Hardy–Sobolev–Maz’ya type equations in bounded domains

M. Bhakta, K. Sandeep

TIFR Centre for Applicable Mathematics, Post Bag No. 6503, Sharadanagar, Chikkabommasandra, Bangalore 560 065, India

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We study the regularity, Palais–Smale characterization and existence/nonexistence of solutions of the Hardy–Sobolev–Maz’ya equation

\[-\Delta u - \lambda \frac{u}{|y|^2} = \left|\frac{u^{p_t-1}u}{|y|^t}\right| \text{ in } \Omega,\]

where $x \in \mathbb{R}^N$ is denoted as $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$ and $p_t = \frac{N+2-2t}{N-2}$. We show different behaviors of PS sequences depending on $t = 0$ or $t > 0$.

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1. Introduction

In this article we study the singular semilinear elliptic problem

\[-\Delta u - \lambda \frac{u}{|y|^2} = \left|\frac{u^{p_t-1}u}{|y|^t}\right| \text{ in } \Omega,\]

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary.

By a solution of the above equation we mean a positive $u \in H^1_0(\Omega)$ satisfying

\[\int_{\Omega} \left( \nabla u \cdot \nabla v - \lambda \frac{u v}{|y|^2} \right) dx = \int_{\Omega} \left|\frac{u^{p_t-1}u v}{|y|^t}\right| dx, \quad \forall v \in H^1_0(\Omega).\]
Equivalently, \( u \) is a critical point of the functional \( J \),

\[
J(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_{\Omega} \frac{u^2}{|y|^2} \, dx - \frac{1}{p_t + 1} \int_{\Omega} \frac{|u|^{p_t+1}}{|y|^t} \, dx, \quad u \in H^1_0(\Omega). \tag{1.2}
\]

\( J \) is a well-defined \( C^1 \) functional on \( H^1_0(\Omega) \) for any open subset of \( \mathbb{R}^N \) thanks to the following Hardy–Sobolev–Maz'ya inequality (see [9]):

\[
S^*_t\left( \int_{\mathbb{R}^N} \frac{|u|^{p_t+1}}{|y|^t} \, dy \, dz \right)^{\frac{2}{p_t+1}} \leq \int_{\mathbb{R}^N} \left[ |\nabla u|^2 - \frac{\lambda u^2}{|y|^2} \right] \, dy \, dz \tag{1.3}
\]

holds for all \( \lambda < \frac{(k-2)^2}{4} \) and \( u \in C_0^\infty(\mathbb{R}^N) \), with the optimal constant \( S^*_t \), where \( N, t, p_t, \ldots \) are as before. Thanks to this inequality, \( \int_{\Omega} |\nabla u|^2 - \frac{\lambda u^2}{|y|^2} \right)^{1/2} \) is a norm equivalent to \( \int_{\Omega} |\nabla u|^2 \) when \( \lambda < \frac{(k-2)^2}{4} \) on \( C_0^\infty(\mathbb{R}^N) \).

When \( \Omega = \mathbb{R}^N \), the existence of critical points for \( J \) has been studied in [11] and [16] and the uniqueness has been studied in [6,8] and [10]. Uniqueness of solutions which lie only in a space which is bigger than \( H^2(\Omega) \) has been studied in [3]. When \( \Omega = \mathbb{R}^N \), the hyperbolic symmetry of the equation (see [5,6,10]) plays a crucial role in the study.

But in a bounded domain the problem (1.1) does not have a solution in general due to the critical nature of the equation. We prove a Pohozaev type nonexistence result in Theorem 4.1. The main difficulty in proving this theorem comes from the fact that the solutions are not regular. In fact from the standard elliptic regularity theory we know that if \( u \) is a solution then \( u \) is smooth in \( \Omega \setminus \{ x : y = 0 \} \). In general we cannot expect the solution to be smooth up to \( \{ x : y = 0 \} \) as the following example shows. For a given \( \lambda < \frac{(k-2)^2}{4} \), if \( t \) is chosen such that \( p_t = 1 + \frac{2}{N-k+\sqrt{(k-2)^2-4\lambda}} \), then the function

\[
u(y, z) = c(\lambda, N, k) \frac{|y|^{\sqrt{12-2\lambda}-2\sqrt{(k-2)}}}{[(1+|y|^2+|z|^2)^{-\frac{1}{2}}]
\]

solves (1.1) with \( \Omega = \mathbb{R}^N \), where \( c(\lambda, N, k) \) is a constant depending on \( \lambda, N, \) and \( k \). Note that \( u \) blows up near the singularity \( y = 0 \) and \( u \) is not even in \( H^2(\Omega) \) for certain \( \lambda \). In Section 2 we study the regularity properties. We prove in Section 2 that the equation has a partial \( H^2 \) regularity (see Theorem 2.1) in \( \mathbb{R}^N \) and the same regularity extends up to the boundary if \( \partial \Omega \) is orthogonal to \( \{ y = 0 \} \) (see Section 2 for definition) in Theorems 2.3 and 2.4. We also prove an \( L^p \) regularity of the solution in Theorem 2.5.

The nonexistence phenomenon for (1.1) is due to the lack of compactness of \( J \) due to a concentration phenomenon. We analyze this noncompactness. We prove in Theorem 3.1, that concentration takes place along a single profile when \( t > 0 \) while concentration takes place along two different profiles when \( t = 0 \). Using this theorem we prove an existence result Theorem 4.2 in the spirit of [7] to give evidence that a nontrivial topology of the singular set \( \Omega \cap \{ x : y = 0 \} \) will imply the existence of solution for (1.1).

**Remark 1.** (i) When \( \lambda = t = 0 \), the problem (1.1) is well studied. Therefore we assume throughout this paper either \( \lambda > 0 \) or \( t > 0 \).

(ii) When \( \lambda = \frac{(k-2)^2}{4} \), results analogous to the ones presented in this paper can be proved, but in a space which is bigger than \( D^{1,2}(\Omega) \) (see [10,16] for a precise definition).
Notations. We denote by $H^1(\Omega)$ the usual Sobolev space and by $D^{1,2}(\Omega)$ the closure of $C_c^{\infty}(\Omega)$ with respect to the norm $\left( \int_\Omega |\nabla u|^2 \right)^{\frac{1}{2}}$. $C$ will denote a general constant which may vary from line to line. We will just write $\int$ when the domain of integration is clear from the contest.

2. Regularity results

In this section we study the regularity properties of Eq. (1.1). When $\lambda = 0$, it follows from [8] that the solutions are in $C^{0,\alpha}_{loc}$. But when $\lambda > 0$, as noted in the introduction the solutions are not even in $H^2(\Omega)$. However we show that the solutions have a partial $H^2$ regularity. We will also prove an $L^p$ estimate for the solution. First we state the result in $\mathbb{R}^N$.

Theorem 2.1. Let $u \in D^{1,2}(\mathbb{R}^N)$ be a solution of the equation

$$-\Delta u - \frac{\lambda}{|y|^2} \frac{u}{|y|^2} = \frac{|u|^{p-1} u}{|y|^2} \quad \text{in } \mathbb{R}^N,$$

then $u_{zi} \in D^{1,2}(\mathbb{R}^N)$ for $1 \leq i \leq N - k$.

Proof. By the definition of solution, we have for any $v \in D^{1,2}(\mathbb{R}^N)$

$$\int \nabla u \cdot \nabla v - \lambda \int \frac{uv}{|y|^2} = \int \frac{|u|^{p-1} uv}{|y|^2}. \quad (2.4)$$

For $|h| > 0$ and $i > k$, define $v = -D_i^{-h}(D_i^h u)$ where $D_i^h u$ denotes the difference quotient

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h} \quad (h \in \mathbb{R}, \ h \neq 0).$$

For this choice of $v$ the LHS of (2.4) simplifies to $\int |\nabla (D_i^h u)|^2 - \lambda \int \frac{(D_i^h u)^2}{|y|^2}$ while the RHS can be simplified as

$$\int \frac{|u|^{p-1} uv}{|y|^2} = \int D_i^h \left( \frac{|u|^{p-1} u}{|y|^2} \right) (D_i^h u)$$

$$= \int \frac{1}{|y|^2} \left( \frac{|u|^{p-1} u(x + he_i) - |u|^{p-1} u(x)}{h} \right) \cdot D_i^h u$$

$$\leq C \int \frac{1}{|y|^2} \left[ |u|^{p-1}(x + he_i) + |u|^{p-1}(x) \right] (D_i^h u)^2$$

$$= C \int \frac{|u|^{p-1}}{|y|^2} \left[ (D_i^{-h} u)^2 + (D_i^h u)^2 \right]$$

$$= C \int_{\Omega_M} \frac{|u|^{p-1}}{|y|^2} \left[ (D_i^{-h} u)^2 + (D_i^h u)^2 \right] + 2CM \int |\nabla u|^2. \quad (2.5)$$

where $\Omega_M = \{ x : |y|^{-1}|u|^{p-1} > M \}$ and we used the estimate $\int (D_i^h u)^2 \leq \int |\nabla u|^2$. Now using the Cauchy–Schwartz inequality and the Hardy–Sobolev–Maz'ya, we get
\[ \int_{\Omega} \frac{|u|^{p_{r-1}}}{|y|^r} \left( (D_i^{-} u)^2 + (D_i^{+} u)^2 \right) \leq \left( \int_{\Omega} \frac{|u|^{p_{r+1}}}{|y|^r} \right)^{\frac{p_{r-1}}{p_{r+1}}} \left( \int_{\Omega} \frac{|D_i^{+} u|^{2}}{|y|^r} \right)^{\frac{p_{r+1}}{2}} \]

\[ \leq \left( \int_{\Omega} \frac{|u|^{p_{r+1}}}{|y|^r} \right)^{\frac{p_{r-1}}{p_{r+1}}} \int |\nabla (D_i^{+} u)|^2 \leq \frac{1}{2} \left( 1 - \frac{4\lambda}{(k-2)^2} \right) \int |\nabla (D_i^{+} u)|^2 \]

by choosing \( M \) large enough. Substituting back in (2.4) and using the Hardy–Sobolev–Maz’ya, we get

\[ \frac{1}{2} \left( 1 - \frac{4\lambda}{(k-2)^2} \right) \int |\nabla (D_i^{+} u)|^2 \leq 2CM \int |\nabla u|^2 \leq C \]

where \( C \) is independent of \( h \). So we have \( D_i^{+} u \) is bounded in \( D^{1,2}(\mathbb{R}^N) \) \( \Rightarrow \) converges weakly and point-wise up to a subsequence to \( u_{z_i} \). Therefore by weak lower semi-continuity we have \( \int_{\mathbb{R}^N} |\nabla u_{z_i}|^2 \leq M \), and this completes the proof. \( \square \)

Next we prove the results in open subsets of \( \mathbb{R}^N \). First a definition:

**Definition 2.2.** Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) with smooth boundary. We say that \( \partial \Omega \) is orthogonal to the singular set if for every \( (0, z_0) \in \partial \Omega \) the normal at \( (0, z_0) \) is in \( \{0\} \times \mathbb{R}^{N-k} \).

**Theorem 2.3.** Let \( \Omega \) be a half-space with \( \partial \Omega \) orthogonal to the singular set. Let \( u \in D^{1,2}(\Omega) \) be a solution of the equation

\[-\Delta u - \lambda \frac{u}{|y|^2} = \frac{|u|^{p_{r-1}} u}{|y|^r} \quad \text{in} \; \Omega, \]

then \( u_{z_i} \in H^1(\Omega) \) for \( 1 \leq i \leq N-k \).

**Proof.** Since \( \Omega \) is orthogonal to the singular set we have \( \Omega = \{(y, z) : z \cdot v > 0\} \) for some \( v \in \mathbb{R}^{N-k} \). Let \( \tilde{u} : \mathbb{R}^N \to \mathbb{R} \) be defined as

\[ \tilde{u}(y, z) = \begin{cases} u(y, z) & \text{if } (y, z) \in \Omega, \\ -u(y, \tilde{z}) & \text{if } (y, z) \notin \Omega \end{cases} \]

where \( \tilde{z} = z - (2z \cdot v)v \). Now we claim:

**Claim.** \( \tilde{u} \in D^{1,2}(\mathbb{R}^N) \) and solves

\[-\Delta \tilde{u} - \lambda \frac{\tilde{u}}{|y|^2} = \frac{|\tilde{u}|^{p_{r-1}} \tilde{u}}{|y|^r} \quad \text{in} \; \mathbb{R}^N. \]

Then the theorem follows from the previous theorem.

**Proof of Claim.** Clearly \( \tilde{u} \in D^{1,2}(\mathbb{R}^N) \). We want to show

\[ \int_{\Omega} \nabla \tilde{u} \cdot \nabla v - \lambda \int_{\Omega} \tilde{u}v = \int_{\Omega} \frac{|\tilde{u}|^{p_{r-1}} \tilde{u}v}{|y|^r}, \quad \forall v \in D^{1,2}(\mathbb{R}^N), \]

\[ \text{LHS} = \left( \int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} uv \right) + \left( \int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} uv \right). \quad (2.6) \]
Now set $\varphi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ to be a $C^\infty$ function s.t.

$$
\varphi(r) = \begin{cases} 
0 & \text{if } r \leq 1, \\
1 & \text{if } r \geq 2.
\end{cases}
$$

Now $\varphi_\varepsilon(x) = \varphi(\frac{|y|}{\varepsilon})$ where $x = (y, z)$. Therefore $\nabla \varphi_\varepsilon = \left( \frac{1}{\varepsilon} \frac{y}{|y|}, \frac{\varphi'(\frac{|y|}{\varepsilon})}{\varepsilon} \right), 0)$. Now

$$
\int_\Omega \nabla u \cdot \nabla (\varphi_\varepsilon v) - \lambda \int_\Omega \frac{uv \varphi_\varepsilon}{|y|^2} = \int_\Omega (\nabla u \cdot \nabla v) \varphi_\varepsilon - \lambda \int_\Omega \frac{uv \varphi_\varepsilon}{|y|^2} + \int_\Omega \nabla u \cdot \nabla \varphi_\varepsilon \cdot v
$$

$$
= \int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega \frac{uv}{|y|^2} + o(1)
$$

where $o(1) \to 0$ in $D^{1,2}(\Omega)$;

$$
\int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega \frac{uv}{|y|^2} = \int_\Omega \nabla u \cdot \nabla (\varphi_\varepsilon v) - \lambda \int_\Omega \frac{uv \varphi_\varepsilon}{|y|^2} + o(1)
$$

$$
= \int_\Omega \left( -\Delta u - \frac{\lambda u}{|y|^2} \right)(\varphi_\varepsilon v) - \int_\partial \frac{\partial u}{\partial v} \varphi_\varepsilon v + o(1)
$$

$$
= \int_\Omega \left| \frac{u^{p-1}uv}{|y|} \right| - \int_\partial \frac{\partial u}{\partial v} \varphi_\varepsilon v + o(1)
$$

as $\varepsilon \to 0$.

Similarly,

$$
\int_{\Omega^c} \nabla \tilde{u} \cdot \nabla v - \lambda \int_{\Omega^c} \frac{\tilde{u} v}{|y|^2} = \int_{\Omega^c} \left( -\Delta \tilde{u} - \frac{\lambda \tilde{u}}{|y|^2} \right)(\varphi_\varepsilon v) + \int_\partial \frac{\partial \tilde{u}}{\partial v} \varphi_\varepsilon v + o(1)
$$

$$
= \int_{\Omega^c} \left| \frac{\tilde{u}^{p-1}\tilde{u}v}{|y|^2} \right| \varphi_\varepsilon v + \int_\partial \frac{\partial \tilde{u}}{\partial v} \varphi_\varepsilon v.
$$

So we have

$$
\int \nabla \tilde{u} \cdot \nabla v - \lambda \int \frac{\tilde{u} v}{|y|^2} = \int \left| \frac{\tilde{u}^{p-1} \tilde{u} v}{|y|^2} \right| + o(1),
$$

$$
-\Delta \tilde{u} - \frac{\tilde{u}}{|y|^2} = \frac{\tilde{u}^{p-1} \tilde{u}}{|y|^2} \text{ in } \Omega.
$$

**Theorem 2.4.** Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with $\partial \Omega$ orthogonal to the singular set and $u \in H^1_0(\Omega)$ solves the equation

$$
-\Delta u - \frac{u}{|y|^2} = \frac{u^{p-1}u}{|y|^2} \text{ in } \Omega,
$$

then $u_{zi} \in H^1(\Omega)$ for all $1 \leq i \leq N - k$. 


Proof. First we show that $u_{z_1} \in H^1_{\text{loc}}(\Omega)$. Let $\Omega_1$ be an open subset satisfying $\overline{\Omega_1} \subset \Omega$. Let $\varphi \in C^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\varphi = 1$ in $\Omega_1$. Define $v \in H^1_0(\Omega)$ by $v = -D_i^ {-\delta} (\varphi^2) D_i^b u$, then we have

$$
\int_{\Omega} \nabla u \cdot \nabla v - \lambda \int_{\Omega} \frac{u v}{|y|^2} = \int_{\Omega} \frac{|u|^{p_i - 1} u v}{|y|^f}.
$$

Now the LHS simplifies to

$$
\int \left| \nabla (\varphi D_i^b u) \right|^2 - \lambda \int \frac{(\varphi D_i^b u)^2}{|y|^2} - \int \left| \nabla (D_i^b u) \right|^2.
$$

The right-hand side can be simplified exactly as in Theorem 2.1. Combining these estimates as before, we get

$$
\frac{1}{2} \left( 1 - \frac{4 \lambda}{(N-2)^2} \right) \int \left| \nabla (\varphi D_i^b u) \right|^2 \leq C \int \left| \nabla u \right|^2 \leq C
$$

and hence $u_{z_1} \in H^1(\Omega_1)$.

Now it remains to show that $\forall x_0 \in \partial \Omega$, there exists $R > 0$ such that $u_{z_1} \in H^1(B(x_0, R) \cap \Omega)$. If $x_0$ is not on the singular set, then this follows from the standard elliptic theory and the Brezis–Kato result (when $t = 0$). Therefore we assume that $x_0 = (0, z_0)$. Since the equation is invariant under orthogonal transformations which fixes the $y$ variable, we may assume that normal at $(0, z_0)$ is $(0, \ldots, 0, 1)$. Since boundary is smooth, there exist $R > 0$ and a smooth function $f$ such that $B(x_0, R) \cap \Omega = \{(y, z) : z_n > f(y, z_1, \ldots, z_{n-1})\}$. Let us flatten the boundary near $x_0$.

Define $\Psi(y, z) = (y, z_1, \ldots, z_{n-1}, z_n + f(y, z_1, \ldots, z_{n-1}))$, $(y, z) \in \Omega_1$, where $\Omega_1 = \{x : \Psi(x) \in B(x_0, R) \cap \Omega\}$. Then

$$
\Psi^{-1}(y, z) = (y, z_1, \ldots, z_{n-1}, z_n - f(y, z_1, \ldots, z_{n-1})).
$$

Now let us define, $v(y, z) = u(\Psi(y, z))$, $(y, z) \in \Omega_1$, then $u(y, z) = v(\Psi^{-1}(y, z))$ for $(y, z) \in B(x_0, R) \cap \Omega$.

Then $v \in H^1(\Omega_1)$, $v = 0$ on $\partial \Omega_1 \cap \{x : z_n = 0\}$ and satisfies the equation

$$
-\Delta v - \frac{\lambda v}{|y|^2} = |\nabla f|^2 \frac{\partial^2 v}{\partial z_n^2} + 2 \sum_{j=1}^{n-1} \frac{\partial f}{\partial x_j} \frac{\partial^2 v}{\partial x_j \partial z_n} + \Delta f \frac{\partial v}{\partial z_n} = \frac{|v|^{p_i - 1} v}{|y|^f}.
$$

(2.7)

in the sense for all $w \in H^1_0(\Omega_1)$

$$
\int_{\Omega_1} \left( \nabla v \cdot \nabla w - \frac{\lambda v w}{|y|^2} + \left| \nabla f \right|^2 \frac{\partial v}{\partial z_n} - 2 \sum_{j=1}^{n-1} \frac{\partial f}{\partial x_j} \frac{\partial v}{\partial x_j} - \Delta f \frac{\partial v}{\partial z_n} \right) \frac{\partial w}{\partial z_n} = \int_{\Omega_1} \frac{|v|^{p_i - 1} v w}{|y|^f}.
$$

Let $\tilde{\Omega}_1$ denote the union of $\Omega$ its reflection with respect to $\{z_{N-k} = 0\}$ and $\partial \Omega \cap \{z_{N-k} = 0\}$ and extend $v$ to $\tilde{\Omega}_1$ by odd reflection. As in the proof of Theorem 2.3 the extended function denoted again by $v$ satisfies the same equation.
Now set \( w = -D^{-h}(\varphi^2 D_i^h v) \) for \( i > k \) where \( \varphi \in C_0^\infty(\Omega_1) \), then for \( h \neq 0 \) and small enough, \( w \in H^1_{01}(\Omega_1) \). Then the RHS can be simplified exactly as before. As seen in the beginning of the proof, integral of the first two terms on the LHS can be simplified as

\[
\int_{\Omega_1} |\nabla (\varphi D_i^h v)|^2 - \lambda \int_{\Omega_1} \frac{(\varphi D_i^h v)^2}{|y|^2} - \int_{\Omega_1} |\nabla \varphi|^2 (D_i^h v)^2.
\]

Now

\[
\int_{\Omega_1} \left[ |\nabla f|^2 \frac{\partial v}{\partial z_n} - 2 \sum_{j=1}^{n-1} \frac{\partial f}{\partial x_j} \frac{\partial v}{\partial x_j} - \nu \Delta f \right] \frac{\partial w}{\partial z_n}
= \int_{\Omega_1} D_i^h \left( |\nabla f|^2 \frac{\partial v}{\partial z_n} - 2 \sum_{j=1}^{n-1} \frac{\partial f}{\partial x_j} \frac{\partial v}{\partial x_j} - \nu \Delta f \right) \frac{\partial (\varphi^2 D_i^h v)}{\partial z_n}.
\]

Let us simplify the terms one by one:

\[
\int_{\Omega_1} D_i^h \left( |\nabla f|^2 \frac{\partial v}{\partial z_n} \right) \frac{\partial (\varphi^2 D_i^h v)}{\partial z_n}
= \int_{\Omega_1} D_i^h \left( |\nabla f|^2 \frac{\partial v}{\partial z_n} \right) \left( \varphi^2 \frac{\partial}{\partial z_n} (D_i^h v) + 2 \varphi \varphi_{zz} D_i^h v \right)
= \int_{\Omega_1} \left( |\nabla f (x + he_i)|^2 \frac{\partial}{\partial z_n} (D_i^h v) + D_i^h (|\nabla f|^2) \frac{\partial v}{\partial z_n} \right) \left( \varphi^2 \frac{\partial}{\partial z_n} (D_i^h v) + 2 \varphi \varphi_{zz} D_i^h v \right)
= \int_{\Omega_1} |\nabla f (x + he_i)|^2 \left[ \varphi \frac{\partial}{\partial z_n} (D_i^h v) \right]^2 + \int_{\Omega_1} D_i^h (|\nabla f|^2) \left( \frac{\partial v}{\partial z_n} \left( \varphi \frac{\partial}{\partial z_n} (D_i^h v) \right) \right)
+ \int_{\Omega_1} |\nabla f (x + he_i)|^2 \left[ \varphi \frac{\partial}{\partial z_n} (D_i^h v) \right] (2 \varphi \varphi_{zz} D_i^h v) + \int_{\Omega_1} D_i^h (|\nabla f|^2) 2 \varphi \varphi_{zz} \frac{\partial v}{\partial z_n} D_i^h v.
\]

For a given \( \epsilon > 0 \), Since the normal at \( x_0 \) is \((0, \ldots, 0, 1)\) we can choose \( r > 0 \) such that \( \sup |\nabla f (x + he_i)|^2 < \epsilon \). Thus the first term can be estimated by \( \epsilon \int |\nabla (\varphi D_i^h v)|^2 \) which can be bounded by \( \epsilon \int |\nabla (\varphi D_i^h v)|^2 \) and \( C \int (D_i^h v)^2 \), which can again be bounded by \( \epsilon \int |\nabla (\varphi D_i^h v)|^2 \) and \( C \int |\nabla v|^2 \). Now using the inequality \( ab \leq a^2 + \frac{b^2}{4} \) and the fact that \( D_i^h |\nabla f|^2 \) is bounded we can estimate the rest of the integrals by \( \epsilon \int |\nabla (\varphi D_i^h v)|^2 \) and \( C \int |\nabla v|^2 \).

Combining all the above estimates and using the Hardy–Sobolev–Maz'ya as before we get \( \int_{\Omega_1} |\nabla (\varphi D_i^h v)|^2 \leq C \) where \( C \) is independent of \( h \) and hence \( \int_{B_1(\psi, \bar{r} \bar{x}, \bar{r})} |\nabla v| \leq C \) for some \( \bar{r} > 0 \) and hence \( \int_{B_{\bar{r}}(\bar{x})} |\nabla u_z| \leq C \) for some \( r > 0 \). This proves the theorem. □

Next we prove the \( L^p \) regularity of the solution. When the singularity is \( \frac{1}{|x|^2} \) the \( L^p \) regularity has been obtained in [1,2] and [13]. Here we prove
Theorem 2.5. Let \( u \in D^{1,2}(\Omega) \) solve the equation

\[
-\Delta u - \lambda \frac{u}{|y|^2} = \frac{|u|^{p-1}u}{|y|^t} \quad \text{in} \; \Omega,
\]

then \( u \in L^p_{\text{loc}}(\Omega) \) for all \( p < \frac{2N}{N-2} \left( \frac{k-2}{2} \right)^2 - 1 \).

Proof. We have

\[
\int_{\Omega} \nabla u \cdot \nabla v = \lambda \int_{\Omega} \frac{uv}{|y|^2} + \int_{\Omega} \frac{|u|^{p-1}uv}{|y|^t}, \quad \forall v \in H_0^1(\Omega).
\]

Now set \( \bar{u} = u^+ + 1 \) and \( u_m \) as follows

\[
u_m = \begin{cases} 
\bar{u} & \text{if } u < m, \\
1 + m & \text{if } u \geq m.
\end{cases}
\]

Now for \( \beta > 0 \) define \( v = v_\beta = \varphi^2 (u_m^\beta \bar{u} - 1) \) where \( \varphi \in C_0^\infty(\Omega) \) s.t. \( 0 \leq \varphi \leq 1 \), \( \varphi \equiv 1 \in B(R) \), \( \text{supp}(\varphi) \subseteq B(2R) \) and \( B(2R) \subseteq \Omega \).

So we have

\[
\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} u_m^\beta \varphi^2 \nabla u \cdot \nabla \bar{u} + 2\beta \int_{\Omega} u_m^{2\beta-1} \bar{u} \varphi^2 \nabla u \cdot \nabla u_m + 2 \int_{\Omega} \varphi (u_m^{2\beta} \bar{u} - 1) \nabla u \cdot \nabla \varphi.
\]

In the support of 1st integral \( \nabla u = \nabla \bar{u} \) and in the support of 2nd integral \( u_m = \bar{u} \), \( \nabla u_m = \nabla u \). So the above expression equals

\[
\int_{\Omega} u_m^\beta \varphi^2 |\nabla \bar{u}|^2 + 2\beta \int_{\Omega} \varphi^2 u_m^{2\beta} |\nabla u_m|^2 + 2 \int_{\Omega} \varphi (u_m^{2\beta} \bar{u} - 1) \nabla u \cdot \nabla \varphi.
\]

Using Cauchy–Schwartz inequality we can write

\[
\int_{\Omega} \nabla u \cdot \nabla v \geq \left( 1 - \frac{\beta}{2 + \beta} \right) \int_{\Omega} u_m^\beta \varphi^2 |\nabla \bar{u}|^2 + 2\beta \int_{\Omega} \varphi^2 u_m^{2\beta} |\nabla u_m|^2 - \frac{2 + \beta}{\beta} \int_{\Omega} u_m^{2\beta} \bar{u}^2 |\nabla \varphi|^2.
\]

So we have

\[
\frac{2}{2 + \beta} \int_{\Omega} u_m^\beta \varphi^2 |\nabla \bar{u}|^2 + 2\beta \int_{\Omega} \varphi^2 u_m^{2\beta} |\nabla u_m|^2
\]

\[
\leq \lambda \int_{\Omega} \frac{u_m^2 \varphi^2}{|y|^2} + \int_{\Omega} \frac{|u|^{p-1} \bar{u}^2 u_m^{2\beta} \varphi^2}{|y|^t} + \frac{2 + \beta}{\beta} \int_{\Omega} u_m^{2\beta} \bar{u}^2 |\nabla \varphi|^2.
\]

Now set \( w = u_m^{\beta} \bar{u} \). So

\[
\int_{\Omega} \varphi^2 |\nabla w|^2 = \beta (\beta + 2) \int_{\Omega} u_m^{2\beta} \varphi^2 |\nabla u_m|^2 + \int_{\Omega} u_m^{2\beta} \varphi^2 |\nabla \bar{u}|^2.
\]
Now

\[
\int_{\Omega} |\nabla (w\varphi)|^2 \leq (\epsilon + 1) \int_{\Omega} |\nabla w|^2 \varphi^2 + \left( 1 + \frac{1}{\epsilon} \right) \int_{\Omega} w^2 |\nabla \varphi|^2
\]

\[
= (1 + \epsilon) \left\{ \frac{\beta + 2}{2} \left[ \int_{\Omega} 2\beta u_m^2 \varphi^2 |\nabla u_m|^2 + 2 \int_{\Omega} u_m^2 \varphi^2 |\nabla \varphi|^2 \right] + \left( 1 + \frac{1}{\epsilon} \right) \int_{\Omega} w^2 |\nabla \varphi|^2 \right\}
\]

\[
\leq (1 + \epsilon) \left\{ \frac{\beta + 2}{2} \left[ \int_{\Omega} (w\varphi)^2 |\nabla \varphi|^2 + \frac{2 + \beta}{\beta} \int_{\Omega} w^2 |\nabla \varphi|^2 \right] \right\}
\]

\[
+ \left( 1 + \frac{1}{\epsilon} \right) \int_{\Omega} w^2 |\nabla \varphi|^2.
\]

So,

\[
\int_{\Omega} |\nabla (w\varphi)|^2 - (1 + \epsilon) \left( \frac{\beta + 2}{2} \right) \lambda \int_{\Omega} \frac{(w\varphi)^2}{|y|^t}
\]

\[
\leq \int_{\Omega} a(x) \frac{(w\varphi)^2}{|y|^t} + C \left( 1 + \frac{1}{\epsilon} + \frac{2 + \beta}{\beta} \right) \int_{B(2R)} w^2
\]

where \( a(x) = |u|^{p+1}_\Omega \in L^{\frac{N}{2}}(\Omega) \).

Now if we take \( s = \frac{Nt}{N-2+t} \) then \( \frac{p+1}{2} = \frac{N}{N-2+t} \) and \( \frac{N-2+t}{N} + \frac{2-t}{N} = 1 \).

So we have

\[
\int_{\Omega} a(x) \frac{(w\varphi)^2}{|y|^t} \leq \int_{|a(x)| \leq K} a(x) \frac{(w\varphi)^2}{|y|^t} + \int_{|a(x)| > K} a(x) \frac{(w\varphi)^2}{|y|^t}
\]

\[
\leq K \int_{\Omega} \frac{(w\varphi)^2}{|y|^t} + \left( \int_{|a(x)| > K} a(x) \frac{1}{|y|^t} \right)^{\frac{2}{p+1}} \left( \int_{\Omega} \frac{(w\varphi)^{p+1}}{|y|^{s+t}} \right)^{2} \leq \left( \frac{N}{N-2+t} \right)^{\frac{2}{p+1}}
\]

\[
\leq K \int_{\Omega} \frac{(w\varphi)^2}{|y|^t} + \delta(K) \left( \int_{\Omega} \frac{(w\varphi)^{p+1}}{|y|^{s+t}} \right)^{2} \leq K \int_{\Omega} \frac{(w\varphi)^2}{|y|^t} + C\delta(K) \int_{\Omega} |\nabla (w\varphi)|^2.
\]

Now choose \( K > 0 \) s.t. \( \delta = C\delta(K) < \left( \frac{\kappa - 2}{2} \right)^{1-\delta} \), with this choice of \( \delta \) choose \( \beta > 0 \) s.t. \( \left( \frac{\beta + 2}{2} \right) \frac{\lambda}{1-\delta} < \left( \frac{\kappa - 2}{2} \right)^{1-\delta} \), i.e. \( \beta < \frac{\lambda}{2} \left( \frac{\kappa - 2}{2} \right)^{1-\delta} - 2 \).

Therefore we have

\[
(1 - \delta) \left\{ \int_{\Omega} |\nabla (w\varphi)|^2 - (1 + \epsilon) \left( \frac{\beta + 2}{2} \right) \frac{\lambda}{1-\delta} \int_{\Omega} \frac{(w\varphi)^2}{|y|^t} \right\}
\]

\[
\leq K \int_{\Omega} \frac{(w\varphi)^2}{|y|^t} + C \left( 1 + \frac{1}{\epsilon} + \frac{2 + \beta}{\beta} \right) \int_{B(2R)} w^2.
\]
Now with that choice of \( \delta, \beta \) we have for \( \epsilon > 0 \) to be small enough

\[
(1 + \epsilon)\left(\frac{\beta + 2}{2}\right)\frac{\lambda}{1 - \delta} \leq \left(\frac{k - 2}{2}\right)^2.
\]

Therefore,

\[
\int_{\Omega} (w\varphi)^{2s} \leq C \left[ \int_{\Omega} \frac{(w\varphi)^2}{|y|^s} + \int_{B(2R)} w^2 \right],
\]
i.e.

\[
\int_{B_R} (u_m^{\beta+1})^{2s} \leq K \int_{B(2R)} \left[ \tilde{u}^{2\beta+2} + \tilde{u}^{2\beta+2} \right] \text{ since } u_m \leq \tilde{u}.
\]

Taking \( m \to \infty \) we get

\[
\int_{B_R} (u^{\beta+1})^{2s} \leq K \int_{B(2R)} \left[ \tilde{u}^{2\beta+2} + \tilde{u}^{2\beta+2} \right].
\]

Now assume \( u \in L_{loc}^{2\beta+2}(\Omega) \). And if we take \( s = \frac{N}{(\beta+1)(N-2)+2} \), then \( p_{ts} + 1 = 2(\beta + 1)s \), so we get

\[
\int_{B(2R)} \tilde{u}^{2\beta+2} \leq \left( \int_{B(2R)} \tilde{u}^{p_{ts}+1} \right)^{\frac{1}{p_{ts}+1}} \cdot B(2R)^{1 - \frac{1}{p_{ts}}} \leq C \int_{\Omega} |\nabla \tilde{u}|^2
\]

where \( |B(2R)| \) is the measure of \( B(2R) \). Then by iteration procedure starting from \( \beta = 0 \) up to the range of \( \beta \) we can get that \( u \in L^p_{loc}(\Omega) \) for \( p < 2^*\left[\frac{2}{k-2}\right]^2(1 - \delta) - 1 \). Now this is true for any \( 0 < \delta < \frac{k-2}{(k-2)^2} \). Therefore we get \( u \in L^p_{loc}(\Omega) \) for \( p < 2^*\left[\frac{2}{k-2}\right]^2 - 1 \). □

3. Palais–Smale characterization

In this section we study the Palais–Smale sequences (in short, PS sequences) of the functional

\[
E_{\lambda}(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda}{2} \int_{\Omega} \frac{u^2}{|y|^s} - \frac{1}{p_t + 1} \int_{\Omega} \frac{|u|^{p_t+1}}{|y|^t}, \quad u \in H^1_0(\Omega),
\]  \tag{3.8}

where \( \Omega \) is a bounded domain with smooth boundary and \( 0 \leq t < 2 \) is fixed. We say that the sequence \( u_n \in H^1_0(\Omega) \) is a PS sequence for \( E_{\lambda} \) at level \( \beta \) if \( E_{\lambda}(u_n) \to \beta \) and \( E_{\lambda}'(u_n) \to 0 \) in \( H^{-1}(\Omega) \). It is easy to see that the weak limit of a PS sequence solves (1.1), except the positivity. However the main difficulty is that the PS sequence may not converge strongly and hence the weak limit can be zero even if \( \beta > 0 \). The main purpose of this section is to classify PS sequences for \( E_{\lambda} \). Classification of PS sequences has been done for various problems having lack of compactness, to quote a few [4,12–14]. While the noncompactness studied in the first two are due to the concentration phenomenon occurring through a single profile, the last one’s noncompactness is due to concentration occurring through two different profiles. We derive a classification theorem for the PS sequences of (3.8) in the same spirit of the above results. Concentration occurs here through one or two profiles depending on \( t > 0 \) or \( t = 0 \) respectively.
Let $V$ be a solution of
$$-\Delta u - \lambda \frac{u}{|y|^2} = \frac{|u|^{p'-1}u}{|y|^t}, \quad u \in D^{1,2}(\mathbb{R}^N). \quad (3.9)$$

Define a sequence $v_n$ as follows:
$$v_n(x) = \varphi(\tilde{R}_n(x - (0, z_n))) \left[ R_n^{2-N} W\left((x - (0, z_n))/R_n\right) \right] \quad (3.10)$$

where $\varphi \in C_0^\infty(B(0,2))$ with $\varphi = 1$ in $B(0,1)$, $(0, z_n) \in \Omega$ and satisfies $R_n \to 0$, $R_n \tilde{R}_n \to 0$ and $\liminf[\tilde{R}_n \text{dist}((0, z_n), \partial \Omega)] > 2$. Then $v_n \in H^1_0(\Omega)$, $v_n \to 0$ and is a PS sequence for $E_\lambda$ at level $\beta = E_{\lambda}(V)$ where $E_{\lambda}(V)$ is as defined in (3.8) with $\Omega = \mathbb{R}^N$.

Suppose $\lambda > 0$ and $t = 0$. Define a sequence $w_n(x)$ as
$$w_n(x) = \varphi(\tilde{R}_n(x - (0, z_n))) \left[ R_n^{2-N} W\left((x - (0, z_n))/R_n\right) \right] \quad (3.11)$$

where $W \in D^{1,2}(\mathbb{R}^N)$ satisfying $-\Delta W = |W|^{p-1}W$, $\varphi$ is as above, $(y_n, z_n) \in \Omega$, $\frac{|y_n|}{\tilde{R}_n} \to \infty$, $R_n \tilde{R}_n \to 0$ and $\liminf[\tilde{R}_n \text{dist}((y_n, z_n), \partial \Omega)] > 2$. Then $w_n \in H^1_0(\Omega)$, $w_n \to 0$ and is also a PS sequence for $E_\lambda$ at level $\beta = E_0(W)$ where $E_0(W)$ is as defined in (3.8) with $\Omega = \mathbb{R}^N$ and $t = 0$.

In fact we see below that the noncompact PS sequences are essentially a finite sum of sequences of the form (3.10) and (3.11) when $t = 0$ and $\lambda > 0$ and a finite sum of sequences of the form (3.10) when $t > 0$ or $\lambda = 0$.

**Theorem 3.1.** Let $\Omega$ be a bounded domain with smooth boundary. Let $u_n$ be a PS sequence for $E_\lambda$ at level $\beta$. Suppose $t = 0$ and $\lambda > 0$, then $\exists n_1, n_2 \in \mathbb{N}$, and functions $v_j^0 \in H^1_0(\Omega)$, $1 \leq j \leq n_1$, and $w_k^0 \in H^1_0(\Omega)$, $1 \leq k \leq n_2$, and $u \in H^1_0(\Omega)$ such that up to a subsequence

1. $u_n = u + \sum_{j=1}^{n_1} v_j^0 + \sum_{k=1}^{n_2} w_k^0 + o(1)$, where $o(1) \to 0$ in $H^1_0(\Omega)$;
2. $\beta = E_{\lambda}(u) + \sum_{j=1}^{n_1} E_{\lambda}(V_j) + \sum_{k=1}^{n_2} E_0(W_k) + o(1)$

where $E_{\lambda}(u) = 0$ and $v_j^0$, $w_k^0$ are PS sequences of the form (3.10) and (3.11) respectively with $V = V_j$ and $W = W_k$.

When $t > 0$ or $\lambda = 0$, the same conclusion holds with $W_k = 0$ for all $k$.

**Proof.** We will prove the theorem in several steps.

**Step 1.** If $u_n$ is a PS sequence for $E_\lambda$ at a level $\beta < \frac{2-t}{2(N-t)} (S^1_t)^{\frac{N-t}{2}}$, then $u_n$ is relatively compact in $H^1_0(\Omega)$.

**Proof.** To begin with, standard arguments tell us that $u_n$ is bounded in $H^1_0(\Omega)$. More precisely, since $E_{\lambda}(u_n) = \beta + o(1)$ and $\langle E_{\lambda}'(u_n), u_n \rangle = o(1)\|u_n\|$, computing $E_{\lambda}(u_n) - \frac{1}{p_t+1} \langle E_{\lambda}'(u_n), u_n \rangle$, we get $\|u_n\|_{H^1_0(\Omega)}^2 \leq C + \|u_n\|_{H^1_0(\Omega)}^{p_t+1} o(1)$ and hence the boundedness follows. Therefore passing to a subsequence if necessary we may assume $u_n \rightharpoonup u$ in $H^1_0(\Omega)$, $u_n \to u$ in $L^p(\Omega)$ for $p < 2^*$ and point-wise. Clearly $E_{\lambda}'(u) = 0$ and hence
$$E_{\lambda}(u) = \left( \frac{1}{2} - \frac{1}{p_t+1} \right) \int_\Omega \frac{|u|^{p_t+1}}{|y|^{t}} \geq 0.$$
Also by Brezis–Lieb lemma we have

$$\int_\Omega |u_n|^{p_t+1}|y|^t = \int_\Omega |u|^{p_t+1}|y|^t + \int_\Omega |u_n - u|^{p_t+1}|y|^t + o(1),$$

$$\int_\Omega u_n^2 = \int_\Omega u^2 + \int_\Omega |u_n - u|^2|y|^t + o(1).$$

So

$$E_\lambda(u_n) = E_\lambda(u) + E_\lambda(u_n - u) + o(1)$$

and hence

$$E_\lambda(v_n) = E_\lambda(u_n) - E_\lambda(u) + o(1) \leq C < \frac{2 - t}{2(N - t)} (S^\lambda_t)^{\frac{N - t}{2}}$$

where $v_n = u_n - u$. Since $u_n$ is a PS sequence and $E'_\lambda(u) = 0$ we get

$$o(1) = \langle E'_\lambda(u_n), v_n \rangle = \langle E'_\lambda(u_n) - E'_\lambda(u), v_n \rangle$$

$$= \int \left[ |\nabla v_n|^2 - \lambda \frac{v_n^2}{|y|^2} \right] + \int \left[ (|u_n|^{p_t-1}u_n - |u|^{p_t-1}u) v_n \right] \frac{1}{|y|^t}.$$ 

Since $\int_\Omega \frac{|u_n|^{p_t-1}u_n u}{|y|^t} \rightarrow \int_\Omega \frac{|u|^{p_t-1}u}{|y|^t}$, the last term can be simplified using Brezis–Lieb lemma to

$$\int \left[ (|u_n|^{p_t+1} - |u_n|^{p_t-1}u_n u) \frac{1}{|y|^t} \right] + o(1) = \int \frac{|v_n|^{p_t+1}}{|y|^t} + o(1).$$

Hence

$$\int |\nabla v_n|^2 - \lambda \int \frac{v_n^2}{|y|^2} - \int \frac{|v_n|^{p_t+1}}{|y|^t} = o(1). \quad (3.12)$$

Simplifying $E_\lambda(v_n)$ using (3.12), the estimate on $E_\lambda(v_n)$ simplifies to

$$\frac{2 - t}{2(N - t)} \int \frac{|v_n|^{p_t+1}}{|y|^t} \leq C < \frac{2 - t}{2(N - t)} (S^\lambda_t)^{\frac{N - t}{2}}.$$ 

Applying (1.3) on $v_n$ and using the above estimate we get

$$\int \frac{|v_n|^{p_t+1}}{|y|^t} \leq \delta \int \left( |\nabla v_n|^2 - \lambda \frac{v_n^2}{|y|^2} \right), \quad \text{where } 0 < \delta < 1. \quad (3.13)$$

Substituting (3.13) in (3.12) we get

$$(1 - \delta) \int \left( |\nabla v_n|^2 - \lambda \frac{v_n^2}{|y|^2} \right) = o(1).$$

This completes the proof of Step 1. \qed
Step 2. If \( \{u_n\} \) is any Palais–Smale sequence for \( E_\beta \) at level \( \beta \) then \( \beta \geq 0 \).

Proof. Since \( u_n \) is a PS sequence \( u_n \) is bounded and hence \( \langle E_\beta'(u_n), u_n \rangle = o(1) \). This implies

\[
\int_\Omega |\nabla u_n|^2 - \lambda \int_\Omega \frac{u_n^2}{y^2} - \int_\Omega \frac{|u_n|^{p+1}}{|y|} \leq o(1).
\]

Substituting back in \( E_\beta \) we get \( E_\beta(u_n) = (\frac{1}{2} - \frac{1}{p+1}) \int_\Omega \frac{|u_n|^{p+1}}{|y|} + o(1) \). This proves Step 2. \( \Box \)

Step 3. Let \( u_n \) be a PS sequence converging weakly to \( 0 \), then up to a subsequence either \( u_n \to 0 \) in \( H_0^1(\Omega) \) or there exists a PS sequence \( \tilde{u}_n \) of \( E_\beta \) such that \( E_\beta(u_n) = E_\beta(\tilde{u}_n) + E_\beta(u_n - \tilde{u}_n) + o(1) \), \( u_n - \tilde{u}_n \) is again a PS sequence for \( E_\beta \) and \( \tilde{u}_n \) is of the form (3.10) or (3.11). If \( t > 0 \), then \( \tilde{u}_n \) must be of the form (3.10).

Proof. In view of Step 1, we may assume \( \liminf_{n \to \infty} E_\beta(u_n) \geq \frac{2t}{2(N-\lambda)} (S_\lambda^{\frac{N-\lambda}{2}})^{\frac{N-\lambda}{2}} \), and this implies up to a subsequence

\[
\lim_{n \to \infty} \int_\Omega \frac{|u_n|^{p+1}}{|y|} \geq (S_\lambda^{\frac{N-\lambda}{2}})^{\frac{N-\lambda}{2}}.
\]

Let \( Q_n(r) \) denote the concentration function

\[
Q_n(r) = \sup_{x = (0, z) \in \Omega} \int_{B_r(x)} \frac{|u_n|^{p+1}}{|y|}.
\]

Now we can choose \( x_n = (0, z_n) \in \Omega \) and \( R_n > 0 \) such that

\[
Q_n(R_n) = \int_{B_{R_n}(x_n)} \frac{|u_n|^{p+1}}{|y|} = \delta
\]

where \( \delta \) is chosen such that \( \delta^{\frac{2-N}{2}} < S_\lambda^{\frac{N-\lambda}{2}} \). Define

\[
v_n(x) = R_n^{\frac{N-2}{2}} u_n(R_n x + x_n), \quad x \in \Omega_n,
\]

where \( \Omega_n = \{x: R_n x + x_n \in \Omega\} \) and extend it to all of \( \mathbb{R}^N \) by putting 0 outside \( \Omega_n \). Then \( v_n \in D^{1,2}(\mathbb{R}^N) \) with support \( v_n \subset \Omega_n \) and satisfies

\[
\sup_{x \in \Omega_n} \int_{B(x, 1)} \frac{|v_n|^{p+1}}{|y|} = \int_{B(0, 1)} \frac{|v_n|^{p+1}}{|y|} = \delta.
\]

Since \( \|v_n\|_{D^{1,2}(\mathbb{R}^N)} = \|u_n\|_{D^{1,2}(\Omega)} \leq c < \infty \), up to a subsequence we may assume \( v_n \to v_0 \) in \( D^{1,2}(\mathbb{R}^N) \).

Now we consider two cases:

**Case 1.** \( v_0 \neq 0 \).

First note that since \( \Omega \) is a bounded domain, the sequence \( R_n \) is bounded. If \( \liminf R_n > 0 \), then this will contradict the fact that \( u_n \to 0 \). Hence \( R_n \to 0 \). Moreover we claim:
Claim. \( \frac{1}{\kappa_n} \text{dist}(x_n, \partial \Omega) \to \infty. \)

**Proof of Claim.** Suppose \( \frac{1}{\kappa_n} \text{dist}(x_n, \partial \Omega) \to C < \infty. \) Then \( \Omega \) exhausts a half-space \( \Omega_\infty. \) Since the points \( x_n \) are of the form \( x_n = (0, z_n) \) we get \( \partial \Omega_\infty \) is orthogonal to the singular set. If \( \varphi \in C_0^\infty(\Omega_\infty) \) then \( \varphi \in C_0^\infty(\Omega_n) \) for large \( n. \) Therefore

\[
\int_{\Omega_n} \left( \nabla v_n \cdot \nabla \varphi - \lambda \frac{v_n \varphi}{|y|^2} \right) = \int_{\Omega_n} \frac{|v_n|^{p-1}v_n \varphi}{|y|^2} + o(1).
\]

Taking the limit as \( n \to \infty \) we find that \( v_0 \) is a nontrivial solution of (4.19) in \( \Omega_\infty \) which contradicts Theorem 4.1. This proves the claim. \( \square \)

The above claim implies \( \Omega_\infty = \mathbb{R}^N \) and as above \( v_0 \) solves (3.9).

Let \( \varphi \in C_0^\infty(\mathbb{R}^N) \) such that \( 0 \leq \varphi \leq 1, \varphi \equiv 1 \) in \( B_1(0), \) supp \( \varphi \subseteq B_2(0). \) Define

\[
\bar{u}_n(x) = \frac{\bar{R}_n}{R} v_0 \left( \frac{x - x_0}{\bar{R}_n} \right) \varphi \left( \bar{R}_n(x - x_0) \right)
\]

(3.15)

where \( \bar{R}_n \) is chosen s.t. \( \bar{R}_n = R_\bar{R}_n \to 0 \) and \( \bar{R}_n \text{dist}(x_0, \partial \Omega) \to \infty \) as \( n \to \infty. \) Clearly \( \bar{u}_n \) is a PS sequence of the form (3.10). Next we show the splitting of energy. First note that using a standard application of Brezis–Lieb lemma, \( E_\lambda(u_n) \) can be written as

\[
\frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla v_n|^2 - \frac{v_n^2}{|y|^2} - \frac{1}{p_t + 1} |v_n|^{p_t+1} \right] = \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla v_0|^2 - \frac{v_0^2}{|y|^2} - \frac{1}{p_t + 1} |v_0|^{p_t+1} \right] + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla (v_n - v_0)|^2 \frac{v_0 - v_n}{|y|^2}^2 + o(1).
\]

Now note that if \( \varphi_n(x) = \varphi(\bar{R}_n x), \) then

\[
\int_{\mathbb{R}^N} |\nabla (v_0 \varphi_n - v_0)|^2 \leq C \int_{\mathbb{R}^N} |\nabla v_0|^2 (\varphi_n - 1)^2 + C \int_{\mathbb{R}^N} |v_0|^2 |\nabla \varphi_n|^2 \leq C \int_{|x| > \frac{1}{\kappa_n}} |v_0|^2 + C \left( \frac{1}{\kappa_n} \leq |x| \leq \frac{2}{\kappa_n} \right)^{2/\nu}.
\]

Hence \( v_0 \varphi_n \to v_0 \) in \( D^{1,2}(\mathbb{R}^N) \) and hence

\[
E_\lambda(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} \left[ |\nabla (\varphi_n v_0)|^2 - (\varphi_n v_0)^2 \right] - \frac{1}{p_t + 1} \int_{\mathbb{R}^N} |\varphi_n v_0|^{p_t+1} |y|^2 + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla (v_n - \varphi_n v_0)|^2 - (v_n - \varphi_n v_0)^2 \frac{v_0 - v_n}{|y|^2}^2 + o(1).
\]

A change of variable will convert the last line into \( E_\lambda(\bar{u}_n) + E_\lambda(u_n - \bar{u}_n) + o(1). \)

Using similar type of arguments we can show \( E_\lambda(u_n - \bar{u}_n) = o(1) \) in \( H^{-1}(\Omega) \).
Case 2. \( v_0 = 0 \).

Let \( \varphi \in C_0^\infty(B((0, z), 1)) \) with \( 0 \leq \varphi \leq 1 \) then \( \psi_n := [\varphi((x - x_n)/R_n)]^2 u_n \) is a bounded sequence in \( H_0^1(\Omega) \) and hence \( (E_{\varphi}(u_n), \psi_n) = o(1) \). Using a change of variable this re-writes as

\[
\int_{\mathbb{R}^N} \left( \nabla v_n \cdot \nabla (\varphi v_n^2) - \lambda \frac{(\varphi v_n^2)^2}{|y|^2} \right) = \int_{\mathbb{R}^N} \frac{|v_n|^p - 1(\varphi v_n^2)^2}{|y|^2} + o(1).
\]

Now the LHS can be simplified as \( \int_{\mathbb{R}^N} (|\nabla (\varphi v_n)|^2 - \lambda \frac{(\varphi v_n^2)^2}{|y|^2}) + o(1) \), while the RHS can be simplified as

\[
\int_{\mathbb{R}^N} \frac{|v_n|^p - 1(\varphi v_n^2)^2}{|y|^2} \leq \left( \int_{B(0, 1)} \frac{|v_n|^p + 1}{|y|^2} \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^N} \frac{|v_n|^{p+1}}{|y|^2} \right)^{\frac{2}{p+1}} \leq \frac{\delta}{S_k^2} \int_{\mathbb{R}^N} \left( |\nabla (\varphi v_n)|^2 - \lambda \frac{(\varphi v_n)^2}{|y|^2} \right).
\]

Substituting back, by the choice of \( \delta \) we get

\[
\int_{\mathbb{R}^N} \left( |\nabla (\varphi v_n)|^2 - \lambda \frac{(\varphi v_n)^2}{|y|^2} \right) = o(1).
\]  
(3.16)

This together with (1.3) gives \( \int_{\mathbb{R}^N} \frac{|v_n|^p - 1}{|y|^2} = o(1) \) and hence \( \int_K \frac{|v_n|^p}{|y|^2} = o(1) \) for any compact set \( K \subset \{(y, z): |y| < 1\} \).

If \( t > 0 \), this implies \( \int_K \frac{|v_n|^p}{|y|^2} = o(1) \) for any compact set \( K \subset \mathbb{R}^N \), which contradicts (3.14). Therefore when \( t > 0 \), Case 2 cannot happen and we are through. So we assume now onwards \( t = 0 \).

The condition (3.14) together with the concentration compactness principle gives

\[
|v_n|^{2^*} dx |_{|x|<1} \rightarrow \sum_j C_j \delta_{x_j}
\]

where \( |x_j| = 1 \). Since \( \int_K |v_n|^{2^*} = o(1) \) for any compact set \( K \subset \{(y, z): |y| < 1\} \), we get \( x_j = (y_j, 0) \) with \( |y_j| = 1 \). Let \( C = \max(C_j) \).

Now define

\[
\tilde{Q}_n(r) = \sup_{x \in \mathbb{R}^N} \int_{B(x, r)} |v_n|^{2^*} dx.
\]

Now for large \( r \), \( \tilde{Q}_n(r) > \frac{C}{2} \) and \( \forall r > 0 \) we have \( \liminf_{n \rightarrow \infty} \tilde{Q}_n(r) > \frac{C}{2} \). Therefore we can find a sequence \( q_n \in \mathbb{R}^N \) and \( s_n > 0 \) s.t. \( s_n \rightarrow 0 \) and \( q_n = (y_n, z_n) \) s.t. \( |y_n| > \frac{1}{2} \) and

\[
\frac{C}{2} = \sup_{q \in \mathbb{R}^N} \int_{B(q, s_n)} |v_n|^{2^*} dx = \int_{B(q, s_n)} |v_n|^{2^*} dx \quad (3.17)
\]

Now define

\[
z_n(x) = s_n^{-\frac{N-2}{2}} v_n(s_n x + q_n) = (s_n R_n)^{-\frac{N-2}{2}} u_n(s_n R_n x + R_n q_n + (0, z_n)).
\]
Therefore up to a subsequence we can assume that $\exists z \in D^{1,2}(\mathbb{R}^N)$ s.t. $z_n \rightarrow z$ in $D^{1,2}(\mathbb{R}^N)$ and $z_n(x) \rightarrow z(x)$ a.e.

First note that $z \neq 0$, otherwise choosing $\varphi \in C^\infty_0(B(x, 1))$ with $0 \leq \varphi \leq 1$ for an arbitrary but fixed $x \in \mathbb{R}^N$ and proceeding exactly like in obtaining (3.16) we can show that $z_n \rightarrow 0$ in $L^2_{loc}(\mathbb{R}^N)$ which contradicts (3.17).

Also observe that $\tilde{z}_n = \tilde{u}_n R_n$ and $R_n \tilde{q}_n + (0, z_n) = \tilde{x}_n = (\tilde{y}_n, \tilde{z}_n)$, with $\tilde{R}_n = o(1)$ and

$$\frac{\tilde{R}_n}{|\tilde{y}_n|} = \frac{s_n R_n}{|\tilde{y}_n R_n|} = \frac{s_n}{|\tilde{y}_n|} < 2s_n = o(1).$$

Support of $z_n \subset \tilde{\Omega}_n := \{x: \tilde{x}_n + \tilde{R}_n x \in \Omega\}$ and $\tilde{\Omega}_n$ exhausts $\tilde{\Omega}_\infty$ which is either a half-space or $\mathbb{R}^N$ depending on $\lim \tilde{R}_n \text{dist}(\tilde{x}_n, \partial \tilde{\Omega}_n)$ is finite or infinite. Next we show that $z$ satisfies (3.9) with $\lambda = t = 0$ in $\tilde{\Omega}_\infty$. We have

$$\int_\Omega \nabla u_n \cdot \nabla \varphi - \lambda \int_\Omega \frac{u_n \varphi}{|y|^2} - \int_\Omega |u_n|^{2^* - 2} u_n \varphi = o(\|\varphi\|), \quad \forall \varphi \in H^1_0(\Omega).$$

Let $\varphi \in C^\infty_0(\tilde{\Omega}_\infty)$. Choose $\varphi = \varphi_n$ as $\varphi_n(x) = \tilde{R}_n^{\frac{2^*}{2}} \varphi((x - \tilde{x}_n)/\tilde{R}_n)$ in the above relation, then a change of variable together with $\|\varphi_n\| = \|\varphi\|$ will give

$$\int_{\mathbb{R}^N} \nabla z_n \cdot \nabla \psi - \lambda \int_{\mathbb{R}^N} \frac{z_n \psi}{|y + \frac{x_n}{R_n}|^2} = \int_{\mathbb{R}^N} |z_n|^{2^* - 2} z_n(x) \psi + o(\|\varphi\|).$$

Taking the limit as $n \rightarrow \infty$ using $\frac{\tilde{y}_n}{\tilde{R}_n} \rightarrow \infty$ we get

$$\int_{\mathbb{R}^N} \nabla z \nabla \psi = \int_{\mathbb{R}^N} |z|^{2^* - 2} z(x) \psi.$$

However we know from the well-known Pohozaev nonexistence result that this is not possible when $\tilde{\Omega}_\infty$ is a half-space. Therefore we get $\lim \tilde{R}_n \text{dist}(\tilde{x}_n, \partial \tilde{\Omega}_n) = \infty$. Now define

$$\tilde{u}_n(x) = \tilde{R}_n^{\frac{2^*}{2}} z \left(\frac{(x - x_n)}{\tilde{R}_n}\right) \varphi\left(\tilde{R}_n(x - x_n)\right)$$

where $\varphi$ is as in (3.15), $\tilde{R}_n$ is chosen s.t. $\tilde{R}_n \tilde{R}_n \rightarrow 0$ and $\tilde{R}_n \text{dist}(x_n, \partial \Omega) \rightarrow \infty$ as $n \rightarrow \infty$. Proceeding exactly as in the case of (3.15), we see that $\tilde{u}_n, u_n - \tilde{u}_n$ are PS sequences and $E_\lambda(u_n) = E_\lambda(\tilde{u}_n) + E_\lambda(u_n - \tilde{u}_n) + o(1)$.

**Step 4.** In this final step we prove the theorem. If $\beta < \frac{2 - t}{2(N - t)}(\mathcal{S}_t^N)^{\frac{N-t}{2-N}}$, then we are done. Otherwise, since $u_n$ is a PS sequence at level $\beta$, $u_n$ is bounded in $H^1_0(\Omega)$ and hence we may assume $u_n$ converges weakly to $u$ in $H^1_0(\Omega)$. Using standard arguments one can show that $u_n - u$ is a PS sequence converging weakly to zero and $\beta = E_\lambda(u) + E_\lambda(u_n - u) + o(1)$. Now either $u_n - u$ is a PS sequence which falls in the case of Step 1, or we can find a $\tilde{u}_n$ as in Step 3. Note that $E_\lambda(\tilde{u}_n)$ converges either to $E_\lambda(V)$ or $E_0(W)$ where $V, W$ are as in (3.10) and (3.11) and $E_\lambda(V), E_0(W) \geq C > 0$. Therefore in finitely many steps we obtain a PS sequence which falls into Step 1. This proves the theorem.
4. Existence and nonexistence

In this section we study the existence and nonexistence of solutions for (1.1) for various domains. We will show that the problem does not have a solution in star shaped domains with boundary orthogonal to the singular set. Then we will show that “a topological hole” in \( \{ x \in \Omega: x = (0, z) \} \) will give existence results for (1.1).

First we will present the nonexistence result whose proof is based on the Pohozaev identity. The difficulty in applying this identity is because of the blowing-up nature of the solution and we overcome this by using the partial \( H^2 \) regularity established in Section 2.

**Theorem 4.1.** Let \( \Omega \subset \mathbb{R}^N \) be an open set with smooth boundary and is star shaped with respect to some point \((0, z_0)\). Suppose in addition \( \partial \Omega \) is orthogonal to the singular set, then the problem

\[
-\Delta u - \lambda \frac{|u|^{p_1-1}u}{|y|^2} = |u|^{p_t-1}u \quad \text{in} \ \Omega, \quad u \in D^{1,2}(\Omega) \tag{4.19}
\]

has a nontrivial solution only if \( \Omega = \mathbb{R}^N \).

**Proof.** We will prove the theorem using the Pohozaev identity. To make the test function smooth we introduce cut-off functions and pass to the limit with the help of the regularity results proved in Section 2. We will assume without loss of generality that \( \Omega \) is star shaped with respect to the origin.

For \( \epsilon > 0 \) and \( R > 0 \), define \( \varphi_{\epsilon,R}(x) = \varphi_{\epsilon}(x) \psi_{R}(x) \) where \( \varphi_{\epsilon}(x) = \varphi(|x|/\epsilon), \ \psi_{R}(x) = \psi(|x|/R) \). \( \varphi \) and \( \psi \) are smooth functions in \( \mathbb{R} \) with the properties \( 0 \leq \varphi, \psi \leq 1 \), with supports of \( \varphi \) and \( \psi \) in \((1, \infty)\) and \((-\infty, 2)\) respectively and \( \varphi(t) = 1 \) for \( t \geq 2 \), and \( \psi(t) = 1 \) for \( t \leq 1 \).

Assume that (4.19) has a nontrivial solution \( u \), then \( u \) is smooth away from the singular set and hence \((x \cdot \nabla)\varphi_{\epsilon,R} \in C_c(\overline{\Omega}) \). Multiplying Eq. (4.19) by this test function and integrating by parts, we get

\[
\int_{\Omega} \nabla u \cdot \nabla ((x \cdot \nabla u)\varphi_{\epsilon,R}) - \lambda \int_{\Omega} \frac{u(x \cdot \nabla u)\varphi_{\epsilon,R}}{|y|^2} - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} (x \cdot \nabla u)\varphi_{\epsilon,R} = \int_{\Omega} \frac{|u|^{p_1-1}u}{|y|^2} (x \cdot \nabla u)\varphi_{\epsilon,R}. \tag{4.20}
\]

Now the RHS of (4.20) can be simplified as

\[
\int_{\Omega} \frac{|u|^{p_1-1}u}{|y|^2} (x \cdot \nabla u)\varphi_{\epsilon,R} = \frac{1}{p_t + 1} \int_{\Omega} (\nabla |u|^{p_t} \cdot x) \varphi_{\epsilon,R} = -\frac{n}{2} \int_{\Omega} \frac{|u|^{p_t+1}}{|y|^2} \varphi_{\epsilon,R} - \frac{1}{p_t + 1} \int_{\Omega} \frac{|u|^{p_1+1}}{|y|^2} \left[ x \cdot (\psi_{R} \nabla \varphi_{\epsilon} + \varphi_{\epsilon} \nabla \psi_{R}) \right].
\]

Note that \( |x \cdot (\psi_{R} \nabla \varphi_{\epsilon} + \varphi_{\epsilon} \nabla \psi_{R})| \leq C \) and hence using the dominated convergence theorem we get

\[
\lim_{R \to \infty} \left[ \lim_{\epsilon \to 0} R H S \right] = -\frac{n}{2} \int_{\Omega} \frac{|u|^{p_t+1}}{|y|^2}. \tag{4.21}
\]

By direct calculation and integration by parts, LHS of (4.20) simplifies as
\[ \text{LHS} = \int_\Omega |\nabla u|^2 \psi_{e,R} + \sum_{i=1}^n \sum_{j=1}^n \frac{1}{2} \int_\Omega (u_{x_i} x_j \psi_{e,R}) + \int_\Omega (x \cdot \nabla u)(\nabla u \cdot \nabla \psi_{e,R}) + \frac{\lambda}{2} \sum_{i=1}^n \int_\Omega u^2 \frac{\psi_{e,R}}{|y|^2} + \frac{1}{2} \sum_{i=1}^k \int_\Omega u^2 \frac{\psi_{e,R}}{|y|^4} - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} (x \cdot \nabla u) \psi_{e,R} \]

\[ = - \frac{n-2}{2} \left[ \int_\Omega |\nabla u|^2 \psi_{e,R} - \lambda \int_\Omega \frac{u^2}{|y|^2} \psi_{e,R} \right] - \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot v) \psi_{e,R} \]

\[ - \frac{1}{2} \left[ \int_\Omega |\nabla u|^2 - \lambda \int_\Omega \frac{u^2}{|y|^2} \right] [(x \cdot \nabla \psi_e) \psi_R + (x \cdot \nabla \psi_R) \psi_e] \]

\[ + \int_\Omega (x \cdot \nabla u) [(\nabla u \cdot \nabla \psi_e) \psi_R + (\nabla u \cdot \nabla \psi_R) \psi_e]. \]

In the last step we used the fact \( x \cdot \nabla u \) = \( x \cdot v \frac{\partial u}{\partial \nu} \) on \( \partial \Omega \) since \( u = 0 \) on \( \partial \Omega \).

Now

\[ \lim_{R \to \infty} \lim_{\epsilon \to 0} \int_\Omega (x \cdot \nabla u)(\nabla u \cdot \nabla \psi_R) \psi_e \leq C \lim_{R \to \infty} \int_{R \leq |x| \leq 2R} |\nabla u|^2 = 0 \]

and

\[ \int_\Omega (x \cdot \nabla u)(\nabla u \cdot \nabla \psi_e) \psi_R \leq \int_\Omega \left( y \cdot \nabla y u + z \cdot \nabla z u \right)(\nabla u \cdot \nabla \psi_e) \psi_R \]

\[ \leq C \int_{\Omega \cap \{ \epsilon \leq |y| \leq 2\epsilon \}} |\nabla u|^2 + R \int_{\Omega \cap \{ \epsilon \leq |y| \leq 2\epsilon \}} |\nabla u| \frac{|\nabla z u|}{\epsilon} \]

\[ \leq (C + R) \int_{\Omega \cap \{ \epsilon \leq |y| \leq 2\epsilon \}} |\nabla u|^2 + R \int_{\Omega \cap \{ \epsilon \leq |y| \leq 2\epsilon \}} \frac{|\nabla z u|^2}{|y|^2}. \]

Clearly the first term goes to 0 as \( \epsilon \to 0 \). The second term tends to 0 as \( \epsilon \to 0 \) using Hardy–Sobolev–Maz’ya inequality as \( u_{x_i} \in H^1(\Omega) \).

Using the above estimates and taking the limit using dominated convergence theorem using the fact \( |x \cdot (\psi_R \nabla \psi_e + \psi_e \nabla \psi_R)| \leq C \), we get

\[ \lim_{R \to \infty} \lim_{\epsilon \to 0} \text{RHS} = - \frac{n-2}{2} \left[ \int_\Omega |\nabla u|^2 - \lambda \int_\Omega \frac{u^2}{|y|^2} \right] - \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot v) \psi_{e,R}. \tag{4.22} \]

Substituting (4.21) and (4.22) in (4.20), and using Eq. (4.19), we get

\[ \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} \right)^2 (x \cdot v) = 0, \]

which implies \( u = 0 \) in \( \Omega \) by the principle of unique continuation. This proves the theorem. \( \square \)

Next we will prove an existence result in the spirit of [7]. We see that an “annulus type topology” of \( \Omega \cap \{ y = 0 \} \) will give rise to a solution of (1.1).
Theorem 4.2. Let $\Omega$ be a bounded domain with smooth boundary in $\mathbb{R}^N$, $N > 3$, and suppose $\Omega$ satisfies 
\[ \{ x: R_1 < |x| < R_2 \} \subset \Omega \text{ and } \{(0,z): |z| < R_1 \} \subset \overline{\Omega} \text{ for some } R_1, R_2 > 0, \] 
then (1.1) has a solution if $\frac{R_2}{R_1}$ is sufficiently large. When $N = 3$ the same conclusion holds provided $\lambda = 0$ and $t = 1$.

Remark 2. The extra assumption on $\lambda$ and $t$ in dimension $N = 3$ comes from the fact that the uniqueness of (1.1) with $\Omega = \mathbb{R}^N$ is known only in this case.

Proof of Theorem 4.2. Let $M = \{ u \in H^1_0(\Omega): \int_{\Omega} |u|^{p+1} \leq 1 \}$ and $I_\lambda$ be the functional,
\[ I_\lambda(u) = \int_{\Omega} \left[ |\nabla u|^2 - \lambda \frac{u^2}{|y|^2} \right], \quad u \in M, \]
then the critical points of $I_\lambda$ give rise to solutions of (1.1), except positivity. Since the equation under study is invariant under dilation we may assume $R_1 = \frac{1}{4R} < 1 < 4R = R_2$. We will prove the theorem by showing that if $I_\lambda$ does not have a critical point, then the unit sphere $S^{N-k-1}$ in $\mathbb{R}^{N-k}$ is contractible in $\{ x \in \Omega: x = (0, z) \}$, which is a contradiction.

We know from [5,8] and [10] that problem (1.1) has a unique solution up to dilation and translation in the $z$ variable when $\Omega = \mathbb{R}^N$, $N > 3$, or when $N = 3$, and $\lambda \leq \frac{2(p+1)}{(p+3)^2}$. Let $u_\lambda$ be a solution of (1.1) when $\Omega = \mathbb{R}^N$.

Define for $\sigma \in S^{N-k-1}$ and $s \in [0, 1)$
\[ u_\sigma^s = \frac{1}{(1-s)^{\frac{N-k}{2}}} u_\lambda \left( \frac{y-z-\sigma}{1-s} \right). \]

Then, $u_\sigma^s \in D^{1,2}(\mathbb{R}^N)$ and $S_\sigma$ is attained for $u_\sigma^s$ as well. Note that $u_\sigma^s$ concentrates at $(0, \sigma)$ as $s \to 1$ and $u_\sigma^s \to u_\lambda$ as $s \to 0$.

Take $\varphi \in C_0^\infty((\frac{1}{4}, 4))$ such that $\varphi = 1$ in $(\frac{1}{2}, 2)$. Define for $R > 1$,
\[ \varphi_R(x) = \begin{cases} 
\varphi(R|x|) & \text{if } 0 < |x| < \frac{1}{R}, \\
1 & \text{if } \frac{1}{R} \leq |x| < R, \\
\varphi\left(\frac{|x|}{R}\right) & \text{if } |x| \geq R.
\end{cases} \]

Define $w_\sigma^s = u_\sigma^s \cdot \varphi_R$ and $w_\lambda = u_\lambda \cdot \varphi_R$, then $w_\sigma^s, w_\lambda \in H^1_0(\Omega)$ and
\[ \int_{\mathbb{R}^N} |\nabla (w_\sigma^s - u_\sigma^s)|^2 \leq C \int_{(B_{2R})^c \cup B_{\frac{1}{4}}} |\nabla u_\sigma^s|^2 + \frac{C}{R^2} \int_{B_{4R} - B_{2R}} \left( u_\sigma^s \right)^2 + cR^2 \int_{B_{4R} - B_{2R}} \int_{\frac{1}{4\pi}} (u_\sigma^s)^2 \]

where $B_R$ denotes the open ball with center 0 and radius $R$. Now 1st term goes to zero as $R \to \infty$ uniformly in $s \in S^{N-k-1}, s \in [0, 1)$,

2nd term $\leq \frac{C}{R^2} \left( \int_{B_{4R} - B_{2R}} (u_\sigma^s)^2 \right)^{\frac{1}{2}} |B_{4R} - B_{2R}|^{-\frac{1}{2}} = C \left( \int_{B_{4R} - B_{2R}} (u_\sigma^s)^2 \right)^{\frac{1}{2}} \to 0$

as $R \to \infty$ uniformly in $\sigma \in S^{N-k-1}, s \in [0, 1)$. Similarly we get 3rd term goes to zero uniformly in $\sigma$ and $s$. Now define
\[ v_\sigma^s = C_\sigma w_\sigma^s \text{ and } v_\lambda = C_\lambda w_\lambda \]
where \( C_i^\alpha, C_\lambda \) are positive constants such that \( v_i^\alpha \), \( v_\lambda \in M \), then \( I_\lambda(v_i^\alpha) \to S_\lambda^\alpha \) uniformly in \( \sigma \) and \( s \). Choose \( R > 0 \) such that \( I_\lambda(v_i^\alpha) < S_1 < S^* \) uniformly for all \( \sigma \) and \( s \) where \( S^* = 2^{\frac{3-\alpha}{\alpha}} S_t^\lambda \) if \( t > 0 \) and \( S^* = \min \{ S_0^\alpha, 2^{\frac{3-\alpha}{\alpha}} S_0^\lambda \} \) if \( t = 0 \).

Next we claim:

**Claim.** There exists a \( u \in M \) with \( I_\lambda(u) < S^* \) and \( I'_\lambda(u) = 0 \).

**Proof of Claim.** Assume the claim is not true. Then using standard arguments (see for instance [15]) one can easily see from Theorem 3.1 and the uniqueness of the problem in \( \mathbb{R}^N \) under the hypothesis of the theorem, that \( I_\lambda \) does not have any PS sequence at level \( \beta \in (S_1^\lambda, S^*) \). Therefore by the standard deformation lemma (see for instance [15]) for any \( \beta \) in this range \( \exists \varepsilon > 0 \) and a flow \( \varphi : M \times [0, 1] \to M \) s.t.

\[
\varphi(M_{\beta+\varepsilon}, 1) \subset M_{\beta-\varepsilon}
\]

where \( M_{\beta} = \{ u \in M : I_\lambda(u) < \beta \} \).

Now given \( \delta > 0 \) we can cover the interval \([S_1^\alpha + \delta, S_1]\) by finite number of \( \varepsilon \) (given by the flow) neighborhoods. Now composing the corresponding deformations we obtain a flow \( \Phi : M \times [0, 1] \to M \) s.t.

\[
\Phi(S_1, 1) \subset M_{S_1^\alpha + \delta}.
\]

We may also assume that \( \Phi(u, t) = u \) for all \( u \in M_{\beta} \) with \( \beta = S_1^\alpha + \frac{\delta}{2} \).

For \( u \in M \) define the center of mass of \( u \) denoted \( F(u) \) by

\[
F(u) = \int_{\Omega} x \frac{|u|^{p_1+1}}{|y|^t} \, dx.
\]

By a standard use of Ekeland variational principle we know that if \( u_n \in M \) and \( I_\lambda(u_n) \to S_\lambda^\alpha \) then \( u_n = \tilde{u}_n + o(1) \) where \( \tilde{u}_n \) is a PS sequence and \( o(1) \to 0 \) in \( H_0^1(\Omega) \). Combining this with Theorem 3.1 we get, for any given neighborhood \( U \) of \( \Omega_0 := \{0, z \in \Omega \} \) \( \exists \delta > 0 \) such that \( F(M_{S_1^\alpha + \delta}) \subseteq U \). Since \( \Omega_0 \) is smooth we can choose a neighborhood \( U \) of \( \Omega_0 \) such that any point \( p \in U \) has a unique nearest neighbor \( q = \pi(p) \in \Omega_0 \) such that \( \pi \) is continuous. Choose \( \delta \) sufficiently small and define

\[
h : S^{N-k-1} \times [0, 1] \to \Omega \quad \text{by}
\]

\[
h(\sigma, s) = \pi \left( F \left( \Phi(v_\lambda^\alpha, 1) \right) \right).
\]

Since \( \delta \) is sufficiently small \( h \) is well defined and continuous and satisfies

\[
h(\sigma, 1) = \sigma, \quad \forall \sigma \in S^{N-k-1},
\]

and

\[
h(\sigma, 0) = \pi \left( F \left( \Phi(v_\lambda, 1) \right) \right) = x_0, \quad \forall \sigma \in S^{N-k-1},
\]

for some \( x_0 \). Hence \( h \) is a contraction of \( S^{N-k-1} \) in \( \Omega_0 \) which contradicts our assumption and hence proves the claim. \( \square \)

It remains to show that either \( u^+ = 0 \) or \( u^- = 0 \). The strict positivity of the solution follows from the strong maximum principle.
Suppose $u^+ \neq 0$ and $u^- \neq 0$. Since $I'_\lambda(u) = 0$, $v = [\int_\Omega |\nabla u|^2 - \lambda \frac{|u|^2}{|y|^2}] \frac{1}{p+1} u$ solves Eq. (1.1). Hence

$$\int_\Omega \left[ |\nabla v^\pm|^2 - \lambda \frac{|v^\pm|^2}{|y|^2} \right] = \int_\Omega \frac{|v^\pm|^{p+1}}{|y|^{p-1}}.$$ 

Hence from (1.3) we get

$$\int_\Omega \left[ |\nabla v^\pm|^2 - \lambda \frac{|v^\pm|^2}{|y|^2} \right] \geq \left[ S^*_t \right]^{\frac{N-1}{2}}$$

and hence

$$\int_\Omega \left[ |\nabla v|^2 - \lambda \frac{v^2}{|y|^2} \right] \geq 2 \left[ S^*_t \right]^{\frac{N-1}{2}}.$$ 

Writing it in terms of $u$ gives $I'_\lambda(u) \geq 2^{\frac{2-q}{2}} S^*_t$ which contradicts the fact that $I'_\lambda(u) < S^*$ and hence either $u^+ = 0$ or $u^- = 0$. $\Box$

References