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Two-Dimensional Domain Representation of Timed Event Graphs[†]

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1. Preliminaries: Basics of Dioid Theory

1.1. Axiomatics and Examples

Definition 1 (Dioid). *Set $\mathfrak{D}$ endowed with two operations denoted $\oplus$ and $\otimes$*

Axiom 1 (Associativity of addition). $(a \oplus b) \oplus c = a \oplus (b \oplus c)$

Axiom 2 (Commutativity of addition). $a \oplus b = b \oplus a$

Axiom 3 (Associativity of multiplication). $(a \otimes b) \otimes c = a \otimes (b \otimes c)$

Axiom 4 (Distributivity of multiplication w.r.t. addition).

$$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c) \qquad c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$$

Axiom 5 (Existence of a zero element ε). $\forall a \ , \quad a \oplus \varepsilon = a$

Axiom 6 (Absorbing zero element). $\forall a \ , \quad a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$

Axiom 7 (Existence of an identity element e). $\forall a \ , \quad a \otimes e = e \otimes a = a$

Axiom 8 (Idempotency of addition). $\forall a \ , \quad a \oplus a = a$

Definition 2 (Commutative dioid). *Multiplication is commutative.*

Generally speaking, neither $\oplus$ nor $\otimes$ are cancellative.

Example 3. $\mathbb{R}_{\max} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ (also $\mathbb{Z}_{\max}$).

Example 4. $\mathbb{Z}_{\min} = (\mathbb{Z} \cup \{+\infty\}, \min, +)$ isomorphic to $\mathbb{Z}_{\max}$ by $x \mapsto -x$.

Example 5. $\mathbb{R}_{\max}$ isomorphic to $(\mathbb{R}^{+}, \max, \times)$ by $x \mapsto \exp(x)$.

Example 6. $\left(\overline{\mathbb{R}}, \max, \min\right)$.

Example 7. Boole algebra $(\{\varepsilon, e\}, \max, \min)$.

Example 8. $\left(2^{\mathbb{R}^2}, \cup, +\right)$.

Example 9. $(\{(-\infty, x]\}, \cup, +)$ isomorphic to $\mathbb{R}_{\max}$ by the bijection

$$\mathbb{R} \to 2^{\mathbb{R}} : x \mapsto \begin{cases} \varnothing & \text{if } x = \varepsilon; \\ (-\infty, x] & \text{otherwise.} \end{cases}$$

Definition 10 (Subdioid). $\mathcal{C} \subset \mathcal{D}$,

- $\varepsilon \in \mathcal{C}$ *and* $e \in \mathcal{C}$;
- $\mathcal{C}$ *is closed for* $\oplus$ *and* $\otimes$.

Definition 11 (Homomorphism). $\Pi : \mathcal{D} \to \mathcal{C}$,

$$\Pi(a \oplus b) = \Pi(a) \oplus \Pi(b) \ , \quad \Pi(\varepsilon) = \varepsilon \qquad (\oplus\text{-morphism})$$

$$\Pi(a \otimes b) = \Pi(a) \otimes \Pi(b) \ , \quad \Pi(e) = e \qquad (\otimes\text{-morphism})$$

Definition 12 (Isomorphism). $\Pi : \mathcal{D} \to \mathcal{C}$ *such that* Π^{-1} *is defined over* $\mathcal{C}$ *and* Π *and* Π^{-1} *are homomorphisms.*

Lemma 13. *If* Π *is a homomorphism from* $\mathcal{D}$ *to* $\mathcal{C}$ *and if it is a bijection, then it is an isomorphism.*

Definition 14 (Congruence). *Equivalence relation (denoted* $\mathcal{R}$*) in* $\mathcal{D}$ *such that*

$$a \,\mathcal{R}\, b \Rightarrow (a \oplus c) \,\mathcal{R}\, (b \oplus c) \ , \quad (a \otimes c) \,\mathcal{R}\, (b \otimes c) \ .$$

Lemma 15. *The quotient of* $\mathcal{D}$ *by a congruence is a dioid for the addition and multiplication induced by those of* $\mathcal{D}$.

Lemma 16. *Let* Π *be a homomorphism from a dioid* $\mathcal{D}$ *to another dioid* $\mathcal{C}$. *The relation* $\mathcal{R}_\Pi$, *defined as* $\{\forall a, b \in \mathcal{D}, a \,\mathcal{R}_\Pi\, b\} \Leftrightarrow \Pi(a) = \Pi(b)$ *is a congruence. The quotient dioid* $\mathcal{D}/\mathcal{R}_\Pi$ *is isomorphic to* $\Pi(\mathcal{D})$.

1.2. Order and Lattice Properties of Dioids

Ordered set notions: least upper bound $\vee$, greatest lower bound $\wedge$, sup-semilattice, inf-semilattice, lattice, complete sup-semilattice (inf-semilattice, lattice), distributive lattice:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \ , \qquad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \ .$$

Isotone mapping Π: $a \geq b \Rightarrow \Pi(a) \geq \Pi(b)$.

Theorem 17. *A complete sup-semilattice having a bottom element is a complete lattice.*

Theorem 18 (and Definition of an order relation). $a = a \oplus b \Leftrightarrow \exists c : a = b \oplus c \ .$

These equivalent statements define a (partial) order relation: $a \geq b \Leftrightarrow a = a \oplus b$ *and*

$$a \geq b \Rightarrow \{\forall c \ , \quad a \oplus c \geq b \oplus c\} \text{ and } ac \geq bc\} \qquad \text{(same for left product).}$$

Also $a \vee b = a \oplus b$ *(hence a dioid is a sup-semilattice), and* ε *is the bottom element of* $\mathfrak{D}$.

Lemma 19 (Total order). *The order relation defined in Theorem 18 is total if and only if*

$$\forall a, b \in \mathfrak{D} \ , \quad a \oplus b = \text{either } a \text{ or } b \ .$$

1.3. Complete Dioids, Archimedian Dioids and Lower Bound

Definition 20 (Complete dioid). *Dioid closed for infinite sums and Axiom 4 extends to infinite sums.*

$\top$ (sum of all elements of $\mathfrak{D}$) always absorbing for addition; $\top \otimes \varepsilon = \varepsilon$ because of Axiom 6.

Do we have $$\forall a \neq \varepsilon \ , \quad \top \otimes a = a \otimes \top = \top \ ?$$

Definition 21 (Archimedian dioid). $\forall a \neq \varepsilon \ , \quad \forall b \ , \quad \exists c$ *and* $d : \quad ac \geq b$ *and* $da \geq b$.

Theorem 22. *In a complete Archimedian dioid,* $\top$ *is absorbing for* $\otimes$.

Theorem 23. *If a dioid is complete, Archimedian, and if it has a cancellative multiplication, then it is isomorphic to the Boole algebra.*

A complete dioid is a complete lattice since $\wedge$ can be constructed for any subset $\mathfrak{C}$ of elements of $\mathfrak{D}$ (Theorem 17). $\wedge$ is associative, commutative, idempotent and has $\top$ as neutral element ($\top \wedge a = a, \forall a$). Also $(a \wedge b)c \leq (ac) \wedge (bc)$ (same for left multiplication). Equality holds true for totally-ordered dioids.

Lemma 24. *If* a *admits a left inverse* b *and a right inverse* c*, then*

- $b = c$ *and this unique inverse is denoted* a^{-1}*;*
- *moreover,* $$a(x \wedge y) = ax \wedge ay \ .$$

The same holds true for right multiplication by a*, and also for right and left multiplication by* a^{-1}*.*

1.4. Distributive Dioids

In general $(a \wedge b) \oplus c \leq (a \oplus c) \wedge (b \oplus c)$, $(a \oplus b) \wedge c \geq (a \wedge c) \oplus (b \wedge c)$.

Example 25. See notes for an example of a nondistributive dioid.

Definition 26 (Distributive dioid). *Dioid which is complete and, for all subsets* $\mathcal{C}$ *of* $\mathcal{D}$*,*

$$\left(\bigwedge_{c \in \mathcal{C}} c\right) \oplus a = \bigwedge_{c \in \mathcal{C}} (c \oplus a) \ , \quad \left(\bigoplus_{c \in \mathcal{C}} c\right) \wedge a = \bigoplus_{c \in \mathcal{C}} (c \wedge a) \ .$$

Remark 27. Distributive dioids are also dioids when endowed with the two operations $\widehat{\oplus} \stackrel{\text{def}}{=} \oplus$ and $\widehat{\otimes} \stackrel{\text{def}}{=} \wedge$. Special features of these dioid structures are that $\forall x, \widehat{\varepsilon} \leq x \leq \widehat{e}$ and that multiplication is commutative and idempotent.

1.5. Matrix Dioids

$\mathcal{D}^{n \times n}$: square $n \times n$ matrices with entries in $\mathcal{D}$ with sum and product of matrices defined conventionally after the sum and product of 'scalars' in $\mathcal{D}$.

STOP Square matrices only are considered.

$\mathcal{D}^{n \times n}$ not a commutative dioid, partially ordered, complete if $\mathcal{D}$ is, not distributive, not Archimedian, no 'circuit of maximum weight' in the graph interpretation when the underlying dioid is not totally ordered.

1.6. Dioids of Power Series

- $\mathfrak{D}[\![z_1,\ldots,z_p]\!]$: formal power series in p commutative variables with coefficients in $\mathfrak{D}$ and exponents $\mathbb{N}$ or in $\mathbb{Z}$, endowed with the conventional sum and product of power series.
- When exponents belong to $\mathbb{Z}$, $\mathfrak{D}$ is assumed to be complete.
- The subset of polynomials is a subdioid of $\mathfrak{D}[\![z_1,\ldots,z_p]\!]$ denoted $\mathfrak{D}[z_1,\ldots,z_p]$.
- $\mathfrak{D}[\![z_1,\ldots,z_p]\!]$ partially ordered, commutative if $\mathfrak{D}$ is, complete if $\mathfrak{D}$ is ($\mathfrak{D}[z_1,\ldots,z_p]$ is not), distributive if $\mathfrak{D}$ is, Archimedian if $\mathfrak{D}$ is and if exponents are in $\mathbb{Z}$ (likewise for $\mathfrak{D}[z_1,\ldots,z_p]$).

1.7. The implicit equation $x = ax \oplus b$

Theorem 28. *In a complete dioid, the least solution of $x = ax \oplus b$, and also of the inequality $x \geq ax \oplus b$, is a^*b where $a^* \stackrel{\text{def}}{=} e \oplus a \oplus a^2 \oplus \ldots$.*

Proof. a^* exists because the dioid is complete. Moreover $a\,(a^*b) \oplus b = a^+b \oplus b = a^*b$. Finally, if x is any solution, by successive substitutions,

$$x \mathop{\geq}^{=} a^k x \oplus \left(e \oplus a \oplus \cdots \oplus a^{k-1}\right) b \geq \left(e \oplus a \oplus \cdots \oplus a^{k-1}\right) b \ , \quad \forall k \ .$$

2. Dater and Counter Representations: Why None Is Fully Satisfactory?

2.1. About the Dater and Counter Representations

- Event: firing of a transition. Firing times are assumed to be zero without loss of generality. Each transition defines a type of event.
- Events of the same type are assigned numbers k sequentially, with $k = 0$ for the first event to occur at or after time 0.
- Dater: $k \mapsto d(k) \in \mathbb{Z}$, where $d(k)$ is the date at which the event numbered k occurs.
- Daters satisfy max-plus linear dynamic equations in the event domain.
- Counter: inverse mapping $t \mapsto c(t) = k$ such that $d(k) \approx t$. Stems from the fact that $d(\cdot)$ is monotonic. Possible definitions:

$$c(t) = \sup\{k \mid d(k) \leq t\} \qquad \text{or} \qquad c(t) = \inf\{k \mid d(k) \geq t\} \ .$$

- Counters satisfy min-plus linear dynamic equations in the time domain.
- γ-transform of $d(\cdot)$: $D(\gamma) = \bigoplus_{k \in \mathbb{Z}} d(k)\gamma^k$.
- Since $\gamma D(\gamma) = \bigoplus_{k \in \mathbb{Z}} d(k)\gamma^{k+1} = \bigoplus_{k \in \mathbb{Z}} d(k-1)\gamma^k$, γ can be interpreted as a backward shift operator.
- Inputs u: source transitions.

- Outputs y: sink transitions.
- 'States' x: internal transitions.

Dater equations: $$x(k) = Ax(k-1) \oplus Bu(k) \ , \qquad y(k) = Cx(k) \ .$$

Assume that $x(k)$ and $u(k)$ are identically ε for $k < 0$. Successive substitutions yield

$$y(k) = \bigoplus_{l=0}^{k} CA^{k-l}Bu(l) = \bigoplus_{l \in \mathbb{Z}} h(k-l)u(l) = \sup_{l \in \mathbb{Z}} \left(h(k-l) + u(k)\right)$$

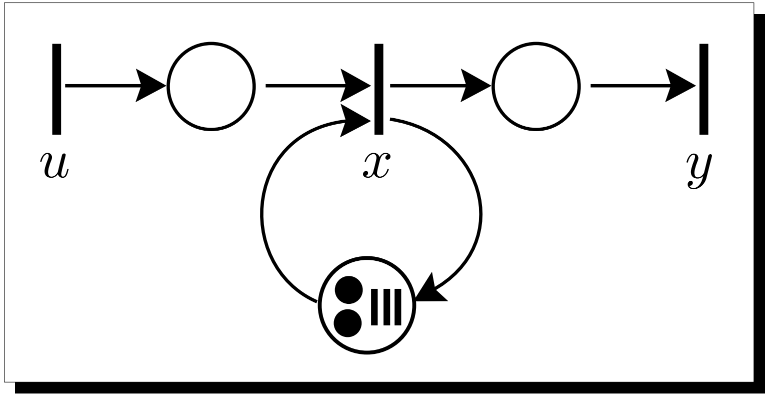
impulse response $$h(k) = \begin{cases} \varepsilon & \text{if } k < 0, \\ CA^kB & \text{otherwise.} \end{cases}$$

$$\{X(\gamma) = \gamma AX(\gamma) \oplus BU(\gamma) \ , \quad Y(\gamma) = CX(\gamma)\} \quad \Rightarrow \quad Y(\gamma) = \underbrace{C(\gamma A)^*B}_{\text{transfer matrix in } \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]} \ U(\gamma) \ .$$

- Theorem 28 has been used to solve the implicit equation for the least solution.
- γ-transform converts (sup)-convolutions into mere products.

Transfer matrix from counter equations. Backward shift operator δ on counters $c(\cdot)$: $\delta c(t) = c(t-1)$; δ-transform $C(\delta) = \bigoplus_{t \in \mathbb{Z}} c(t)\delta^t$; counter equations in $\overline{\mathbb{Z}}_{\min}$ are transformed via the δ-transform $\rightsquigarrow$ transfer matrix with entries in $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$. In the case of counters, convolutions are inf-convolutions.

2.2. Order of the System



Dater equations:

$$y(k) = 3y(k-2) \oplus u(k)$$

Counter equations:

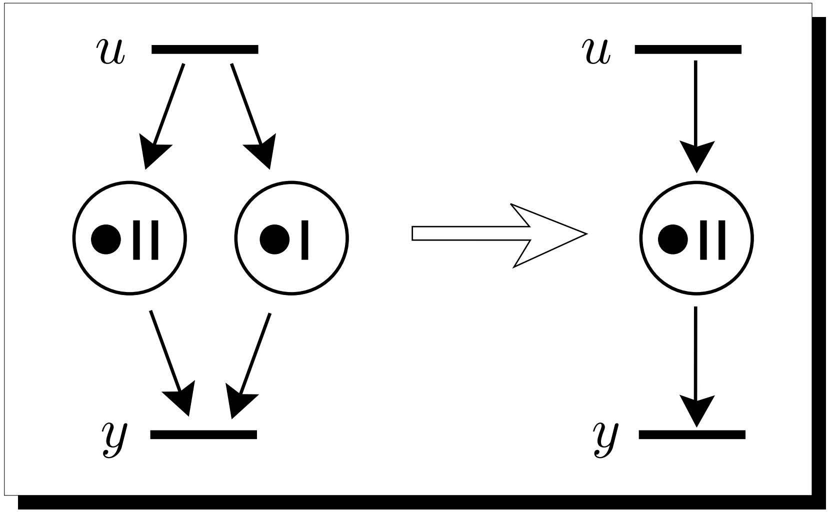
$$y(t) = 2y(t-3) \oplus u(t)$$

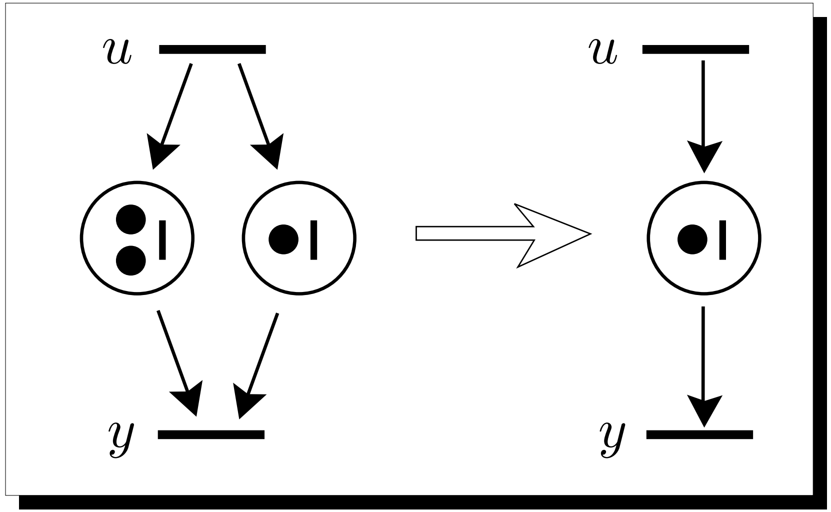
2.3. Monotonicity of Trajectories

Monotonicity is not an intrinsic property of solutions of dater or counter equations. For example, for the above dater equation,

k	$\dots$	-2	-1	0	1	2	$\dots$
$u(k)$	$\dots$	ε	ε	e	e	e	$\dots$
$y(k)$	$\dots$	-2	-3	1	4	3	$\dots$

2.4. Simplifications





Left-hand $y(k) = 2u(k-1) \oplus 1u(k-1) = (2 \oplus 1)u(k-1) = 2u(k-1)$

hence $y(t) = 1u(t-2) \oplus 1u(t-1) = 1(u(t-2) \oplus u(t-1)) = 1u(t-2)$

therefore $t\gamma^k \oplus \tau\gamma^k = \max(t,\tau)\gamma^k \;, \qquad k\delta^t \oplus k\delta^\tau = k\delta^{\max(t,\tau)}$

Right-hand $y(t) = 2u(t-1) \oplus 1u(t-1) = (2 \oplus 1)u(t-1) = 1u(t-1)$

hence $y(k) = 1u(k-2) \oplus 1u(k-1) = 1(u(k-2) \oplus u(k-1)) = 1u(k-1)$

therefore $k\delta^t \oplus \kappa\delta^t = \min(k,\kappa)\delta^t \;, \qquad t\gamma^k \oplus t\gamma^\kappa = t\gamma^{\min(k,\kappa)}$

Finally $\gamma^k \oplus \gamma^\kappa = \gamma^{\min(k,\kappa)} \;, \qquad \delta^t \oplus \delta^\tau = \delta^{\max(t,\tau)}$

2.5. How a Transition Firing Changes the Model

Before firing

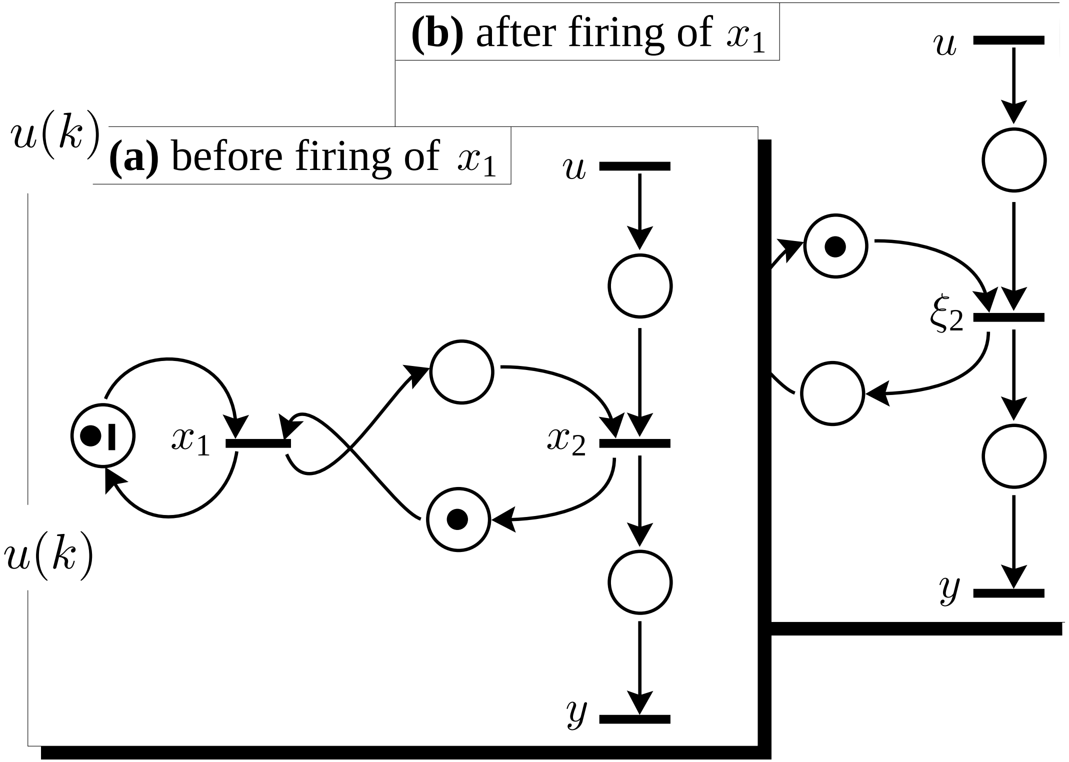
$$\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} 1 & e \\ 1 & e \end{pmatrix} \begin{pmatrix} x_1(k-1) \\ x_2(k-1) \end{pmatrix} \oplus \begin{pmatrix} \varepsilon \\ e \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} \varepsilon & e \end{pmatrix} \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$$

After firing

$$\begin{pmatrix} \xi_1(k) \\ \xi_2(k) \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon \\ e & \varepsilon \end{pmatrix} \begin{pmatrix} \xi_1(k-1) \\ \xi_2(k-1) \end{pmatrix} \oplus \begin{pmatrix} e \\ e \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} \varepsilon & e \end{pmatrix} \begin{pmatrix} \xi_1(k) \\ \xi_2(k) \end{pmatrix}$$



These are two state space realizations of the same γ-transfer $e \oplus \gamma(1\gamma)^*$

$$x(k) = Ax(k-1) \oplus Bu(k) \ , \quad y(k) = Cx(k)$$

$$\xi(k) = \overline{A}\xi(k-1) \oplus \overline{B}u(k) \ , \quad y(k) = \overline{C}\xi(k)$$

No linear transformation T such that $B = T\overline{B}$

A two-dimensional representation is now introduced:

- monomials such as $t\gamma^k$ in γ-transforms and $k\delta^t$ in δ-transforms will be represented by monomials of the form $\gamma^k\delta^t$;
- the basic objects will be power series in (γ, δ) with <u>Boolean</u> coefficients, with the conventional sum and product of power series;
- the additional rules $\gamma^k \oplus \gamma^\kappa = \gamma^{\min(k,\kappa)}$ and $\delta^t \oplus \delta^\tau = \delta^{\max(t,\tau)}$ will receive an algebraic foundation.

Two main steps to achieve this:

1. (γ-transforms of) monotonic trajectories (daters) will be ‘filtered’ out of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ by a simple congruence;
2. $\overline{\mathbb{Z}}_{\max}$ itself can be ‘filtered’ out of a larger dioid using another congruence of the same type.

The result is the so-called $\mathfrak{M}_{\mathfrak{in}}^{\mathfrak{ax}}[\![\gamma,\delta]\!]$ algebra which is the natural dioid in which the theory should be developed.

3. Construction of the $\mathfrak{M}_{\mathfrak{in}}^{\mathfrak{ax}}[\![\gamma,\delta]\!]$ Algebra

3.1. A Fundamental Theorem

Theorem 29. *Let* $\mathfrak{D}$ *be a* commutative *and* complete *dioid. Let* p *be any given element in* $\mathfrak{D}$*.*

1. *Let* Π_p *be the mapping from* $\mathfrak{D}$ *into itself defined by* $\Pi_p : x \mapsto p^* x$*. This* Π_p *is a dioid homomorphism, and the relation*

   $$x \,\mathfrak{R}_p\, y \iff p^* x = p^* y$$

   is a congruence.

2. *The subset* $\Pi_p(\mathfrak{D})$*, also denoted* $p^*\mathfrak{D}$*, is a dioid with* ε *as zero element and* p^* *as identity element.*

3. *Each equivalence class contains one and only one element belonging to* $p^*\mathfrak{D}$ *and this element is explicitly given by* $p^* x$ *for any* x *in the class. This* $p^* x$ *is the greatest element in the class of* x*, and it is the least element among those elements of* $p^*\mathfrak{D}$ *which are greater than* x*. Therefore* $p^* x$ *may be called the 'best approximation from above' of* x *by an element of* $p^*\mathfrak{D}$*, and the equivalence relation* $\mathfrak{R}_p$ *could alternatively have been defined as: "*x *and* y *are equivalent if they have the same 'best approximation from above' in* $p^*\mathfrak{D}$*".*

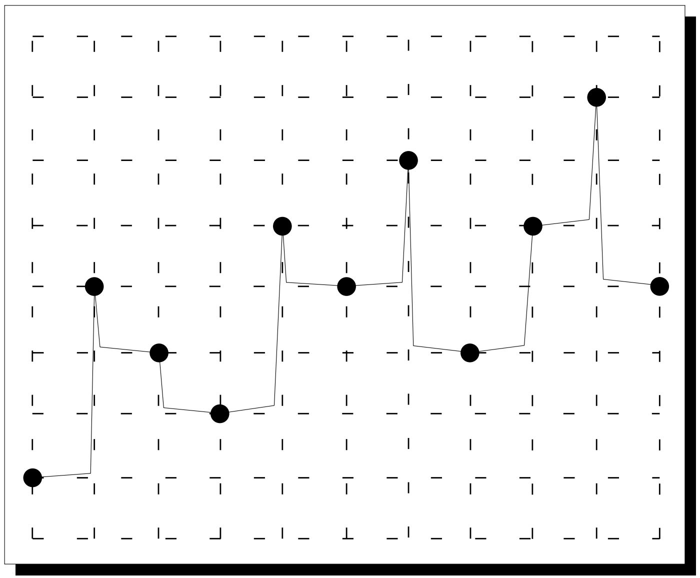
4. *Finally,* $p^*\mathfrak{D}$ *and* $\mathfrak{D}/\mathfrak{R}_p$ *are isomorphic dioids.*

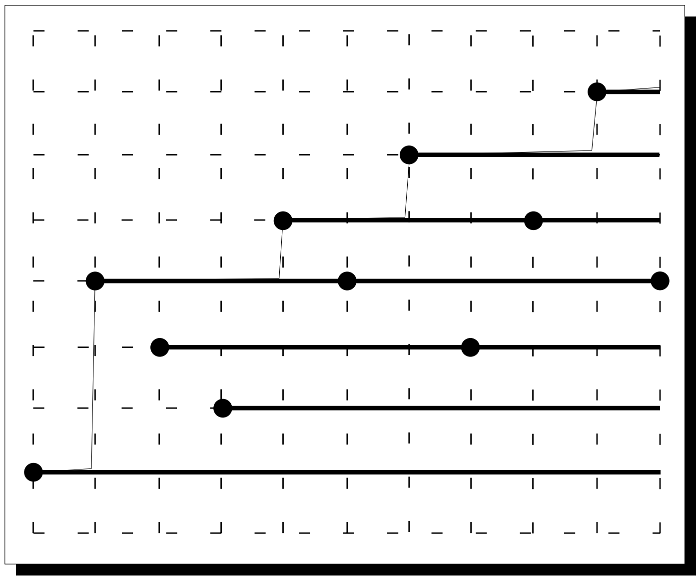
3.2. Filtering Monotonic Elements of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$

Let $D(\gamma) = \bigoplus d(k)\gamma^k \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$:

$$\forall k \in \mathbb{Z} \ , \quad \{d(k) \geq d(k-1)\} \iff \{d(k) = d(k) \oplus d(k-1)\} \iff$$
$$\{D(\gamma) = D(\gamma) \oplus \gamma D(\gamma)\} \iff \{D(\gamma) = \gamma^* D(\gamma)\} \iff D(\gamma) \in \gamma^* \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$$

The 'best approximation from above' of any element $D'(\gamma) \in \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ by a dater is given by $D(\gamma) = \gamma^* D'(\gamma)$, that is, $d(k) = \sup_{l \leq k} d'(l) = \bigoplus_{l \leq k} d'(l)$.





In $\gamma^* \overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$,

$$\gamma^k \oplus \gamma^\kappa = \gamma^{\min(k,\kappa)}$$

means

$$\left(\gamma^k \oplus \gamma^\kappa\right) \gamma^* = \gamma^{\min(k,\kappa)} \gamma^*$$

3.3. Another Representation of $\overline{\mathbb{Z}}_{\max}$

- $\mathcal{L} \subset 2^{\mathbb{Z}}$ consists of half lines extending towards $-\infty$

- $\overline{\mathbb{Z}}_{\max} \sim (\mathcal{L}, \cup, +)$ by the bijection $\overline{\mathbb{Z}} \to 2^{\mathbb{Z}} : x \mapsto \begin{cases} \varnothing & \text{if } x = \varepsilon = -\infty; \\ \mathbb{Z} & \text{if } x = \top = +\infty; \\ \{y \in \mathbb{Z} \mid y \leq x\} & \text{otherwise.} \end{cases}$

- $\mathbb{B}[\![\delta]\!] =$ dioid of formal power series in a variable δ with exponents in $\mathbb{Z}$ and coefficients in the Boolean dioid $\mathbb{B} = \{\varepsilon, e\}$

- $\left(2^{\mathbb{Z}}, \cup, +\right) \sim \mathbb{B}[\![\delta]\!]$ by the bijection

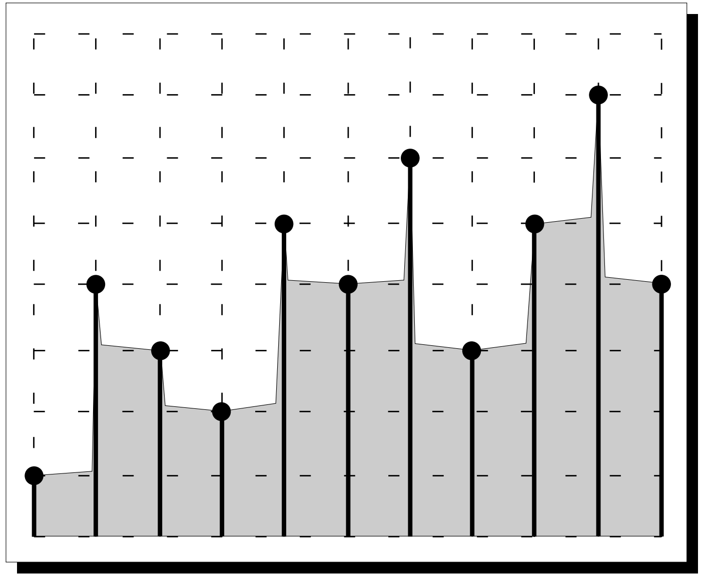
$$A \mapsto A(\delta) = \bigoplus_{t \in \mathbb{Z}} a(t)\delta^t \quad \text{with} \quad a(t) = \begin{cases} e & \text{if} t \in A \\ \varepsilon & \text{otherwise} \end{cases}$$

- $L \in \mathcal{L} \Leftrightarrow \delta^{-1}L \subset L \Leftrightarrow \delta^{-1}L(\delta) \leq L(\delta) \Leftrightarrow L(\delta) \oplus \delta^{-1}L(\delta) = L(\delta) \Leftrightarrow L(\delta) = \left(\delta^{-1}\right)^* L(\delta)$

Finally, $\boxed{\left(\delta^{-1}\right)^* \mathbb{B}[\![\delta]\!] \sim (\mathcal{L}, \cup, +) \sim \overline{\mathbb{Z}}_{\max}}$

$$\delta^t \oplus \delta^\tau = \delta^{\max(t,\tau)} \quad \text{means} \quad \left(\delta^t \oplus \delta^\tau\right)\left(\delta^{-1}\right)^* = \delta^{\max(t,\tau)}\left(\delta^{-1}\right)^*$$

Remark 30. $\overline{\mathbb{Z}}_{\max}$ is represented vertically along the y-axis. Considering $\left[\left(\delta^{-1}\right)^{*}\mathbb{B}[\![\delta]\!]\right][\![\gamma]\!]$ instead of $\overline{\mathbb{Z}}_{\max}[\![\gamma]\!]$ amounts to 'hanging a half line' which hangs downwards at each point of the graph. Hence graphs are replaced by 'hypographs', and $\oplus$ corresponds to the union of hypographs (equivalent to the upper hull of functions).



3.4. The Dioid $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

Ultimately,

$$\gamma^{*}\overline{\mathbb{Z}}_{\max}[\![\gamma]\!] \sim \gamma^{*}\left[\left(\delta^{-1}\right)^{*}\mathbb{B}[\![\delta]\!]\right][\![\gamma]\!] \sim \gamma^{*}\left(\delta^{-1}\right)^{*}\mathbb{B}[\![\gamma,\delta]\!]$$

This dioid is called $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$.

A useful formula: in any commutative dioid, $a^{*}b^{*}=(a\oplus b)^{*} \Rightarrow \gamma^{*}\left(\delta^{-1}\right)^{*}=\left(\gamma\oplus\delta^{-1}\right)^{*}$

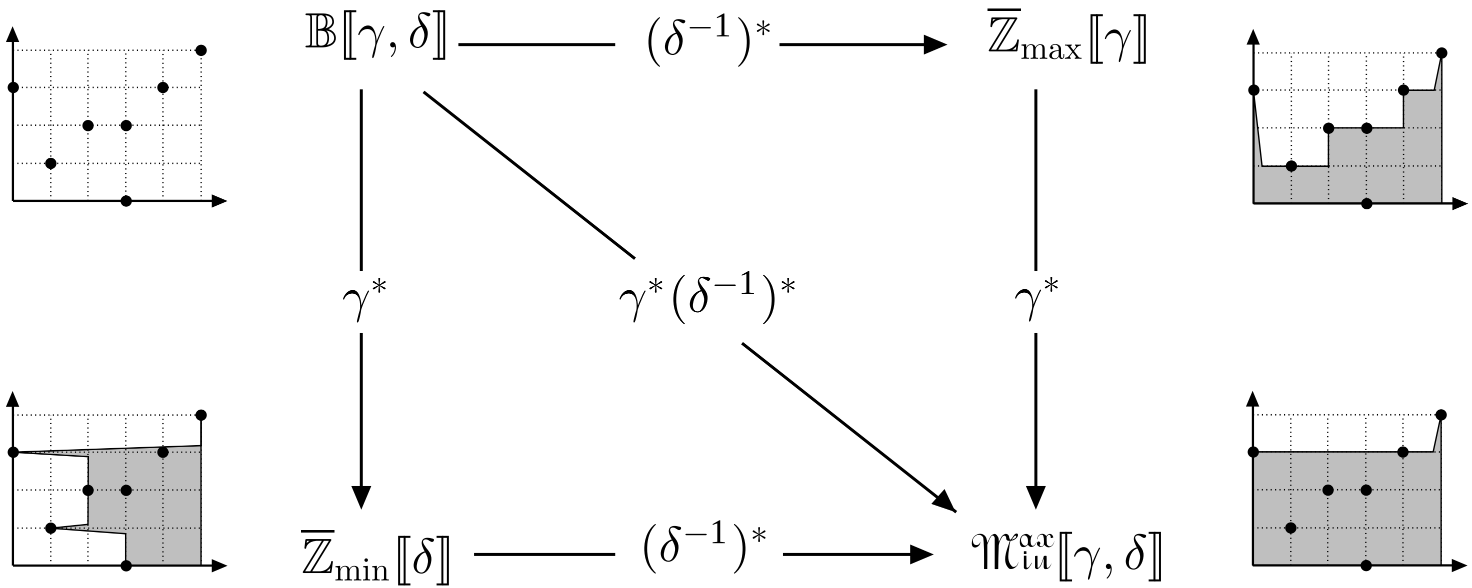
Another path from $\mathbb{B}[\![\gamma,\delta]\!]$ to $\mathfrak{M}_{\mathfrak{in}}^{\mathfrak{ax}}[\![\gamma,\delta]\!]$

1. Start with $\overline{\mathbb{Z}}_{\min}[\![\delta]\!]$ which encompasses δ-transforms of counters. Nondecreasing counters $c(\cdot)$ are characterized by

$$\forall t \in \mathbb{Z}\ ,\quad \{c(t) \geq c(t-1)\} \iff \{c(t-1) = c(t-1) \oplus c(t)\} \iff$$
$$\left\{C(\delta) = C(\delta) \oplus \delta^{-1}C(\delta)\right\} \iff \left\{C(\delta) = \left(\delta^{-1}\right)^* C(\delta)\right\} \iff C(\delta) \in \left(\delta^{-1}\right)^* \overline{\mathbb{Z}}_{\min}[\![\delta]\!]$$

2. Then, $\overline{\mathbb{Z}}_{\min} \sim \gamma^*\mathbb{B}[\![\gamma]\!]$ by a construction similar to that of the previous subsection.

3. Finally, $$(\delta^{-1})^* \,\overline{\mathbb{Z}}_{\min}[\![\delta]\!] \sim (\delta^{-1})^* \left[\gamma^*\mathbb{B}[\![\gamma]\!]\right] [\![\delta]\!] \sim \mathfrak{M}_{\mathfrak{in}}^{\mathfrak{ax}}[\![\gamma,\delta]\!].$$



Cones and algebra of information

- $\gamma^* \left(\delta^{-1}\right)^* = \left(\gamma \oplus \delta^{-1}\right)^*$ encodes the 'South-East' cone with vertex at the origin.
- $X(\gamma, \delta) \otimes \gamma^* \left(\delta^{-1}\right)^*$ can be achieved geometrically by 'hanging' a South-East cone to all points of X and taking the union of all the points of $\mathbb{Z}^2$ so covered.
- A monomial $\gamma^k \delta^t$ is interpreted as: *"the transition x incurs its firing numbered k at the earliest at time t"*.
- If $(\kappa, \tau) \preceq (k, t)$ in the sense that $\kappa \geq k$ and $\tau \leq t$, then $\gamma^k \delta^t \oplus \gamma^\kappa \delta^\tau = \gamma^k \delta^t$ because $\gamma^\kappa \delta^\tau$ conveys less information than $\gamma^k \delta^t$. Geometrically, the former cone is 'in the shadow' of the latter.
- The algebra $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ can be viewed as an algebra of information. The timed event graph transmits the information (about the whole history) available at the input transitions to the output transitions through the internal transitions. The places of the Petri net introduce 'shifts' in the information, both along the event x-axis and along the time y-axis according to their numbers of 'bars' (holding times) and tokens of the initial marking, respectively. At internal transitions, the information transmitted by all the incoming arcs is summed up and possibly compacted by using the simplification rules.

Lemma 31. *For any element of $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$, all representatives in $\mathbb{B}[\![\gamma, \delta]\!]$ have the same valuation in γ and the same degree in δ.*

A polynomial P in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ is an element which has finite valuation and degree (apart from ε) and then $P \leq \gamma^{\mathrm{val}(P)} \delta^{\deg(P)}$. A monomial is such that the equality holds true.

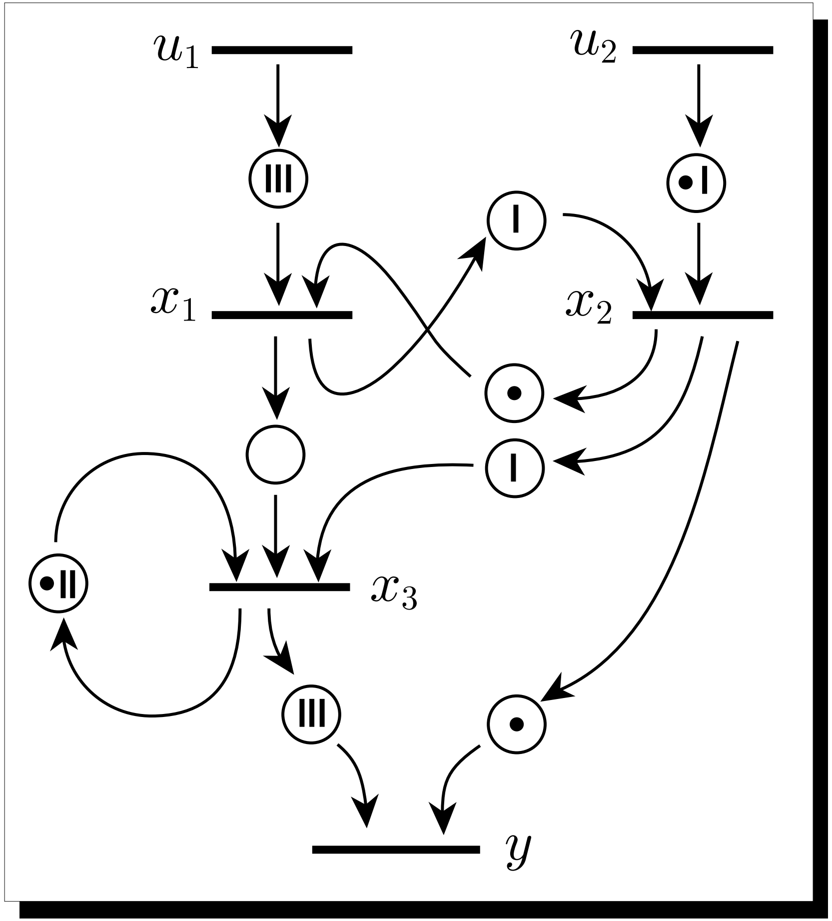
3.5. Equations and Transfer Matrices in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$

$$X_1 = \gamma X_2 \oplus \delta^3 U_1$$

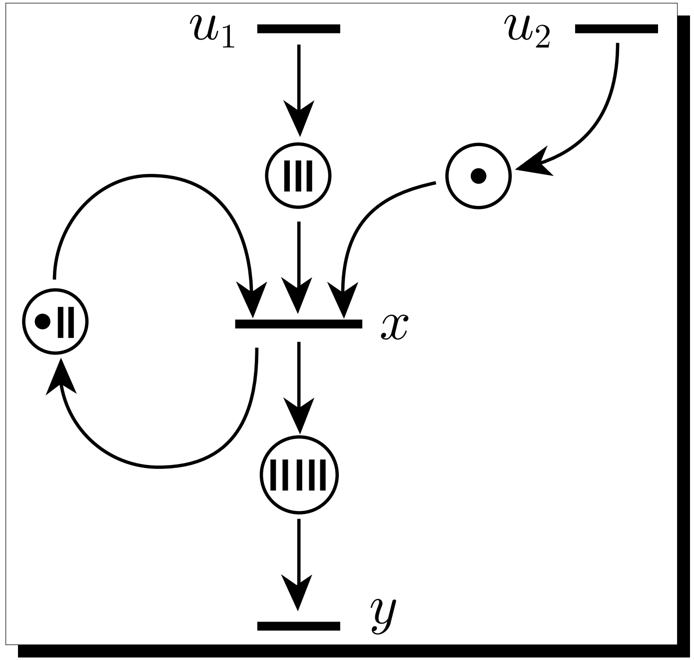
$$X_2 = \delta X_1 \oplus \gamma\delta U_2$$

$$X_3 = X_1 \oplus \delta X_2 \oplus \gamma\delta^2 X_3$$

$$Y = \gamma X_2 \oplus \delta^3 X_3$$



$$Y = \delta^5 (\gamma\delta^2)^* (\delta^3 U_1 \oplus \gamma U_2)$$



Some calculations in $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$ (exercise)

$$X_1 = \gamma X_2 \oplus \delta^3 U_1$$
$$X_2 = \delta X_1 \oplus \gamma\delta U_2$$
$$X_3 = X_1 \oplus \delta X_2 \oplus \gamma\delta^2 X_3$$
$$Y = \gamma X_2 \oplus \delta^3 X_3$$

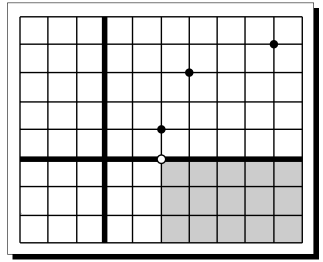
$$\begin{aligned}
Y &= \gamma X_2 \oplus \delta^3 X_1 \oplus \delta^4 X_2 \oplus \gamma\delta^5 X_3 \\
&= \delta^3 X_3 \\
X_3 &= (\gamma\delta^2)^* \,(X_1 \oplus \delta X_2) \\
&= (\gamma\delta^2)^* \left(X_1 \oplus \delta^2 X_1 \oplus \gamma\delta^2 U_2\right) \\
&= (\gamma\delta^2)^* \left(\delta^2 X_1 \oplus \gamma\delta^2 U_2\right) \\
X_1 &= \gamma\delta X_1 \oplus \gamma^2\delta U_2 \oplus \delta^3 U_1 \\
&= (\gamma\delta)^*\delta \left(\delta^2 U_1 \oplus \gamma^2 U_2\right) \\
X_3 &= (\gamma\delta^2)^*(\gamma\delta)^* \left(\delta^5 U_1 \oplus \gamma^2\delta^3 U_2\right) \oplus (\gamma\delta^2)^*\gamma\delta^2 U_2 \\
&= \left(\gamma\delta^2\right)^* \left(\delta^5 U_1 \oplus \gamma\delta^2(e \oplus \gamma\delta) U_2\right)
\end{aligned}$$

note:

$$\left(\gamma\delta^2\right)^* \gamma\delta^2 \leq \left(\gamma\delta^2\right)^* \gamma\delta^2(e \oplus \gamma\delta) \leq \left(\gamma\delta^2\right)^* \gamma\delta^2(\gamma\delta)^* = \left(\gamma\delta^2\right)^* \gamma\delta^2,$$

$$\begin{aligned}
X_3 &= \left(\gamma\delta^2\right)^* \delta^2 \left(\delta^3 U_1 \oplus \gamma U_2\right) \\
Y &= \delta^5 \left(\gamma\delta^2\right)^* \left(\delta^3 U_1 \oplus \gamma U_2\right)
\end{aligned}$$

4. Rationality, Realizability and Periodicity



Definition 32 (Causality). $h \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ *is causal either if* $h = \varepsilon$ *or if* $\mathrm{val}(h) \geq 0$ *and* $h \geq \gamma^{\mathrm{val}(h)}$*.*

Definition 33 (Rationality). $h \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ *is rational if it belongs to the rational closure of the subset* $\mathfrak{T} \stackrel{\text{def}}{=} \{\varepsilon, e, \gamma, \delta\}$*.*

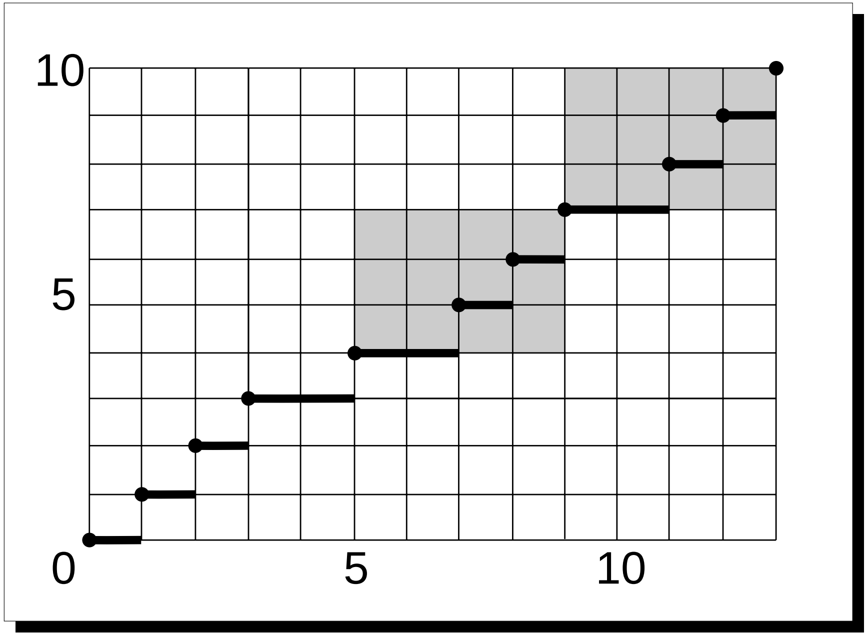
Definition 34 (Realizability). $H \in (\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!])^{p\times m}$ *is realizable if* $H = C(\gamma A_1 \oplus \delta A_2)^* B$ *where* A_1 *and* A_2 *are* $n \times n$ *matrices,* n *being an arbitrary but finite integer (depending on* H*),* C *and* B *are* $n \times m$ *and* $p \times n$ *matrices respectively, and every entry of these matrices is equal to either* ε *or* e*.*

Definition 35 (Periodicity). $h \in \mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$ *is periodic if there exist two polynomials* p *and* q *and a monomial* m *(all causal) such that* $h = p \oplus q m^*$*.*

Another more complex, but equivalent, definition is: there exists a representative of h in $\mathbb{B}[\![\gamma,\delta]\!]$ equal to $\overline{p} \oplus (\gamma^\nu \delta^\tau)\overline{q}\,(\gamma^r \delta^s)^*$, where ν, τ, r, s are nonnegative integers, and $\overline{p}$ and $\overline{q}$ are polynomials in (γ,δ) with nonnegative exponents and with a degree in (γ,δ) less than or equal to $(\nu-1, \tau-1)$, resp. $(r-1, s-1)$.

Example: $\overline{p} \oplus (\gamma^{\nu}\delta^{\tau})\overline{q}\,(\gamma^{r}\delta^{s})^{*}$ with $\nu = 5, \tau = 4, r = 4, s = 3;$

$$h = e \oplus \gamma\delta \oplus \gamma^2\delta^2 \oplus \gamma^3\delta^3 \oplus \gamma^5\delta^4 \left(e \oplus \gamma^2\delta \oplus \gamma^3\delta^2\right) \left(\gamma^4\delta^3\right)^*$$



Theorem 36. *For $H \in \left(\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]\right)^{p\times m}$, the following three statements are equivalent*

(i) H is realizable;

(ii) H is rational;

(iii) H is periodic.

5. Frequency Response of Event Graphs

In conventional system theory (stable linear time-invariant system in continuous time-domain), a signal input of pure frequency ω yields a signal output of the same frequency ω, but phase-shifted by $\arg H(j\omega)$ and amplified by $|H(j\omega)|$.

5.1. Numerical Functions Associated with Elements of $\mathbb{B}[\![\gamma,\delta]\!]$ and $\mathcal{M}_{in}^{ax}[\![\gamma,\delta]\!]$

$$F \in \mathbb{B}[\![\gamma,\delta]\!] : \quad F = \bigoplus_{(k,t)\in F} \gamma^k\delta^t \text{ or } \bigoplus_{(k,t)\in\mathbb{Z}^2} F(k,t)\gamma^k\delta^t \text{ with } F(k,t) = \begin{cases} e & \text{if } (k,t)\in F; \\ \varepsilon & \text{otherwise.} \end{cases}$$

The **evaluation homomorphism** $\mathfrak{F} : \mathbb{B}[\![\gamma,\delta]\!] \to \left(\overline{\mathbb{Z}}\right)^{\mathbb{Z}^2}$ associates with F a numerical function $\mathfrak{F}(F)$:

$$\mathfrak{F}(F) : (g,d) \mapsto \bigoplus_{(k,t)\in\mathbb{Z}^2} F(k,t) g^k d^t = \sup_{(k,t)\in\mathbb{Z}^2} (F(k,t) + gk + dt) = \sup_{(k,t)\in F} (gk + dt)$$

Lemma 37. *The set of mappings from $\mathbb{Z}^2$ into $\mathbb{Z}$ endowed with the pointwise maximum as addition, and the pointwise (conventional) addition as multiplication, is a complete dioid. The mapping $\mathfrak{F}$ is a dioid homomorphism from $\mathbb{B}[\![\gamma,\delta]\!]$ into this dioid of numerical functions.*

Properties of $\mathfrak{F}(F)$

- $\mathfrak{F}(F)$, extended to $\mathbb{R}^2$, is convex as the upper hull of a family of linear functions.
- $\mathfrak{F}(F)$ is positively homogeneous of degree 1: it suffices to know the value of $\mathfrak{F}(F)$ for all values of the ratio g/d ranging in $\overline{\mathbb{Q}}$.
- Geometrically, $\mathfrak{F}(F)$ is the 'support function' of the subset F. Support functions characterize only the convex hulls of subsets $\Rightarrow \mathfrak{F}$ is non injective; $\mathfrak{F}(F)$ depends only on the extreme points of the subset F.

$$\left[\mathfrak{F}\left(\gamma^*(\delta^{-1})^*\right)\right](g,d) = \begin{cases} 0 & \text{if } g \leq 0 \text{ and } d \geq 0 \\ +\infty & \text{otherwise} \end{cases} \qquad \text{hence}$$

$$\forall (g,d) \in (-\mathbb{N}) \times \mathbb{N} \;, \quad F\gamma^*(\delta^{-1})^* = G\gamma^*(\delta^{-1})^* \Rightarrow [\mathfrak{F}(F)](g,d) = [\mathfrak{F}(G)](g,d)$$

Remark 38. For $(g,d) \notin (-\mathbb{N}) \times \mathbb{N}$, $\mathfrak{F}(F)(g,d)$ is set to $+\infty$ whenever F is considered as an element of $\mathfrak{M}_{\mathfrak{in}}^{\mathfrak{ax}}[\![\gamma,\delta]\!]$. The subset of numerical functions equal to $+\infty$ outside $(-\mathbb{N}) \times \mathbb{N}$, plus the function ε equal to $-\infty$ everywhere, is also a complete dioid for the operations defined at Lemma 37.

Definition 39 (Evaluation homomorphism). $\mathfrak{F} : \mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!] \rightarrow \left(\left(\overline{\mathbb{Z}}\right)^{\mathbb{Z}^2}, \underset{\max}{\text{pointwise}}, \underset{+}{\text{pointwise}} \right)$

$$\mathfrak{F}(F) \overset{\text{def}}{=} \begin{cases} \varepsilon & \text{if } F = \varepsilon \\ +\infty & \text{if } F \neq \varepsilon \text{ and if } (g,d) \notin (-\mathbb{N}) \times \mathbb{N} \\ \text{after any representative} & \text{otherwise} \end{cases}$$

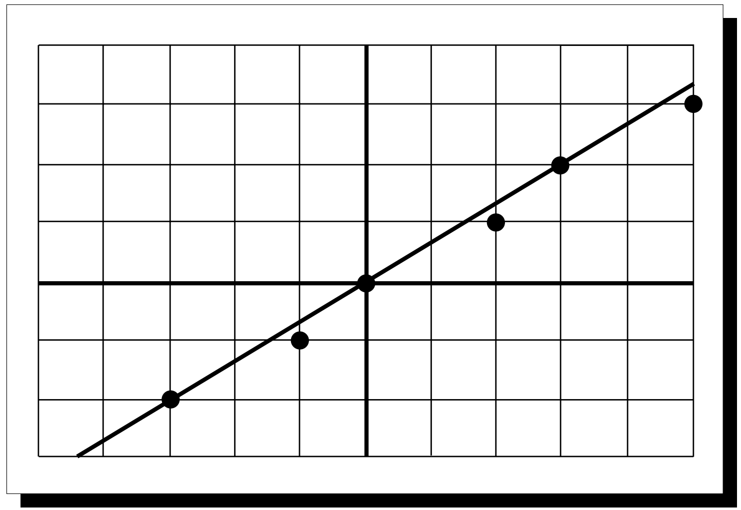
Lemma 40. *This mapping $\mathfrak{F}$ is a dioid homomorphism over $\mathcal{M}_{in}^{ax}[\![\gamma, \delta]\!]$.*

5.2. Eigenfunctions of Rational Transfer Functions

Let k and $t \in \mathbb{N}$, let $c = t/k$ and $L_c = \bigoplus_{\tau \leq c \times \ell} \gamma^\ell \delta^\tau$.

This describes a sequence of events which occurs at the average rate of $1/c$ events per unit of time.

Example: $L_{2/3} = (e \oplus \gamma^2 \delta) \left(\gamma^3 \delta^2\right)^* \left(\gamma^{-3} \delta^{-2}\right)^*$



If a SISO event graph has the transfer function $H = P \oplus Q \left(\gamma^r \delta^s\right)^*$, the ratio r/s is the limit of the rate of events the system can process.

If L_c is used as input, and if $1/c$ exceeds r/s, then there will be an indefinite accumulation of tokens inside the system and the output is indefinitely delayed with respect to the input. Otherwise, L_c is an eigenfunction of H.

Theorem 41. *Let $H = P \oplus Q\,(\gamma^r \delta^s)^*$, with $Q \neq \varepsilon$, and $r > 0$, $s > 0$. Then,*

1. *$\forall g \leq 0$ and $d \geq 0$, $[\mathfrak{F}(H)](g,d) \neq +\infty$ if and only if*

$$c = -g/d \geq s/r \ , \tag{1}$$

   and then

$$\exists \kappa_c \geq 0 \ , \quad \theta_c \geq 0 \ : \\ \sup_{(k,t)\in H} (gk + dt) = [\mathfrak{F}(H)](g,d) = [\mathfrak{F}(P \oplus Q)](g,d) = g\kappa_c + d\theta_c \ ;$$

2. *κ_c and θ_c are not necessarily unique, but any selection produces nonincreasing functions κ_c and θ_c of c;*

3. *let $c \overset{\text{def}}{=} t/k$ and assume that c satisfies (1); then*

$$H L_c = \gamma^{\kappa_c} \delta^{\theta_c} L_c \ .$$

Example 42.

- Let $H = \gamma\delta^2 \oplus (\gamma^2\delta)^*$. The system can process 2 events per time unit at most.
- $(\kappa_c, \theta_c) = \begin{cases} (0,0) & \text{if } 2 \leq c < +\infty; \\ (1,2) & \text{if } 1/2 \leq c \leq 2\,. \end{cases}$

  This is the 'Black plot' (in conventional system theory, it is the locus of points $(\arg H(j\omega), \log |H(j\omega)|)$ when ω varies).
- Consider the input L_c with $c = 2/3$ (in black) and the response of the system (in gray).
  - In the dater point of view $\kappa_c = 1$ is the 'phase shift', $\theta_c = 2$ is the 'amplification gain'.
  - In the counter point of view, these roles are interchanged.