k)$ ‘spare time’ or ‘margin’ available at transition x_i

for firing number k

$P(k) = \xi(k) \not\circ x(k)$ Riccati matrix ?

Differences between input and output daters (resp. counters)
of a place or a group of places evaluate the sojourn time of tokens
(resp. the accumulated stock of tokens):
algebraically, they behave as correlations.

Towards geometric theory

In ordinary vector spaces with linear operators $\mathcal{U} \xrightarrow{B} \mathcal{X} \xrightarrow{C} \mathcal{Y}$

under some conditions, for given x, there exists a unique

$y \in \operatorname{im} B$ such that $(x - y) \in \ker C$ (projection of x onto $\operatorname{im} B$ parallel to $\ker C$).

$$y = B(CB)^{-1}Cx$$

In dioids, for nonlinear residuated operators,

$$y = B\circ(C\circ B)\circ C(x)$$

is the unique $y \in \operatorname{im} B$ such that

$$C(y) = C(x)$$

under some conditions

Example

$$B = \begin{pmatrix} e & 1 \\ 0.5 & e \\ 2 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} e & e & e \\ 2 & 1 & e \end{pmatrix}$$